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On a degenerate boundary value problem to the two-dimensional self-similar nonlinear wave system

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Abstract

This paper focuses on a degenerate boundary value problem arising from the study of the two-dimensional Riemann problem to the nonlinear wave system. In order to deal with the parabolic degeneracy, we introduce a partial hodograph transformation to transform the nonlinear wave system into a new system, which displays a clear regularity–singularity structure. The local existence of classical solutions for the new system is established in a weighted metric space. Returning the solution to the original variables, we obtain the existence of classical solutions to the degenerate boundary value problem for the nonlinear wave system.

MSC: 35L65; 35L80; 76H05

Keywords: Nonlinear wave system; Degenerate hyperbolic; Characteristic decomposition; Classical solution

1 Introduction

We are interested in a degenerate boundary value problem arising from the study of the two-dimensional four-constant Riemann problem to the nonlinear wave system

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + p_x = 0, \\ (\rho v)_t + p_y = 0, \end{cases} \tag{1.1}$$

where ρ, (u, v), are, respectively, the density and the velocity, and $p = p(\rho)$ is a given function of ρ. This system is obtained either by ignoring the quadratic terms in the velocity (u, v) to the two-dimensional isentropic compressible Euler equations in the gas dynamics, or by writing the nonlinear wave equation as a first-order system. We refer the reader to references [3, 4] for the background information.

The multidimensional Riemann problem of the quasilinear hyperbolic conservation laws is one of important problems in mathematical fluid dynamics containing in particular the oblique shock reflection problem and the dam collapse problem. Most importantly, the Riemann problem performs the role of 'building blocks' for all fields of theory, numerics, and applications, see the survey [18] and the references therein. The study of the two-dimensional Riemann problem to the Euler equations was initiated by Zhang and Zheng [32]. The authors provided a set of conjectures on the structure of solutions by using the generalized characteristic analysis method and numerical experiments. However, until now, none of them has been completely proved due to the existence of transonic structures [7, 19, 36]. Many pieces of work have been contributed on understanding these transonic structures for the Euler equations and its related systems. We refer the reader to [17, 20–23, 25–27, 30] and references cited there, and especially the monographs [19, 36] for the results of the Euler and pressure-gradient systems. In particular, for the relevant results of the two-dimensional Riemann problem to the nonlinear wave system (1.1), one may consult [6, 10–16] and references therein.

We consider in this paper system (1.1) together with a smooth state function $p = p(\rho)$ satisfying

$$\forall \rho > 0, \quad p(\rho) \gtrless 0, \qquad p'(\rho) > 0, \quad \text{and} \quad p''(\rho) \gtrless 0. \tag{1.2}$$

It is clear that the well-known equations of state of the polytropic gas $p = A\rho^{\gamma}$ $(\gamma > 1)$ and of the Chaplygin gas $p = -1/\rho$ satisfy (1.2). The equation of state of the Chaplygin gas was introduced by Chaplygin [5] and was taken as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics by Tsien [28] and von Karman [29]. Moreover, this equation of state has been advertised as a possible model for dark energy; see, e.g., [2, 8]. We are looking for the self-similar solutions of (1.1), that is, the solutions depend only on the self-similar variables $(\xi, \eta) = (x/t, y/t)$. In terms of variables (ξ, η), system (1.1) can be transformed to

$$\begin{cases} -\xi\rho_\xi - \eta\rho_\eta + (\rho u)_\xi + (\rho v)_\eta = 0, \\ -\xi(\rho u)_\xi - \eta(\rho u)_\eta + p_\xi = 0, \\ -\xi(\rho v)_\xi - \eta(\rho v)_\eta + p_\eta = 0. \end{cases} \tag{1.3}$$

The eigenvalues of (1.3) are

$$\Lambda_0 = \frac{\eta}{\xi}, \qquad \Lambda_\pm = \frac{\xi\eta \pm \sqrt{p'(\rho)(\xi^2 + \eta^2 - p'(\rho))}}{\xi^2 - p'(\rho)}. \tag{1.4}$$

We see from (1.2) and (1.4) that system (1.3) is hyperbolic at the infinity (i.e., $|\xi| + |\eta| = \infty$) and changes type to elliptic near the origin. The hyperbolic and elliptic regions may be separated by a boundary curve composed by degenerate curves and shocks, which are free boundaries to be determined together with the solutions [1]. To construct a global solution of nonlinear mixed-type system, iterative methods seem to be the most likely choices.

The purpose of the present paper is to establish the local existence of classical solutions to the two-dimensional self-similar nonlinear wave system (1.3) with degenerate boundary data, which is an essential step for using an iterative process to construct a global solution of mixed type equation. The local existence of classical sonic-supersonic solutions was investigated for compressible Euler equations with polytropic gases in [9, 33, 35] and for a pressure-gradient system in [34]. We consider the degenerate boundary value problem of (1.1) with the convex equation of state satisfying (1.2) in the current paper and will explore that problem for the general nonconvex equation of state in the next work.

The rest of the paper is organized as follows. In Sect. 2, we describe the problem in detail and then state our main result. Section 3 is devoted to reformulating the problem in the new dependent and independent variables. In Sect. 4, we complete the proof of the main results by solving the new problem and then converting the solution to the original coordinates.

2 The problem and the main results

We first decouple p from (ρu) and (ρv) to obtain a second-order quasilinear equation,

$$\big(a(p)-\xi^2\big)p_{\xi\xi}-2\xi\eta p_{\xi\eta}+\big(a(p)-\eta^2\big)p_{\eta\eta}+b(p)(\xi p_\xi+\eta p_\eta)^2-2(\xi p_\xi+\eta p_\eta)=0, \quad (2.1)$$

where

$$a(p)=\frac{1}{\rho'(p)}>0, \qquad b(p)=-\frac{\rho''(p)}{\rho'(p)}=\frac{a'(p)}{a(p)} \quad (2.2)$$

for all $p>0$ by (1.2). The two eigenvalues of (2.1) are

$$\tilde{\Lambda}_\pm=\frac{\xi\eta\pm\sqrt{a(p)(\xi^2+\eta^2-a(p))}}{\xi^2-a(p)}. \quad (2.3)$$

For convenience to deal with our problem, we rewrite (2.1) in terms of the polar coordinates (r,θ) as

$$P_{\theta\theta}-\frac{r^2(r^2-a(P))}{a(P)}P_{rr}+rP_r+\frac{b(P)r^3}{a(P)}P_r^2-\frac{2r^2}{a(P)}P_r=0, \quad (2.4)$$

where $r=\sqrt{\xi^2+\eta^2}$, $\theta=\arctan(\eta/\xi)$ and $P(r,\theta)=p(r\cos\theta,r\sin\theta)$. The two family of characteristics are defined as

$$\Gamma_\pm:\frac{dr}{d\theta}=\pm\lambda, \qquad \lambda=\sqrt{\frac{r^2(r^2-a(P))}{a(P)}}. \quad (2.5)$$

It is clear that Eq. (2.4) is of mixed type: hyperbolic for $r^2>a(P)$, elliptic for $r^2<a(P)$ and parabolic degenerate for $r^2=a(P)$.

Let $r_a<r_b$ be two positive constants and $\Gamma:\theta=\varphi(r)$ be a smooth curve defined on $[r_a,r_b]$ satisfying $|\varphi'(r)|\le\varphi_0$ for same positive constant φ_0. That means the curve Γ can not be a circular arc. We assign the boundary data on Γ as follows:

$$P\big(\varphi(r),r\big)=P_0(r), \qquad P_\theta\big(\varphi(r),r\big)=P_1(r) \quad \text{with } a\big(P_0(r)\big)=r^2. \quad (2.6)$$

The aim of the paper is to look for a classical solution of the boundary value problem (2.4) (2.6). Since the wave speed $\lambda=0$ on Γ, the hyperbolic problem (2.4) (2.6) is parabolic degenerate.

In the hyperbolic region, Eq. (2.4) has the interesting characteristic decomposition [10]

$$\begin{cases}\partial_+\partial_-P=Q(\partial_+P-\partial_-P)\partial_-P,\\ \partial_-\partial_+P=Q(\partial_-P-\partial_+P)\partial_+P,\end{cases} \quad (2.7)$$

where

$$\partial_\pm := \partial_\theta \pm \lambda\partial_r, \quad \text{and} \quad Q = \frac{a'(P)r^2}{4a(P)(r^2 - a(P))}.$$

Introduce

$$R = \partial_+ P, \qquad S = \partial_- P,$$

from this and (2.5) we have

$$P_\theta = \frac{R+S}{2}, \qquad P_r = \frac{R-S}{2\lambda} = \frac{\sqrt{a(P)}}{2r} \cdot \frac{R-S}{\sqrt{r^2 - a(P)}}. \tag{2.8}$$

Moreover, from (2.7), we obtain the system for (P,R,S)

$$\begin{cases} P_\theta = \frac{R+S}{2}, \\ R_\theta - \lambda R_r = \frac{a'(P)r^2 R}{4a(P)} \cdot \frac{S-R}{r^2 - a(P)}, \\ S_\theta + \lambda S_r = \frac{a'(P)r^2 S}{4a(P)} \cdot \frac{R-S}{r^2-a(P)}, \end{cases} \tag{2.9}$$

where λ is defined in (2.5). Then we look for a local classical solution of system (2.9) with the boundary data

$$(P,R,S)\bigl(\varphi(r),r\bigr) = \bigl(P_0(r), P_1(r), P_1(r)\bigr), \quad \forall r \in [r_a, r_b]. \tag{2.10}$$

We point out that the local existence of the degenerate boundary value problem (2.9) (2.10) cannot be obtained directly by the classical local existence theory of nonlinear hyperbolic equations (see, e.g. [24, 31]). This is because system (2.9) is not a continuously differentiable system by the degeneracy. The idea we employed here is inspired by the work of Zhang and Zheng [33] for studying the steady Euler equations. The main technique is to isolate possible singularities by introducing a partial hodograph transformation. We establish the local existence and uniqueness of classical solutions for a new system under a suitable function class by using the fixed point method. Converting the solution to the original coordinates, we thus obtain a local classical solution to the problem (2.9) and (2.10). The results of this paper can be stated as follows.

Theorem 1 *Suppose that the equation of state $p = p(\cdot) \in C^4$ satisfies (1.2). Moreover, we assume the functions $(\varphi, P_0, P_1)(r)$ satisfy*

$$\begin{aligned} &\varphi(r) \in C^4\bigl([r_a, r_b]\bigr) \quad \textit{with } \bigl|\varphi'(r)\bigr| \le \varphi_0, \\ &P_0(r) \in C^4\bigl([r_a, r_b]\bigr), \qquad P_1(r) \in C^3\bigl([r_a, r_b]\bigr) \quad \textit{with } \bigl|P_1(r)\bigr| \ge k_0, \end{aligned} \tag{2.11}$$

where φ_0 and k_0 are two positive constants. Then the degenerate boundary value problem (2.9) (2.10) has a classical solution in the hyperbolic region near Γ.

From Theorem 1, we have the following.

Theorem 2 *Let the assumptions in Theorem 1 hold. Then there is a classic solution to the degenerate boundary value problem (2.4) and (2.6) in the hyperbolic region near Γ.*

3 The problem in new coordinates

In this section, we introduce new dependent and independent variables to reformulate the problem. To deal with the singularities caused by the degenerate, we first introduce a partial hodograph transformation as follows:

$$t = \sqrt{r^2 - a\big(P(r,\theta)\big)}, \qquad \tilde{r} = r. \tag{3.1}$$

Note that the sonic curve $\varGamma$ is transformed to a segment on $t = 0$ with $\tilde{r} \in [r_a, r_b]$. From (3.1) one has

$$\partial_\theta = -\frac{a'(P)(R+S)}{4t}\partial_t, \qquad \partial_r = \partial_{\tilde{r}} + \frac{4r^2t - a'(P)\sqrt{a(P)}(R-S)}{4rt^2}\partial_t. \tag{3.2}$$

In terms of $(t, \tilde{r})$, system (2.9) can be rewritten as

$$\begin{cases} R_t + \frac{2rt^2}{a'(P)\sqrt{a(P)}S+2r^2t}R_r = \frac{a'(P)r^2R}{2\sqrt{a(P)}[a'(P)\sqrt{a(P)}S+2r^2t]}\cdot\frac{R-S}{t}, \\ S_t - \frac{2rt^2}{a'(P)\sqrt{a(P)}R-2r^2t}S_r = \frac{a'(P)r^2S}{2\sqrt{a(P)}[a'(P)\sqrt{a(P)}R-2r^2t]}\cdot\frac{S-R}{t}, \end{cases} \tag{3.3}$$

together with a decoupled trivial equation

$$\partial_t P = -\frac{2t}{a'(P)}. \tag{3.4}$$

Here and below we still use r to represent $\tilde{r}$ and denote $P(t,r) = P(r,\theta)$, $R(t,r) = R(r,\theta)$, $S(t,r) = S(r,\theta)$, which will not cause confusion in understanding. Furthermore, we find by (1.2) that

$$a\big(P_0(r)\big) \geq k_1 > 0, \qquad \big|a'\big(P_0(r)\big)\big| \geq k_2 > 0, \quad \forall r \in [r_a, r_b], \tag{3.5}$$

for some positive constants k_1, k_2. Hence we can solve $P(t,r)$ from Eq. (3.4) with the initial data $P(0,r) = P_0(r)$. That means system (3.3) is closed in the coordinates (t,r).

Corresponding to the boundary data (2.10), the initial data of (3.3) are

$$(R,S)(0,r) = \big(P_1(r), P_1(r)\big), \quad \forall r \in [r_a, r_b]. \tag{3.6}$$

In addition, by (2.6) we see that

$$P_r\big(\varphi(r), r\big) = P_0'(r) - \varphi'(r)P_1(r) := P_2(r),$$

which along with (3.3) yields

$$(R_t, S_t)(0,r) = \big(P_2(r), -P_2(r)\big), \quad \forall r \in [r_a, r_b], \tag{3.7}$$

for smooth solutions. Therefore, we look for a classic solution to system (3.3) with the initial data (3.6) and (3.7).

Next we introduce two new dependent variables to homogenize the initial data,

$$U(t,r) = R(t,r) - P_1(r) - P_2(r)t, \qquad V(t,r) = S(t,r) - P_1(r) + P_2(r)t. \tag{3.8}$$

Thus by (3.6) and (3.7) we get the homogeneous initial condition

$$U(0,r) = V(0,r) = U_t(0,r) = V_t(0,r) = 0, \quad \forall r \in [r_a, r_b]. \tag{3.9}$$

Moreover, system (3.3) is transformed into

$$\begin{cases} U_t + \frac{2rt^2}{a'\sqrt{a}V+f}U_r = \frac{U-V}{2t} + b_1(U,V,t,r), \\ V_t - \frac{2rt^2}{a'\sqrt{a}U+g}V_r = \frac{V-U}{2t} + b_2(U,V,t,r), \end{cases} \tag{3.10}$$

where $f = f(t,r) = a'\sqrt{a}(P_1(r) - P_2(r)t) + 2r^2t$, $g = g(t,r) = a'\sqrt{a}(P_1(r) + P_2(r)t) - 2r^2t$, and

$$\begin{aligned} b_1(U,V,t,r) &= \left(\frac{a'r^2[U + P_1(r) + P_2(r)t)]}{2\sqrt{a}[a'\sqrt{a}V + f]} - \frac{1}{2}\right)\left(\frac{U-V}{t} + 2P_2(r)\right) \\ &\quad - \frac{2t^2r}{a'\sqrt{a}V + f}\left(P_1'(r) + P_2'(r)t\right), \\ b_2(U,V,t,r) &= \left(\frac{a'r^2[V - P_1(r) + P_2(r)t)]}{2\sqrt{a}[a'\sqrt{a}U + g]} - \frac{1}{2}\right)\left(\frac{V-U}{t} - 2P_2(r)\right) \\ &\quad - \frac{2t^2r}{a'\sqrt{a}U + g}\left(P_1'(r) - P_2'(r)t\right). \end{aligned}$$

Then the previous problem is reformulated as follows.

Problem 3.1 Under the assumptions in Theorem 1, we seek a classical solution to initial data problem (3.9) (3.10) in the region $t > 0$.

To solve Problem (3.1), we first define a suitable function space. Let δ be a small positive constant. Set

$$D(\delta) := \left\{(t,r) | 0 \le t \le \delta, r_1(t) \le r \le r_2(t)\right\},$$

where $r_1(t)$, $r_2(t)$ are continuously differentiable on $0 \le t \le \delta$, $r_1(0) = r_a$, $r_2(0) = r_b$ and $r_1(t) < r_2(t)$ for $0 \le t \le \delta$.

Definition 3.1 The domain $D(\delta)$ is called a strong domain of determinacy to system (3.10) if for any $(\xi,\eta) \in D(\delta)$ and any smooth functions (U,V) satisfying the homogeneous initial condition (3.9), the curves $r_\pm(t;\xi,\eta)$ defined by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}r_+ = \Lambda_+(V(t,r_+)), \\ r_+|_{t=\xi} = \eta, \end{cases} \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}r_- = \Lambda_-(U(t,r_-)), \\ r_-|_{t=\xi} = \eta, \end{cases} \tag{3.11}$$

are also inside $D(\delta)$ for $0 < t \le \xi$. Here

$$\Lambda_+(V) = \frac{2rt^2}{a'\sqrt{a}V + f}, \qquad \Lambda_-(U) = -\frac{2rt^2}{a'\sqrt{a}U + g}. \tag{3.12}$$

Next we define a suitable class of functions. Let $S = S(M,\delta)$ be a function class consisting of all continuously differentiable functions $F = (f_1, f_2)^T : D(\delta) \to \mathbb{R}^2$ satisfying the following properties

(S_1) $f_1(0,r) = f_2(0,r) = \partial_t f_1(0,r) = \partial_t f_2(0,r) = 0$,
(S_2) $\|\frac{f_1}{t^2}\|_{L^\infty} + \|\frac{f_2}{t^2}\|_{L^\infty} \le M$,
(S_3) $\|\frac{\partial_r f_1}{t^2}\|_{L^\infty} + \|\frac{\partial_r f_2}{t^2}\|_{L^\infty} \le M$,
(S_4) $\partial_r F$ is Lipschitz continuous with respect to r with $\|\frac{\partial^2_{rr} f_1}{t^2}\|_{L^\infty} + \|\frac{\partial^2_{rr} f_2}{t^2}\|_{L^\infty} \le M$,

where δ and M are two positive constants. Denote by $\mathcal{W}$ the function class containing only continuous functions on $D(\delta)$ satisfying (S_1) and (S_2). It is easy to see that $\mathcal{S}$ and $\mathcal{W}$ are subsets of $C^0(D(\delta);\mathbb{R}^2)$. Moreover, we define a weighted metric on $\mathcal{S}$ and $\mathcal{W}$ as follows:

$$d(\mathbf{F},\mathbf{G}) := \left\|\frac{f_1 - g_1}{t^2}\right\|_{L^\infty} + \left\|\frac{f_2 - g_2}{t^2}\right\|_{L^\infty}.$$

It is not difficult to check that $(\mathcal{W},d)$ is a completed metric space, while $(\mathcal{S},d)$ is not a closed subset in $(\mathcal{W},d)$.

Theorem 1 follows directly from the following theorem.

Theorem 3 *Suppose that the conditions listed in Theorem 1 hold and that $D(\delta_0)$ is a strong domain of determinacy to the system (3.10) for some positive constant δ_0. Then there exit constants $\delta \in (0,\delta_0)$ and M such that the degenerate hyperbolic problem (3.10) and (3.9) has a classical solution in the function class $\mathcal{S}(M,\delta)$.*

4 Proof of the main theorems

In this section, we use the fixed point method to prove Theorem 3 and then complete the proof of Theorem 1. The proof is divided into five steps. In Step 1, we construct an integration iteration mapping in the function class $\mathcal{S}(M,\delta)$ by the differential system (3.10). In Step 2, we establish a series of a priori estimates for b_1, b_2 and $\Lambda_\pm$. We show that the above iteration mapping is a contraction in Step 3. In Step 4, we show that this limit vector function also belongs to $\mathcal{S}(M,\delta)$. Finally, in Step 5 we return the solution to the original coordinates (r,θ).

Step 1. The iteration mapping. Denote

$$\frac{d}{d_+(V)} := \partial_t + \Lambda_+(V)\partial_r, \qquad \frac{d}{d_-(U)} := \partial_t + \Lambda_-(U)\partial_r. \tag{4.1}$$

Then system (3.10) can be rewritten as

$$\frac{d}{d_+(V)}U = \frac{U-V}{2t} + b_1(U,V,r,t), \qquad \frac{d}{d_-(U)}V = \frac{V-U}{2t} + b_2(U,V,r,t). \tag{4.2}$$

Assume the vector functions $(u,v)^T(t,r)$ are in the set $\mathcal{S}$, we consider the linear system of (3.11)

$$\frac{d}{d_+(v)}U = \frac{u-v}{2t} + b_1(u,v,r,t), \qquad \frac{d}{d_-(u)}V = \frac{v-u}{2t} + b_2(u,v,r,t), \tag{4.3}$$

which combined with the property S_1 gives

$$U(\xi,\eta) = \int_0^\xi \left\{\frac{u-v}{2t} + b_1\big(t, r_+(t;\xi,\eta)\big)\right\} \mathrm{d}t, \tag{4.4}$$

$$V(\xi,\eta)=\int_0^{\xi}\left\{\frac{v-u}{2t}+b_2\bigl(t,r_-(t;\xi,\eta)\bigr)\right\}\mathrm{d}t, \tag{4.5}$$

where $r_\pm$ are defined as in (3.11) and

$$\begin{aligned}
&b_1\bigl(t,r_+(t;\xi,\eta)\bigr)=b_1\bigl(u\bigl(t,r_+(t;\xi,\eta)\bigr),v\bigl(t,r_+(t;\xi,\eta)\bigr),t,r_+(t;\xi,\eta)\bigr),\\
&b_2\bigl(t,r_-(t;\xi,\eta)\bigr)=b_2\bigl(u\bigl(t,r_-(t;\xi,\eta)\bigr),v\bigl(t,r_-(t;\xi,\eta)\bigr),t,r_-(t;\xi,\eta)\bigr).
\end{aligned}$$

It is clear that Eqs. (4.4) and (4.5) define a mapping

$$\mathcal{T}\left(\begin{pmatrix}u\\ v\end{pmatrix}\right)=\begin{pmatrix}U\\ V\end{pmatrix}.$$

Therefore, our problem is changed to find a fixed point of the mapping $\mathcal{T}$ in the set $\mathcal{S}(M,\delta)$ for some suitable constants M and δ.

Step 2. A priori estimates. We derive a series of estimates about b_1, b_2 and $\Lambda_\pm$ for later use. We will use $K>1$ to denote a constant depending only on C^3 norms of a, P_1, P_2, φ' and k_0, k_1, k_2, r_a, r_b. Since $(u,v)^T\in\mathcal{S}$, we see by (2.11) and (3.5) that there exists a small constant δ_0 such that for $t\le\delta_0$

$$\bigl|a'\sqrt{a}v+f\bigr|\ge\frac{k_0k_1k_2}{2}\ge\frac{1}{K},\qquad \bigl|a'\sqrt{a}u+g\bigr|\ge\frac{k_0k_1k_2}{2}\ge\frac{1}{K}. \tag{4.6}$$

Moreover, we have

$$|u-v|\le Mt^2,\qquad |u_r-v_r|\le Mt^2,\qquad |u_{rr}-v_{rr}|\le Mt^2. \tag{4.7}$$

To estimate b_1, we first note that

$$\begin{aligned}
&\frac{a'r^2[u+P_1(r)+P_2(r)t]}{2\sqrt{a}(a'\sqrt{a}v+f)}-\frac{1}{2}\\
&\quad=\frac{\sqrt{a}r^2(a'\sqrt{a}u+g+2r^2t)-\sqrt{a}(a\sqrt{a}v+f)}{2\sqrt{a}(a'\sqrt{a}v+f)}\\
&\quad=t\cdot\frac{t\cdot\frac{aa'(u-v)}{t^2}+2aa'P_2(r)+a't[u+P_1(r)+P_2(r)t]-2\sqrt{a}r^2t}{2\sqrt{a}(a'\sqrt{a}v+f)},
\end{aligned} \tag{4.8}$$

from which one has

$$\left|\frac{a'r^2[u+P_1(r)+P_2(r)t]}{2\sqrt{a}(a'\sqrt{a}v+f)}-\frac{1}{2}\right|\le tK\cdot(1+Mt). \tag{4.9}$$

In addition, differentiating (4.8) with respect to r obtains

$$\left|\partial_r\left(\frac{a'r^2[u+P_1(r)+P_2(r)t]}{2\sqrt{a}(a'\sqrt{a}v+f)}-\frac{1}{2}\right)\right|\le tK\cdot(1+Mt), \tag{4.10}$$

$$\left|\partial_{rr}\left(\frac{a'r^2[u+P_1(r)+P_2(r)t]}{2\sqrt{a}(a'\sqrt{a}v+f)}-\frac{1}{2}\right)\right|\le tK\cdot(1+Mt). \tag{4.11}$$

Furthermore, we denote the last term in b_1 by $\Phi = \frac{2rt^2}{a'\sqrt{av+f}}(P_1'(r) + P_2'(r)t)$ and then obtain

$$|\Phi| + |\Phi_r| + |\Phi_{rr}| \le Kt^2(1+Mt)^2. \tag{4.12}$$

Combining (4.9)–(4.12) and using the expression of b_1 yield

$$|b_1| + |b_{1r}| + |b_{1rr}| \le Kt(1+Mt)^2. \tag{4.13}$$

By similar arguments for b_2 get

$$|b_2| + |b_{2r}| + |b_{2rr}| \le Kt(1+Mt)^2. \tag{4.14}$$

For Λ_+, we use the fact

$$\left|a'\sqrt{a}v+f\right| + \left|\partial_r\left(a'\sqrt{a}v+f\right)\right| + \left|\partial_{rr}\left(a'\sqrt{a}v+f\right)\right| \le K(1+Mt),$$

to obtain

$$|\Lambda_+| + |\partial_r\Lambda_+| + |\partial_{rr}\Lambda_+| \le Kt^2(1+Mt)^2. \tag{4.15}$$

Similarly, one has

$$|\Lambda_-| + |\partial_r\Lambda_-| + |\partial_{rr}\Lambda_-| \le Kt^2(1+Mt)^2. \tag{4.16}$$

Step 3. Contraction of the mapping. We have the following lemma.

Lemma 4.1 *Under the assumptions of Theorem 3, there exist positive constants δ, M and $0 < \beta < 1$ such that*

(1) *$\mathcal{T}$ maps $\mathcal{S}$ into $\mathcal{S}$;*

(2) *for any pair $\mathbf{F}, \hat{\mathbf{F}}$ in $\mathcal{S}$,*

$$d\big(\mathcal{T}(\mathbf{F}), \mathcal{T}(\hat{\mathbf{F}})\big) \le \beta d(\mathbf{F}, \hat{\mathbf{F}}). \tag{4.17}$$

Here the constants M, δ, β depend only on the C^3 norms of a, $P_1(r)$, $P_2(r)$, φ' and k_0, k_1, k_2, r_a, r_b.

Proof Let $\mathbf{F} = (u, v)$, $\hat{\mathbf{F}} = (\hat{u}, \hat{v})$ be in set $\mathcal{S}$ and $\mathbf{G} = \mathcal{T}(\mathbf{F}) = (U, V)$ and $\hat{G} = \mathcal{T}(\hat{\mathbf{F}}) = (\hat{U}, \hat{V})$. It is obvious that $U(0,\eta) = V(0,\eta) = 0$.

Moreover, we use (4.7) and (4.13)–(4.14) to obtain

$$|U| \le \int_0^\xi \left\{\left|\frac{u-v}{2t}\right| + |b_1|\right\} \mathrm{d}t \le \int_0^\xi \frac{Mt}{2} + Kt(1+M\delta)^2 \,\mathrm{d}t \le \frac{M}{4}\xi^2 + K\xi^2(1+M\delta)^2,$$
$$|V| \le \int_0^\xi \left\{\left|\frac{v-u}{2t}\right| + |b_2|\right\} \mathrm{d}t \le \int_0^\xi \frac{Mt}{2} + Kt(1+M\delta)^2 \,\mathrm{d}t \le \frac{M}{4}\xi^2 + K\xi^2(1+M\delta)^2,$$

from which we have

$$\left|\frac{U(\xi,\eta)}{\xi^2}\right| + \left|\frac{V(\xi,\eta)}{\xi^2}\right| \le \frac{M}{2} + K(1+M\delta)^2. \tag{4.18}$$

In order to establish the bound of U_r/t^2, we differentiate (4.4) with respect to η to find that

$$\frac{\partial U}{\partial \eta}(\xi,\eta) = \int_0^{\xi} \left(\frac{u_r - v_r}{2t} + \frac{\partial b_1}{\partial r} \right) \cdot \frac{\partial r_+}{\partial \eta}\,\mathrm{d}t, \tag{4.19}$$

where

$$\frac{\partial r_+}{\partial \eta}(t;\xi,\eta) = \exp\left\{ \int_{\xi}^{t} \frac{\partial \Lambda_+(v)}{\partial r}\big(\tau, r_+(\tau;\xi,\eta)\big)\,\mathrm{d}\tau \right\}. \tag{4.20}$$

Applying (4.7), (4.13) and (4.15), we derive

$$\begin{aligned} \left|\frac{\partial U}{\partial \eta}\right| &\le \int_0^{\xi} \left(\left|\frac{u_r - v_r}{2t}\right| + \left|\frac{\partial b_1}{\partial r}\right| \right) \cdot \left|\frac{\partial r_+}{\partial \eta}\right| \mathrm{d}t \\ &\le \int_0^{\xi} \left(\frac{M}{2}t + Kt(1+M\delta)^2 \right) \exp\left\{Kt^3(1+M\delta)^2\right\} \mathrm{d}t \\ &\le \xi^2 \left(\frac{M}{4} + K(1+M\delta)^2 \right) \exp\left\{K\delta^3(1+M\delta)^2\right\}. \end{aligned}$$

A similar estimate holds for V. Hence we arrive at

$$\left|\frac{U_\eta}{\xi^2}\right| + \left|\frac{V_\eta}{\xi^2}\right| \le \left(\frac{M}{2} + K(1+M\delta)^2 \right) \exp\left\{K\delta^3(1+M\delta)^2\right\}. \tag{4.21}$$

To estimate U_r/t^2 and V_r/t^2, we differentiate (4.19) with respect to η to obtain

$$\frac{\partial^2 U}{\partial \eta^2}(\xi,\eta) = \int_0^{\xi} \left\{ \left(\frac{u_{rr} - v_{rr}}{2t} + \frac{\partial^2 b_1}{\partial r^2} \right) \left(\frac{\partial r_+}{\partial \eta} \right)^2 + \left(\frac{u_r - v_r}{2t} + \frac{\partial b_1}{\partial r} \right) \frac{\partial^2 r_+}{\partial \eta^2} \right\} \mathrm{d}t, \tag{4.22}$$

where

$$\frac{\partial^2 r_+}{\partial \eta^2} = \frac{\partial r_+}{\partial \eta} \cdot \int_{\xi}^{t} \frac{\partial^2 \Lambda_+}{\partial r^2} \cdot \frac{\partial r_+}{\partial \eta}\,\mathrm{d}\tau.$$

It follows by (4.15) and (4.20) that

$$\left|\frac{\partial^2 r_+}{\partial \eta^2}\right| \le K\delta^3(1+M\delta)^2 \exp\left\{K\delta^3(1+M\delta)^2\right\}.$$

Therefore, we have

$$\begin{aligned} \left|\frac{\partial^2 U}{\partial \eta^2}\right| &\le \int_0^{\xi} \left\{ \left(\left|\frac{u_{rr} - v_{rr}}{2t}\right| + \left|\frac{\partial^2 b_1}{\partial r^2}\right| \right) \left|\frac{\partial r_+}{\partial \eta}\right|^2 + \left(\left|\frac{u_r - v_r}{2t}\right| + \left|\frac{\partial b_1}{\partial r}\right| \right) \left|\frac{\partial^2 r_+}{\partial \eta^2}\right| \right\} \mathrm{d}t \\ &\le \int_0^{\xi} \{ \left(\frac{M}{2}t + Kt(1+M\delta)^2 \right) \left[1 + K\delta^3(1+M\delta)^2\right] \exp\left\{K\delta^3(1+M\delta)^2\right\} \mathrm{d}t \\ &\le \xi^2 \left(\frac{M}{4} + K(1+M\delta)^2 \right) \left[1 + K\delta^3(1+M\delta)^2\right] \exp\left\{K\delta^3(1+M\delta)^2\right\}. \end{aligned} \tag{4.23}$$

In the same way we have the bound for $V_{\eta\eta}$

$$\left|\frac{\partial^2 V}{\partial \eta^2}\right| \le \xi^2\left(\frac{M}{4}+K(1+M\delta)^2\right)\left[1+K\delta^3(1+M\delta)^2\right]\exp\left\{K\delta^3(1+M\delta)^2\right\},$$

which together with (4.23) yields

$$\left|\frac{U_{\eta\eta}}{\xi^2}\right|+\left|\frac{V_{\eta\eta}}{\xi^2}\right| \le \left(\frac{M}{2}+K(1+M\delta)^2\right)\left[1+K\delta^3(1+M\delta)^2\right]\exp\left\{K\delta^3(1+M\delta)^2\right\}. \quad (4.24)$$

We choose $M \ge 16K > 16$ and then set $\delta \le \min\{\delta_0, 1/M\}$ to see that

$$\begin{aligned}&\left(\frac{M}{2}+K(1+M\delta)^2\right)\left[1+K\delta^3(1+M\delta)^2\right]\exp\left\{K\delta^3(1+M\delta)^2\right\}\\ &\quad \le \left(\frac{M}{2}+\frac{M}{4}\right)\left(1+\frac{1}{64}\right)\exp\left(\frac{1}{64}\right) < \frac{5}{6}M < M.\end{aligned} \quad (4.25)$$

Therefore, we combine (4.18), (4.21) and (4.24) to conclude that (S_2)–(S_4) are preserved by the mapping $\mathcal{T}$.

To prove $\mathcal{T}(\mathbf{F}) \in \mathcal{S}$, it is enough to show that $U_\xi(0,\eta) = V_\xi(0,\eta) = 0$. We differentiate (4.4) with respect to ξ to arrive at

$$\frac{\partial U}{\partial \xi}(\xi,\eta) = \frac{u-v}{2\xi} + b_1 + \int_0^\xi \left(\frac{u_r - v_r}{2t} + \frac{\partial b_1}{\partial r}\right)\frac{\partial r_+}{\partial \xi}\,\mathrm{d}t, \quad (4.26)$$

where

$$\frac{\partial r_+}{\partial \xi}(t;\xi,\eta) = -\Lambda_+\big(\xi,\eta,V(\xi,\eta)\big)\cdot\frac{\partial r_+}{\partial \eta}(t;\xi,\eta). \quad (4.27)$$

It is easily seen by (4.7), (4.13), (4.15) and (4.20) that $U_\xi(0,\eta) = 0$. Similarly, we have $V_\xi(0,\eta) = 0$, which means that $\mathcal{T}$ maps $\mathcal{S}$ into itself.

Next we we show that the inequality (4.17) holds for some positive constant $\beta < 1$. According to the definition of the mapping $\mathcal{T}$, we have

$$\frac{d}{d_+(v)}U = \frac{u-v}{2t} + b_1(u,v,t,r), \qquad \frac{d}{d_+(\hat v)}\hat U = \frac{\hat u - \hat v}{2t} + b_1(\hat u,\hat v,t,r),$$

and from this and (4.1) one gets

$$\begin{aligned}&\frac{d}{d_+(v)}(U-\hat U)\\ &\quad = \frac{(u-\hat u)-(v-\hat v)}{2t} + \left[b_1(u,v,t,r) - b_1(\hat u,\hat v,t,r)\right] + \big(\Lambda_+(v)-\Lambda_+(\hat v)\big)\partial_r \hat U.\end{aligned} \quad (4.28)$$

Recalling the expression of b_1 suggests

$$\begin{aligned}&b_1(u,v,t,r) - b_1(\hat u,\hat v,t,r)\\ &\quad = \left(\frac{a' r^2[u+P_1(r)+P_2(r)t]}{2\sqrt{a}(a'\sqrt{a}v+f)} - \frac{1}{2}\right)\left(\frac{u-\hat u}{t} - \frac{v-\hat v}{t}\right)\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\hat{u}-\hat{v}}{t}+2P_2(r)\right)\left\{\frac{a'r^2[u+P_1(r)+P_2(r)t]}{2\sqrt{a}(a'\sqrt{a}v+f)}-\frac{a'r^2[\hat{u}+P_1(r)+P_2(r)t]}{2\sqrt{a}(a'\sqrt{a}\hat{v}+f)}\right\}\\
&-\left(\Phi(v)-\Phi(\hat{v})\right):=I+II+III.
\end{aligned}
\tag{4.29}
$$

For I, we find by (4.9) that

$$
|I|\le Kt(1+M\delta)\left(\left|\frac{u-\hat{u}}{t}\right|-\left|\frac{v-\hat{v}}{t}\right|\right)\le Kt^2(1+M\delta)d(\mathbf{F},\hat{\mathbf{F}}).
\tag{4.30}
$$

For II, one has

$$
\begin{aligned}
|II|&\le K(1+M\delta)\left|\frac{u+P_1(r)+P_2(r)t}{a'\sqrt{a}v+f}-\frac{\hat{u}+P_1(r)+P_2(r)t}{a'\sqrt{a}\hat{v}+f}\right|\\
&=K(1+M\delta)\left|\frac{(a'\sqrt{a}\hat{v}+f)(u-\hat{u})-[a'\sqrt{a}(P_1+P_2t)+\hat{u}](v-\hat{v})}{(a'\sqrt{a}v+f)(a'\sqrt{a}\hat{v}+f)}\right|\\
&\le Kt^2(1+M\delta)d(\mathbf{F},\hat{\mathbf{F}}).
\end{aligned}
\tag{4.31}
$$

By the definition of Φ in (4.12), it is easy to obtain

$$
|III|=\left|\Phi(v)-\Phi(\hat{v})\right|\le Kt^4d(\mathbf{F},\hat{\mathbf{F}}).
\tag{4.32}
$$

Putting (4.30)–(4.32) into (4.29) yields

$$
\left|b_1(u,v,t,r)-b_1(\hat{u},\hat{v},t,r)\right|\le Kt^2(1+M\delta)d(\mathbf{F},\hat{\mathbf{F}}).
\tag{4.33}
$$

In addition, we use the definition of Λ_+ to obtain

$$
\left|\Lambda_+(v)-\Lambda_+(\hat{v})\right|=\left|\frac{2rt^2}{a'\sqrt{a}v+f}-\frac{2rt^2}{a'\sqrt{a}\hat{v}+f}\right|\le Kt^4d(\mathbf{F},\hat{\mathbf{F}}).
\tag{4.34}
$$

Combining (4.28) and (4.33)–(4.34), we have

$$
\begin{aligned}
|U-\hat{U}|&\le\int_0^t\left(\frac{t}{2}+Kt^2(1+M\delta)+KMt^6\right)d(\mathbf{F},\hat{\mathbf{F}})\,\mathrm{d}t\\
&\le t^2\left\{\frac{1}{4}+K\delta(1+M\delta)\right\}d(\mathbf{F},\hat{\mathbf{F}}),
\end{aligned}
$$

from which one gets

$$
\left|\frac{U-\hat{U}}{t^2}\right|\le\left\{\frac{1}{4}+K\delta(1+M\delta)\right\}d(\mathbf{F},\hat{\mathbf{F}}).
$$

Following the same argument as above one obtains for the estimate $|V-\hat{V}|/t^2$,

$$
\left|\frac{U-\hat{U}}{t^2}\right|+\left|\frac{V-\hat{V}}{t^2}\right|\le\left\{\frac{1}{2}+2K\delta(1+M\delta)\right\}d(\mathbf{F},\hat{\mathbf{F}})=:\beta d(\mathbf{F},\hat{\mathbf{F}}).
$$

For choosing δ as before, we see that $\beta<1$, which concludes the proof of (4.17). The proof of Lemma 4.1 is complete. □

Step 4. Properties of the limit function. We claim that the limit of the iteration sequence $\mathbf{F}^{(n)} = \mathcal{T}\mathbf{F}^{(n-1)}$ is in the space $\mathcal{S}$. The claim follows directly from the lemma.

Lemma 4.2 *Under the assumptions of Theorem 1, for the iteration sequence* $\mathbf{F}^{(\mathbf{n})}$, $\partial_t\mathbf{F}^{(\mathbf{n})}(t,r)$ *and* $\partial_r\mathbf{F}^{(\mathbf{n})}(t,r)$ *are uniformly Lipschitz continuous on* $D(\delta)$.

Proof Assume $(u,v)^T \in \mathcal{S}$, we know by Lemma 4.1 that $(U,V)^T = \mathcal{T}(u,v)^T$ is also in $\mathcal{S}$. The proof is divided into three steps.

Firstly, we prove that $|U_t| + |V_t| \le 2Mt$. This follows directly from (4.4) and (4.5). In fact, we recall the expression of U_ξ given in (4.26) and use (4.7), (4.13) and (4.15) to obtain

$$\begin{aligned}\left|\frac{\partial U}{\partial \xi}\right| &\le \left|\frac{u-v}{2\xi}\right| + |b_1| + \int_0^\xi \left(\left|\frac{u_r - v_r}{2t}\right| + \left|\frac{\partial b_1}{\partial r}\right|\right)\left|\frac{\partial r_+}{\partial \xi}\right| \mathrm{d}t \\ &\le \frac{M\xi}{2} + K\xi(1+M\delta)^2 \\ &\quad + \int_0^\xi \left(\frac{Mt}{2} + Kt(1+M\delta)^2\right) \cdot K\delta^2(1+M\delta)^2 \exp\bigl\{K\delta^3(1+M\delta)^2\bigr\}\,\mathrm{d}t \le M\end{aligned}$$

for choosing M and δ as in (4.25).

Secondly, we show that $|U_{tr}| + |V_{tr}| \le 2Mt$. To prove it, we differentiate (4.19) with respect to ξ to obtain

$$\begin{aligned}\frac{\partial^2 U}{\partial\xi\partial\eta} &= \left(\frac{u_r - v_r}{2\xi} + \frac{\partial b_1}{\partial r}\right)\frac{\partial r_+}{\partial \eta} \\ &\quad + \int_0^\xi \left\{\left(\frac{u_{rr} - v_{rr}}{2t} + \frac{\partial^2 b_1}{\partial r^2}\right)\frac{\partial r_+}{\partial\eta}\frac{\partial r_+}{\partial\xi} + \left(\frac{u_r - v_r}{2t} + \frac{\partial b_1}{\partial r}\right)\frac{\partial^2 r_+}{\partial\xi\partial\eta}\right\}\mathrm{d}t,\end{aligned} \tag{4.35}$$

where

$$\frac{\partial^2 r_+}{\partial\xi\partial\eta} = \frac{\partial r_+}{\partial\eta}\left\{\int_\xi^t \frac{\partial^2 \Lambda_+(v)}{\partial r^2}\cdot\frac{\partial r_+}{\partial\xi}\,\mathrm{d}\tau - \frac{\partial\Lambda_+(v)}{\partial r}\right\}.$$

By employing (4.7), (4.13), (4.15), (4.20) and (4.27), we find that

$$\begin{aligned}\left|\frac{\partial^2 U}{\partial\xi\partial\eta}\right| &\le \left(\frac{M\xi}{2} + K\xi(1+M\delta)^2\right)\exp\bigl\{K\delta^3(1+M\delta)^2\bigr\} \\ &\quad + \int_0^\xi \left(\frac{Mt}{2} + Kt(1+M\delta)^2\right)\exp\bigl\{2K\delta^3(1+M\delta)^2\bigr\}\bigl(Kt^2 + Kt^2(1+M\delta)^2\bigr)\,\mathrm{d}t \\ &\le M\xi,\end{aligned}$$

if M and δ are chosen as in (4.25), from which and the corresponding estimate for $V_{\xi\eta}$ we have $|U_{\xi\eta}| + |V_{\xi\eta}| \le 2M\xi$.

Finally, we claim that $|U_{tt}| + |V_{tt}| \le 10M$. Differentiating (4.26) with respect to ξ leads to

$$\begin{aligned}\frac{\partial^2 U}{\partial\xi^2} &= \frac{u_\xi - v_\xi}{2\xi} - \frac{u-v}{2\xi^2} + \frac{\partial b_1}{\partial\xi} + \left(\frac{u_r - v_r}{2\xi} + \frac{\partial b_1}{\partial r}\right)\frac{\partial r_+}{\partial\xi} \\ &\quad + \int_0^\xi \left\{\left(\frac{u_{rr} - v_{rr}}{2t} + \frac{\partial^2 b_1}{\partial r^2}\right)\left(\frac{\partial r_+}{\partial\xi}\right)^2 + \left(\frac{u_r - v_r}{2t} + \frac{\partial b_1}{\partial r}\right)\frac{\partial^2 r_+}{\partial\xi^2}\right\}\mathrm{d}t,\end{aligned} \tag{4.36}$$

where

$$\frac{\partial^2 r_+}{\partial \xi^2} = -\frac{\partial \Lambda_+}{\partial \xi}\frac{\partial r_+}{\partial \eta} - \Lambda_+ \frac{\partial^2 r_+}{\partial \xi \partial \eta}.$$

By a direct calculation, one has

$$\begin{aligned}\left|\frac{\partial^2 r_+}{\partial \xi^2}\right| &\leq K\xi(1+M\delta)\exp\bigl\{K\delta^3(1+M\delta)^2\bigr\} + K\xi^4(1+M\delta)^2\exp\bigl\{2K\delta^3(1+M\delta)^2\bigr\} \\ &\leq 2K\xi(1+M\delta)\exp\bigl\{K\delta^3(1+M\delta)^2\bigr\}. \end{aligned} \tag{4.37}$$

Furthermore, according the expression of b_1 arrives at

$$\begin{aligned}\frac{\partial b_1}{\partial t} = {}& \left(\frac{a' r^2(u+P_1+P_2t)}{2\sqrt{a}[a'\sqrt{a}v+f]} - \frac{1}{2}\right)\left(\frac{u_t - v_t}{t} - \frac{u-v}{t^2}\right) \\ &+ \frac{\partial}{\partial t}\left(\frac{a' r^2(u+P_1+P_2t)}{2\sqrt{a}[a'\sqrt{a}v+f]}\right)\left(\frac{u-v}{t} + 2P_2\right) \\ &- \frac{2t^2 r P_2'}{a'\sqrt{a}v+f} - \frac{4tr(P_1'+P_2't)}{a'\sqrt{a}v+f} + \frac{2t^2 r(P_1'+P_2't)}{(a'\sqrt{a}v+f)^2}\partial_t\bigl(a'\sqrt{a}v+f\bigr),\end{aligned}$$

from which and (4.7), (4.9) and the fact $|u_t| + |v_t| \leq 2Mt$ we obtain the estimate of b_{1t} by a straight forward calculation

$$\left|\frac{\partial b_1}{\partial t}\right| \leq K(1+M\delta)^3.$$

Inserting the above and (4.37) into (4.36), we see that

$$\begin{aligned}\left|\frac{\partial^2 U}{\partial \xi^2}\right| &\leq M + M + K(1+M\delta)^3 + K\delta^2\bigl(M\delta + K\delta(1+M\delta)\bigr)\exp\bigl\{2K\delta^3(1+M\delta)^2\bigr\} \\ &\quad + K\delta^2(1+M\delta)^2\exp\bigl\{2K\delta^3(1+M\delta)^2\bigr\} \leq 5M \end{aligned} \tag{4.38}$$

for choosing M and δ as in (4.25), i.e., $M \geq 16K$ and $\delta \leq 1/M$. Repetition of the same argument for V obtains $|V_{\xi\xi}| \leq 5M$, which together with (4.38) gets $|U_{\xi\xi}| + |V_{\xi\xi}| \leq 10M$.

The proof of Lemma 4.2 is completed by Lemma 4.1 and the above estimates. □

With the help of Lemma 4.1 and Lemma 4.2, we conclude Theorem 3.

Step 5. Convert solution back to original variables. We now return the solution in the coordinate plane $(t,\tilde{r})$ into the original coordinate plane (r,θ). By the definitions of U and V in (3.8), we first know the functions $R(t,\tilde{r})$ and $S(t,\tilde{r})$. In addition, we see that the coordinate transformation $(r,\theta) \mapsto (t,\tilde{r})$ is a one-to-one mapping. Indeed, the Jacobian is

$$J = \frac{\partial(t,\tilde{r})}{\partial(r,\theta)} = \frac{a'(P)[U+V+2P_1(r)]}{4t},$$

which is strictly positive or strictly negative. Therefore, we can obtain R and S as functions of r and θ. We integrate the first equation of (2.9) to obtain the function $P(r,\theta)$. Thus the proof of Theorem 1 is complete. In order to conclude Theorem 2, it suffices to check that the relation $P_r = (R - S)/(2\lambda)$ holds on $D(\delta)$. By a direct calculation, we find that

$$\partial_\theta(R - S - 2\lambda P_r) = -\frac{a'(P)r^2(R+S)}{4a(P)[r^2 - a(P)]}(R - S - 2\lambda P_r). \tag{4.39}$$

Since $P_r = P_2(r)$ is uniformly bounded and $R = S = P_1(r)$ on Γ, $G := R - S - 2\lambda P_r = 0$ on Γ. We note that, in terms of $(t, \tilde{r})$, Eq. (4.39) can be rewritten as

$$\partial_t G = \frac{r^2}{a} \cdot \frac{G}{t}. \tag{4.40}$$

Moreover, we have

$$\frac{G}{t}\bigg|_{t=0} = \frac{U - V + 2P_2(r)t - \frac{2rt}{\sqrt{r^2-t^2}}P_r}{t}\bigg|_{t=0} = \frac{U-V}{t}\bigg|_{t=0} = 0,$$

which together with (4.40) leads to $G \equiv 0$ on $D(\delta)$, which completes the proof of Theorem 2.

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