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Blow-up rates and uniqueness of entire large solutions to a semilinear elliptic equation with nonlinear convection term

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Abstract

In this paper, we analyze the blow-up rates and uniqueness of entire large solutions to the equation $\Delta u = a(x)f(u) + \mu b(x)|\nabla u|^q$, $x \in \mathbb{R}^N$ ($N \geq 3$), where $\mu > 0$, $q > 0$ and $a, b \in C_{loc}^\alpha(\mathbb{R}^N)$ ($\alpha \in (0, 1)$). The weight a is nonnegative, b is able to change sign in \mathbb{R}^N , and $f \in C^1[0, \infty)$ is positive and nondecreasing on $(0, \infty)$ and rapidly or regularly varying at infinity. Additionally, we investigate the uniqueness of entire large solutions.

Keywords: Blow-up rates; Entire large solutions; Convection term

1 Introduction and the main results

The purpose of this paper is to study the blow-up rates and uniqueness of entire large solutions to the following elliptic equation:

$$\Delta u = a(x)f(u) + \mu b(x)|\nabla u|^q, \quad x \in \mathbb{R}^N, u(x) > 0, \tag{1.1}$$

where $\mu > 0$, $q \in (0, 2]$ and u is an entire large solution of Eq. (1.1) when it solves Eq. (1.1) and $u(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. The nonlinearity f is such that

- (f₁) $f \in C^1[0, \infty)$, $f(0) = 0$, $f'(t) \geq 0$ and $f(t) > 0$ for $t > 0$ are satisfied; and
- (f₂) the Keller–Osserman condition

$$\int_1^\infty (F(t))^{-1/2} dt < \infty \tag{1.2}$$

holds, and there exists $C_f \in [1, \infty)$ such that

$$\lim_{t \rightarrow \infty} ((F(t))^{1/2})' \int_t^\infty (F(s))^{-1/2} ds = C_f, \quad F(t) = \int_0^t f(s) ds. \tag{1.3}$$

The weight a is such that

- (a₁) $a \in C_{loc}^\alpha(\mathbb{R}^N)$ ($\alpha \in (0, 1)$) is positive in \mathbb{R}^N ; and
- (a₂) there exist $k_1 \in \mathcal{K}$ and positive constant $\lambda_1 \in [2, 2(N - 1)]$ such that

$$0 < a_1 := \liminf_{|x| \rightarrow \infty} \frac{a(x)}{|x|^{-\lambda_1} k_1(|x|)} \leq a_2 := \limsup_{|x| \rightarrow \infty} \frac{a(x)}{|x|^{-\lambda_1} k_1(|x|)} < \infty,$$

where

$$\int_{t_0}^{\infty} \frac{k_1(s)}{s} ds < \infty, \quad \text{if } \lambda_1 = 2.$$

\mathcal{K} denotes the set of Karamata functions k defined on $[t_0, \infty)$ by

$$k(t) := c \exp\left(\int_{t_0}^t \frac{y(s)}{s} ds\right), \quad t \geq t_0 > 0$$

with $c > 0$ and $y \in C([t_0, \infty))$ such that $\lim_{t \rightarrow \infty} y(t) = 0$.

The weight b is such that

- (b₁) $b \in C_{loc}^{\alpha}(\mathbb{R}^N)$ ($\alpha \in (0, 1)$) may change sign in \mathbb{R}^N ; and
- (b₂) there exist $k_2 \in \mathcal{K}$ and positive constant λ_2 such that

$$0 < b_1 := \liminf_{|x| \rightarrow \infty} \frac{|b(x)|}{|x|^{-\lambda_2} k_2(|x|)} \leq b_2 := \limsup_{|x| \rightarrow \infty} \frac{|b(x)|}{|x|^{-\lambda_2} k_2(|x|)} < \infty,$$

where \mathcal{K} is defined in (a₂).

Let Ω be a bounded domain with a C^2 -boundary in \mathbb{R}^N or $\Omega = \mathbb{R}^N$. The existence, nonexistence, uniqueness and asymptotic behavior of large solutions to Eq. (1.1) have been investigated extensively by many authors in different contexts. When Ω is a bounded domain, a “large solution” is understood as a solution u of Eq. (1.1) on Ω such that $\lim_{x \rightarrow \partial\Omega} u(x) = \infty$.

Now, we review these studies and divide them into the following two sets:

Part I ($\mu = 0$). When Ω is a bounded domain with a C^2 -boundary, $b \equiv 1$ in $\Omega \subseteq \mathbb{R}^N$ ($N = 2$) and $f(u) = e^u$; Bieberbach [5] first studied the existence, uniqueness and asymptotic behavior of large solutions to Eq. (1.1). In this case, Eq. (1.1) plays an important role in the theory of Riemannian surfaces of constant negative curvature and in the theory of automorphic functions. Later, Rademacher [40] used the ideas of Bieberbach to prove that the results still hold for $N = 3$. When $f(u) = e^u$ and b is continuous and strictly positive on $\bar{\Omega}$, Lazer and McKenna [30] extended the above results to a bounded domain Ω in \mathbb{R}^N ($N \geq 1$) with a uniform outer sphere. However, Keller [26] and Osserman [39] conducted systematic research on Eq. (1.1) and obtained the following important results:

- (i) If $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $b \equiv 1$ on $\bar{\Omega}$, and f satisfies (f₁), then Eq. (1.1) has a classical large solution if and only if the Keller–Osserman condition (f₂) holds.
- (ii) If $b \equiv 1$ in \mathbb{R}^N and f satisfies (f₁), then Eq. (1.1) has an entire large solution if and only if the following condition holds:

$$\int_1^{\infty} (2F(t))^{-1/2} dt < \infty.$$

When Ω is a bounded domain, $b \equiv 1$ in Ω and $f(u) = u^{\frac{N+2}{N-2}}$ with $N > 2$, Loewner and Nirenberg [32] were motivated by certain geometric problems to prove that Eq. (1.1) has a unique positive classical large solution u satisfying

$$\lim_{d(x) \rightarrow 0} u(x)(d(x))^{\frac{N-2}{2}} = \left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}}.$$

Later, Bandle and Marcus [4] further studied the exact asymptotic behavior and uniqueness of a large solution to Eq. (1.1) when f is convex on $(0, \infty)$, i.e., $s \mapsto f(s)$ is increasing on $(0, \infty)$. However, when b is a large constant, by obtaining good boundary layer estimates for b , Du and Guo [14] proved that the large solution is unique for much weaker assumptions as regards f , i.e., $s \mapsto f(s)$ is increasing on (T, ∞) for some large constant T . For other insights into the existence, uniqueness, multiplicity and asymptotic behavior of large solutions to Eq. (1.1) with $\mu = 0$ in bounded domains, please refer to [8, 15, 17, 24, 29, 31, 33–36, 50, 52–54] and the references therein.

Now, let us return to entire large solutions of Eq. (1.1) with $\mu = 0$. Oleinik [38] and Sattinger [44] studied the nonexistence of entire positive solutions of Eq. (1.1). Later, the existence and asymptotic behavior of entire large solutions was first described by Ni in [37]. For $f(u) = u^\gamma$ with $\gamma > 1$, Cheng and Ni [7] studied the existence and uniqueness of the maximum positive solution and completely classified all possible positive solutions. Later, Lair [27], Cîrstea and Rădulescu [10], and Tao and Zhang [46] further investigated the existence of entire large solutions of Eq. (1.1). In particular, when f exhibits sublinear growth at infinity, Lair [28] proved that if b satisfies

$$\int_0^\infty r(\phi_1(r) - \phi_2(r))\varphi(r) \, dr < \infty,$$

where

$$\varphi(r) = \exp\left(\hbar \int_0^r \tau \phi_2(\tau) \, d\tau\right)$$

with

$$\phi_1(r) := \max_{|x|=r} b(x), \quad \phi_2(r) := \min_{|x|=r} b(x),$$

then Eq. (1.1) has entire large solutions if and only if

$$\int_0^\infty r\phi_2(r) \, dr = \infty.$$

In [16], Dupaigne et al. established the existence of a minimum solution u and noted that if u is an unbounded solution, then it blows up at infinity. In particular, they also showed that the entire large solution of Eq. (1.1) is unique for some appropriate assumptions as regards f and b . Inspired by the ideas of the authors in [1, 17, 18], Wan [47, 48] and Wan et al. [49] used a perturbation method and a truncation technique to study the asymptotic behavior and uniqueness of entire large solutions of Eq. (1.1). In particular, the authors in [49] found some necessary and sufficient conditions for these rapidly and regularly varying functions and proved some sharp uniqueness results for entire large solutions when f belongs to a more general class of functions.

Part II ($\mu \neq 0$). When $f(u) = u^\gamma$ ($\gamma > 1$) or $f(u) = e^u$, $\mu = 1$ and the weights $a, b \equiv 1$ in Ω , Bandle and Giarrusso [3] studied the existence and asymptotic behavior of large solutions to Eq. (1.1). Later, Giarrusso [21] and [22] further extended the above results when f belongs to a more general class of functions. When $\Omega = \mathbb{R}^N$, $f(u) = u^\gamma$ ($\gamma > 1$) and $\mu b = \pm 1$ in \mathbb{R}^N , Lair investigated the existence of entire large solutions of Eq. (1.1). When f satisfies

(\mathbf{f}_1), Ghergu and Rădulescu [19] established the existence and nonexistence of entire large solutions of an one-dimensional logistic problem. When Ω is a smooth bounded domain or $\Omega = \mathbb{R}^N$, Ghergu and Rădulescu [20] studied the existence and nonexistence of large solutions of Eq. (1.1) provided the nonlinearity f satisfies $\sup_{s \geq 1} f(s)/s < \infty$ and the weight $\mu b \equiv -1$ on Ω . In particular, they supplied, for the case $\Omega = \mathbb{R}^N$, a necessary and sufficient condition for the existence of a positive solution that blows up at infinity. In a different direction, Cîrstea and Rădulescu [9, 11–13] first introduced Karamata regular variation theory to study the boundary behavior and uniqueness of large solutions of Eq. (1.1) in a bounded domain. This contributed to knowledge in some new directions and enabled us to obtain a significant amount of information about the qualitative behavior of large solutions in a general framework. Using this approach, Zhang [51] and Huang et al. [25] determined the exact asymptotic behavior of large solutions to Eq. (1.1) with $\mu = \pm 1$ in a bounded domain.

Recently, Rhouma and Drissi [42] studied the existence of positive radial solutions and entire large solutions to a class of p -Laplacian equations. Santos et al. [43] studied the existence of entire large solutions to a class of variable-index Laplacian equations. Alves, Santos and Zhou [2] established a new comparison principle to investigate the existence of entire large solutions to Eq. (1.1) that depend on a nonnegative real parameter. Moreover, inspired by the idea of Mitidieri and Pohozaev [34], they established the nonexistence of entire large solutions by constructing a test function, which is null in the exterior of appropriate balls in \mathbb{R}^N .

Motivated by the work of [2] and [47–49], in this paper, we determine the exact asymptotic behavior of entire large solutions to Eq. (1.1) at infinity in \mathbb{R}^N , and we show that the convection term $\mu b(x)|\nabla u(x)|^q$ does not affect the asymptotic behavior under certain conditions. Moreover, we prove that the entire large solution is unique using its asymptotic behavior.

Our results are summarized as follows.

Theorem 1.1 *Let f satisfy (\mathbf{f}_1) – (\mathbf{f}_2) with $(q - 1)C_f < 1$, weight a satisfy (\mathbf{a}_1) – (\mathbf{a}_2) , weight b satisfy (\mathbf{b}_1) – (\mathbf{b}_2) , and $\lambda_2 > \frac{\lambda_1(2-q)}{2}$. Moreover, we further assume that one of the following conditions holds:*

- (\mathbf{S}_1) $b \leq 0$ in \mathbb{R}^N and $q \in (0, 2]$; and
- (\mathbf{S}_2) b can change sign in \mathbb{R}^N , $q = 1$ and $\min\{\lambda_1, \lambda_2\} \geq 2$. In particular, if $\lambda_2 = 2$, we need to verify that

$$\int_{t_0}^{\infty} \frac{k_2(s)}{s} ds < \infty.$$

Then any entire large solution u of problem (1.1) satisfies

$$\begin{aligned} \xi_2^{1-C_f} &\leq \liminf_{|x| \rightarrow \infty} \frac{u(x)}{\psi\left(\left(\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds\right)^{1/2}\right)} \\ &\leq \limsup_{|x| \rightarrow \infty} \frac{u(x)}{\psi\left(\left(\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds\right)^{1/2}\right)} \leq \xi_1^{1-C_f}, \end{aligned} \tag{1.4}$$

where ψ is uniquely determined by

$$\int_{\psi(t)}^{\infty} (2F(s))^{-1/2} ds = t, \quad t > 0, \tag{1.5}$$

and

$$\xi_i = \left(\frac{4a_i C_f}{(\lambda_1 - 2)(C_f - 1) + 2(N - 2)} \right)^{1/2}, \quad i = 1, 2.$$

In particular, if $C_f = 1$ in (\mathbf{f}_2) , then any entire large solution u of problem (1.1) satisfies

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\psi\left(\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds\right)^{1/2}} = 1. \tag{1.6}$$

Remark 1.1 In Theorem 1.1, (\mathbf{S}_1) and (\mathbf{S}_2) are auxiliary conditions that guarantee the existence of entire large solutions to Eq. (1.1) (please refer to [2]).

Remark 1.2 In Theorem 1.1, $C_f(q - 1) < 1$ implies that if $C_f = 1$, then $q \neq 2$.

Theorem 1.2 Let a satisfy (\mathbf{a}_1) , b satisfy (\mathbf{b}_1) and the following condition:

(\mathbf{b}_3) $b \geq 0$ in \mathbb{R}^N , $q \geq 1$ or $b \leq 0$ in \mathbb{R}^N , $0 < q < 1$.

Additionally, let f satisfy (\mathbf{f}_1) and the following condition:

(\mathbf{f}_3) $t \mapsto \frac{f(t)}{t}$ is nondecreasing on $(0, \infty)$.

Assume that u_1 and u_2 are arbitrary entire large solutions of Eq. (1.1) that satisfy

$$\lim_{|x| \rightarrow \infty} \frac{u_1(x)}{u_2(x)} = 1.$$

Then $u_1 \equiv u_2$ in \mathbb{R}^N .

The outline of this paper is as follows. The proofs of Theorem 1.1 and Theorem 1.2 are given in Sects. 2 and 3. Then we list some bases for Karamata regular variation theory and present some auxiliary results in Appendices 1 and 2.

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Proof Let $\varepsilon \in (0, b_1/2)$ and

$$\tau_1 = (\xi_1^2 - \varepsilon \xi_1^2/b_1)^{1/2}, \quad \tau_2 = (\xi_2^2 + \varepsilon \xi_2^2/b_2)^{1/2}.$$

We see that

$$(1/2)^{1/2} \xi_1 < \tau_1 < \tau_2 < (3/2)^{1/2} \xi_2$$

and

$$\lim_{\varepsilon \rightarrow 0} \tau_1 = \xi_1, \quad \lim_{\varepsilon \rightarrow 0} \tau_2 = \xi_2.$$

For any constant $R > t_0$, we define

$$\Omega_R := \{x \in \mathbb{R}^N : |x| > R\},$$

where t_0 is given by the definition of \mathcal{K} in **(a₂)**.

From **(f₁)–(f₂)** with $(q - 1)C_f < 1$, **(a₁)–(a₂)** and **(b₁)–(b₂)** with $\lambda_2 > \frac{\lambda_1(2-q)}{2}$, Propositions 3–4, and Lemma B.2(iii), we see that there exists a sufficiently large $R_\varepsilon > 0$ and a sufficiently small $\delta_\varepsilon > 0$ corresponding to $\varepsilon > 0$ such that, for $(x, r) \in \Omega_{R_\varepsilon} \times (0, 2\delta_\varepsilon)$,

$$\frac{|x|^{2-\lambda_1}k_1(|x|)}{\int_{|x|}^\infty s^{1-\lambda_1}k_1(s)ds} + 2(N - \lambda_1) + \frac{2|x|k'_1(|x|)}{k_1(|x|)} > 0, \tag{2.1}$$

$$\begin{aligned} & \left| \frac{\tau_i^2}{4} \frac{|x|^{2-\lambda_1}k_1(|x|)}{\int_{|x|}^\infty s^{1-\lambda_1}k_1(s)ds} - \frac{\tau_i^2(\lambda_1 - 2)}{4} \right| + \left| -\frac{\tau_i^2}{4} \left(\frac{|x|^{2-\lambda_1}k_1(|x|)}{\int_{|x|}^\infty s^{1-\lambda_1}k_1(s)ds} + 2(N - \lambda_1) \right) \right. \\ & \left. + \frac{2|x|k'_1(|x|)}{k_1(|x|)} \frac{\psi'(r)}{r\psi''(r)} - \frac{(2(N - 1) - \lambda_1)\tau_i^2}{4C_f} \right| < \varepsilon/3, \end{aligned} \tag{2.2}$$

$$\begin{aligned} & \left| \mu \left(\frac{\tau_i}{2} \right)^q \frac{b(x)}{|x|^{-\lambda_2}k_2(|x|)} \frac{(-\psi'(r))^q \left(\int_{|x|}^\infty s^{1-\lambda_1}k_1(s)ds \right)^{-\frac{q}{2}} |x|^{q(1-\lambda_1)+\lambda_1-\lambda_2}k_1^q(|x|)k_2(|x|)}{\psi''(r)} \frac{k_2(|x|)}{k_1(|x|)} \right| \\ & = \mu \left(\frac{\tau_i}{2} \right)^q \frac{b(x)}{|x|^{-\lambda_2}k_2(|x|)} \frac{k_2(|x|)}{k_1(|x|)} \cdot \frac{(-\psi'(r))}{r\psi''(r)} \cdot r(-\psi'(r))^{q-1} \cdot \left(\int_{|x|}^\infty s^{1-\lambda_1}k_1(s)ds \right)^{-\frac{q}{2}} \\ & \quad \times |x|^{q(1-\lambda_1)+\lambda_1-\lambda_2}k_1^q(|x|) < \varepsilon/3 \end{aligned} \tag{2.3}$$

and

$$|x|^{-\lambda_1}k_1(|x|)(a_1 - \varepsilon/3) < a(x) < |x|^{-\lambda_1}k_1(|x|)(a_2 + \varepsilon/3). \tag{2.4}$$

Take

$$\sigma \in (0, \delta_\varepsilon) \quad \text{with } \sigma < (1/2)^{1/2} \xi_1 \left(\int_{R_\varepsilon}^\infty s^{1-\lambda_1}k_1(s)ds \right)^{1/2}$$

and let u be an arbitrary entire large solution of problem (1.1).

Define

$$D_-^\sigma := \Omega_{R_\varepsilon} \setminus \Omega_-^\sigma, \quad D_+^\sigma := \Omega_{R_\varepsilon} \setminus \Omega_+^\sigma,$$

where

$$\Omega_-^\sigma := \left\{ x \in \Omega_{R_\varepsilon} : \tau_1 \left(\int_{|x|}^\infty s^{1-\lambda_1}k_1(s)ds \right)^{1/2} \leq \sigma \right\} \tag{2.5}$$

and

$$\Omega_+^\sigma := \left\{ x \in \Omega_{R_\varepsilon+r_0} : \psi \left(\tau_2 \left(\int_{|x|}^\infty s^{1-\lambda_1}k_1(s)ds \right)^{1/2} + \sigma \right) \leq u(x) \right\}, \tag{2.6}$$

where r_0 is a sufficiently large constant that Ω_+^σ is an exterior domain of the ball with $R_\varepsilon + r_0$ as the radius and the origin as the center. By the definitions of Ω_\pm^σ , we see that D_\pm^σ are annular domains.

Let

$$\bar{u}_\varepsilon(x) = \psi \left(\tau_1 \left(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds \right)^{1/2} - \sigma \right), \quad x \in D_-^\sigma,$$

and

$$\underline{u}_\varepsilon = \psi \left(\tau_2 \left(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds \right)^{1/2} + \sigma \right), \quad x \in D_+^\sigma.$$

Indeed, we can always adjust the above $\delta_\varepsilon > 0$ to ensure that

$$(3/2)^{1/2} \xi_2 \left(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds \right)^{1/2} < \delta_\varepsilon, \quad x \in \Omega_{R_\varepsilon}.$$

By a straightforward calculation combined with (2.1)–(2.4), we see that, for $x \in D_-^\sigma$,

$$\begin{aligned} & \Delta \bar{u}_\varepsilon - a(x)f(\bar{u}_\varepsilon) - \mu b(x)|\nabla \bar{u}_\varepsilon|^q \\ &= \psi'' \left(\tau_1 \left(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds \right)^{1/2} - \sigma \right) |x|^{-\lambda_1} k(|x|) \left[\frac{\tau_1^2}{4} \frac{|x|^{2-\lambda_1} k_1(|x|)}{\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds} \right. \\ & \quad \left. + \frac{\tau_1^2}{4} \left(\frac{|x|^{2-\lambda_1} k_1(|x|)}{\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds} + 2(N - \lambda_1) + \frac{2|x|k_1'(|x|)}{k_1(|x|)} \right) \right. \\ & \quad \times \frac{-\psi'(\tau_1(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2} - \sigma)}{\tau_1(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2} \psi''(\tau_1(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2} - \sigma)} - \frac{a(x)}{|x|^{-\lambda_1} k_1(|x|)} \\ & \quad \left. - \frac{(-\psi'(\tau_1(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2} - \sigma)^q (\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{-\frac{q}{2}} |x|^{q(1-\lambda_1)+\lambda_1-\lambda_2} k_1^q(|x|)}{\psi''(\tau_1(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2} - \sigma)} \right. \\ & \quad \left. \times \frac{k_2(|x|)}{k_1(|x|)} \mu \left(\frac{\tau_1}{2} \right)^q \frac{b(x)}{|x|^{-\lambda_2} k_2(|x|)} \right] \\ & \leq \psi''(r) |x|^{-\lambda_1} k_1(|x|) \left[\frac{\tau_1^2}{4} \left(\frac{|x|^{2-\lambda_1} k_1(|x|)}{\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds} - (\lambda_1 - 2) \right) \right. \\ & \quad \left. - \frac{\tau_1^2}{4} \left(\frac{|x|^{2-\lambda_1} k_1(|x|)}{\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds} + 2(N - \lambda_1) + \frac{2|x|k_1'(|x|)}{k_1(|x|)} \right) \frac{\psi'(r)}{r\psi''(r)} - \frac{(2(N - 1) - \lambda_1)\tau_1^2}{4C_f} \right. \\ & \quad \left. - \left(\frac{a(x)}{|x|^{-\lambda_1} k_1(|x|)} - a_1 \right) + \tau_1^2 \left(\frac{2(N - 2) + (C_f - 1)(\lambda_1 - 2)}{4C_f} \right) - a_1 \right] \leq 0, \end{aligned}$$

where

$$r = \tau_1 \int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds - \sigma,$$

i.e., \bar{u}_ε is a supersolution of Eq. (1.1) in D_-^σ .

Similarly, we can show that $\underline{u}_\varepsilon$ is a subsolution of Eq. (1.1) in D_σ^+ . We assert that there exists a large constant $M > 0$ independent of σ such that

$$u(x) \leq \bar{u}_\varepsilon(x) + M, \quad x \in D_-^\sigma \tag{2.7}$$

and

$$\underline{u}_\varepsilon(x) \leq u(x) + M, \quad x \in \Omega_{R_\varepsilon}. \tag{2.8}$$

In fact, we can choose a positive constant M independent of σ such that $x \in \{x \in \mathbb{R}^N : |x| = R_\varepsilon\}$

$$u(x) \leq \bar{u}_\varepsilon(x) + M \tag{2.9}$$

and

$$\underline{u}_\varepsilon(x) \leq u(x) + M. \tag{2.10}$$

Because

$$u(x) < \bar{u}_\varepsilon = \infty, \quad x \in \left\{x \in \mathbb{R}^N : \tau_1 \left(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds \right)^{1/2} = \sigma \right\},$$

we can always choose a sufficiently small $\rho > 0$ such that

$$\sup_{x \in D_-^\sigma} u(x) \leq \bar{u}_\varepsilon(x), \quad x \in D_-^\sigma \setminus \tilde{D}_-^\sigma, \tag{2.11}$$

where

$$\tilde{D}_-^\sigma = \Omega_{R_\varepsilon} \setminus \tilde{\Omega}_-^\sigma$$

with

$$\tilde{\Omega}_-^\sigma = \left\{x \in \Omega_{R_\varepsilon} : \tau_1 \left(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds \right)^{1/2} \leq \sigma(1 + \rho) \right\}.$$

Combining (2.9) with (2.11), we obtain

$$u(x) \leq \bar{u}_\varepsilon(x) + M, \quad x \in \partial(\tilde{D}_-^\sigma).$$

In contrast, we conclude, using (2.10) and the definition of Ω_+^σ (please refer to (2.6)), that

$$\underline{u}_\varepsilon(x) \leq u(x) + M, \quad x \in \partial(D_+^\sigma).$$

We note that u and $\underline{u}_\varepsilon$ are both subsolutions in \tilde{D}_-^σ and D_+^σ . Therefore, using (f_1) , we see that $\bar{u}_\varepsilon + M$ and $u + M$ are supersolutions in \tilde{D}_-^σ and D_+^σ , respectively. It follows by Lemma B.3 that

$$u(x) \leq \bar{u}_\varepsilon(x) + M, \quad x \in \tilde{D}_-^\sigma \tag{2.12}$$

and

$$u_\varepsilon(x) \leq u(x) + M, \quad x \in D_+^\sigma. \tag{2.13}$$

Indeed, (2.12) combined with (2.11) implies that (2.7) holds, and (2.13) and (2.6) together imply that (2.8) holds.

Now, using (2.7) and (2.8), we show that, for any $x \in \Omega_{R_\varepsilon}$, the following hold:

$$\begin{aligned} \frac{u(x)}{\psi(\tau_1(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2})} &\leq 1 + \frac{M}{\psi(\tau_1(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2})}; \\ \frac{u(x)}{\psi(\tau_2(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2})} &\geq 1 - \frac{M}{\psi(\tau_2(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2})}. \end{aligned} \tag{2.14}$$

By the definition of D_-^σ , we see that, for fixed $x_1 \in \Omega_{R_\varepsilon}$, there exists a sufficiently small σ_1 such that $x_1 \in D_-^{\sigma_1}$. However, we know that, for any $\sigma < \sigma_1$, $D_-^{\sigma_1} \not\subseteq D_-^\sigma$. This finding implies that $x_1 \in D_-^\sigma$. This fact, combined with (2.7) and (2.8), shows that

$$u(x_1) \leq \psi\left(\tau_1\left(\int_{|x_1|}^\infty s^{1-\lambda_1} k_1(s) ds\right)^{1/2} - \sigma\right) + M \tag{2.15}$$

and

$$\psi\left(\tau_2\left(\int_{|x_1|}^\infty s^{1-\lambda_1} k_1(s) ds\right)^{1/2} + \sigma\right) \leq u(x_1) + M. \tag{2.16}$$

Since (2.15)–(2.16) hold for any $\sigma < \sigma_1$, using the rank preservation of the limit of the function ψ , we obtain

$$u(x_1) \leq \psi\left(\tau_1\left(\int_{|x_1|}^\infty s^{1-\lambda_1} k_1(s) ds\right)^{1/2}\right) + M$$

and

$$\psi\left(\tau_2\left(\int_{|x_1|}^\infty s^{1-\lambda_1} k_1(s) ds\right)^{1/2}\right) \leq u(x_1) + M.$$

These findings imply that, for any $x \in \Omega_{R_\varepsilon}$, the conditions in (2.14) hold.

Consequently, by Lemma B.2(ii), we have

$$\begin{aligned} \limsup_{|x| \rightarrow \infty} \frac{u(x)}{\psi(\tau_1(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2})} &\leq 1, \\ \liminf_{|x| \rightarrow \infty} \frac{u(x)}{\psi(\tau_2(\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2})} &\geq 1. \end{aligned}$$

For $C_f \in [1, \infty)$, it follows by Lemma B.2(iv) that

$$\limsup_{|x| \rightarrow \infty} \frac{u(x)}{\psi((\int_{|x|}^\infty s^{1-\lambda_1} k_1(s) ds)^{1/2})}$$

$$\begin{aligned}
 &= \limsup_{|x| \rightarrow \infty} \frac{u(x)}{\psi(\tau_1(\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds)^{1/2})} \cdot \lim_{|x| \rightarrow \infty} \frac{\psi(\tau_1(\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds)^{1/2})}{\psi((\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds)^{1/2})} \\
 &\leq \tau_1^{1-C_f}; \\
 &\liminf_{|x| \rightarrow \infty} \frac{u(x)}{\psi((\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds)^{1/2})} \\
 &= \liminf_{|x| \rightarrow \infty} \frac{u(x)}{\psi(\tau_2(\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds)^{1/2})} \cdot \lim_{|x| \rightarrow \infty} \frac{\psi(\tau_2(\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds)^{1/2})}{\psi((\int_{|x|}^{\infty} s^{1-\lambda_1} k_1(s) ds)^{1/2})} \\
 &\geq \tau_2^{1-C_f}.
 \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we obtain (1.4). □

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Proof Since

$$\lim_{|x| \rightarrow \infty} \frac{u_1(x)}{u_2(x)} = 1,$$

for arbitrary fixed $\varepsilon > 0$, there exists a sufficiently large R_ε such that

$$(1 - \varepsilon)u_2(x) \leq u_1(x) \leq (1 + \varepsilon)u_2(x), \quad x \in \Omega_{R_\varepsilon}. \tag{3.1}$$

Set

$$u^+(x) = (1 + \varepsilon)u_2(x), \quad u^-(x) = (1 - \varepsilon)u_2(x), \quad x \in \mathbb{R}^N.$$

A simple calculation shows that

$$\begin{aligned}
 \Delta u^+ &= (1 + \varepsilon)\Delta u_2 = (1 + \varepsilon)a(x)f(u_2) + \mu(1 + \varepsilon)b(x)|\nabla u_2|^q, \\
 \Delta u^- &= (1 - \varepsilon)\Delta u_2 = (1 - \varepsilon)a(x)f(u_2) + \mu(1 - \varepsilon)b(x)|\nabla u_2|^q.
 \end{aligned}$$

By (f_3) , we have

$$(1 + \varepsilon)f(u_2) \leq f((1 + \varepsilon)u_2), \quad (1 - \varepsilon)f(u_2) \geq f((1 - \varepsilon)u_2).$$

In contrast, we apply (b_1) and (b_3) and conclude that

$$(1 + \varepsilon)b(x)|\nabla u_2|^q \leq b(x)|(1 + \varepsilon)\nabla u_2|^q, \quad (1 + \varepsilon)b(x)|\nabla u_2|^q \leq b(x)|(1 + \varepsilon)\nabla u_2|^q.$$

It follows by combining these with (a_1) that

$$\Delta u^+ \leq a(x)f(u^+) + \mu b(x)|u^+|^q; \quad \Delta u^- \geq a(x)f(u^-) + \mu b(x)|u^-|^q, \quad x \in \mathbb{R}^N.$$

Let u_0 be the unique solution of

$$\Delta u = a(x)f(u) + \mu b(x)|\nabla u|^q, \quad x \in \Omega_0, u|_{\partial\Omega_0} = u_1,$$

where $\Omega_0 = \mathbb{R}^N \setminus \Omega_{R_\varepsilon}$. Using the comparison principle, we obtain

$$u^-(x) \leq u_0(x) \leq u^+(x), \quad x \in \Omega_0. \tag{3.2}$$

Noting that $u_0 = u_1$ on Ω_0 and combining (3.1) with (3.2), we obtain

$$(1 - \varepsilon)u_2(x) \leq u_1(x) \leq (1 + \varepsilon)u_2(x), \quad x \in \mathbb{R}^N. \tag{3.3}$$

Letting $\varepsilon \rightarrow 0$ completes the proof. □

Appendix 1

In this section, we introduce some preliminaries to Karamata regular variation theory.

Definition A.1 ([45], Definition 1.1) A positive continuous function f defined on $[a, \infty)$ for some $a > 0$ is called *regularly varying at infinity* with index ρ , written as $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s \rightarrow \infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{A.1}$$

In particular, when $\rho = 0$, f is called *slowly varying at infinity*.

Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Some basic examples of functions that are slowly varying at infinity are

- (i₁) every measurable function on $[a, \infty)$ with a positive limit at infinity;
- (i₂) $(\ln s)^\beta$ and $(\ln(\ln s))^\beta$, $\beta \in \mathbb{R}$; and
- (i₃) $e^{(\ln s)^p}$, $0 < p < 1$.

Definition A.2 ([52], Definition 2.1) A positive continuous function f defined on $[a, \infty)$ for some $a > 0$ is called *rapidly varying to infinity at infinity* if for each $\rho > 0$

$$\lim_{s \rightarrow \infty} \frac{f(s)}{s^\rho} = \infty$$

Some basic examples of functions that are rapidly varying to infinity at infinity are

- (ii₁) e^s and e^{e^s} ;
- (ii₂) $e^{e^{(\ln s)^p}}$, e^{s^p} and $e^{e^{s^p}}$, $p > 0$;
- (ii₃) $s^p e^{(\ln s)^p}$ and $(\ln s)^\beta e^{(\ln s)^p}$, $p > 1$, $\beta \in \mathbb{R}$; and
- (ii₄) $(\ln s)^\beta e^{s^p}$ and $s^\beta e^{s^p}$, $p > 0$, $\beta \in \mathbb{R}$.

Definition A.3 ([54], Definition 2.2) A positive continuous function f defined on $(0, a]$ for some $a > 0$ is called *rapidly varying to infinity at zero* if for each $\rho > 0$

$$\lim_{s \rightarrow 0^+} h(s)s^\rho = \infty.$$

Obviously, if we replace the variable s in functions (ii₁)–(ii₄) with the variable s^{-1} , we obtain basic examples of functions that are rapidly varying to infinity at zero.

In addition, it is not difficult to understand that a positive continuous function g defined on $(0, a)$ for some $a > 0$ is regularly varying at zero with index ρ (written as $g \in RVZ_\rho$) if $s \rightarrow g(1/s) \in RV_{-\rho}$. Similarly, g is called rapidly varying at zero if $s \rightarrow g(1/s)$ is rapidly varying at infinity.

Proposition 1 (Uniform convergence theorem; [41], Proposition 0.5) *If $f \in RV_\rho$, then (A.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.*

Proposition 2 (Representation theorem; [45], Theorem 1.2; [41], Corollary, p. 17) *A function L is slowly varying at infinity if and only if it can be written in the form*

$$L(s) = \varphi(s) \exp\left(\int_{a_1}^s \frac{y(\tau)}{\tau} d\tau\right), \quad s \geq a_1,$$

for some $a_1 \geq a$, where the functions φ and y are continuous and $s \rightarrow \infty$, $y(s) \rightarrow 0$ and $\varphi(s) \rightarrow c_0$ with $c_0 > 0$. If $\varphi \equiv c_0$, then L is said to be normalized slowly varying at infinity and

$$f(s) = s^\rho \hat{L}(s), \quad s \geq a_1$$

is said to be normalized regularly varying at infinity with index ρ (written as $f \in NRV_\rho$).

A function $f \in C^1[a_1, \infty)$ for some $a_1 > 0$ belongs to NRV_ρ if and only if

$$\lim_{s \rightarrow \infty} \frac{sf'(s)}{f(s)} = \rho.$$

Proposition 3 (Asymptotic behavior; [6], Propositions 1.5.8 and 1.5.10) *If a function L is slowly varying at infinity, then, for $s \rightarrow \infty$,*

$$\int_s^\infty \tau^\rho L(\tau) d\tau \sim (-\rho - 1)^{-1} \tau^{1+\rho} L(\tau), \quad \text{for } \rho < -1.$$

Proposition 4 ([6], Proposition 1.3.6) *If a function L is slowly varying at zero, then, for $s \rightarrow +\infty$,*

$$s^\rho L(s) \rightarrow +\infty, \quad s^{-\rho} L(s) \rightarrow 0, \quad \text{for } \rho > 0.$$

Appendix 2

In this section, we collect some useful results.

Lemma B.1 ([53], Lemma 2.2) *Let f satisfy (\mathbf{f}_1) .*

- (i) *If f satisfies (\mathbf{f}_2) , then $C_f \geq 1$.*
- (ii) *(\mathbf{f}_2) holds for $C_f \in (1, +\infty)$ if and only if $f \in RV_{\frac{C_f+1}{C_f-1}}$.*
- (iii) *If (\mathbf{f}_2) holds for $C_f = 1$, then f is rapidly varying to infinity at infinity.*

Lemma B.2 ([53], Lemma 2.3) *Let f satisfy (f_1) – (f_2) and ψ be the solution of problem (1.5).*

Then

- (i) $\psi'(t) = -(2F(\psi(t)))^{1/2}$, $\psi''(t) = f(\psi(t))$, $\psi(t) > 0$, $t > 0$;
- (ii) $\lim_{t \rightarrow 0^+} \psi(t) = \infty$;
- (iii) $\lim_{t \rightarrow 0^+} \frac{t\psi''(t)}{\psi'(t)} = -C_f$; and
- (iv) $\lim_{t \rightarrow 0^+} \frac{t\psi'(t)}{\psi(t)} = 1 - C_f$, where $C_f \in [1, \infty)$.

Lemma B.3 (Comparison principle; [23], Theorems 10.1, 10.2) *Let $\Psi(x, s, \xi)$ satisfy the following two conditions:*

(D₁) Ψ is nonincreasing in s for all $(x, \xi) \in \Omega \times \mathbb{R}^N$; and

(D₂) Ψ is continuously differentiable with respect to the variable ξ in $\Omega \times (0, \infty) \times \mathbb{R}^N$.

If $u, v \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies $\Delta u + \Psi(x, u, \nabla u) \geq \Delta v + \Psi(x, v, \nabla v)$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω .

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Availability of data and materials

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Authors' contributions

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