Wang and Ru *Boundary Value Problems* (2019) 2019:9
https://doi.org/10.1186/s13661-019-1124-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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Some existence results of positive solutions for p-Laplacian systems

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Abstract

In this paper, we establish several new existence theorems for positive solutions of systems of $(2n, 2m)$-order of two p-Laplacian equations. The results are based on the Krasnosel'skii fixed point theorem and mainly complement those of Djebali, Moussaoui, and Precup.

Keywords: p-Laplacian; Positive solution; Fixed point theorem

1 Introduction

Quasilinear elliptic systems have been used in a great variety of applications, and existence results and a priori estimates of positive solutions for quasilinear elliptic systems have been broadly investigated. For instance, D'Ambrosio and Mitidieri [1] studied Liouville theorems for a class of possibly singular quasilinear elliptic equations and inequalities in the framework of Carnot groups, and their results are new even in the canonical Euclidean setting. In [2] the authors proved a priori estimates for the solutions of elliptic systems involving quasilinear operators in divergence form in an open set $\Omega \subset R^N$ and, as a consequence, obtained theorems on nonexistence of positive solutions in the case $\Omega = R^N$. More related results can be found in [3–5]. Equations of the p-Laplacian form occur in the study of non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium. To our best knowledge, there are many papers devoted to the study of differential equations with p-Laplacian. We refer the readers to [6] (one-dimensional p-Laplacian), [7–10] (fourth-order p-Laplacian), and [11] (2nd-order p-Laplacian). Most of these results are based upon the quadrature method, topological degree, the fixed point theorem on a cone, or the lower and upper solution method. Especially, Djebali et al. [12] have shown some existence results for the fourth-order p-Laplacian nonlinear system

$$\begin{cases} [\varphi_p(u'')]'' = h_1(u,v), & t \in (0,1), \\ [\varphi_p(v'')]'' = h_2(u,v), & t \in (0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & i = 0,1, \\ v^{(2j)}(0) = v^{(2j)}(1) = 0, & j = 0,1. \end{cases}$$

The first existence result is obtained via the classical Krasnosel'skii fixed point theorem of cone compression and expansion under the following notation and assumptions.

(H1) For $i = 1, 2$, there exist nonnegative constants h_i^0 and h_i^∞ defined as

$$h_i^0 = \lim_{u+v\to 0} \frac{h_i(u,v)}{(u+v)^{p-1}} \quad \text{and} \quad h_i^\infty = \lim_{u+v\to\infty} \frac{h_i(u,v)}{(u+v)^{p-1}}.$$

The second existence result is obtained via the vector versions of the Krasnosel'skii fixed point theorem [13] under the following notation and assumptions.

(H2) For $\lambda = 0$ or $\lambda = +\infty$, there exist nonnegative constants H_1^λ and H_2^λ defined as

$$H_1^\lambda = \lim_{u\to\lambda} \frac{h_1(u,v)}{u^{p-1}} \quad \text{uniformly with respect to } v \text{ on compact subsets of } R^+,$$

$$H_2^\lambda = \lim_{v\to\lambda} \frac{h_2(u,v)}{v^{p-1}} \quad \text{uniformly with respect to } u \text{ on compact subsets of } R^+.$$

A comparison of the obtained results to those from the literature is provided.

In addition, there are some papers concerned with the existence and multiplicity of positive solutions of systems of $(2n, 2m)$th-order equations under assumption (H1); see [14–17]. The proof is based on the classical Krasnosel'skii fixed point theorem of cone compression and expansion [14, 15, 17], fixed point index arguments and upper and lower solutions method [16]. However, in [12, 14–17] the uniqueness of positive solutions is not considered. Therefore, via the classical Krasnosel'skii fixed point theorem of cone compression and expansion, the results we are going to present reveal how the behavior of the functions f_i $(i = 1, 2)$ at zero and infinity have a profound effect on the existence, uniqueness, and multiplicity of a positive solution of the following system:

$$\begin{cases} (-1)^n [\varphi_p(u^{(2n_1)})]^{(2n_2)} = f_1(t,v), & t \in (0,1), \\ (-1)^m [\varphi_p(v^{(2m_1)})]^{(2m_2)} = f_2(t,u), & t \in (0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & i = 0,1,\ldots,n-1, \\ v^{(2j)}(0) = v^{(2j)}(1) = 0, & j = 0,1,\ldots,m-1, \end{cases} \tag{1.1}$$

where $n_1 + n_2 = n$, $m_1 + m_2 = m$, $n, m \in \mathbf{N}$, $n, m \geq 2$. Here $\varphi_p(s) = s|s|^{p-2}$ $(p > 1)$ refers to the p-Laplacian, and $f_i : [0,1] \times R^+ \to R^+$ are continuous $(i = 1, 2)$ with $f_2(t, 0) = 0$.

Furthermore, in Sect. 4, we consider the existence of positive solutions of the system

$$\begin{cases} (-1)^n [\varphi_p(u^{(2n_1)})]^{(2n_2)} = f(u,v), & t \in (0,1), \\ (-1)^m [\varphi_p(v^{(2m_1)})]^{(2m_2)} = g(u,v), & t \in (0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & i = 0,1,\ldots,n-1, \\ v^{(2j)}(0) = v^{(2j)}(1) = 0, & j = 0,1,\ldots,m-1, \end{cases} \tag{1.2}$$

under the following assumption:

(H3) there exist two pairs of nonnegative functions F_1, F_2 and G_1, G_2 such that

$$\begin{cases} F_1(v) \leq f(u,v) \leq F_2(v) & \text{for all } u \in [0,+\infty], \\ G_1(u) \leq g(u,v) \leq G_2(u) & \text{for all } v \in [0,+\infty], \end{cases}$$

or

($\widetilde{\mathrm{H}}3$) there exist two pairs of nonnegative functions F_1, F_2 and G_1, G_2 such that

$$\begin{cases} F_1(u) \le f(u,v) \le F_2(u), & \text{for all } v \in [0,+\infty], \\ G_1(v) \le g(u,v) \le G_2(v), & \text{for all } u \in [0,+\infty]. \end{cases}$$

At the end of Sect. 4, we also give examples where $f(u,v)$ and $g(u,v)$ satisfying assumption (H3) do not satisfy assumptions (H1) and (H2).

2 Preliminaries

Let $G_n(t,s)$ be the Green function of the linear boundary value problem

$$\begin{cases} (-1)^{(n)}\omega^{(2n)}(t) = 0, & t \in [0,1], \\ \omega^{(2i)}(0) = \omega^{(2i)}(1) = 0, & 0 \le i \le n-1. \end{cases}$$

By induction the Green function $G_n(t,s)$ can be expressed as (see[15])

$$G_i(t,s) = \int_0^1 G(t,\xi)G_{i-1}(\xi,s)\,d\xi, \quad 2 \le i \le n,$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Lemma 2.1 ([15]) *The function $G_n(t,s)$ has the following properties*:

(1) $G_n(t,s) > 0$, $(t,s) \in (0,1) \times (0,1)$.

(2) *For any* $(t,s) \in [0,1] \times [0,1]$, $G_n(t,s) \le \frac{1}{6^{n-1}} s(1-s)$.

(3) *If* $\delta \in (0,\frac{1}{2})$, *then for any* $(t,s) \in [\delta, 1-\delta] \times [0,1]$,

$$\begin{aligned} G_n(t,s) &\ge \delta^n \left(\frac{4\delta^3 - 6\delta^2 + 1}{6}\right)^{n-1} s(1-s) \\ &\ge 6^{n-1}\delta^n \left(\frac{4\delta^3 - 6\delta^2 + 1}{6}\right)^{n-1} \max_{0\le t\le 1} G_n(t,s). \end{aligned}$$

Lemma 2.2 *For any $m,n \in \mathbf{N}^+$ and $h \in C[0,1]$ with $h(t) \ge 0$, the solution of the boundary value problem*

$$\begin{cases} (-1)^n[\varphi_p(u^{(2n)})]^{(2m)} = h(t), & t \in (0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & 0 \le i \le n-1, \end{cases}$$

can be expressed by $u(t) = \int_0^1 G_n(t,s)\varphi_q(\int_0^1 G_m(s,\tau)h(\tau)\,d\tau)\,ds$, where φ_q stands for the inverse function $\varphi_q = \varphi_p^{-1}$ with conjugates p, q, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, the solution satisfies the estimate

$$\min_{t\in[\delta,1-\delta]} u(t) \ge 6^{n-1}\delta^n \left(\frac{4\delta^3 - 6\delta^2 + 1}{6}\right)^{n-1} \|u\|,$$

where $\|u\| = \max_{t\in[0,1]} |u(t)|$, the norm of $C[0,1]$.

Proof For $t \in [\delta, 1-\delta]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G_n(t,s)\varphi_q\left(\int_0^1 G_m(s,\tau)h(\tau)\,d\tau\right)ds \\ &\geq \max_{t\in[0,1]}\int_0^1 6^{n-1}\delta^n\left(\frac{4\delta^3-6\delta^2+1}{6}\right)^{n-1} G_n(t,s)\varphi_q\left(\int_0^1 G_m(s,\tau)h(\tau)\,d\tau\right)ds \\ &\geq 6^{n-1}\delta^n\left(\frac{4\delta^3-6\delta^2+1}{6}\right)^{n-1}\max_{t\in[0,1]}\int_0^1 G_n(t,s)\varphi_q\left(\int_0^1 G_m(s,\tau)h(\tau)\,d\tau\right)ds \\ &= 6^{n-1}\delta^n\left(\frac{4\delta^3-6\delta^2+1}{6}\right)^{n-1}\|u\|. \end{aligned}$$

□

Definition 2.3 Let K be a cone in a real Banach space X. With some positive $u_0 \in K\backslash\{0\}$, $A: K \to K$ is called u_0-sublinear if

(a) for any $x > 0$, there exist $\theta_1 > 0$ and $\theta_2 > 0$ such that $\theta_1 u_0 \leq Ax \leq \theta_2 u_0$;

(b) for any $\theta_1 u_0 \leq x \leq \theta_2 u_0$ and $t \in (0,1)$, there exists $\eta = \eta(x,t) > 0$ such that $A(tx) \geq (1+\eta)tAx$.

Lemma 2.4 ([14], Theorem 2.2.2) *An increasing and u_0-sublinear operator A has at most one positive fixed point.*

Lemma 2.5 ([18]) *Let E be a Banach space, and let $K \subset E$ be a cone in E. Let Ω_1 and Ω_2 be open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and let $A: K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

At the end of this section, for any α_i, $\tilde{\alpha}_i$, β_i, $\tilde{\beta}_i \in R^+$ $(i = 1,2)$, we give some notations:

$$\begin{aligned} &\underline{\varphi}_i^0 = \liminf_{c\to 0^+}\min_{t\in[\delta,1-\delta]}\frac{f_i(t,c)}{c^{\alpha_i}}, \qquad \overline{\varphi}_i^0 = \limsup_{c\to 0^+}\max_{t\in[0,1]}\frac{f_i(t,c)}{c^{\alpha_i}}, \\ &\underline{\psi}_i^\infty = \liminf_{c\to\infty}\min_{t\in[\delta,1-\delta]}\frac{f_i(t,c)}{c^{\beta_i}}, \qquad \overline{\psi}_i^\infty = \limsup_{c\to\infty}\max_{t\in[0,1]}\frac{f_i(t,c)}{c^{\beta_i}}, \\ &\underline{F}_1^0 = \liminf_{c\to 0^+}\frac{F_1(c)}{c^{\alpha_1}}, \qquad \overline{F}_2^\infty = \limsup_{c\to+\infty}\frac{F_2(c)}{c^{\beta_1}}, \\ &\underline{G}_1^0 = \liminf_{d\to 0^+}\frac{G_1(d)}{d^{\alpha_2}}, \qquad \overline{G}_2^\infty = \limsup_{d\to+\infty}\frac{G_2(d)}{d^{\beta_2}}, \\ &\underline{F}_1^\infty = \liminf_{c\to+\infty}\frac{F_1(c)}{c^{\tilde{\alpha}_1}}, \qquad \overline{F}_2^0 = \limsup_{c\to 0^+}\frac{F_2(c)}{c^{\tilde{\beta}_1}}, \\ &\underline{G}_1^\infty = \liminf_{d\to+\infty}\frac{G_1(d)}{d^{\tilde{\alpha}_2}}, \qquad \overline{G}_2^0 = \limsup_{d\to 0^+}\frac{G_2(d)}{d^{\tilde{\beta}_2}}. \end{aligned}$$

For any $\alpha, \beta, \sigma \in R^+$ and $m, n, l \in \mathbf{N}^+$, let

$$\Gamma = \int_\delta^{1-\delta} t(1-t)\,dt, \qquad \theta_n(\delta) = \delta^n\left(\frac{4\delta^3-6\delta^2+1}{6}\right)^{n-1},$$

$$\sigma(n) = 6^{n-1}\theta_n(\delta), \qquad L(m,n) = \frac{1}{6^m}\frac{1}{6^{n(q-1)}},$$
$$S(m,n,l,\alpha) = \theta_m(\delta)\theta_n^{q-1}(\delta)\sigma(l)^{\alpha(q-1)}\Gamma^q.$$

3 Main results of Problem (1.1)

Define the mapping $A : C[0,1] \to C[0,1]$ by

$$A(u)(t) = A_1 \circ A_2(u)(t),$$

where

$$A_1(v)(t) = \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\big(\tau,v(\tau)\big)\,d\tau\right)ds,$$
$$A_2(u)(t) = \int_0^1 G_{m_1}(t,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)f_2\big(\tau,u(\tau)\big)\,d\tau\right)ds.$$

Let $P = \{u \in C[0,1], u \geq 0\}$. The pair $(u,v) \in C[0,1] \times C[0,1]$ is a positive solution of (1.1) if and only if (u,v) belongs to $P \setminus \{0\} \times P \setminus \{0\}$ and satisfies $u = A_1 v$, $v = A_2 u$. If $u \in P \setminus \{0\}$ is a fixed point of A, then define $v = A_2 u$. Then $v \in P \setminus \{0\}$, so that $(u,v) \in C[0,1] \times C[0,1]$ solves (1.1). So our main goal is to look for nonzero fixed points of A in the subcone

$$K = \left\{u(t) \in P : \min_{\delta \leq t \leq 1-\delta} u(t) \geq \sigma(n_1)\|u\|\right\}.$$

Since $u(t) \in K$ with $u(t) \geq 0$, this means that the corresponding solutions of (1.1) are nonnegative.

For any given $r > 0$, let

$$\Omega_r = \big\{u \in C[0,1] : \|u\| < r\big\},$$
$$\partial\Omega_r = \big\{u \in C[0,1] : \|u\| = r\big\}.$$

Lemma 3.1 *For any $0 < r < R$, the operator $A : K \cap (\overline{\Omega}_R \setminus \Omega_r) \to K$ is completely continuous.*

Proof For any $u(t) \in K \cap (\overline{\Omega}_R \setminus \Omega_r)$, by Lemma 2.2 we get

$$\begin{aligned}
\min_{t\in[\delta,1-\delta]} A(u)(t) &= \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\big(\tau,A_2(u)(\tau)\big)\,d\tau\right)ds \\
&\geq 6^{n_1-1}\delta^{n_1}\left(\frac{4\delta^3 - 6\delta^2 + 1}{6}\right)^{n_1-1} \\
&\quad \times \max_{t\in[0,1]} \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\big(\tau,A_2(u)(\tau)\big)\,d\tau\right)ds \\
&= \sigma(n_1)\big\|A(u)(t)\big\|,
\end{aligned}$$

which implies that $A : K \cap (\overline{\Omega}_R \setminus \Omega_r) \to K$.

Now, we verify that A is completely continuous.

First, for the continuity of A, we only need to prove that $\|A(u_n) - A(u)\| \to 0$ if $u_n \to u$ as $n \to \infty$. Let us consider

$$\begin{aligned}
\left|A(u_n)(t) - A(u)(t)\right| &= \left| \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u_n)(\tau)\right)d\tau\right)ds \right. \\
&\quad \left. - \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds \right| \\
&\le \int_0^1 G_{n_1}(t,s)\left|\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u_n)(\tau)\right)d\tau\right)\right. \\
&\quad \left. - \varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)\right| ds.
\end{aligned}$$

From the continuity of f_1 and f_2 it follows that $\|A(u_n) - A(u)\| \to 0$ as $n \to \infty$.

Second, we show that the operator A is uniformly bounded. For any $u(t) \in K \cap (\overline{\Omega}_R \setminus \Omega_r)$, by Lemma 2.1(2) we have

$$\begin{aligned}
\left|A(u)(t)\right| &= \left|\int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds\right| \\
&\le \left|\int_0^1 \frac{1}{6^{n_1-1}}s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1-\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds\right|.
\end{aligned}$$

Since f_1 and f_2 are continuous, it is clear that $A(u)(t)$ is uniformly bounded on $K \cap (\overline{\Omega}_R \setminus \Omega_r)$.

Finally, we show the equicontinuity of the operator A. From the expression of $G(t,s)$ we easily obtain that

$$\frac{\partial G(t,\tau)}{\partial t} = \begin{cases} 1-s, & 0 \le t \le s \le 1, \\ -s, & 0 \le s \le t \le 1, \end{cases}$$

which implies that $|\frac{\partial G(t,\tau)}{\partial t}|$ is bounded. There exists a constant $M > 0$ such that $|\frac{\partial G(t,\tau)}{\partial t}| < M$. For any $u(t) \in K \cap (\overline{\Omega}_R \setminus \Omega_r)$, $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
&\left|A(u)(t_2) - A(u)(t_1)\right| \\
&\quad = \left|\int_0^1 G_{n_1}(t_2,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds\right. \\
&\qquad \left. - \int_0^1 G_{n_1}(t_1,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds\right| \\
&\quad = \left|\int_0^1 G_{n_1}(t_2,\xi)\left[\int_0^1 G_{n_1-1}(\xi,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds\right]d\xi\right. \\
&\qquad \left. - \int_0^1 G_{n_1}(t_1,\xi)\left[\int_0^1 G_{n_1-1}(\xi,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds\right]d\xi\right| \\
&\quad = \left|\int_0^1 \frac{\partial G(t,\tau)}{\partial t}(t_2 - t_1)\left[\int_0^1 G_{n_1-1}(\xi,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds\right]d\xi]\right| \\
&\quad \le K\int_0^1\left[\int_0^1 G_{n_1-1}(\xi,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds\right]d\xi]|t_2 - t_1|,
\end{aligned}$$

which implies that A is equicontinuous. By the Arzelà–Ascoli theorem we get that $A : K \cap (\overline{\Omega}_R \setminus \Omega_r) \to K$ is compact. Consequently, it follows that $A : K \cap (\overline{\Omega}_R \setminus \Omega_r) \to K$ is completely continuous. □

Theorem 3.2 *Assume that $\alpha_i, \beta_i > 0$ $(i = 1, 2)$ with*

$$\alpha_1\alpha_2 \geq (p-1)^2, \qquad \beta_1\beta_2 \geq (p-1)^2.$$

In addition, let the functions f_i $(i = 1, 2)$ satisfy the following assumptions:

$$\overline{\varphi}_1^0 < +\infty, \qquad \overline{\varphi}_2^0 = 0, \qquad \underline{\psi}_1^\infty > 0, \qquad \underline{\psi}_2^\infty = +\infty.$$

Then (1.1) *has at least one positive solution.*

Proof On one hand, from the assumption $\overline{\varphi}_1^0 < +\infty$ we have that there exist $\varepsilon > 0$ and $\widehat{r} \in (0, 1)$ such that

$$f_1(t, v) \leq \left(\overline{\varphi}_1^0 + \varepsilon\right) v^{\alpha_1} \quad \text{for } t \in [0, 1], v \in [0, \widehat{r}].$$

Furthermore, from the assumption $\overline{\varphi}_2^0 = 0$ we have that there exist $\varepsilon_1 > 0$ and $\overline{r} \in (0, \widehat{r})$ such that

$$f_2(t, u) \leq \varepsilon_1 u^{\alpha_2} \quad \text{for } t \in [0, 1], u \in [0, \overline{r}],$$

where ε_1 satisfies

$$\begin{aligned}
&\varepsilon_1^{q-1} L(m_1, m_2) \leq \widehat{r}, \\
&\varepsilon_1^{\alpha_1(q-1)^2} L^{\alpha_1(q-1)}(m_1, m_2) L(n_1, n_2)\left(\overline{\varphi}_1^0 + \varepsilon\right)^{q-1} \leq \widehat{r}.
\end{aligned}$$

Set $r = \bar{r}$. Then for any $u \in K \cap \partial\Omega_r$, we have

$$\begin{aligned}
v(t) &= A_2(u)(t) = \int_0^1 G_{m_1}(t, s)\varphi_q\left(\int_0^1 G_{m_2}(s, \tau) f_2(\tau, u)\, d\tau\right) ds \\
&\leq \frac{1}{6^{m_1}}\varphi_q\left(\frac{1}{6^{m_2}}\right)\varphi_q\left(\varepsilon_1 \|u\|^{\alpha_2}\right) \\
&= \varepsilon_1^{q-1} L(m_1, m_2)\|u\|^{\alpha_2(q-1)} \leq 1.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
A_1(v)(t) &= \int_0^1 G_{n_1}(t, s)\varphi_q\left(\int_0^1 G_{n_2}(s, \tau) f_1(\tau, v)\, d\tau\right) ds \\
&\leq \int_0^1 \frac{1}{6^{n_1-1}} s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1-\tau)\left(\overline{\varphi}_1^0 + \varepsilon\right) v^{\alpha_1}(\tau)\, d\tau\right) ds \\
&\leq \left(\overline{\varphi}_1^0 + \varepsilon\right)^{q-1}\frac{1}{6^{n_1}}\varphi_q\left(\frac{1}{6^{n_2}}\right)\varphi_q\left(\|v\|^{\alpha_1}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\overline{\varphi}_1^0 + \varepsilon\right)^{q-1} L(n_1, n_2) \left\| \varepsilon_1^{q-1} L(m_1, m_2) \|u\|^{\alpha_2(q-1)} \right\|^{\alpha_1(q-1)} \\
&= \varepsilon_1^{\alpha_1(q-1)^2} L^{\alpha_1(q-1)}(m_1, m_2) L(n_1, n_2) \left(\overline{\varphi}_1^0 + \varepsilon\right)^{q-1} \|u\|^{\alpha_1\alpha_2(q-1)^2} \\
&\leq \|u\|^{\alpha_1\alpha_2(q-1)^2} \leq \|u\|,
\end{aligned}
$$

that is, $\|A(u)(t)\| \leq \|u\|$ for $u \in K \cap \partial\Omega_r$.

On the other hand, from the assumptions $\underline{\psi}_1^\infty > 0$ and $\underline{\psi}_2^\infty = +\infty$ it follows that there exist $C_1 > 0$ and $\overline{R} > 1$ such that

$$
\begin{aligned}
&f_1(t, v) \geq \left(\underline{\psi}_1^\infty - \varepsilon\right) v^{\beta_1} \quad \text{for } t \in [0, 1], v \geq \overline{R}, \\
&f_2(t, u) \geq C_1 u^{\beta_2} \quad \text{for } t \in [0, 1], u \geq \overline{R},
\end{aligned}
$$

where C_1 satisfies

$$
\begin{aligned}
&C_1^{q-1} S(m_1, m_2, n_1, \beta_2) \geq 1, \\
&C_1^{\beta_1(q-1)^2} \left(\underline{\psi}_1^\infty - \varepsilon\right)^{q-1} S(n_1, n_2, m_1, \beta_1) S^{\beta_1(q-1)}(m_1, m_2, n_1, \beta_2) \geq 1.
\end{aligned}
$$

Set $R = \max\{\frac{\overline{R}}{\sigma}, \overline{R}^{\frac{1}{\beta_1(q-1)}}\}$. Then for any $u \in K \cap \partial\Omega_R$, we have $u(t) \geq \sigma R \geq \sigma\frac{\overline{R}}{\sigma} = \overline{R}$ for $t \in [\delta, 1-\delta]$, and

$$
\begin{aligned}
v(t) = A_2(u)(t) &= \int_0^1 G_{m_1}(t, s)\varphi_q\left(\int_0^1 G_{m_2}(s, \tau) f_2(\tau, u)\, d\tau\right) ds \\
&\geq \int_\delta^{1-\delta} \theta_{m_1}(\delta) s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{m_2}(\delta)\tau(1-\tau) C_1 \sigma^{\beta_2}(n_1)\|u\|^{\beta_2}\, d\tau\right) ds \\
&\geq C_1^{q-1} S(m_1, m_2, n_1, \beta_2)\|u\|^{\beta_2(q-1)} \\
&\geq \|u\|^{\beta_2(q-1)} = R^{\beta_2(q-1)} \geq \overline{R}.
\end{aligned}
$$

Furthermore, for $t \in [\delta, 1-\delta]$, we also get

$$
\begin{aligned}
A_1(v)(t) &= \int_0^1 G_{n_1}(t, s)\varphi_q\left(\int_0^1 G_{n_2}(s, \tau) f_1(\tau, v)\, d\tau\right) ds \\
&\geq \int_\delta^{1-\delta} \theta_{n_1}(\delta) s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{n_2}(\delta)\tau(1-\tau)\left(\underline{\psi}_1^\infty - \varepsilon\right)\sigma^{\beta_1}(m_1)\|v\|^{\beta_1}\, d\tau\right) ds \\
&\geq \left(\underline{\psi}_1^\infty - \varepsilon\right)^{q-1} S(n_1, n_2, m_1, \beta_1)\|v\|^{\beta_1(q-1)}.
\end{aligned}
$$

Then from the above inequalities, for $t \in [\delta, 1-\delta]$, we have

$$
\begin{aligned}
A(u)(t) &\geq C_1^{q-1} S(n_1, n_2, n_1, \beta_1)\Gamma^q \left\| C_2^{q-1} S(m_1, m_2, n_1, \beta_2)\Gamma^q \|u\|^{\beta_2(q-1)} \right\|^{\beta_1(q-1)} \\
&= C_1^{\beta_1(q-1)^2} \left(\underline{\psi}_1^\infty - \varepsilon\right)^{q-1} S(n_1, n_2, m_1, \beta_1) S^{\beta_1(q-1)}(m_1, m_2, n_1, \beta_2)\|u\|^{\beta_1\beta_2(q-1)^2} \\
&\geq \|u\|^{\beta_1\beta_2(q-1)^2} \geq \|u\|,
\end{aligned}
$$

which yields that $\|A(u)(t)\| \geq \|u\|$ for $u \in K \cap \partial\Omega_R$.

Therefore, by Lemma 2.5 the operator A has at least one fixed point in $K \cap (\overline{\Omega}_R \setminus \Omega_r)$. $\square$

Theorem 3.3 *Assume that $\alpha_i, \beta_i > 0$ ($i = 1, 2$) with*

$$\alpha_1\alpha_2 \le (p-1)^2, \qquad \beta_1\beta_2 \le (p-1)^2.$$

In addition, let the functions f_i ($i = 1, 2$) satisfy the following assumptions:

$$\underline{\varphi}_1^0 > 0, \qquad \underline{\varphi}_2^0 = +\infty, \qquad \overline{\psi}_1^\infty < +\infty, \qquad \overline{\psi}_2^\infty = 0.$$

Then (1.1) *has at least one positive solution.*

Proof On one hand, from the assumptions $\underline{\varphi}_1^0 > 0$ and $\underline{\varphi}_2^0 = +\infty$ we have that there exist $C_3 > 0$ and $0 < \overline{r} < 1$ such that

$$f_1(t, v) \ge \left(\underline{\varphi}_1^0 - \varepsilon\right)v^{\alpha_1} \quad \text{for } t \in [0, 1], 0 \le v \le \overline{r},$$
$$f_2(t, u) \ge C_3 u^{\alpha_2} \quad \text{for } t \in [0, 1], 0 \le u \le \overline{r},$$

where C_3 satisfies

$$C_3^{\alpha_1(q-1)^2}\left(\underline{\varphi}_1^0 - \varepsilon\right)^{q-1} S(n_1, n_2, n_1, \alpha_1) S^{\alpha_1(q-1)}(m_1, m_2, n_1, \alpha_2) \ge 1.$$

Since f_2 is continuous and $f_2(t, 0) = 0$, there exists $\widehat{r} \in (0, \overline{r})$ such that

$$f_2(t, u) \le \overline{r}^{p-1} \quad \text{for } t \in [0, 1], u \in [0, \widehat{r}].$$

Set $r = \widehat{r}$. For any $u(t) \in K \cap \partial\Omega_r$, we have

$$\begin{aligned} v(t) = A_2(u)(t) &= \int_0^1 G_{m_1}(t, s)\varphi_q\left(\int_0^1 G_{m_2}(s, \tau) f_2(\tau, u)\, d\tau\right) ds \\ &\le L(m_1, m_2)\varphi_q\left(\overline{r}^{p-1}\right) \le \overline{r}. \end{aligned}$$

Then, for $t \in [\delta, 1-\delta]$, we obtain

$$\begin{aligned} A_1(v)(t) &= \int_0^1 G_{n_1}(t, s)\varphi_q\left(\int_0^1 G_{n_2}(s, \tau) f_1(\tau, v)\, d\tau\right) ds \\ &\ge \int_\delta^{1-\delta} \theta_{n_1}(\delta)s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{n_2}(\delta)\tau(1-\tau)\left(\underline{\varphi}_1^0 - \varepsilon\right)\sigma^{\alpha_1}(m_1)\|v\|^{\alpha_1}\, d\tau\right) ds \\ &\ge \left(\underline{\varphi}_1^0 - \varepsilon\right)^{q-1} S(n_1, n_2, m_1, \alpha_1)\|v\|^{\alpha_1(q-1)} \end{aligned}$$

and

$$\begin{aligned} v(t) = A_2(u)(t) &= \int_0^1 G_{m_1}(t, s)\varphi_q\left(\int_0^1 G_{m_2}(s, \tau) f_2(\tau, u)\, d\tau\right) ds \\ &\ge \int_\delta^{1-\delta} \theta_{m_1}(\delta)s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{m_2}(\delta)\tau(1-\tau)C_3\sigma^{\alpha_2}(n_1)\|u\|^{\alpha_2}\, d\tau\right) ds \\ &\ge C_3^{q-1} S(m_1, m_2, n_1, \alpha_2)\|u\|^{\alpha_2(q-1)}. \end{aligned}$$

From these inequalities we have

$$\|A(u)(t)\| \geq A(u)(t)|_{t\in[\delta,1-\delta]} \geq \|u\|^{\alpha_1\alpha_2(q-1)^2} \geq \|u\| \quad \text{for } u(t) \in K \cap \partial\Omega_r.$$

On the other hand, by the assumptions $\overline{\psi}_1^{\infty} < +\infty$ and $\overline{\psi}_2^{\infty} = 0$ there exist $\varepsilon_2 > 0$ and $\overline{R} > 0$ such that

$$\begin{aligned} &f_1(t,v) \leq \left(\overline{\psi}_1^{\infty} + \varepsilon\right)v^{\beta_1} \quad \text{for } t \in [0,1], v \geq \overline{R}, \\ &f_2(t,u) \leq \varepsilon_2 u^{\beta_2} \quad \text{for } t \in [0,1], u \geq \overline{R}, \end{aligned}$$

where ε_2 satisfies $\varepsilon_2^{\beta_1(q-1)^2}(\overline{\psi}_1^{\infty} + \varepsilon)^{q-1}L(n_1,n_2)L^{\beta_1(q-1)}(m_1,m_2) < 1$. Since f_i is continuous, let

$$\begin{aligned} &N_1 = \max\left\{f_1(t,v) : 0 \leq t \leq 1, 0 \leq v \leq \overline{R}\right\}, \\ &N_2 = \max\left\{f_2(t,u) : 0 \leq t \leq 1, 0 \leq u \leq \overline{R}\right\}. \end{aligned}$$

Then we have the estimates

$$f_1(t,v) \leq \left(\overline{\psi}_1^{\infty} + \varepsilon\right)v^{\beta_1} + N_1, \qquad f_2(t,u) \leq \varepsilon_2 u^{\beta_2} + N_2.$$

Via some computations we obtain the inequalities

$$\begin{aligned} A(u)(t) &= A_1 \circ A_2(u)(t) \\ &= \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\left(\tau, A_2(u)(\tau)\right)d\tau\right)ds \\ &\leq \int_0^1 \frac{1}{6^{n_1-1}}s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1-\tau)\left[\left(\overline{\psi}_1^{\infty} + \varepsilon\right)\left(A_2(u)(\tau)\right)^{\beta_1} + N_1\right]d\tau\right)ds \\ &= L(n_1,n_2)\left(\left(\overline{\psi}_1^{\infty} + \varepsilon\right)\left(A_2(u)(\tau)\right)^{\beta_1} + N_1\right) \end{aligned}$$

and

$$\begin{aligned} A_2(u)(t) &= \int_0^1 G_{m_1}(t,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)f_2\left(\tau, u(\tau)\right)d\tau\right)ds \\ &\leq \int_0^1 \frac{1}{6^{m_1-1}}s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{m_2-1}}\tau(1-\tau)\left[\varepsilon_2 u(\tau))^{\beta_2} + N_2\right]d\tau\right)ds \\ &= L(m_1,m_2)\varphi_q\left(\varepsilon_2 u^{\beta_2}(\tau) + N_2\right). \end{aligned}$$

It is clear that the term with the highest index is

$$\varepsilon_2^{\beta_1(q-1)^2}\left(\overline{\psi}_1^{\infty} + \varepsilon\right)^{q-1}L(n_1,n_2)L^{\beta_1(q-1)}(m_1,m_2)u^{\beta_1\beta_2(q-1)^2} < u^{\beta_1\beta_2(q-1)^2}.$$

Thus there exists a sufficiently large $R > 0$ such that

$$\|A(u)(t)\| \leq \|u\|, \quad u(t) \in K \cap \partial\Omega_R.$$

Therefore by Lemma 2.5 the operator A has at least one fixed point in $K \cap (\overline{\Omega}_R \setminus \Omega_r)$. □

Theorem 3.4 *Assume that the functions f_i $(i = 1, 2)$ satisfy the following assumptions*:

(i) *$f_i(t, c)$ is nondecreasing on c uniformly for $t \in [0, 1]$;*

(ii) *there exist four positive constants $k_1 < k_2$, $l_1 < l_2$ such that*

$$k_1 v^{\alpha_1} \le f_1(t, v) \le k_2 v^{\alpha_1} \quad \text{uniformly in } t \in [0, 1], v \in [0, +\infty);$$
$$l_1 u^{\alpha_2} \le f_2(t, u) \le l_2 u^{\alpha_2} \quad \text{uniformly in } t \in [0, 1], u \in [0, +\infty).$$

(iii) *there exist two positive constants α_1, α_2 with $\alpha_1\alpha_2 < (p - 1)^2$ such that*

$$f_i(t, \xi c) \ge \xi^{\alpha_i} f_i(t, c) \quad \text{for all } \xi \in (0, 1).$$

Then (1.1) *has a unique positive solution.*

Proof First, we give the existence result. On one hand, for $u \in K$ and $t_0 \in [\delta, 1 - \delta]$, we have

$$\begin{aligned} A_1(v)(t_0) &= \int_0^1 G_{n_1}(t_0, s)\varphi_q\left(\int_0^1 G_{n_2}(s, \tau) f_1\bigl(\tau, v(\tau)\bigr)\,d\tau\right) ds \\ &\ge \int_\delta^{1-\delta} \theta_{n_1}(\delta) s(1 - s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{n_2}(\delta)\tau(1 - \tau) k^{\alpha_1}\sigma^{\alpha_1}(m_1)\|v\|^{\alpha_1}\,d\tau\right) ds \\ &\ge S(n_1, n_2, m_1, \alpha_1) k_1^{\alpha_1(q-1)}\|v\|^{\alpha_1(q-1)}. \end{aligned}$$

In the similar way, we also have

$$\begin{aligned} \bigl\|v(t)\bigr\| \ge A_2(u)(t_0) &= \int_0^1 G_{m_1}(t_0, s)\varphi_q\left(\int_0^1 G_{m_2}(s, \tau)\right) f\bigl(\tau, u(\tau)\bigr)\,d\tau)\,ds \\ &\ge S(m_1, m_2, n_1, \alpha_2) l_1^{\alpha_2(q-1)}\|u\|^{\alpha_2(q-1)}. \end{aligned}$$

Combining these inequalities, for $u \in K$, we get

$$\bigl\|A(u)(t)\bigr\| \ge S(n_1, n_2, m_1, \alpha_1) S^{\alpha_1(q-1)}(m_1, m_2, n_1, \alpha_2) k_1^{\alpha_1(q-1)} l_1^{\alpha_1\alpha_2(q-1)^2}\|u\|^{\alpha_1\alpha_2(q-1)^2}.$$

Since $\alpha_1\alpha_2 < (p - 1)^2$, there exists a sufficiently small $r > 0$ such that $\|A(u)(t)\| > \|u\|$ for $u \in K \cap \partial\Omega_r$.

On the other hand, for $u \in K$,

$$\begin{aligned} A_1(v)(t) &= \int_0^1 G_{n_1}(t, s)\varphi_q\left(\int_0^1 G_{n_2}(s, \tau) f_1\bigl(\tau, v(\tau)\bigr)\,d\tau\right) ds \\ &\le \int_0^1 \frac{1}{6^{n_1-1}} s(1 - s)\varphi_q\left(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1 - \tau) v^{\alpha_1}(\tau)\,d\tau\right) ds \\ &\le \frac{1}{6^{n_1}} \frac{1}{6^{n_2(q-1)}} k_2^{\alpha_1(q-1)}\|v\|^{\alpha_1(q-1)}. \end{aligned}$$

In a similar way, we have

$$\begin{aligned}\|v(t)\| &= \|A_2(u)(t)\| = \max_{t\in[0,1]}\int_0^1 G_{m_1}(t,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)f_2\big(\tau,u(\tau)\big)\,d\tau\right)ds\\ &\le \frac{1}{6^{m_1}}\frac{1}{6^{m_2(q-1)}}l_2^{\alpha_2(q-1)}\|u\|^{\alpha_2(q-1)}.\end{aligned}$$

Combining these inequalities, for $u\in K$, we get

$$\|A(u)(t)\| \le L(n_1,n_2)L^{\alpha_1(q-1)}(m_1,m_2)k_2^{\alpha_1(q-1)}l_2^{\alpha_1\alpha_2(q-1)^2}\|u\|^{\alpha_1\alpha_2(q-1)^2}.$$

Since $\alpha_1\alpha_2 < (p-1)^2$, there exists a sufficiently large $R > 0$ such that $\|A(u)(t)\| < \|u\|$ for $u \in K\cap\partial\Omega_R$.

Therefore by Lemma 2.5 the operator A has at least one fixed point in $K\cap(\overline{\Omega}_R\setminus\Omega_r)$. Finally, we prove that A has at most one fixed point in $P\setminus\{0\}$. It is easy to see that A_1 and A_2 are increasing operators with respect to the partial order induced by K. So is $A = A_1A_2$. By Lemma 2.4 we only need to verify that A is u_0-sublinear for some positive $u_0\in C[0,1]$. Take $u_0 = t(1-t)$. Set $M = \max_{t\in[0,1]} F(t)$, where

$$\begin{aligned}F(t) = \int_0^1 &G_{n_1-1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)\right.\\ &\left.f_1\left(\tau,\int_0^1 G_{m_1}(\tau,\nu)\varphi_q\left(\int_0^1 G_{m_2}(\nu,\mu)f_2\big(,\mu,u(\mu)\big)\,d\mu\right)d\nu\right)d\tau\right)ds.\end{aligned}$$

Then we have

$$A(u)(t) = \int_0^1 G(t,s)F(s)\,ds \le M\int_0^1 G(t,s)\,ds = \frac{M}{2}t(1-t).$$

Furthermore, we have

$$A(u)(t) = \int_0^1 G(t,s)F(s)\,ds \ge \int_0^1 CG(t,t)G(s,s)F(s)\,ds = \int_0^1 CG(s,s)F(s)\,dsG(t,t).$$

So we can choose $\theta_1 = \int_0^1 CG(s,s)F(s)\,ds$ and $\theta_2 = \frac{M}{2}$.

From the above discussion we know that $A = A_1A_2$ satisfies (a) of Definition 2.3. The proof is complete if A satisfies (b) of Definition 2.3. To this end, let $\theta_1u_0 \le u \le \theta_2u_0$, $\xi\in(0,1)$. Then a direct calculation gives $A_2(\xi u) = \xi^{\frac{\alpha_2}{p-1}}A_2(u)$, $A_1(\xi v) = \xi^{\frac{\alpha_1}{p-1}}A_1(v)$. Since $\xi\in(0,1)$ and $\alpha_1\alpha_2 < (p-1)^2$, we get $A(\xi u) = A_1(\xi^{\frac{\alpha_2}{p-1}}A_2(u)) = \xi^{\frac{\alpha_1\alpha_2}{(p-1)^2}}A(u) \ge (1+\eta)\xi A(x)$ for some $\eta > 0$. □

Theorem 3.5 *Assume that* $\alpha_i,\beta_i > 0$ $(i = 1,2)$ *with*

$$\alpha_1\alpha_2 \le (p-1)^2,\qquad \beta_1\beta_2 \ge (p-1)^2.$$

In addition, let the functions f_i $(i = 1,2)$ *satisfy the following assumptions::*

(i) $\underline{\varphi}_1^0 > 0$, $\underline{\varphi}_2^0 = +\infty$, $\underline{\psi}_1^\infty > 0$, $\underline{\psi}_2^\infty = +\infty$;

(ii) *there exists* $\widetilde{R}$ *such that* $\frac{1}{6^{n_1}}\varphi_q(\frac{1}{6^{n_2}})\varphi_q(N^1_{\widetilde{R}}) \le \widetilde{R}$, *where*

$$N^1_{\widetilde{R}} = \max\{f_1(t,v) : 0 \le t \le 1, 0 \le v \le L(m_1,m_2)\varphi_q(N^2_{\widetilde{R}})\},$$
$$N^2_{\widetilde{R}} = \max\{f_2(t,u) : 0 \le t \le 1, \sigma(n_1)\widetilde{R} \le u \le \widetilde{R}\}.$$

Then (1.1) *has at least two positive solutions.*

Proof For any $u \in K \cap \partial\Omega_{\widetilde{R}}$, we have

$$\begin{aligned} v(t) &= A_2(u)(t) = \int_0^1 G_{m_1}(t,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)f_2(\tau,u(\tau))\,d\tau\right)ds \\ &\le \int_0^1 \frac{1}{6^{m_1-1}}s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{m_2-1}}\tau(1-\tau)f_2(\tau,u(\tau))\,d\tau\right)d\tau)\,ds \\ &= L(m_1,m_2)\varphi_q(N^2_{\widetilde{R}}). \end{aligned}$$

Furthermore, we get

$$\begin{aligned} A(u)(t) &= A_1 \circ A_2(u)(t) \\ &= \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1(\tau,A_2(u)(\tau))\,d\tau\right)ds \\ &\le \int_0^1 \frac{1}{6^{n_1-1}}s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1-\tau)N^1_{\widetilde{R}}\,d\tau\right)ds \\ &= L(n_1,n_2)\varphi_q(N^1_{\widetilde{R}}) \le \widetilde{R}, \end{aligned}$$

that is, $\|A(u)(t)\| \le \|u\|$ for $u \in K \cap \partial\Omega_{\widetilde{R}}$.

Since $\alpha_1\alpha_2 \le (p-1)^2$, $\beta_1\beta_2 \ge (p-1)^2$, $\underline{\varphi}^0_1 > 0$, $\underline{\varphi}^0_2 = +\infty$, $\underline{\psi}^\infty_1 > 0$, and $\underline{\psi}^\infty_2 = +\infty$, from the proofs of Theorems 3.2 and 3.3 it follows that there exist $r > 0$ (sufficiently small) and $R > 0$ (sufficiently large) such that $\|A(u)(t)\| \ge \|u\|$, $u \in K \cap \partial\Omega_r$, and $\|A(u)(t)\| \ge \|u\|$, $u \in K \cap \partial\Omega_R$. Therefore by Lemma 2.5 the operator A has at least two fixed points in $K \cap (\overline{\Omega}_{\widetilde{R}} \setminus \Omega_r)$ and $K \cap (\overline{\Omega}_R \setminus \Omega_{\widetilde{R}})$. □

Theorem 3.6 *Assume that* $\alpha_i, \beta_i > 0$ $(i = 1,2)$ *with*

$$\alpha_1\alpha_2 \ge (p-1)^2, \qquad \beta_1\beta_2 \le (p-1)^2.$$

In addition, let the functions f_i $(i = 1,2)$ *satisfy the following assumptions*:

(i) $\overline{\varphi}^0_1 < +\infty$, $\overline{\varphi}^0_2 = 0$, $\overline{\psi}^\infty_1 < \infty$, $\overline{\psi}^\infty_2 = 0$;

(ii) *there exists* $\widehat{R}$ *such that* $S(n_1,n_2,m_1,0)\varphi_q(K^1_{\widehat{R}}) \ge \widehat{R}$, *where*

$$\begin{aligned} K^1_{\widehat{R}} = \min\{&f_1(t,v) : \delta \le t \le 1-\delta, \\ &S(m_1,m_2,n_1,0)\varphi_q(K^2_{\widehat{R}}) \le v \le L(m_1,m_2)\varphi_q(N^2_{\widehat{R}})\}, \end{aligned}$$
$$N^2_{\widehat{R}} = \max\{f_2(t,u) : 0 \le t \le 1, \sigma(n_1)\widetilde{R} \le u \le \widetilde{R}\},$$
$$K^2_{\widehat{R}} = \min\{f_2(t,u) : \delta \le t \le 1-\delta, \sigma(n_1)\widetilde{R} \le u \le \widetilde{R}\}.$$

Then (1.1) *has at least two positive solutions.*

Proof For any $u \in K \cap \partial\Omega_{\widehat{R}}$, we have

$$\begin{aligned} v(t) &= A_2(u)(t) = \int_0^1 G_{m_1}(t,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)f_2\big(\tau,u(\tau)\big)\,d\tau\right)ds \\ &\le \int_0^1 \frac{1}{6^{m_1-1}}s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{m_2-1}}\tau(1-\tau)f_2\big(\tau,u(\tau)\big)\,d\tau\right)d\tau)\,ds \\ &= L(m_1,m_2)\varphi_q\big(N^2_{\widehat{R}}\big). \end{aligned}$$

For $t \in [\delta, 1-\delta]$,

$$\begin{aligned} v(t) &= A_2(u)(t) = \int_0^1 G_{m_1}(t,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)f_2(\tau,u)\,d\tau\right)ds \\ &\ge \int_\delta^{1-\delta} \theta_{m_1}(\delta)s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{m_2}(\delta)\tau(1-\tau)f_2(\tau,u)\,d\tau\right)ds \\ &\ge S(m_1,m_2,n_1,0)\Gamma^q\varphi_q\big(K^2_{\widehat{R}}\big). \end{aligned}$$

Thus, for $t \in [\delta, 1-\delta]$, we have the estimates

$$S(m_1,m_2,n_1,0)\varphi_q\big(K^2_{\widehat{R}}\big) \le v(t) \le L(m_1,m_2)\varphi_q\big(N^2_{\widehat{R}}\big)$$

and

$$\begin{aligned} A(u)(t) &= A_1 \circ A_2(u)(t) \\ &= \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f_1\big(\tau,A_2(u)(\tau)\big)\,d\tau\right)ds \\ &\ge \int_\delta^{1-\delta} \theta_{n_1}(\delta)s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{n_2}(\delta)\tau(1-\tau)f_1(\tau,v)\,d\tau\right)ds \\ &\ge S(n_1,n_2,m_1,0)\varphi_q\big(K^1_{\widehat{R}}\big), \end{aligned}$$

that is, $\|A(u)(t)\| \ge \|u\|$ for $u \in K \cap \partial\Omega_{\widetilde{R}}$.

Since $\alpha_1\alpha_2 \ge (p-1)^2$, $\beta_1\beta_2 \le (p-1)^2$, $\overline{\varphi}_1^0 < +\infty$, $\overline{\varphi}_2^0 = 0$, $\overline{\psi}_1^\infty < \infty$, and $\overline{\psi}_2^\infty = 0$, from the proof of Theorem 3.2 and Theorem 3.3 it follows that there exist $r > 0$ (sufficiently small) and $R > 0$ (sufficiently large) such that $\|A(u)(t)\| \le \|u\|$, $u \in K \cap \partial\Omega_r$, and $\|A(u)(t)\| \le \|u\|$, $u \in K \cap \partial\Omega_R$. Therefore by Lemma 2.5 the operator A has at least two fixed points in $K \cap (\overline{\Omega}_{\widehat{R}} \setminus \Omega_r)$ and $K \cap (\overline{\Omega}_R \setminus \Omega_{\widehat{R}})$. □

Example 1 Assume that $\alpha, \beta > 0$. Then for the problem

$$\begin{cases} (-1)^n[\varphi_p(u^{(2n_1)})]^{(2n_2)} = v^\alpha(t), & t \in (0,1), \\ (-1)^m[\varphi_p(v^{(2m_1)})]^{(2m_2)} = u^\beta(t), & t \in (0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & i = 0,1,\ldots,n-1, \\ v^{(2j)}(0) = v^{(2j)}(1) = 0, & j = 0,1,\ldots,m-1, \end{cases} \tag{3.1}$$

we have the following existence, uniqueness, and nonexistence results:

(I) If $\alpha\beta \neq (p-1)^2$, then (3.1) has at least a positive solution.
(II) If $\alpha\beta < (p-1)^2$, then (3.1) has a unique positive solution.
(III) If $\alpha\beta = (p-1)^2$, then (3.1) has no positive solutions.

Proof First, we give the proof of existence results.

(I), (II) On one hand, for $u \in K$ and $t_0 \in [\delta, 1-\delta]$, we have

$$\begin{aligned} A_1(v)(t_0) &= \int_0^1 G_{n_1}(t_0,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)v^\alpha(\tau)\,d\tau\right)ds \\ &\geq \int_\delta^{1-\delta} \theta_{n_1}(\delta)s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{n_2}(\delta)\tau(1-\tau)\sigma^\alpha(m_1)\|v\|^\alpha\,d\tau\right)ds \\ &\geq S(n_1,n_2,m_1,\alpha)\|v\|^{\alpha(q-1)}. \end{aligned}$$

In a similar way, we also have

$$\begin{aligned} \|v(t)\| &\geq A_2(u)(t_0) = \int_0^1 G_{m_1}(t_0,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)\right)u^\beta(\tau)\,d\tau)\,ds \\ &\geq S(m_1,m_2,n_1,\beta)\|u\|^{\beta(q-1)}. \end{aligned}$$

Combining these inequalities, for $u \in K$, we get

$$\|A(u)(t)\| \geq S(n_1,n_2,m_1,\alpha)S^{\alpha(q-1)}(m_1,m_2,n_1,\beta)\|u\|^{\alpha\beta(q-1)^2}. \tag{3.2}$$

On the other hand, for $u \in K$,

$$\begin{aligned} A_1(v)(t) &= \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)v^\alpha(\tau)\,d\tau\right)ds \\ &\leq \int_0^1 \frac{1}{6^{n_1-1}}s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1-\tau)v^\alpha(\tau)\,d\tau\right)ds \\ &\leq \frac{1}{6^{n_1}}\frac{1}{6^{n_2(q-1)}}\|v\|^{\alpha(q-1)}. \end{aligned}$$

In a similar way, we have

$$\begin{aligned} \|v(t)\| &= \|A_2(u)(t)\| = \max_{t\in[0,1]}\int_0^1 G_{m_1}(t,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)u^\beta(\tau)\,d\tau\right)ds \\ &\leq \frac{1}{6^{m_1}}\frac{1}{6^{m_2(q-1)}}\|u\|^{\beta(q-1)}. \end{aligned}$$

Combining these inequalities, for $u \in K$, we get

$$\|A(u)(t)\| \leq L(n_1,n_2)L^{\alpha(q-1)}(m_1,m_2)\|u\|^{\alpha\beta(q-1)^2}. \tag{3.3}$$

We take into account the following two cases.

Case 1: If $\alpha\beta < (p-1)^2$, then from Theorem 3.4 it follows that (3.1) has only a positive solution.

Case 2: If $\alpha\beta > (p-1)^2$, then from (3.2) it follows that there exists $R > 1$ such that, for any $u \in K \cap \partial\Omega_R$,

$$\|A(u)(t)\| \geq S(n_1,n_2,m_1,\alpha)S^{\alpha(q-1)}(m_1,m_2,n_1,\beta)\|u\|^{\frac{\alpha\beta}{(p-1)^2}} > \|u\|.$$

By (3.3) there exists $0 < r < 1$ such that, for any $u \in K \cap \partial\Omega_r$

$$\|A(u)(t)\| \leq L(n_1,n_2)L^{\alpha(q-1)}(m_1,m_2)\|u\|^{\frac{\alpha\beta}{(p-1)^2}} < \|u\|.$$

Therefore by Lemma 2.5 the operator A has at least one fixed point in $K \cap (\overline{\Omega}_R \setminus \Omega_r)$.

(III) We only need to show that A has no positive fixed point in K. On the contrary, if A has a positive fixed point $u^* \in K$, then we have

$$\begin{aligned} \|u^*\| &= \|A(u^*)(t)\| \\ &\leq \frac{1}{6^{n_1}}\frac{1}{6^{n_2(q-1)}}\left[\frac{1}{6^{m_1}}\frac{1}{6^{m_2(q-1)}}\|u^*\|^{\beta(q-1)}\right]^{\alpha(q-1)} \\ &= L(n_1,n_2)L^{\alpha(q-1)}(m_1,m_2)\|u^*\|^{\frac{\alpha\beta}{(p-1)^2}} \\ &< \|u^*\|^{\frac{\alpha\beta}{(p-1)^2}} = \|u^*\|, \end{aligned}$$

which yields a contradiction. □

Example 2 If $p = 3$ and $n_1 = n_2 = m_1 = m_2 = 1$, then (1.1) is related to the fourth-order system

$$\begin{cases} [\varphi_3(u''(t))]'' = f_1(t,v), & t \in (0,1), \\ [\varphi_3(v''(t))]'' = f_2(t,u), & t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases} \tag{3.4}$$

where $f_1(t,v) = tv^2$ and $f_2(t,u) = tu + tu^3$. Choosing $\alpha_1 = 2$, $\alpha_2 = \frac{3}{2}$, $\beta_1 = 2$, $\beta_2 = \frac{5}{2}$, and $\delta = \frac{1}{4}$, it is easy to verify that

$$\begin{aligned} \underline{\varphi}_1^0 &= \liminf_{v\to 0^+}\min_{t\in[\frac14,\frac34]}\frac{f_1(t,v)}{v^{\alpha_1}} = \liminf_{v\to 0^+}\frac{\frac14 v^2}{v^2} = 1, \\ \underline{\varphi}_2^0 &= \liminf_{u\to 0^+}\min_{t\in[\frac14,\frac34]}\frac{f_2(t,u)}{u^{\alpha_2}} = \liminf_{u\to 0^+}\frac{\frac14(u+u^3)}{u^{\frac32}} = +\infty, \\ \underline{\psi}_1^\infty &= \liminf_{v\to\infty}\min_{t\in[\frac14,\frac34]}\frac{f_1(t,v)}{v^{\beta_1}} = \liminf_{v\to\infty}\frac{\frac14 v^2}{v^2} = 1, \\ \underline{\psi}_2^\infty &= \liminf_{u\to+\infty}\min_{t\in[\frac14,\frac34]}\frac{f_2(t,u)}{u^{\beta_2}} = \liminf_{u\to+\infty}\frac{\frac14(u+u^3)}{u^{\frac52}} = +\infty, \end{aligned}$$

which implies that (i) of Theorem 3.5 holds.

Choosing $\widetilde{R} = 46{,}000$, via some computations we can get

$$N_{\widetilde{R}}^2 = \max\{f_2(t,u) : 0 \le t \le 1, \sigma\widetilde{R} \le u \le \widetilde{R}\} = \widetilde{R} + \widetilde{R}^3,$$
$$N_{\widetilde{R}}^1 = \max\left\{f_1(t,v) : 0 \le t \le 1, 0 \le v \le \frac{1}{6}\varphi_{\frac{3}{2}}\left(\frac{1}{6}\right)\varphi_{\frac{3}{2}}\left(N_{\widetilde{R}}^2\right)\right\} = \left(\frac{1}{6}\right)^3\left(\widetilde{R} + \widetilde{R}^3\right),$$
$$\frac{1}{6}\varphi_{\frac{3}{2}}\left(\frac{1}{6}\right)\varphi_{\frac{3}{2}}\left(N_{\widetilde{R}}^1\right) = \left(\frac{1}{6}\right)^3\left(\widetilde{R} + \widetilde{R}^3\right)^{\frac{1}{2}} \le \widetilde{R},$$

which yields that (ii) of Theorem 3.5 holds. Therefore (3.4) has at least two positive solutions.

Example 3 If $p = 2$, $n_1 = m_1 = 1$, and $n_2 = m_2 = 1$, then (1.1) is related to the second-order system

$$\begin{cases} u^{(4)}(t) = f_1(t,v), & t \in (0,1), \\ v^{(4)}(t) = f_2(t,u), & t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases} \tag{3.5}$$

where $f_1(t,v) = (t + \frac{2^{58}\cdot 3^6}{11^6})v^2$ and

$$f_2(t,u) = \begin{cases} tu^3, & 0 \le u \le 1; \\ tu^{\frac{1}{8}}, & 1 \le u. \end{cases}$$

Choosing $\alpha_1 = 2$, $\alpha_2 = \frac{3}{2}$, $\beta_1 = 2$, $\beta_2 = \frac{1}{4}$, and $\delta = \frac{1}{4}$, it is easy to verify that

$$\overline{\varphi}_1^0 = \limsup_{v\to 0^+} \max_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_1(t,v)}{v^{\alpha_1}} = \limsup_{v\to 0^+} \frac{(\frac{3}{4} + \frac{2^{58}\cdot 3^6}{11^6})v^2}{v^2} = \frac{3}{4} + \frac{2^{58}\cdot 3^6}{11^6},$$
$$\overline{\varphi}_2^0 = \limsup_{u\to 0^+} \max_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_2(t,u)}{u^{\alpha_2}} = \limsup_{u\to 0^+} \frac{\frac{3}{4}(u^3)}{u^{\frac{3}{2}}} = 0,$$
$$\overline{\psi}_1^\infty = \limsup_{v\to \infty} \max_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_1(t,v)}{v^{\beta_1}} = \limsup_{v\to \infty} \frac{(\frac{3}{4} + \frac{2^{58}\cdot 3^6}{11^6})v^2}{v^2} = \frac{3}{4} + \frac{2^{58}\cdot 3^6}{11^6},$$
$$\overline{\psi}_2^\infty = \limsup_{u\to +\infty} \max_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_2(t,u)}{u^{\beta_2}} = \limsup_{u\to +\infty} \frac{\frac{3}{4}(u^{\frac{1}{8}})}{u^{\frac{1}{4}}} = 0,$$

which implies that (i) of Theorem 3.6 holds.

Now, we will show that there exists a $\widehat{R}$ such that (ii) of Theorem 3.5 holds. For convenience, choose $\widehat{R} < 1$. Via some computations we can get

$$\theta_{n_1}(\delta) = \theta_{m_1}(\delta) = \frac{1}{4}, \qquad \theta_{n_2}^{q-1}(\delta) = \theta_{m_2}^{q-1}(\delta) = \frac{1}{4}, \qquad \sigma = \frac{1}{4}, \qquad \Gamma^q = \left(\frac{11}{96}\right)^2,$$
$$K_{\widehat{R}}^2 = \min\{f_2(t,u) : \delta \le t \le 1-\delta, \sigma\widehat{R} \le u \le \widehat{R}\}$$

$$
\begin{aligned}
&= \min\left\{f_2(t,u) : \frac{1}{4} \le t \le \frac{3}{4}, \frac{1}{4}\widehat{R} \le u \le \widehat{R}\right\} \\
&= \min\left\{tu^3 : \frac{1}{4} \le t \le \frac{3}{4}, \frac{1}{4}\widehat{R} \le u \le \widehat{R}\right\} = \left(\frac{1}{4}\right)^4 \widehat{R}^3, \\
N_{\widehat{R}}^2 &= \max\left\{f_2(t,u) : 0 \le t \le 1, \sigma\widehat{R} \le u \le \widehat{R}\right\} \\
&= \max\left\{f_2(t,u) : 0 \le t \le 1, \frac{1}{4}\widehat{R} \le u \le \widehat{R}\right\} \\
&= \max\left\{tu^3 : 0 \le t \le 1, \frac{1}{4}\widehat{R} \le u \le \widehat{R}\right\} = \widetilde{R}^3, \\
K_{\widehat{R}}^1 &= \min\left\{f_1(t,v) : \delta \le t \le 1-\delta, \theta_{m_1}(\delta)\theta_{m_2}^{q-1}(\delta)\Gamma^q\varphi_q\left(K_{\widehat{R}}^2\right) \le v \le \frac{1}{6^{m_1}}\varphi_q\left(\frac{1}{6^{m_2}}\right)\varphi_q\left(N_{\widehat{R}}^2\right)\right\} \\
&= \min\left\{f_1(t,v) : \frac{1}{4} \le t \le \frac{3}{4}, \frac{1}{4^2}\left(\frac{11}{96}\right)^2 \frac{1}{4^4}\widehat{R}^3 \le v \le \frac{1}{6^2}\widehat{R}^3\right\} \\
&= \min\left\{\left(t + \frac{2^{58}\cdot 3^6}{11^6}\right)v^2 : \frac{1}{4} \le t \le \frac{3}{4}, \frac{1}{4^2}\left(\frac{11}{96}\right)^2 \frac{1}{4^4}\widehat{R}^3 \le v \le \frac{1}{6^2}\widehat{R}^3\right\} \\
&= \left(\frac{1}{4} + \frac{2^{58}\cdot 3^6}{11^6}\right)\frac{1}{4^{12}}\left(\frac{11}{96}\right)^4 \widehat{R}^6.
\end{aligned}
$$

Choosing $\left(\frac{\frac{2^{58}\cdot 3^6}{11^6}}{\frac{1}{4}+\frac{2^{58}\cdot 3^6}{11^6}}\right)^{\frac{1}{5}} < \widehat{R} < 1$, we have

$$
\begin{aligned}
\theta_{n_1}(\delta)\theta_{n_2}^{q-1}(\delta)\Gamma^q\varphi_q\left(K_{\widehat{R}}^1\right) &= \frac{1}{4^2}\left(\frac{11}{96}\right)^2\left(\frac{1}{4} + \frac{2^{58}\cdot 3^6}{11^6}\right)\frac{1}{4^{12}}\left(\frac{11}{96}\right)^4 \widehat{R}^6 \\
&= \left(\frac{1}{4} + \frac{2^{58}\cdot 3^6}{11^6}\right)\frac{11^6}{2^{58}\cdot 3^6}\widehat{R}^6 \ge \widehat{R},
\end{aligned}
$$

which yields that (ii) of Theorem 3.6 holds. Therefore (3.5) has at least two positive solutions.

4 Main results of Problem (1.2)

Theorem 4.1 *Assume that* (H3) *or* ($\widetilde{\mathrm{H}}3$) *holds. In addition, assume that the functions* F_i, G_i $(i = 1,2)$ *satisfy the following condition*:

there exist $\alpha_i, \beta_i > 0$ $(i = 1,2)$ *with* $\alpha_i \le (p-1)$ *and* $\beta_i \le (p-1)$ *such that*

$$
\underline{F}_1^0 = \underline{G}_1^0 = +\infty \quad \textit{and} \quad \overline{F}_2^\infty = \overline{G}_2^\infty = 0.
$$

Then (1.2) *has at least one positive solution.*

Proof Let E denote the Banach space $C[0,1] \times C[0,1]$ with norm $\|(u,v)\| = \max\{|u(t)|_1, |v(t)|_1\}$, where $|u|_1 = \max_{t\in[0,1]}|u(t)|$. Define the mapping $A : E \to E$ by

$$
A(u,v)(t) = \bigl(A_1(u,v)(t), A_2(u,v)(t)\bigr),
$$

where

$$A_1(u,v)(t) = \int_0^1 G_{n_1}(t,s)\varphi_q\biggl(\int_0^1 G_{n_2}(s,\tau)f\bigl(u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds,$$
$$A_2(u,v)(t) = \int_0^1 G_{m_1}(t,s)\varphi_q\biggl(\int_0^1 G_{m_2}(s,\tau)g\bigl(u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds.$$

Let $P = \{u \in C[0,1], u \geq 0\}$, and let K be its subcone defined by

$$K = \Bigl\{u \in P: \min_{\delta\leq t\leq 1-\delta} u(t) \geq \sigma(n_1)|u|_1\Bigr\} \times \Bigl\{v: \min_{\delta\leq t\leq 1-\delta} v(t) \geq \sigma(m_1)|v|_1\Bigr\}.$$

As in the proof of Lemma 3.1, it is clear that $A: K \to K$ is completely continuous.

On one hand, from the assumption $\overline{F}_2^\infty = \overline{G}_2^\infty = 0$ it follows that there exist $\varepsilon > 0$ and $\overline{R} > 0$ such that

$$F_2(v) \leq \varepsilon v^{\beta_1} \quad \text{for } v \geq \overline{R} \quad \text{and} \quad G_2(u) \leq \varepsilon u^{\beta_2} \quad \text{for } u \geq \overline{R},$$

where ε satisfies

$$\varepsilon^{q-1}\max\bigl\{L(n_1,n_2),L(m_1,m_2)\bigr\} < 1.$$

For given $\overline{R}$, let

$$N_1 = \max_{0\leq v\leq\overline{R}} F_2(v), \qquad N_2 = \max_{0\leq u\leq\overline{R}} G_2(u).$$

Then we have

$$f(u,v) \leq F_2(v) \leq \varepsilon v^{\beta_1} + N_1, \qquad g(u,v) \leq G_2(u) \leq \varepsilon u^{\beta_2} + N_2 \quad \text{for } u,v \geq 0.$$

Furthermore, we have the the estimates

$$\begin{aligned} A_1(u,v)(t) &= \int_0^1 G_{n_1}(t,s)\varphi_q\biggl(\int_0^1 G_{n_2}(s,\tau)f\bigl(u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\leq \int_0^1 \frac{1}{6^{n_1-1}}s(1-s)\varphi_q\biggl(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1-\tau)\bigl[\varepsilon v^{\beta_1}(\tau)+N_1\bigr]\,d\tau\biggr)\,ds \\ &= L(n_1,n_2)\bigl(\varepsilon|v|_1^{\beta_1}+N_1\bigr)^{q-1} \end{aligned}$$

and

$$\begin{aligned} A_2(u,v)(t) &= \int_0^1 G_{m_1}(t,s)\varphi_q\biggl(\int_0^1 G_{m_2}(s,\tau)g\bigl(u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\leq \int_0^1 \frac{1}{6^{m_1-1}}s(1-s)\varphi_q\biggl(\int_0^1 \frac{1}{6^{m_2-1}}\tau(1-\tau)\bigl[\varepsilon u^{\beta_2}(\tau)+N_2\bigr]\,d\tau\biggr)\,ds \\ &= L(m_1,m_2)\bigl(\varepsilon|u|_1^{\beta_2}+N_2\bigr)^{q-1}. \end{aligned}$$

Therefore, combining them with the assumption $\beta_i \leq p-1$, we get that there exists a sufficiently large $R > 0$ such that, for any $(u,v) \in \partial\Omega_R \cap K$,

$$\begin{aligned}\|A(u,v)\| &\leq \max\bigl\{L(n_1,n_2)\bigl(\varepsilon|v|_1^{\beta_1}+N_1\bigr)^{q-1}, L(m_1,m_2)\bigl(\varepsilon|u|_1^{\beta_2}+N_2\bigr)^{q-1}\bigr\} \\ &\leq \max\bigl\{L(n_1,n_2)\bigl(\varepsilon R^{\beta_1}+N_1\bigr)^{q-1}, L(m_1,m_2)\bigl(\varepsilon R^{\beta_2}+N_2\bigr)^{q-1}\bigr\} \leq R.\end{aligned}$$

On the other hand, from the assumption $\underline{F}_1^0 = \underline{G}_1^0 = +\infty$ it follows that there exist $M > 0$ and $r < 1$ such that

$$F_1(v) \geq Mv^{\alpha_1} \quad \text{for } 0 \leq v \leq r \quad \text{and} \quad G_1(u) \geq Mu^{\alpha_2} \quad \text{for } 0 \leq u \leq r,$$

where M satisfies

$$M^{(q-1)} \min\bigl\{S(n_1,n_2,m_1,\alpha_1), S(m_1,m_2,n_1,\alpha_2)\bigr\} \geq 1.$$

Then for any $(u,v) \in \partial\Omega_r \cap K$ and $t_0 \in [\delta, 1-\delta]$, we have

$$\begin{aligned}A_1(u,v)(t_0) &= \int_0^1 G_{n_1}(t_0,s)\varphi_q\biggl(\int_0^1 G_{n_2}(s,\tau)f\bigl(u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\geq \int_\delta^{1-\delta} \theta_{n_1}(\delta)s(1-s)\varphi_q\biggl(\int_\delta^{1-\delta} \theta_{n_2}(\delta)\tau(1-\tau)F_1\bigl(v(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\geq \int_\delta^{1-\delta} \theta_{n_1}(\delta)s(1-s)\varphi_q\biggl(\int_\delta^{1-\delta} \theta_{n_2}(\delta)\tau(1-\tau)M\sigma^{\alpha_1}(m_1)|v|_1^{\alpha_1}\,d\tau\biggr)\,ds \\ &\geq M^{(q-1)}S(n_1,n_2,m_1,\alpha_1)|v|_1^{\alpha_1(q-1)} > |v|_1^{\alpha_1(q-1)}.\end{aligned}$$

In a similar way, we also have

$$\begin{aligned}A_2(u,v)(t_0) &= \int_0^1 G_{m_1}(t_0,s)\varphi_q\biggl(\int_0^1 G_{m_2}(s,\tau)\biggr)g\bigl(u(\tau),v(\tau)\bigr)\,d\tau)\,ds \\ &\geq M^{(q-1)}S(m_1,m_2,n_1,\alpha_2)|u|_1^{\alpha_2(q-1)} > |u|_1^{\alpha_2(q-1)}.\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}\max\bigl\{A_1(u,v)(t_0), A_2(u,v)(t_0)\bigr\} &> \max\bigl\{|v|_1^{\alpha_1(q-1)}, |u|_1^{\alpha_2(q-1)}\bigr\} \\ &= \max\bigl\{r^{\alpha_1(q-1)}, r^{\alpha_2(q-1)}\bigr\} \geq r.\end{aligned}$$

Therefore, for any $(u,v) \in \partial\Omega_r \cap K$, we have $\|A(u,v)\| > \|(u,v)\|$.

Therefore by Lemma 2.5 the operator A has at least one fixed point in $K \cap (\overline{\Omega}_R \setminus \Omega_r)$. □

Theorem 4.2 *Assume that* (H3) *or* ($\widetilde{\mathrm{H}}3$) *holds. In addition, let the functions* F_i, G_i ($i = 1, 2$) *satisfy the following assumption*:

there exist $\tilde{\alpha}_i, \tilde{\beta}_i > 0$ *with* $\tilde{\alpha}_i \geq (p-1)$ *and* $\tilde{\beta}_i \geq (p-1)$ *such that*

$$\underline{F}_1^\infty = \underline{G}_1^\infty = +\infty, \qquad \overline{F}_2^0 = \overline{G}_2^0 = 0.$$

Then (1.2) *has at least one positive solution.*

Proof On one hand, from the assumption $\overline{F}_2^0 = \overline{G}_2^0 = 0$ it follows that there exist $\varepsilon > 0$ and $r < 1$ such that

$$F_2(v) \le \varepsilon v^{\tilde{\beta}_1} \quad \text{for } 0 \le v \le r \quad \text{and} \quad G_2(u) \le \varepsilon u^{\tilde{\beta}_2} \quad \text{for } 0 \le u \le r,$$

where ε satisfies

$$\varepsilon^{q-1} \max\bigl\{L(n_1,n_2), L(m_1,m_2)\bigr\} < 1.$$

Then, for any $(u,v) \in \partial\Omega_r \cap K$, we have

$$\begin{aligned} A_1(u,v)(t) &= \int_0^1 G_{n_1}(t,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f\bigl(u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\ &\le \int_0^1 \frac{1}{6^{n_1-1}}s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1-\tau)\bigl[\varepsilon v^{\tilde{\beta}_1}(\tau)\bigr]\,d\tau\right)ds \\ &= L(n_1,n_2)\bigl(\varepsilon|v|_1^{\tilde{\beta}_1}\bigr)^{q-1} \end{aligned}$$

and

$$\begin{aligned} A_2(u,v)(t) &= \int_0^1 G_{m_1}(t,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)g\bigl(u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\ &\le \int_0^1 \frac{1}{6^{m_1-1}}s(1-s)\varphi_q\left(\int_0^1 \frac{1}{6^{m_2-1}}\tau(1-\tau)\bigl[\varepsilon u^{\tilde{\beta}_2}(\tau)\bigr]\,d\tau\right)ds \\ &= L(m_1,m_2)\bigl(\varepsilon|u|_1^{\tilde{\beta}_2}\bigr)^{q-1}. \end{aligned}$$

Therefore, combining these inequalities with the assumption $\tilde{\beta}_i \ge p-1$, we have

$$\bigl\|A(u,v)\bigr\| < \max\bigl\{|v|_1^{\tilde{\beta}_1(q-1)}, |u|_1^{\tilde{\beta}_2(q-1)}\bigr\} \le r.$$

On the other hand, from the assumption $\underline{F}_1^\infty = \underline{G}_1^\infty = +\infty$ it follows that there exist $M > 0$ and $\overline{R} > r$ such that

$$F_1(v) \ge Mv^{\tilde{\alpha}_1} \quad \text{for } v \ge \overline{R} \quad \text{and} \quad G_1(u) \ge Mu^{\tilde{\alpha}_2} \quad \text{for } u \ge \overline{R},$$

where M satisfies

$$M^{(q-1)} \min\bigl\{S(n_1,n_2,n_1,\tilde{\alpha}_1), S(m_1,m_2,m_1,\tilde{\alpha}_2)\bigr\} \ge 1.$$

Set $R = \max\{\frac{\overline{R}}{\sigma(n_1)} + 1, \frac{\overline{R}}{\sigma(m_1)} + 1\}$. Let

$$D_1 = \min_{0\le u\le R} F_1\bigl(v(\tau)\bigr), \qquad D_2 = \min_{0\le u\le R} G_1\bigl(u(\tau)\bigr).$$

Then, for any $(u,v) \in \partial\Omega_R \cap K$ and $t_0 \in [\delta, 1-\delta]$, we consider two cases.

Case i: If $\|(u,v)\| = |v|_1 = R$, then $\min_{t\in[\delta,1-\delta]} v(t) \geq \sigma(m_1)|v|_1 > \overline{R}$, and we have

$$\begin{aligned} A_1(u,v)(t_0) &= \int_0^1 G_{n_1}(t_0,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)f\big(u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\geq \int_\delta^{1-\delta} \theta_{n_1}(\delta)s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{n_2}(\delta)\tau(1-\tau)F_1\big(v(\tau)\big)\,d\tau\right)ds \\ &\geq \int_\delta^{1-\delta} \theta_{n_1}(\delta)s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{n_2}(\delta)\tau(1-\tau)M\sigma^{\tilde{\alpha}_1}(m_1)|v|_1^{\tilde{\alpha}_1}\,d\tau\right)ds \\ &\geq M^{(q-1)}S(n_1,n_2,m_1,\tilde{\alpha}_1)|v|_1^{\tilde{\alpha}_1(q-1)} > |v|_1^{\tilde{\alpha}_1(q-1)} \end{aligned}$$

and

$$\begin{aligned} A_2(u,v)(t_0) &= \int_0^1 G_{m_1}(t_0,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)\right)g\big(u(\tau),v(\tau)\big)\,d\tau)\,ds \\ &\geq \int_0^1 G_{m_1}(t_0,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)\right)G_1\big(u(\tau)\big)\,d\tau)\,ds \\ &\geq \int_0^1 G_{m_1}(t_0,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)\right)D_2\,d\tau)\,ds. \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} &\max\left\{A_1(u,v)(t_0),A_2(u,v)(t_0)\right\} \\ &\quad > \max\left\{|v|_1^{\tilde{\alpha}_1(q-1)}, \int_0^1 G_{m_1}(t_0,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)\right)D_2\,d\tau)\,ds\right\} \\ &\quad \geq |v|_1^{\tilde{\alpha}_1(q-1)} = R^{\tilde{\alpha}_1(q-1)} \geq R. \end{aligned}$$

Case ii: If $\|(u,v)\| = |u|_1 = R$, then $\min_{t\in[\delta,1-\delta]} u(t) \geq \sigma(n_1)|u|_1 > \overline{R}$, and we have

$$\begin{aligned} A_1(u,v)(t_0) &= \int_0^1 G_{mn1}(t_0,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)\right)f\big(u(\tau),v(\tau)\big)\,d\tau)\,ds \\ &\geq \int_0^1 G_{n_1}(t_0,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)\right)F_1\big(v(\tau)\big)\,d\tau)\,ds \\ &\geq \int_0^1 G_{n_1}(t_0,s)\varphi_q\left(\int_0^1 G_{n_2}(s,\tau)\right)D_1\,d\tau)\,ds \end{aligned}$$

and

$$\begin{aligned} A_2(u,v)(t_0) &= \int_0^1 G_{m_1}(t_0,s)\varphi_q\left(\int_0^1 G_{m_2}(s,\tau)g\big(u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\geq \int_\delta^{1-\delta} \theta_{m_1}(\delta)s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{m_2}(\delta)\tau(1-\tau)G_1\big(u(\tau)\big)\,d\tau\right)ds \\ &\geq \int_\delta^{1-\delta} \theta_{m_1}(\delta)s(1-s)\varphi_q\left(\int_\delta^{1-\delta} \theta_{m_2}(\delta)\tau(1-\tau)M\sigma^{\tilde{\alpha}_2}(n_1)|u|_1^{\tilde{\alpha}_2}\,d\tau\right)ds \\ &\geq M^{(q-1)}S(m_1,m_2,n_1,\tilde{\alpha}_2)|u|_1^{\tilde{\alpha}_2(q-1)} > |u|_1^{\tilde{\alpha}_2(q-1)}. \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
&\max\bigl\{A_1(u,v)(t_0),A_2(u,v)(t_0)\bigr\}\\
&\quad>\max\biggl\{\int_0^1 G_{n_1}(t_0,s)\varphi_q\biggl(\int_0^1 G_{n_2}(s,\tau)\biggr)D_1\,d\tau)\,ds,|u|_1^{\tilde{\alpha}_2(q-1)}\biggr\}\\
&\quad\geq |u|_1^{\tilde{\alpha}_2(q-1)}=R^{\tilde{\alpha}_2(q-1)}\geq R.
\end{aligned}$$

So, for any $(u,v)\in\partial\Omega_R\cap K$, we have $\|A(u,v)\|>\|(u,v)\|$.

Therefore by Lemma 2.5 the operator A has at least one fixed point in $K\cap(\overline{\Omega}_R\setminus\Omega_r)$. □

Example 4 Let $\alpha,\beta>0$ and $p=4$. Then, for the problem

$$\begin{cases}
(-1)^n[\varphi_4(u^{(2n_1)})]^{(2n_2)}=(\sin(u+v)+1)v^{\alpha}(t), & t\in(0,1),\\
(-1)^m[\varphi_4(v^{(2m_1)})]^{(2m_2)}=(e^{-v}+\arctan(u+1))u^{\beta}(t), & t\in(0,1),\\
u^{(2i)}(0)=u^{(2i)}(1)=0, & i=0,1,\ldots,n-1,\\
v^{(2j)}(0)=v^{(2j)}(1)=0, & j=0,1,\ldots,m-1,
\end{cases}$$

it is obvious that

$$\begin{aligned}
&F_1(v)=v^{\alpha}(t)\leq f(u,v)=\bigl(\sin(u+v)+1\bigr)v^{\alpha}(t)\leq F_2(v)=2v^{\alpha}(t),\\
&G_2(u)=\frac{\pi}{4}u^{\beta}(t)\leq g(u,v)=\bigl(e^{-v}+\arctan(u+1)\bigr)u^{\beta}(t)\leq G_2(u)=\biggl(1+\frac{\pi}{2}\biggr)u^{\beta}(t).
\end{aligned}$$

We choose $\alpha_1=\widetilde{\alpha}_1=3$, $\alpha_2=\tilde{\alpha}_2=\frac{\beta+3}{2}$, $\beta_1=\tilde{\beta}_1=\frac{3\alpha+3}{4}$, $\beta_2=\tilde{\beta}_2=\frac{4\beta+3}{5}$.

Case I. If $\alpha,\beta<3$, then it is easy to verify that

$$\begin{aligned}
&\underline{F}_1^0=\liminf_{v\to0^+}\frac{F_1(v)}{c^{\alpha_1}}=\liminf_{v\to0^+}\frac{v^{\alpha}}{v^3}=+\infty,\\
&\overline{F}_2^{\infty}=\limsup_{v\to+\infty}\frac{F_2(v)}{v^{\beta_1}}=\limsup_{v\to+\infty}\frac{v^{\alpha}}{v^{\frac{3\alpha+3}{4}}}=0,\\
&\underline{G}_1^0=\liminf_{u\to0^+}\frac{G_1(u)}{u^{\alpha_2}}=\liminf_{u\to0^+}\frac{u^{\beta}}{u^{\frac{\beta+3}{2}}}=+\infty,\\
&\overline{G}_2^{\infty}=\limsup_{u\to+\infty}\frac{G_2(u)}{u^{\beta_2}}=\limsup_{u\to+\infty}\frac{u^{\beta}}{u^{\frac{4\beta+3}{5}}}=0.
\end{aligned}$$

So by Theorem 4.1 the problem has at least one positive solution.

Case II. If $\alpha,\beta>3$, then it is easy to verify that

$$\begin{aligned}
&\underline{F}_1^{\infty}=\liminf_{v\to+\infty}\frac{F_1(v)}{v^{\tilde{\alpha}_1}}=\liminf_{v\to+\infty}\frac{v^{\alpha}}{v^3}=+\infty,\\
&\overline{F}_2^0=\limsup_{v\to0^+}\frac{F_2(v)}{v^{\tilde{\beta}_1}}=\limsup_{v\to0^+}\frac{v^{\alpha}}{v^{\frac{3\alpha+3}{4}}}=0,
\end{aligned}$$

$$\underline{G}_1^\infty = \liminf_{u\to+\infty} \frac{G_1(u)}{u^{\tilde{\alpha}_2}} = \liminf_{u\to+\infty} \frac{u^\beta}{u^{\frac{\beta+3}{2}}} = +\infty,$$

$$\overline{G}_2^0 = \limsup_{u\to 0^+} \frac{G_2(u)}{u^{\tilde{\beta}_2}} = \limsup_{u\to 0^+} \frac{u^\beta}{u^{\frac{4\beta+3}{5}}} = 0.$$

So by Theorem 4.2 the problem has at least one positive solution.

Acknowledgements
We would like to thank the referee for his/her valuable comments and helpful suggestions, which have led to an improvement of the presentation of this paper.

Funding
The authors were supported by the Fundamental Research Funds for the Central University (No. 2018B58614).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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Publisher's Note

Received: 6 November 2018 Accepted: 7 January 2019 Published online: 15 January 2019

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