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Existence results for a generalization of the time-fractional diffusion equation with variable coefficients

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Abstract

In this paper we consider the Cauchy problem of a generalization of time-fractional diffusion equation with variable coefficients in $\mathbb{R}^{n+1}_+$, where the time derivative is replaced by a regularized hyper-Bessel operator. The explicit solution of the inhomogeneous linear equation for any $n \in \mathbb{Z}^+$ and its uniqueness in a weighted Sobolev space are established. The key tools are Mittag-Leffler functions, *M*-Wright functions and Mikhlin multiplier theorem. At last, we obtain the existence of solution of the semilinear equation for $n = 1$ by using a fixed point theorem.

MSC: 34A08; 35R11

Keywords: Fractional diffusion equation; Mittag-Leffler function; *M*-Wright function; Mikhlin multiplier theorem; Weighted Sobolev space; Existence

1 Introduction

In this paper we study the existence of solutions for the following generalization of the time-fractional diffusion equation with variable coefficients:

$$\begin{cases} {}^{C}(t^{\theta}\frac{\partial}{\partial t})^{\alpha}u - \triangle u = f & \text{in } \mathbb{R}^{n+1}_+, \\ u(0,x) = \varphi(x), \end{cases} \tag{1}$$

where $\mathbb{R}^{n+1}_+ = (0,+\infty) \times \mathbb{R}^n$, $\triangle = \sum_{i=1}^{n} \partial^2_{x_i}$ is the Laplace differential operator, ${}^{C}(t^{\theta}\frac{\partial}{\partial t})^{\alpha}$ stands for a Caputo-like counterpart to hyper-Bessel operator of order $\alpha \in (0,1)$ and the parameter $\theta < 1$.

Fractional models are proved to be more adequate than those of integer order for some problems in science and engineering. Fractional differential equations play a very important role in the mathematical modeling of various physical systems [8, 10, 14, 20, 30]. The investigation of (1) is inspired by the fractional extension of the diffusion equation governing the law of the fractional Brownian motion [3, 22]:

$$\left(t^{1-2H}\frac{\partial}{\partial t}\right)^{\alpha}u(t,x) = H^{\alpha}\frac{\partial^2}{\partial x^2}u(t,x), \quad \alpha \in (0,1), H \in (0,1), x \in \mathbb{R}, \tag{2}$$

where $(t^{1-2H}\frac{\partial}{\partial t})^{\alpha}$ is a hyper-Bessel type operator. Set $y = H^{\frac{\alpha}{2}}x$ and $1 - 2H = \theta$, then (2) is reduced into

$$\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha}u(t,y) - \frac{\partial^2}{\partial y^2}u(t,y) = 0, \quad \alpha \in (0,1), \theta \in (-1,1), x \in \mathbb{R}, \tag{3}$$

which is a special case of (1). For the general case, [11, 12] provided the definition of the operator $(t^{\theta}\frac{\partial}{\partial t})^{\alpha}$ for $\alpha \in (0,1]$ and $\theta \in \mathbb{R}$ when studying the fractional diffusions and fractional relaxation.

The hyper-Bessel operator reads

$$L = t^{a_1}\frac{d}{dt}t^{a_2}\frac{d}{dt}\cdots\frac{d}{dt}t^{a_{n+1}}, \quad t > 0, \tag{4}$$

where a_i, $i = 1,2,\ldots,n+1$, are real numbers and $n \in \mathbb{Z}^+$. To the best of our knowledge, the fractional power L^{α} of the hyper-Bessel operator was first introduced by Dimovski [9] and developed by McBride and Lamb [19, 23, 24]. The theory of L^{α} has been applied to solve various problems, such as diffusive transport [11, 12, 29], Brownian motion [3, 22, 25–28]. Recently, Al-Musalhi, Al-Salti, and Karimov generalized $(t^{\theta}\frac{d}{dt})^{\alpha}$ to the Caputo-like counterpart of hyper-Bessel operator ${}^{C}(t^{\theta}\frac{\partial}{\partial t})^{\alpha}$ in [1] defined by

$${}^{C}\left(t^{\theta}\frac{d}{dt}\right)^{\alpha}f(t) = \left(t^{\theta}\frac{d}{dt}\right)^{\alpha}f(t) - \frac{f(0)t^{-\alpha(1-\theta)}}{(1-\theta)^{-\alpha}\Gamma(1-\theta)}, \quad 0 < \alpha < 1, \theta < 1.$$

They used Erdélyi–Kober fractional integral to express the hyper-Bessel operator and established the series solution by considering both direct and inverse source problem in a rectangular domain. In [2], Al-Saqabi and his collaborators considered Volterra integral equation of the second kind and a fractional differential equation, involving Erdélyi–Kober fractional integral or differential operator. The explicit solutions of these equations were derived by use of transmutation method. For a special case of $\theta = 0$ and $\alpha > 1$, the existence of unique solution was established by use of a perturbation argument and Green's function in [4, 5]. In [13], applying a direct variational approach and the theory of the fractional derivative spaces, the existence of infinitely many distinct positive solutions were given. For more results related to hyper-Bessel operator and Erdélyi–Kober fractional integral or differential operator, see [6, 29, 31] and references therein. However, these methods and techniques cannot be directly employed to the multidimensional or the nonlinear case in Sobolev space. In this paper, we will go a step further to form the explicit solution in multidimensional space, then use Mittag-Leffler functions and Mikhlin's multiplier theorem to obtain the weighted $\dot{H}^{s,p}$, $1 < p < +\infty$ and L^{∞} estimate of the solution. At last, we form a contractible mapping to show the existence of solution of the semilinear problem in a suitable fractional derivative Sobolev space. The main idea is motivated in the proof of [32, 33]. The existence of solutions in Banach spaces were also investigated in [7, 13, 34–38] and the necessary and sufficient conditions on the initial data for the solvability of a space-fractional semilinear parabolic equation were obtained in [17].

This paper is organized as follows: In Sect. 2, the related results of Mittag-Leffler functions and M-Wright functions are recalled. The explicit solution of a related time-fractional ordinary differential equation is established. In Sect. 3, in terms of the explicit solution given in Sect. 2, we derive the existence and uniqueness of solution $u \in$

$C([0,+\infty),L^p(\mathbb{R}^n)) \cap C((0,+\infty),\dot{H}^{k,p}(\mathbb{R}^n)) \cap C^\alpha((0,+\infty),L^p(\mathbb{R}^n))$, $k = 1,2$ of the corresponding linear problem. In the last section, by use a fixed point theorem we show the existence of solution $u \in C([0,T),L^p(\mathbb{R})) \cap C((0,T),\dot{H}^{k,p}(\mathbb{R})) \cap C^\alpha((0,T),L^p(\mathbb{R}))$, $k = 1,2$ of the semilinear problem for a fixed positive number T.

2 Preliminaries

In this section we present some necessary definitions and auxiliary results for the convenience of the reader, then establish the explicit solution of the Cauchy problem of a time-fractional ordinary differential equation.

First, we recall Mittag-Leffler function $E_{\delta,\beta}(z)$ with two parameters, which can be found in [15, 16] or [30],

$$E_{\delta,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k+\beta)}, \quad \Re(\delta) > 0, \Re(\beta) > 0. \tag{5}$$

Lemma 2.1

$$\frac{d}{dy}E_{\delta,\beta}(y) = \frac{E_{\delta,\beta-1}(y) - (\beta-1)E_{\delta,\beta}(y)}{\delta y}, \tag{6}$$

$$\frac{d^m}{dy^m}\bigl(y^{\beta-1}E_{\delta,\beta}(y^\delta)\bigr) = y^{\beta-m-1}E_{\delta,\beta-m}(y^\alpha), \quad \Re(\beta-m) > 0, m \in \mathbb{N}. \tag{7}$$

Lemma 2.2 *Let $\delta < 2$, $\beta \in \mathbb{R}$ and $\frac{\pi\delta}{2} < \mu < \min\{\pi,\pi\delta\}$. Then we have the following estimate:*

$$\bigl|E_{\delta,\beta}(y)\bigr| \le \frac{M}{1+|y|}, \quad \mu \le |\arg y| \le \pi.$$

where M denotes a positive constant.

Lemma 2.3 *For each $k \in \mathbb{Z}^+$ and any $\Re(\alpha) > 0$, $\beta \in \mathbb{R}$, $0 \le \delta \le 1$, there exists a positive constant C_k such that*

$$|y|^k \left|\frac{d^k}{dy^k}\bigl(y^\delta E_{\alpha,\beta}(y)\bigr)\right| \le C_k. \tag{8}$$

Proof For $k = 1$, (8) directly follows from (6) in Lemma 2.1 and Lemma 2.2.

For $k = 2$, $y^2\frac{d^2}{dy^2} = (y\frac{d}{dy})^2 - y\frac{d}{dy}$. Then it is enough to show $(y\frac{d}{dy})^2(y^\delta E_{\alpha,\beta}(y))$ is bounded. By a direct computation in terms of (6), we get that

$$\begin{aligned}\left(y\frac{d}{dy}\right)^2 &\bigl(y^\delta E_{\alpha,\beta}(y)\bigr) \\ &= \frac{1}{\alpha}y\frac{d}{dy}\bigl(y^\delta\bigl(E_{\alpha,\beta-1}(y) - (\beta-1)E_{\alpha,\beta}(y)\bigr)\bigr) + \delta y\frac{d}{dy}\bigl(y^\delta E_{\alpha,\beta}(y)\bigr).\end{aligned}$$

This reduces to $k = 1$. Hence, (8) holds for $k = 2$. Furthermore, following the same idea, we conclude that $(y\frac{d}{dy})^k(y^\delta E_{\alpha,\beta}(y))$ is bounded for any $k \in \mathbb{Z}^+$.

By induction, assume for $k-1$ that

$$|y|^{k-1}\left|\frac{d^{k-1}}{dy^{k-1}}\bigl(y^\delta E_{\alpha,\beta}(y)\bigr)\right| \le C_{k-1}, \tag{9}$$

$$y^{k-1}\frac{d^{k-1}}{dy^{k-1}}=\sum_{i=1}^{k-1}b_i\left(y\frac{d}{dy}\right)^i, \tag{10}$$

where b_i are constants. Then by use of (6) or (7), we have

$$\begin{aligned} & y^k\left(\frac{d}{dy}\right)^k\left(y^\delta E_{\alpha,\beta}(y)\right) \\ &= y\frac{d}{dy}\left(\sum_{i=1}^{k-1}b_i\left(y\frac{d}{dy}\right)^i\left(y^\delta E_{\alpha,\beta}(y)\right)\right) \\ &= \sum_{i=1}^{k}d_i\left(y\frac{d}{dy}\right)^i\left(y^\delta E_{\alpha,\beta}(y)\right). \end{aligned} \tag{11}$$

It follows from (9) and (11) that (8) holds. □

From (8) we can prove the following.

Corollary 2.4 *For each $\gamma\in Z^+$ and any $\alpha>0$, $\beta\in\mathbb{R}$, $0\le\delta\le1$, there exists a positive constant C_γ such that*

$$\left||\xi|^\gamma\frac{\partial^\gamma}{\partial\xi^\gamma}\left(y^\delta E_{\alpha,\beta}(y)\right)\right|\le C_\gamma, \tag{12}$$

where $y=-\rho^{-\alpha}|\xi|^2t^{\rho\alpha}$.

Next, we choose the version of Mikhlin's multiplier theorem given in [18] as our lemma.

Lemma 2.5 *Let $a(\xi)$ be the symbol of a singular integral operator A in $\mathbb{R}^n$. Suppose that $a(\xi)\in C^\infty(\mathbb{R}^n\setminus\{0\})$, and there is some positive constant M for all $\xi\neq0$ such that*

$$|\xi|^{|\gamma|}\left|\frac{\partial^\gamma a(\xi)}{\partial\xi^\gamma}\right|\le M,\quad 0\le|\gamma|\le1+\frac{[n]}{2}.$$

Then, A is a bounded linear operator from $L^p(\mathbb{R}^n)$ into itself for $1<p<+\infty$, and its operator norm depends only on M, n and p.

Based on expression (5), the explicit solution of the following problem of the inhomogeneous time-fractional differential equation

$$\begin{cases} {}^C(t^\theta\frac{d}{dt})^\alpha u(t)=-\lambda u(t)+f(t), & t>0,\\ u(0)=u_0, \end{cases} \tag{13}$$

is obtained, where u_0 is a constant number, $\theta<1$, $0<\alpha<1$.

Theorem 2.6 *Consider problem (13). Then there is an explicit solution, which is given in the integral form*

$$u(t)=u_0E_{\alpha,1}\left(\lambda^*t^{\rho\alpha}\right)+\frac{1}{\rho^\alpha}\int_0^t\left(t^\rho-s^\rho\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda^*\left(t^\rho-s^\rho\right)^\alpha\right)f(s)\,d\left(s^\rho\right), \tag{14}$$

where $\rho=1-\theta$ and $\lambda^*=-\frac{\lambda}{\rho^\alpha}$.

Proof In terms of Lemma 2.7 given in [1], the expression of $u(t)$ is written as

$$\begin{aligned} u(t) &= u_0 E_{\alpha,1}\big(\lambda^* t^{\rho\alpha}\big) + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t \big(t^\rho - s^\rho\big)^{\alpha-1} f(s)\, d\big(s^\rho\big) \\ &\quad + \frac{\lambda^*}{\rho^\alpha} \int_0^t \big(t^\rho - s^\rho\big)^{2\alpha-1} E_{\alpha,2\alpha}\big(\lambda^*\big(t^\rho - s^\rho\big)^\alpha\big) f(s)\, d\big(s^\rho\big) \\ &= u_0 E_{\alpha,1}\big(\lambda^* t^{\rho\alpha}\big) + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t \big(t^\rho - s^\rho\big)^{\alpha-1} \\ &\quad \times \big(1 + \Gamma(\alpha)\lambda^*\big(t^\rho - s^\rho\big)^\alpha E_{\alpha,2\alpha}\big(\lambda^*\big(t^\rho - s^\rho\big)^\alpha\big)\big) f(s)\, d\big(s^\rho\big), \end{aligned} \tag{15}$$

Besides, the integrand in the last integral of (16) satisfies

$$\begin{aligned} &1 + \Gamma(\alpha) y^\alpha E_{\alpha,2\alpha}\big(y^\alpha\big) \\ &\quad = 1 + \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{y^{(k+1)\alpha}}{\Gamma(k\alpha + 2\alpha)} \\ &\quad = 1 + \Gamma(\alpha) \sum_{k=1}^{\infty} \frac{y^{k\alpha}}{\Gamma(k\alpha + \alpha)} \\ &\quad = \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{y^{k\alpha}}{\Gamma(k\alpha + \alpha)} \\ &\quad = \Gamma(\alpha) E_{\alpha,\alpha}\big(y^\alpha\big). \end{aligned} \tag{16}$$

Then substituting (16) into (15) with $y^\alpha = \lambda^*(t^\rho - s^\rho)^\alpha$, the explicit solution (14) is established.

Hence, we complete the proof of Theorem 2.6. □

Last, we recite the asymptotic behavior of M-Wright function derived in [21], which is defined as

$$M_\nu(y) = \sum_{n=0}^{\infty} \frac{(-y)^n}{n!\,\Gamma(-n\nu + 1 - \nu)}, \quad \nu \in (0,1).$$

Lemma 2.7 *Given* $a(\nu) = \frac{1}{\sqrt{2\pi(1-\nu)}} > 0$, $b(\nu) = \frac{1-\nu}{\nu} > 0$ *for some* ν, *the asymptotic representation of* M*-Wright function for large* y *is*

$$M_\nu\left(\frac{y}{\nu}\right) \sim a(\nu) y^{\frac{\nu - \frac{1}{2}}{1-\nu}} e^{-b(\nu) y^{\frac{1}{1-\nu}}}.$$

3 Existence and uniqueness of solution of the linear problem

In this section, based on Theorem 2.6, Mattag-Leffler function, M-Wright functions and Mikhlin multiplier theorem, we show the existence of L^p solution of the corresponding linear problem (1) for any $n \in \mathbb{Z}^+$.

We first consider the linear problem

$$\begin{cases} {}^C\big(t^\theta \frac{\partial}{\partial t}\big)^\alpha u - \triangle u = f(t,x) & \text{in } \mathbb{R}_+^{n+1}, \\ u(0,x) = \varphi(x). \end{cases} \tag{17}$$

Taking partial Fourier transformation with respect to x in Eq. (17) yields the following problem:

$$\begin{cases} {}^{C}(t^{\theta}\frac{\partial}{\partial t})^{\alpha}\hat{u}(t,\xi) = -|\xi|^2\hat{u}(t,\xi) + \hat{f}(t,\xi) & \text{in } \mathbb{R}^{n+1}_{+}, \\ \hat{u}(0,\xi) = \hat{\varphi}(\xi), \end{cases}$$

where $\hat{u}(t,\xi) = \mathfrak{F}(u(t,x)) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(t,x)\,dx$.

Set $\lambda = |\xi|^2$ in (11). According to Theorem 2.6, the solution of (17) is given by

$$u(t,x) = u_0(t,x) + \frac{1}{\rho^{\alpha}}\int_0^t \bigl(t^{\rho} - s^{\rho}\bigr)^{\alpha-1}\mathfrak{F}^{-1}\bigl(E_{\alpha,\alpha}\bigl(-\rho^{-\alpha}|\xi|^2\bigl(t^{\rho} - s^{\rho}\bigr)^{\alpha}\bigr)f(s,\xi)\bigr)\,d\bigl(s^{\rho}\bigr), \tag{18}$$

where

$$u_0(t,x) = \mathfrak{F}^{-1}\bigl(\hat{\varphi}(\xi)E_{\alpha,1}\bigl(-\rho^{-\alpha}|\xi|^2 t^{\rho\alpha}\bigr)\bigr). \tag{19}$$

Theorem 3.1 *Set $1 < p < +\infty$, $\alpha \in (0,1)$, $\theta < 1$. Suppose $\varphi \in C_0^{\infty}(\mathbb{R}^n)$, $f \in C_0^{\infty}(\mathbb{R}^{n+1}_{+})$, then there exists a unique solution $u \in C([0,+\infty), L^p(\mathbb{R}^n)) \cap C((0,+\infty), \dot{H}^{k,p}(\mathbb{R}^n)) \cap C^{\alpha}((0,+\infty), L^p(\mathbb{R}^n))$ of problem (17), which is represented by (18) under Fourier transformation and satisfies*

$$\begin{aligned} &\sum_{k=0}^{2}\bigl\|t^{\delta_k}u(t,\cdot)\bigr\|_{\dot{H}^{k,p}(\mathbb{R}^n)} + \left\|t^{\delta_2}{}^{C}\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha}u(t,\cdot)\right\|_{L^p(\mathbb{R}^n)} \\ &\quad\lesssim \|\varphi\|_{L^p(\mathbb{R}^n)} + t^{\delta_2}\int_0^1\sum_{k=0}^{2}\bigl(1 - s^{\rho}\bigr)^{\alpha-1-\frac{k\alpha}{2}}\bigl\|f(st,\cdot)\bigr\|_{L^p(\mathbb{R}^n)}\,d\bigl(s^{\rho}\bigr), \end{aligned} \tag{20}$$

where $\dot{H}^{k,p}(\mathbb{R}^n)$ denotes the homogeneous Sobolev space, $\delta_k = \frac{\rho\alpha k}{2}$, $\rho = 1 - \theta$.

Proof It follows from (18)–(19) that

$$\begin{aligned} &\bigl\|u(t,\cdot)\bigr\|_{\dot{H}^{\delta,p}(\mathbb{R}^n)} \\ &\quad= \bigl\|\mathfrak{F}^{-1}\bigl(|\xi|^{\delta}\hat{u}(t,\xi)\bigr)\bigr\|_{L^p(\mathbb{R}^n)} \\ &\quad\le \bigl\|\mathfrak{F}^{-1}\bigl(|\xi|^{\delta}\hat{u}_0(t,\xi)\bigr)\bigr\|_{L^p(\mathbb{R}^n)} \\ &\qquad+ \left\|\mathfrak{F}^{-1}\left(\frac{|\xi|^{\delta}}{\rho^{\alpha}}\int_0^t\bigl(t^{\rho} - s^{\rho}\bigr)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\rho^{-\alpha}|\xi|^2\bigl(t^{\rho} - s^{\rho}\bigr)^{\alpha}\bigr)\hat{f}(s,\xi)\,d\bigl(s^{\rho}\bigr)\right)\right\|_{L^p(\mathbb{R}^n)} \\ &\quad\lesssim \bigl\|\mathfrak{F}^{-1}\bigl(\hat{\varphi}(\xi)t^{-\frac{\rho\alpha\delta}{2}}\bigl(-\rho^{-\alpha}|\xi|^2t^{\rho\alpha}\bigr)^{\frac{\delta}{2}}E_{\alpha,1}\bigl(-\rho^{-\alpha}|\xi|^2t^{\rho\alpha}\bigr)\bigr)\bigr\|_{L^p(\mathbb{R}^n)} + \int_0^t\bigl(t^{\rho} - s^{\rho}\bigr)^{\alpha-1-\frac{\delta\alpha}{2}} \\ &\qquad\times \bigl\|\mathfrak{F}^{-1}\bigl(\bigl(-\rho^{-\alpha}|\xi|^2\bigl(t^{\rho} - s^{\rho}\bigr)^{\alpha}\bigr)^{\frac{\delta}{2}}E_{\alpha,\alpha}\bigl(-\rho^{-\alpha}|\xi|^2\bigl(t^{\rho} - s^{\rho}\bigr)^{\alpha}\bigr)\hat{f}(s,\xi)\bigr)\bigr\|_{L^p(\mathbb{R}^n)}\,d\bigl(s^{\rho}\bigr). \end{aligned} \tag{21}$$

Let $y = -\rho^{-\alpha}|\xi|^2(t^{\rho} - s^{\rho})^{\alpha}$, then (12) yields

$$|\xi|^{\gamma}\left|\frac{\partial^{\gamma}}{\partial\xi^{\gamma}}\bigl(y^{\frac{\delta}{2}}E_{\alpha,\beta}(y)\bigr)\right| \le C_{\gamma}.$$

According to Lemma 2.5, we have

$$\left\|\mathfrak{F}^{-1}\left(\hat{\varphi}(\xi)t^{-\frac{\rho\alpha\delta}{2}}y^{\frac{\delta}{2}}E_{\alpha,1}(y)\right)\right\|_{L^p(\mathbb{R}^n)} \lesssim t^{-\frac{\rho\alpha\delta}{2}}\|\varphi\|_{L^p(\mathbb{R}^n)}, \tag{22}$$

$$\left\|\mathfrak{F}^{-1}\left(y^{\frac{\delta}{2}}E_{\alpha,\alpha}(y)\hat{f}(s,\xi)\right)\right\|_{L^p(\mathbb{R}^n)} \lesssim \left\|f(s,\cdot)\right\|_{L^p(\mathbb{R}^n)}. \tag{23}$$

Substituting (22)–(23) into (21), we get

$$\left\|u(t,\cdot)\right\|_{\dot{H}^{\delta,p}(\mathbb{R}^n)} \lesssim t^{-\frac{\rho\alpha\delta}{2}}\left(\|\varphi\|_{L^p(\mathbb{R}^n)} + t^{\rho\alpha}\int_0^1\left(1-s^\rho\right)^{\alpha-1-\frac{\delta\alpha}{2}}\left\|f(st,\cdot)\right\|_{L^p(\mathbb{R}^n)}\,d\left(s^\rho\right)\right).$$

Summing up with $\delta = 0,1,2$, we arrive at the following estimate:

$$\begin{aligned}\sum_{k=0}^{2}\left\|t^{\delta_k}u(t,\cdot)\right\|_{\dot{H}^{k,p}(\mathbb{R}^n)} &\lesssim \|\varphi\|_{L^p(\mathbb{R}^n)} \\ &\quad + t^{\rho\alpha}\int_0^1\sum_{k=0}^{2}\left(1-s^\rho\right)^{\alpha-1-\frac{k\alpha}{2}}\left\|f(st,\cdot)\right\|_{L^p(\mathbb{R}^n)}\,d\left(s^\rho\right)\end{aligned} \tag{24}$$

with $\delta_k = \frac{\rho\alpha k}{2}$.

For the term ${}^C(t^\theta\frac{\partial}{\partial t})^\alpha u(t,\cdot)$, we will use Eq. (17) to estimate as follows:

$$\begin{aligned}&\left\|{}^C\left(t^\theta\frac{\partial}{\partial t}\right)^\alpha u(t,\cdot)\right\|_{L^p(\mathbb{R}^n)} \\ &\quad = \left\|\triangle u + f(t,x)\right\|_{L^p(\mathbb{R}^n)} \\ &\quad \lesssim \left\|u(t,\cdot)\right\|_{\dot{H}^{2,p}(\mathbb{R}^n)} + \left\|f(t,\cdot)\right\|_{L^p(\mathbb{R}^n)} \\ &\quad \lesssim t^{-\rho\alpha}\left(\|\varphi\|_{L^p(\mathbb{R}^n)} + t^{\rho\alpha}\int_0^1\sum_{k=0}^{2}\left(1-s^\rho\right)^{\alpha-1-\frac{k\alpha}{2}}\left\|f(st,\cdot)\right\|_{L^p(\mathbb{R}^n)}\,d\left(s^\rho\right)\right).\end{aligned} \tag{25}$$

Combing (24) and (25), we arrive at (20), which implies the existence and uniqueness of solution $u \in C([0,+\infty),L^p(\mathbb{R}^n)) \cap C((0,+\infty),\dot{H}^{k,p}(\mathbb{R}^n)) \cap C^\alpha((0,+\infty),L^p(\mathbb{R}^n))$, $k = 1,2$.

Thus, we complete the proof of Theorem 3.1. □

4 Existence of solution of the semilinear problem

In this section, we consider the semilinear problem (1) in the half-space $\mathbb{R}^2_+$ and show the existence of a solution by use of a fixed point theorem.

We assume a condition on the nonlinear term with a positive constant C so that

$$\left|f(u)\right| \lesssim |u|^\mu, \qquad \left|f^{(k)}(u)\right| \lesssim C, \quad \mu > 1, k = 1,2. \tag{26}$$

The L^∞-norm estimate of $u_0(t,x)$ is necessary, with $u_0(t,x)$ defined in (19).

Theorem 4.1

$$\left\|u_0(t,\cdot)\right\|_{L^\infty(\mathbb{R}^2_+)} \lesssim \|\varphi\|_{L^\infty(\mathbb{R})}. \tag{27}$$

Proof It follows from (19) that

$$u_0(t,x) = \mathfrak{F}^{-1}\left(E_{\alpha,1}\left(-\rho^{-\alpha}|\xi|^2 t^{\rho\alpha}\right)\right) * \varphi(x),$$

and then we arrive

$$\left\|u_0(t,\cdot)\right\|_{L^\infty(\mathbb{R}^2_+)} \lesssim \left\|\mathfrak{F}^{-1}\left(E_{\alpha,1}\left(-\rho^{-\alpha}|\xi|^2 t^{\rho\alpha}\right)\right)\right\|_{L^\infty((0,+\infty),L^1(\mathbb{R}))}\|\varphi\|_{L^\infty(\mathbb{R})}. \tag{28}$$

The Fourier transformation of M-Wright function given by (4.15) in [12] is

$$\mathfrak{F}\left(M_\nu\left(|x|\right)\right) = 2E_{2\nu,1}\left(-|\xi|^2\right),$$

which implies

$$\mathfrak{F}^{-1}\left(E_{\alpha,1}\left(-\rho^{-\alpha}|\xi|^2 t^{\rho\alpha}\right)\right) = \frac{\rho^{\frac{\alpha}{2}}}{2t^{\frac{\rho\alpha}{2}}} M_{\frac{\alpha}{2}}\left(\rho^{\frac{\alpha}{2}}|x|t^{-\frac{\rho\alpha}{2}}\right).$$

Then by a direct computation in terms of the analytic expression of M-Wright function and the asymptotics for large variables given in Lemma 2.7, we have

$$\begin{aligned}
&\left\|\mathfrak{F}^{-1}\left(E_{\alpha,1}\left(-\rho^{-\alpha}|\xi|^2 t^{\rho\alpha}\right)\right)\right\|_{L^\infty((0,+\infty),L^1(\mathbb{R}))} \\
&\quad\le \left\|\frac{\rho^{\frac{\alpha}{2}}}{2t^{\frac{\rho\alpha}{2}}} M_{\frac{\alpha}{2}}\left(\rho^{\frac{\alpha}{2}}|x|t^{-\frac{\rho\alpha}{2}}\right)\right\|_{L^\infty((0,+\infty),L^1(\mathbb{R}))} \\
&\quad\le C.
\end{aligned} \tag{29}$$

Substituting (29) into (28), we obtain (27).

This concludes the proof of Theorem 4.1. □

Theorem 4.2 *Set* $1 < p < +\infty$, $\alpha \in (0,1)$, $\theta < 1$. *Suppose* $\varphi \in C_0^\infty(\mathbb{R})$ *and let* $f(t,x,\cdot)$ *satisfy* (26), *then there exists a solution* $u \in C([0,T),L^p(\mathbb{R})) \cap C((0,T),\dot{H}^{k,p}(\mathbb{R})) \cap C^\alpha((0,T),L^p(\mathbb{R}))$, $k = 1,2$ *to problem* (1) *for some positive constant* T.

Proof Set S_M denote a closed set given by

$$\begin{aligned}
S_M \equiv \Big\{ &u \in C\big([0,T),L^p(\mathbb{R})\big) \cap C\big((0,T),\dot{H}^{k,p}(\mathbb{R})\big) \\
&\cap C^\alpha\big((0,T),L^p(\mathbb{R})\big) : \sup_{t\in(0,T)} \left\|u(t,\cdot)\right\|_{S_M} \le M\Big\},
\end{aligned}$$

where

$$\left\|u(t,\cdot)\right\|_{S_M} = \sum_{k=0}^{2}\left\|t^{\delta_k}u(t,\cdot)\right\|_{\dot{H}^{k,p}(\mathbb{R})} + \left\|t^{\delta_2 C}\left(t^\theta\frac{\partial}{\partial t}\right)^\alpha u(t,\cdot)\right\|_{L^p(\mathbb{R})}$$

and $\delta_k = \frac{\rho\alpha k}{2}$, $\rho = 1-\theta$, the positive constants T and M will be given in the following.

Consider the nonlinear mapping F in S_M such that

$$\begin{aligned} Fu &= \mathfrak{F}^{-1}\big(\hat{\varphi}(\xi)E_{\alpha,1}\big(-\rho^{-\alpha}|\xi|^2 t^{\rho\alpha}\big) \\ &\quad + \frac{1}{\rho^\alpha}\int_0^t \big(t^\rho - s^\rho\big)^{\alpha-1}E_{\alpha,\alpha}\big(-\rho^{-\alpha}|\xi|^2\big(t^\rho - s^\rho\big)^\alpha \hat{f}\big(s,\xi,u(s,\xi)\big)\,d\big(s^\rho\big)\big). \end{aligned}$$

On the one hand, in terms of a modified result of Theorem 3.1 and Theorem 4.1, we arrive at

$$\begin{aligned} \big\|Fu(t,\cdot)\big\|_{S_M} &\lesssim \|\varphi\|_{L^p(\mathbb{R})} + t^{\delta_2}\int_0^1 \left(\sum_{k=0}^{2}\big(1-s^\rho\big)^{\alpha-1-\frac{k\alpha}{2}}\big\|f(u)(st,\cdot)\big\|_{L^p(\mathbb{R})}\right)d\big(s^\rho\big) \\ &\lesssim \|\varphi\|_{L^p(\mathbb{R})} + t^{\delta_2}\int_0^1 \left(\sum_{k=0}^{1}\big(1-s^\rho\big)^{\alpha-1-\frac{k\alpha}{2}}\big\|u(st,\cdot)\big\|_{L^p(\mathbb{R})}\big\|u(st,\cdot)\big\|^{\mu-1}_{L^\infty(\mathbb{R}^2_+)}\right. \\ &\quad \left. + \big(1-s^\rho\big)^{\frac{\alpha}{2}-1}(st)^{-\delta_1}\big\|(st)^{\delta_1}\partial_i u(st,\cdot)\big\|_{L^p(\mathbb{R})}\right)d\big(s^\rho\big) \\ &\lesssim \|\varphi\|_{L^p(\mathbb{R})} + t^{\delta_2}\|\varphi\|^{\mu-1}_{L^\infty(\mathbb{R})}\sum_{k=0}^{1}\int_0^1 \big(1-s^\rho\big)^{\alpha-1-\frac{k\alpha}{2}}\big\|u(st,\cdot)\big\|_{L^p(\mathbb{R})} \\ &\quad + t^{\delta_1}\int_0^1 \big(1-s^\rho\big)^{\frac{\alpha}{2}-1}s^{-\delta_1}\big\|(st)^{\delta_1}\partial_i u(st,\cdot)\big\|_{L^p(\mathbb{R})}\,d\big(s^\rho\big) \\ &\le C_0\|\varphi\|_{L^p(\mathbb{R})} + C_1\big(t^{\delta_2}\|\varphi\|^{\mu-1}_{L^\infty(\mathbb{R})} + t^{\delta_1}\big)\sup_{t\in(0,T)}\big\|u(t,\cdot)\big\|_{S_M}. \end{aligned} \tag{30}$$

Take T such that

$$\frac{1}{2} - C_1\big(T^{\delta_2}\|\varphi\|^{\mu-1}_{L^\infty(\mathbb{R})} + T^{\delta_1}\big) > 0, \tag{31}$$

then for $M = 2C_0\|\varphi\|_{L^p(\mathbb{R}^n)}$, (30)–(31) yield

$$\sup_{t\in(0,T)}\big\|Fu(t,\cdot)\big\|_{S_M} \le M. \tag{32}$$

This demonstrates that the mapping F maps S_M into itself.

On the other hand, for any $u \in S_M$, $v \in S_M$, by a direct computation, we have

$$\begin{aligned} &\big\|(Fu - Fv)(t,\cdot)\big\|_{S_M} \\ &\quad \lesssim t^{\delta_2}\int_0^1 \sum_{k=0}^{1}\big(1-s^\rho\big)^{\alpha-1-\frac{k\alpha}{2}}\big\|f(u)(st,\cdot) - f(v)(st,\cdot)\big\|_{L^p(\mathbb{R})}\,d\big(s^\rho\big) \\ &\qquad + t^{\delta_2}\int_0^1 \big(1-s^\rho\big)^{\frac{\alpha}{2}-1}\big\|\partial_i\big(f(u) - f(v)\big)(st,\cdot)\big\|_{L^p(\mathbb{R})}\,d\big(s^\rho\big) \\ &\quad \lesssim t^{\delta_2}\int_0^1 \sum_{k=0}^{1}\big(1-s^\rho\big)^{\alpha-1-\frac{k\alpha}{2}}\big\|(u-v)(st,\cdot)\big\|_{L^p(\mathbb{R})}\big(\|u\|^{\mu-1}_{L^\infty(\mathbb{R}^2_+)} + \|v\|^{\mu-1}_{L^\infty(\mathbb{R}^2_+)}\big)\,d\big(s^\rho\big) \end{aligned}$$

$$+ t^{\delta_1} \int_0^1 (1-s^\rho)^{\frac{\alpha}{2}-1} s^{\delta_1} \left(\left\| \partial_i (u-v)(st,\cdot) \right\|_{L^p(\mathbb{R})} + \left\| (u-v)(st,\cdot) \right\|_{L^p(\mathbb{R})} \right) d(s^\rho)$$
$$\leq C_1 \left(T^{\delta_2} \|\varphi\|^{\mu-1}_{L^\infty(\mathbb{R})} + T^{\delta_1} \right) \sup_{t\in(0,T)} \left\| (u-v)(t,\cdot) \right\|_{S_M}. \tag{33}$$

According to (31) and (33), one has

$$\sup_{t\in(0,T)} \left\| (Fu - Fv)(t,\cdot) \right\|_{S_M} < \sup_{t\in(0,T)} \left\| (u-v)(t,\cdot) \right\|_{S_M}, \tag{34}$$

which implies that mapping F is a contraction.

In terms of (32) and (34), we confirm that mapping F has one fixed point in S_M. This concludes the proof of Theorem 4.2. □

5 Conclusions

In this paper, the Cauchy problem (1) has been considered. By means of Mikhlin's multiplier theorem, in terms of Mittag-Leffler functions and M-Wright functions, we obtained an explicit solution $u \in C([0,+\infty), L^p(\mathbb{R}^n)) \cap C((0,+\infty), \dot{H}^{k,p}(\mathbb{R}^n)) \cap C^\alpha((0,+\infty), L^p(\mathbb{R}^n))$, $k = 1, 2$ for the linear equation with a source term. Meanwhile, the local existence of a solution of the semilinear equation in $\mathbb{R}^2_+$ was obtained by a fixed point theorem.

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