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Periodic solutions for second order damped boundary value problem with nonnegative Green's functions

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Abstract

In this article, we study the existence of positive periodic solutions of second order damped boundary value problem $u'' + p(t)u' + q(t)u = g(t, u, u')$, $u(0) = u(T)$, $u'(0) = u'(T)$. The main tools are the nonlinear alternative principle of Leray–Schauder and Schauder's fixed point theorem. We emphasize that the damped term and nonnegative Green's functions are the key points. We also apply the results to examples for testing. Some recent results in the literature are improved and generalized.

MSC: 34B15; 34B16; 34D20

Keywords: Second order damped boundary value problem; Nonnegative Green's function; Positive periodic solutions; Schauder's fixed point theorem; Leray–Schauder alternative principle

1 Introduction

The main purpose of the article is to research the existence of positive T-periodic solutions for the nonlinear damped boundary value problem

$$u'' + p(t)u' + q(t)u = g(t, u, u'), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{1.1}$$

where $p, q \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, $g \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}) \times (0, \infty) \times \mathbb{R}, \mathbb{R})$.

When $p(t) = 0$ and the nonlinearity excludes the derivative term, (1.1) becomes the following boundary value problem:

$$u'' + q(t)u = g(t, u), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{1.2}$$

which has been widely studied in many literatures during the previous two decades. See, for example, [9, 21, 27] for the regular cases and [1, 10, 22, 31, 32, 34] for the singular cases. Usually, variational methods or topological methods have been widely applied in the research. Particularly, the nonlinear alternative principle of Leray–Schauder [5, 11, 20], cone fixed point theorems [3, 6, 12, 16, 29], Schauder's fixed point theorem [10, 15, 30], degree theory [14, 32, 33], and the method of upper and lower solutions [18, 27] are several classical tools.

However, it has not attracted much attention to the research of damped boundary value problem (1.1) due to the difficulty of damped term and the derivative dependence in the nonlinearity. Several existence results can be found in [5, 23] for scalar linear differential equation and [8, 33] for system, the stability result of differential equation (1.1) can be found in [4]. Some other results about damped problem can be found in [13, 26, 28].

In the article, we research the existence of positive T-periodic solutions of (1.1) using a nonlinear alternative of Leray–Schauder and Schauder's fixed point theorem, which have been used in [7, 10, 11, 20] when the nonlinearity excludes the derivative term. In order to overcome technical difficulty, we choose the right function space and the associated norm which are required to deal with the derivative dependence in the nonlinearity. Furthermore, we obtain two different existence results of T-periodic solutions of problem (1.1) when the nonlinearity is dependent on the derivative. These new results generalize some recent results in [10, 11, 20, 29, 33] in several aspects.

In the above mentioned articles, when some fixed point theorems or the nonlinear alternative principle of Leray–Schauder in cones are applied to study the existence of periodic solutions of boundary value problem (1.1) or (1.2), one main assumption is very important, that is: the associated Green's function $G(t,s)$ for the linear boundary value problem

$$u'' + q(t)u = 0, \qquad u(0) = u(T), \qquad u'(0) = u'(T) \tag{1.3}$$

or

$$u'' + p(t)u' + q(t)u = 0, \qquad u(0) = u(T), \qquad u'(0) = u'(T) \tag{1.4}$$

is positive, which implies that equations (1.3) or (1.4) satisfy the strict anti-maximum principle.

However, in this paper, we loosen the assumption condition and consider the following case:

(H_1) For all $(t,s) \in [0,T] \times [0,T]$, the corresponding Green's function $G(t,s)$ of (1.4) is nonnegative.

The main motivation of this article is from the recent papers [17, 24], in which the second order systems have been studied in the case where the associated nonnegative Green's functions may have zeros. Especially, we observe that even when the Green's function vanishes, the following fact also holds:

$$\delta = \min_{0\le s\le T} \int_0^T G(t,s)\,dt > 0.$$

Based on this fact, Graef, Kong, and Wang constructed the following cone in [17]:

$$K = \left\{ u \in X : u(t) \ge 0 \text{ and } \int_0^T u(t)\,dt \ge \frac{\delta}{M}\|u\| \right\}. \tag{1.5}$$

Using the above cone, they proved that the equation

$$u'' + q(t)u = g(t)f(u)$$

had at least a nontrivial T-periodic solution for the superlinear or sublinear case in [17]. Liao [24] used it to establish that the scalar problem (1.2) had at least two positive T-periodic solutions under some conditions.

In this paper, we also use the cone defined in [17] to study the existence and multiplicity of nontrivial T-periodic solutions for the damped boundary value problem (1.1). The main tools are the nonlinear alternative principle of Leray–Schauder and Schauder's fixed point theorem. We emphasize that there are two new technical difficulties. One is that the corresponding nonnegative Green's function of nonlinear system of (1.1) may have zeros, the other is that the nonlinearity of system (1.1) includes the derivation term.

To test the new result, we apply it to an example. We obtain new existence results of positive T-periodic solutions for the following boundary value problem:

$$\begin{aligned} &u'' + p(t)u' + q(t)u = \bigl(h(t) + k(t)\bigl|u'\bigr|^{\alpha}\bigr)\bigl(\bigl(a(t)u^{\beta} + \nu b(t)u^{\gamma}\bigr) + e(t)\bigr), \\ &u(0) = u(T), \qquad u'(0) = u'(T), \end{aligned} \tag{1.6}$$

where $p, q, h, k, a, b, e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, α, β, γ, and ν are positive parameters. The equation of (1.6) unifies many of the special characters of examples: damping term, depending on the derivative, and so on. So it is usually considered for academic purposes to illustrate how our way is applicable to complication problem. The damping term often appears in many of physics models.

The rest of the article is organized as follows. In the next section, we give two known results concerning the sign of Green's function of the corresponding linear damped boundary value problem (1.4). There are two classes of functions p, q which can guarantee that the Green's function of (1.4) is nonnegative.

In Sect. 3, we establish and prove one existence result for problem (1.1) if the Green's function of (1.4) is nonnegative. We also give application of the new results to (1.6). We have generalized those results in [24] and improved those in [5].

In Sect. 4, by applying Schauder's fixed point theorem, we establish another different existence result of (1.1).

2 Preliminaries and notation

For a given function $p \in L^1[0, T]$, let p_* and p^* be the essential infimum and supremum if they exist. The usual L^p-norm is denoted by $\|\cdot\|_p$. If $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$, we call $\tilde{p}$ the conjugate exponent of p. We define the set $\tilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}) := \{p(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}) : \bar{p} = \frac{1}{T}\int_0^T p(s)\,ds = 0\}$. Given $\psi \in L^1[0, T]$ if $\psi \geq 0$ for all $t \in [0, T]$ and it is positive in a set of positive measures, we write as $\psi \succ 0$.

Let $G(t, s)$ be the Green's function of problem (1.4). We say that the unique T-periodic solution of (1.4) is nonresonant when it is the trivial one. Therefore, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$u'' + p(t)u' + q(t)u = e(t) \tag{2.1}$$

has a unique T-periodic solution denoted by

$$u(t) = \int_0^T G(t, s)e(s)\,ds,$$

where $p, q, e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$.

Definition 2.1 For any $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, if (2.1) has a unique T-periodic solution x_e which satisfies $x_e > 0$ for all t if $e \succ 0$, we say that equation (1.4) satisfies the anti-maximum principle.

Theorem 2.2 *Suppose that equation* (1.4) *satisfies the anti-maximum principle. Then the corresponding Green's function* $G(t,s)$ *of* (1.4) *is nonnegative for all* $(t,s) \in [0,T] \times [0,T]$.

Proof The proof process is similar to the proof of [36, Theorem 4.1], for the case $p(t) \equiv 0$. □

In Sects. 3 and 4, we suppose that hypothesis (H_1) is satisfied, which means that equation (1.4) satisfies the anti-maximum principle.

In order to guarantee that (H_1) is satisfied, we present two known results.

Firstly, Hakl and Torres have given an explicit criterion to prove that (1.4) satisfies the anti-maximum principle in [19] for the general case $p, q \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$. To describe these simply, let us write the functions

$$\sigma(p)(t) = \exp\left(\int_0^t p(s)\,ds\right)$$

and

$$\sigma_1(p)(t) = \sigma(p)(T)\int_0^t \sigma(p)(s)\,ds + \int_t^T \sigma(p)(s)\,ds.$$

Theorem 2.3 ([19]) *Suppose that* $q \not\equiv 0$ *and*

$$\int_0^T q(s)\sigma(p)(s)\sigma_1(-p)(s)\,ds \geq 0, \tag{2.2}$$

$$\sup_{0\leq t\leq T}\left\{\int_t^{t+T}\sigma(-p)(s)\,ds\int_t^{t+T}\big[q(s)\big]_+\sigma(p)(s)\,ds\right\} \leq 4, \tag{2.3}$$

where $[q(s)]_+ = \max\{q(s), 0\}$. *Then equation* (1.4) *satisfies the anti-maximum principle.*

Remark 2.4 According to Theorem 2.2, under the condition of Theorem 2.3, the corresponding Green's function G(t, s) of (1.4) is nonnegative for all $(t,s) \in [0,T] \times [0,T]$.

Secondly, if $\int_0^T q(s)\sigma(p)(s)\,ds > 0$ and $\bar{p} = 0$, Cabada and Cid in [2] established another criterion. Before describing these, some preliminaries are listed. Given an exponent $q \in [1, \infty]$, let $\mathbf{M}(q)$ be the best constant in the Sobolev inequality

$$C\|u\|_{q,[0,1]} \leq \|u'\|_{2,[0,1]} \quad \text{for all } u \in H_0^1(0,1).$$

The explicit formula for $\mathbf{M}(q)$ is known. See [35]. That is,

$$\mathbf{M}(q) = \begin{cases} (\frac{2\pi}{q})^{1/2}(\frac{2}{2+q})^{1/2-1/q}\frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})} & \text{if } 1 \leq q < \infty, \\ 2 & \text{if } q = \infty, \end{cases}$$

where $\Gamma(\cdot)$ is the gamma function of Euler. Especially, $\mathbf{M}(1) = \sqrt{12}$, $\mathbf{M}(2) = \pi$.

Theorem 2.5 ([2]) *Suppose that* $\bar{p} = 0$, $\int_0^T q(s)\sigma(p)(s) > 0$. *Furthermore, we assume that there exists* $p \in [1, \infty]$ *such that*

$$\big(B(T)\big)^{1+1/q} \|Q_+\|_{p,T} \leq \mathbf{M}^2(2q),$$

where

$$B(T) = \int_0^T \sigma(-p)(s)\,ds$$

and

$$Q_+(t) = \big[q(t)\big]_+ \big(\sigma(p)(t)\big)^{2-1/p}.$$

Then the associated Green's function $G(t, s)$ *of* (1.4) *is nonnegative.*

Under hypothesis (H_1), we usually denote

$$M = \max_{0 \leq s,t \leq T} G(t,s), \qquad w(t) = \int_0^T G(t,s)\,ds.$$

The following nonlinear alternative principle of Leray–Schauder is the main tool for us to prove the first existence result, which can be seen in [25].

Theorem 2.6 *Assume that* Ω *is a relatively compact subset of a convex set* K *in a normed space* X. *Let* $T : \bar{\Omega} \to K$ *be a compact map with* $0 \in \Omega$. *Then one of the following two conclusions holds*:

(I) T *has at least one fixed point in* $\bar{\Omega}$.

(II) *There exist* $x \in \partial\Omega$ *and* $0 < \lambda < 1$ *such that* $x = \lambda Tx$.

3 Existence result (I)

We establish the first existence result of (1.1) using nonlinear alternative principle of Leray–Schauder in this section. Here we use cone (1.5) even if the Green's function is nonnegative. We also have that the following fact holds:

$$\delta = \min_{0 \leq s \leq T} \int_0^T G(t,s)\,dt > 0.$$

Theorem 3.1 *Assume that* (1.4) *satisfies* (H_1) *and* $\bar{p} = 0$,

$$\int_0^T q(t)\sigma(p)(t)\,dt > 0. \tag{3.1}$$

Furthermore, we suppose that there exists a positive constant r *such that, for all* $(t, u, v) \in [0, T] \times (0, r] \times \mathbb{R}$:

(H_2) *There exists a continuous function* $\psi \succ 0$ *satisfying* $g(t, u, v) \geq \psi(t)$.

(H_3) *There exist nonnegative continuous and nondecreasing functions* $h(\cdot)$ *in* $(0, r]$ *and* $\rho(\cdot)$ *in* $(0, \infty)$ *such that*

$$0 \leq g(t, u, v) \leq h(u)\rho\big(|v|\big).$$

(H_4)

$$h(r)M\omega^* T\rho(Nr) < \delta r,$$

where

$$N = \frac{2\int_0^T q(t)\sigma(p)(t)\,dt}{\min_{0\le t\le T}\sigma(p)(t)}. \tag{3.2}$$

Then the damped boundary value problem (1.1) *has at least a positive* T*-periodic solution* u *with* $0 < \|u\| < r$.

Proof We will apply Theorem (2.6). To do this, we first consider the family of equations

$$u'' + p(t)u' + q(t)u = \lambda g\bigl(t, u, u'\bigr), \tag{3.3}$$

where $\lambda \in [0,1]$. Note that a T-periodic solution of (3.3) is just a fixed point of the operator equation

$$u(t) = \lambda(\mathrm{A}u)(t), \tag{3.4}$$

where A is defined by

$$(\mathrm{A}u)(t) = \int_0^T G(t,s)g\bigl(s, u(s), u'(s)\bigr)\,ds,$$

it is a continuous and completely continuous operator.

We claim that, for any $\lambda \in [0,1]$, equation (3.4) has no fixed point u with $u \in \partial\Omega$, where

$$\Omega = \bigl\{u \in X : \|u\| < r\bigr\},$$

and $X = \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ is a Banach space by taking the norm $\|u\| = \max_{t\in[0,T]}|u(t)|$ for $u \in X$. Obviously, Ω is an open subset in X with $0 \in \Omega$.

Otherwise, assume that there exists $\lambda \in [0,1]$ such that equation (3.4) has a fixed point u with $\|u\| = r$. Notice that

$$\begin{aligned}
\int_0^T u(t)\,dt &= \lambda\int_0^T\int_0^T G(t,s)g\bigl(s,u(s),u'(s)\bigr)\,ds\,dt \\
&= \lambda\int_0^T g\bigl(s,u(s),u'(s)\bigr)\int_0^T G(t,s)\,dt\,ds \\
&\ge \lambda\delta\int_0^T g\bigl(s,u(s),u'(s)\bigr)\,ds \\
&= \frac{\delta}{M}M\lambda\int_0^T g\bigl(s,u(s),u'(s)\bigr)\,ds \\
&\ge \frac{\delta}{M}\max_{t\in[0,T]}\left\{\lambda\int_0^T G(t,s)g\bigl(s,u(s),u'(s)\bigr)\,ds\right\} \\
&= \frac{\delta}{M}\|u\|.
\end{aligned}$$

Hence, for all t, we obtain

$$\int_0^T u(t)\,dt \ge \frac{\delta}{M}\|u\| = \frac{\delta}{M}r.$$

Next multiplying by an integral factor $\sigma(p)(t)$ for equation (3.3), we obtain that equations (3.3) are equivalent to

$$\bigl(\sigma(p)(t)u'\bigr)' + q(t)\sigma(p)(t)u = \sigma(p)(t)\bigl(\lambda g\bigl(t,u(t),u'(t)\bigr)\bigr) \tag{3.5}$$

for any T-periodic solution $u(t)$.

Integrating (3.5) from 0 to T, using the period boundary conditions and $\bar{p} = 0$, we have

$$\int_0^T q(t)\sigma(p)(t)u(t)\,dt = \int_0^T \sigma(p)(t)\bigl(\lambda g\bigl(t,u(t),u'(t)\bigr)\bigr).$$

Furthermore, using $u(0) = u(T)$, we obtain that there exist some $t_0 \in [0,T]$ satisfying $u'(t_0) = 0$. Therefore

$$\begin{aligned}
\bigl|\sigma(p)(t)u'(t)\bigr| &= \left|\int_{t_0}^t \bigl(\sigma(p)(s)u'(s)\bigr)'\,ds\right| \\
&= \left|\int_{t_0}^t \sigma(p)(s)\bigl(\lambda g\bigl(t,u(s),u'(s)\bigr) - q(s)u(s)\bigr)\,ds\right| \\
&\le \int_0^T \sigma(p)(s)\bigl(\lambda g\bigl(t,u(s),u'(s)\bigr) + q(s)u(s)\bigr)\,ds \\
&= 2\int_0^T q(s)\sigma(p)(s)u(s)\,ds \\
&\le 2r\int_0^T q(s)\sigma(p)(s)\,ds,
\end{aligned}$$

where assumption (3.1) is used. Therefore,

$$\Bigl(\min_{0\le t\le T}\sigma(p)(t)\Bigr)\bigl|u'(t)\bigr| \le 2r\int_0^T q(s)\sigma(p)(s)\,ds.$$

So we have that

$$\bigl|u'(t)\bigr| \le \frac{2r\int_0^T q(t)\sigma(p)(t)\,dt}{\min_{0\le t\le T}\sigma(p)(t)} = Nr. \tag{3.6}$$

Thus due to condition (H_3), for all $t \in [0,T]$, we have

$$\begin{aligned}
\int_0^T u(t)\,dt &= \lambda\int_0^T\int_0^T G(t,s)g\bigl(s,u(s),u'(s)\bigr)\,ds\,dt \\
&\le \int_0^T\int_0^T G(t,s)g\bigl(s,u(s),u'(s)\bigr)\,ds\,dt \\
&\le \int_0^T\int_0^T G(t,s)h\bigl(u(s)\bigr)\rho\bigl(\bigl|u'(s)\bigr|\bigr)\,ds\,dt
\end{aligned}$$

$$\begin{aligned}&\le \int_0^T \int_0^T G(t,s)h(r)\rho(Nr)\,ds\,dt\\&\le \omega^* Th(r)\rho(Nr).\end{aligned}$$

Therefore,

$$\frac{\delta}{M}r \le \omega^* Th(r)\rho(Nr).$$

This is a contradiction to condition (H_4). By applying Theorem(2.6), we get that $u = \mathrm{A}u$ has a fixed point u in Ω, i.e., the damped boundary value problem (1.1) has one T-periodic solution u with $\|u\| < r$.

Finally, by condition (H_2), we have that

$$\begin{aligned}\int_0^T u(t)\,dt &= \int_0^T \int_0^T G(t,s)g\bigl(s,u(s),u'(s)\bigr)\,ds\,dt\\&\ge \int_0^T \int_0^T G(t,s)\psi(s)\,ds\,dt\\&= \int_0^T \psi(s)\int_0^T G(t,s)\,dt\,ds\\&\ge \delta\int_0^T \psi(s)\,ds > 0,\end{aligned}$$

which means that u is one positive T-periodic solution of the damped boundary value problem (1.1). □

In what follows, we give an application of the first existence Theorem 3.1.

Example 3.2 Suppose that (1.4) satisfies (H_1), consider the damped boundary value problem

$$\begin{aligned}&u'' + p(t)u' + q(t)u = \bigl(1 + |u'|^\alpha\bigr)\bigl(u^\beta + \nu u^\gamma + e(t)\bigr),\\&u(0) = u(T), \qquad u'(0) = u'(T),\end{aligned} \tag{3.7}$$

where $p \in \tilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z},\mathbb{R})$, $q, e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z},\mathbb{R})$, $e \succ 0$, $\alpha > 0$, $0 < \beta < 1$, $\gamma > \beta$, and $\nu > 0$ is a parameter. Then

(i) for each $\nu > 0$, (3.7) has at least a nontrivial T-periodic solution if $\alpha + \gamma < 1$;

(ii) for each $0 < \nu < \nu_1$, (3.7) has at least a nontrivial T-periodic solution if $\alpha + \gamma \ge 1 > \alpha + \beta$, where ν_1 is some constant.

Proof In order to apply Theorem 3.1, we let

$$\psi(t) = e(t), \qquad h(u) = u^\beta + \nu u^\gamma + e^*, \qquad \rho(u) = 1 + |u|^\alpha,$$

where $e^* = \max_t e(t)$. Since $e \succ 0$, (H_2) is satisfied. Clearly, $\rho(x)$, $h(x)$ are nondecreasing in $x \in \mathbb{R}_+$ for $0 < \beta < \gamma$. Then (H_3) is satisfied. Moreover, existence condition (H_4) is equivalent to

$$\nu < \frac{\delta r - L^\alpha MT\omega^* r^{\alpha+\beta} - MT\omega^* r^\beta - L^\alpha MT\omega^* e^* r^\alpha - MT\omega^* e^*}{L^\alpha MT\omega^* r^{\alpha+\gamma} + MT\omega^* r^\gamma}$$

for some $r > 0$. Therefore, the damped boundary value problem (3.7) has at least a non-trivial T-periodic solution for

$$0 < \nu < \nu_1 := \sup_{r>0} \frac{\delta r - L^\alpha MT\omega^* r^{\alpha+\beta} - MT\omega^* r^\beta - L^\alpha MT\omega^* e^* r^\alpha - MT\omega^* e^*}{L^\alpha MT\omega^* r^{\alpha+\gamma} + MT\omega^* r^\gamma}.$$

We notice that if $\alpha + \gamma < 1$, $\nu_1 = \infty$; if $\alpha + \gamma \geq 1 > \alpha + \beta$, $\nu_1 < \infty$. Thus the desired results (i) and (ii) are obtained. □

4 Existence result (II)

By using Schauder's fixed point theorem, we establish the second existence result for (1.1) in this section.

Theorem 4.1 *Consider the damped boundary value problem* (1.1). *Assume that* (H_1) *is satisfied for* (1.4), $p \in \tilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z})$ *and* (3.1) *is satisfied. Furthermore, we suppose that the following conditions hold*:

(H_5) *For each* $S > 0$, *there exists a continuous function* $\phi_S \succ 0$, *which satisfies* $g(t,u,v) \geq \phi_S(t)$ *for all* $(t,u,v) \in [0,T] \times (0,S] \times \mathbb{R}$.

(H_6) *There exist nonnegative continuous functions* $h(\cdot)$, $l(\cdot)$, *and* $\rho(\cdot)$, *which satisfy*

$$0 \leq g(t,u,v) \leq l(t)h(u)\rho\bigl(|v|\bigr) \quad \textit{for all } (t,u,v) \in [0,T] \times (0,\infty) \times \mathbb{R},$$

and $h(\cdot)$ *and* $\rho(\cdot)$ *are nondecreasing in* $(0,\infty)$.

(H_7) *There exists a constant* $R > 0$ *which satisfies* $R > \Phi_{R*} = r > 0$ *and*

$$R \geq h(R)L^* \rho(NR),$$

where N *is the same constant as in* (3.2) *and*

$$\Phi_R(t) = \int_0^T G(t,s)\phi_R(s)\,ds,$$

$$L(t) = \int_0^T G(t,s)l(s)\,ds.$$

Then the damped boundary value problem (1.1) *has at least a positive* T*-periodic solution* u *with* $r \leq \|u\| \leq R$.

Proof Let $\mathbb{C}_T$ denote the set of all continuous T-periodic functions. Define a completely continuous map $\mathrm{A} : \mathbb{C}_T \to \mathbb{C}_T$:

$$(\mathrm{A}u)(t) = \int_0^T G(t,s)g\bigl(s,u(s),u'(s)\bigr)\,ds. \tag{4.1}$$

Then a periodic solution of the damped boundary value problem (1.1) is just a fixed point of A.

Denote R to be the positive constant which satisfies (H_7). Define the set

$$\mathcal{D} = \bigl\{u \in \mathbb{C}_T : r \leq u(t) \leq R \text{ for all } t \in [0,T]\bigr\}.$$

Clearly, $\mathcal{D}$ is a convex closed set. Furthermore, on the one hand, for each $x \in \mathcal{D}$ and for all $t \in [0, T]$, due to condition (H_5) and the nonnegative $G(t,s)$ for all $(t,s) \in [0,T] \times [0,T]$, we obtain

$$(\mathrm{A}u)(t) \geq \int_0^T G(t,s)\phi_R(s)\,ds \geq \Phi_{R*} = r > 0.$$

On the other hand, using the similar deduction in Sect. 3, the following inequality can be obtained:

$$\left|u'(t)\right| \leq NR,$$

where N is the constant in (3.2). Therefore, by conditions (H_6) and (H_7), we obtain that

$$\begin{aligned}(\mathrm{A}u)(t) &\leq \int_0^T G(t,s)l(s)h\bigl(x(s)\bigr)\rho\bigl(\left|u'(s)\right|\bigr)\,ds \\ &\leq h(R)L^*\rho(NR) \leq R.\end{aligned}$$

In conclusion, $\mathrm{A}(\mathcal{D}) \subset \mathcal{D}$. To apply Schauder's fixed point theorem, the damped boundary value problem (1.1) has at least a positive T-periodic solution u with $r \leq \|u\| \leq R$. □

In what follows, we also give an example for an application of the second existence Theorem 4.1.

Example 4.2 Suppose that (1.4) satisfies (H_1). We consider the damped boundary value problem (3.7) again, where $p \in \tilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, $q, e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, $e \succ 0$, $\alpha > 0$, $0 < \beta < 1$, $\gamma > \beta$, and ν is a positive parameter. Then
- (i) if $\alpha + \gamma < 1$, (3.7) has at least one nontrivial T-periodic solution for each $\nu > 0$;
- (ii) if $\alpha + \gamma \geq 1 > \alpha + \beta$, (3.7) has at least one nontrivial T-periodic solution for each $0 < \nu < \nu_2$, where ν_2 is some positive constant.

Proof We take

$$\phi(t) = e(t), \qquad h(u) = u^\beta + \nu u^\gamma + e^*, \qquad l(t) = 1, \qquad \rho(u) = 1 + |u|^\alpha.$$

Clearly, (H_5) is satisfied due to $e \succ 0$. Moreover, since $0 < \beta < \gamma$, it is easy to verify that $h(\cdot)$, $\rho(\cdot)$, and $l(\cdot)$ are nondecreasing in $(0, +\infty)$. So (H_6) is satisfied. Now condition (H_7) becomes

$$\nu \leq \frac{R - \omega^* R^\beta - R^{\alpha+\beta}\omega^* L^\alpha - R^\alpha \omega^* L^\alpha e^* - e^*\omega^*}{\omega^* R^\gamma + \omega^* R^{\alpha+\gamma} L^\alpha}$$

for some $R > 0$. Therefore, by applying Theorem 4.1, the damped boundary value problem (3.7) has at least one nontrivial T-periodic solution for

$$0 < \nu < \nu_2 := \sup_{R>0} \frac{R - \omega^* R^\beta - R^{\alpha+\beta}\omega^* L^\alpha - R^\alpha \omega^* L^\alpha e^* - e^*\omega^*}{\omega^* R^\gamma + \omega^* R^{\alpha+\gamma} L^\alpha}.$$

Notice that if $\alpha + \gamma < 1$, $\nu_2 = \infty$; if $\alpha + \gamma \geq 1 > \alpha + \beta$, $\nu_2 < \infty$. So the desired results (i) and (ii) are obtained. □

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