# Regularity criteria for the two-and-half-dimensional magnetic Bénard system with partial dissipation, magnetic diffusion, and thermal diffusivity 

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#### Abstract

This paper concentrates on the global regularity of classical solution to the $2 \frac{1}{2} \mathrm{D}$ magnetic Bénard system with partial dissipation, magnetic diffusion, and thermal diffusivity (i.e., horizontal dissipation, horizontal magnetic diffusion, and horizontal thermal diffusivity; vertical dissipation, vertical magnetic diffusion, and vertical thermal diffusivity). For the $2 \frac{1}{2} D$ magnetic Bénard system with full dissipation, magnetic diffusion, and thermal diffusivity, the global existence and uniqueness can be obtained by the standard energy method. However, can the classical solution for the $2 \frac{1}{2} \mathrm{D}$ incompressible magnetic Bénard system still keep its global regularity when losing some partial dissipation, magnetic diffusion, and thermal diffusivity terms? We will give a rigorous proof to the global regularity for the $2 \frac{1}{2} \mathrm{D}$ magnetic Bénard system with horizontal and vertical dissipation, magnetic diffusion, and thermal diffusivity respectively in this paper. Furthermore, we also show that any possible finite time blow-up can be controlled by the $L^{\infty}$-norm of the vertical velocity and magnetic components, not include the temperature component (see Theorems 1.1 and 1.2). The results extend the recent work by Cheng and Du (J. Math. Fluid Mech. 17:769-797, 2015), and generalize the recent works by Regmi (Math. Methods Appl. Sci. 40:1497-1504, 2017), and Ma and Zhang (Bound. Value Probl. 2018:79, 2018).


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## 1 Introduction

The 2D incompressible magnetic Bénard system can be represented in the form

$$
\left\{\begin{array}{l}
\partial_{t} u+(u \cdot \nabla) u+\nabla p=\mu_{1} \partial_{x x} u+\mu_{2} \partial_{y y} u+(b \cdot \nabla) b+\theta e_{2},  \tag{1.1}\\
\partial_{t} b+(u \cdot \nabla) b=v_{1} \partial_{x x} b+v_{2} \partial_{y y} b+(b \cdot \nabla) u, \\
\partial_{t} \theta+(u \cdot \nabla) \theta=\kappa_{1} \partial_{x x} \theta+\kappa_{2} \partial_{y y} \theta+u \cdot e_{2}, \\
\nabla \cdot u=\nabla \cdot b=0, \\
u(x, y, 0)=u_{0}(x, y), \quad b(x, y, 0)=b_{0}(x, y), \quad \theta(x, y, 0)=\theta_{0}(x, y),
\end{array}\right.
$$

where $t>0,(x, y) \in \mathbb{R}^{2}, u(x, y, t)=\left(u_{1}(x, y, t), u_{2}(x, y, t)\right)$ is a vector field denoting the velocity, $b(x, y, t)=\left(b_{1}(x, y, t), b_{2}(x, y, t)\right)$ is a vector field denoting the magnetic, $\theta=\theta(x, y, t)$ is a scalar function denoting the temperature, and $p$ is the scalar pressure. The forcing term $\theta e_{2}$ in the velocity equation models the acting of the buoyancy force on the fluid motion and $e_{2}=(0,1)$ the unit vector in the vertical direction. $u \cdot e_{2}$ models the Rayleigh-Bénard convection in a heated inviscid fluid. The parameters $\mu_{1} \geq 0, \mu_{2} \geq 0, \nu_{1} \geq 0, \nu_{2} \geq 0, \kappa_{1} \geq 0$, and $\kappa_{2} \geq 0$ are six non-dimensional constants. When $\mu_{1}=\mu_{2}, \nu_{1}=\nu_{2}$, and $\kappa_{1}=\kappa_{2}$, (1.1) reduces to the 2 D standard incompressible magnetic Bénard system.
The magnetic Bénard problem comes from the convection motions in a heated and incompressible fluid. In a homogeneous, viscous, and electrically conducting fluid, the convection will occur if the temperature gradient passes a certain critical threshold in two horizontal layers and the convection is permeated by an imposed uniform magnetic field, normal to the layers, and heated from below. The magnetic Bénard problem illuminates the heat convection phenomenon under the presence of the magnetic field (e.g., [21]).

The global regularity to 2D magnetic Bénard problem with full dissipation and magnetic diffusion (i.e., $\mu_{1}=\mu_{2}=v_{1}=v_{2}=1, \kappa_{1}=\kappa_{2}=0$ ) was proved by Zhou, Fan, and Nakamura [26]. Later, Cheng and Du improved this result in [5]. Recently, Ma [16] established the global well-posedness and conditional regularity for the 2D incompressible magnetic Bénard fluid system with mixed partial viscosity. It is currently unknown whether the solutions of the 3D magnetic Bénard fluid system are globally regular (in time). The author in [15] dealt with the Cauchy problem to the 3D system of incompressible magnetic Bénard fluids. He proved that as the initial data satisfy $\left\|u_{0}\right\|_{H^{1}\left(\mathbb{R}^{3}\right)}^{2}+\left\|b_{0}\right\|_{H^{1}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\theta_{0}\right\|_{H^{1}\left(\mathbb{R}^{3}\right)}^{2} \leq \varepsilon$, where $\varepsilon$ is a suitably small positive number, the 3D magnetic Bénard system with mixed partial dissipation, magnetic diffusion, and thermal diffusivity admits global smooth solutions. In [13], we investigated the blow-up criteria of strong solutions and a regularity criterion of weak solutions for the magnetic Bénard fluid system in $\mathbb{R}^{3}$ in a sense of scaling invariant by employing a different decomposition for nonlinear terms. More precisely, the strong solution $(u, b, \theta)$ of a magnetic Bénard fluid system is proved to be smooth on $(0, T]$ provided the velocity field $u$ satisfies

$$
u \in L^{\frac{2}{1-r}}\left(0, T ; \dot{\mathbb{X}}_{r}\left(\mathbb{R}^{3}\right)\right) \quad \text { with } 0 \leq r<1
$$

or the gradient field of velocity $\nabla u$ satisfies

$$
\nabla u \in L^{\frac{2}{2-\gamma}}\left(0, T ; \dot{\mathbb{X}}_{\gamma}\left(\mathbb{R}^{3}\right)\right) \quad \text { with } 0 \leq \gamma \leq 1
$$

Moreover, we proved that if the following conditions hold:

$$
u \in L^{\infty}\left(0, T ; \dot{\mathbb{X}}_{1}\left(\mathbb{R}^{3}\right)\right) \quad \text { and } \quad\|u\|_{L^{\infty}\left(0, T ; \dot{\mathbb{X}}_{1}\left(\mathbb{R}^{3}\right)\right)}<\varepsilon
$$

where $\varepsilon>0$ is a suitable small constant, then the strong solution $(u, b, \theta)$ of magnetic Bénard fluid system can also be extended beyond $t=T$. Finally, we showed that if some partial derivatives of the velocity components, magnetic components, and temperature components (i.e., $\tilde{\nabla} \tilde{u}, \tilde{\nabla} \tilde{b}, \tilde{\nabla} \theta)$ belong to the multiplier space, the solution $(u, b, \theta)$ actually is smooth on $(0, T)$.

If we neglect the thermal effects in the fluid motion, the 2 D magnetic Bénard system can be specialized to the well-known 2D magnetohydrodynamics (MHD) system

$$
\left\{\begin{array}{l}
\partial_{t} u+(u \cdot \nabla) u+\nabla p=\mu_{1} \partial_{x x} u+\mu_{2} \partial_{y y} u+(b \cdot \nabla) b,  \tag{1.2}\\
\partial_{t} b+(u \cdot \nabla) b=v_{1} \partial_{x x} b+v_{2} \partial_{y y} b+(b \cdot \nabla) u, \\
\nabla \cdot u=\nabla \cdot b=0, \\
u(x, y, 0)=u_{0}(x, y), \quad b(x, y, 0)=b_{0}(x, y)
\end{array}\right.
$$

where $(x, y) \in \mathbb{R}^{2}, t>0, u=\left(u_{1}(x, y, t), u_{2}(x, y, t)\right), p=p(x, y, t)$, and $b=\left(b_{1}(x, y, t), b_{2}(x, y, t)\right)$ denote the velocity vector, scalar pressure, and the magnetic field of the fluid, respectively. The MHD system has attracted quite a lot of attention lately from various authors. Actually, there is a considerable body of literature on the global regularity of the MHD system. We recall here, without any claim of completeness, $[3,6,7,24]$ and the references cited therein.
Suppose that the magnetic field $b \equiv 0$, system (1.1) is nothing but the so-called Bénard system

$$
\begin{cases}\partial_{t} u+(u \cdot \nabla) u+\nabla p=\mu_{1} \partial_{x x} u+\mu_{2} \partial_{y y} u+\theta e_{2}, & (x, y) \in \mathbb{R}^{2}, t>0  \tag{1.3}\\ \partial_{t} \theta+(u \cdot \nabla) \theta=\kappa_{1} \partial_{x x} \theta+\kappa_{2} \partial_{y y} \theta+u \cdot e_{2}, & (x, y) \in \mathbb{R}^{2}, t>0 \\ \nabla \cdot u=0, & (x, y) \in \mathbb{R}^{2}, t>0 \\ u(x, y, 0)=u_{0}(x, y), \quad \theta(x, y, 0)=\theta_{0}(x, y), & (x, y) \in \mathbb{R}^{2},\end{cases}
$$

where $u=u(x, y, t)=\left(u_{1}(x, y, t), u_{2}(x, y, t)\right): \mathbb{R}^{2} \times[0, \infty) \rightarrow \mathbb{R}^{2}$, denotes the velocity field of a 2D incompressible fluid. The term $p=p(x, y, t): \mathbb{R}^{2} \times[0, \infty) \rightarrow \mathbb{R}$ denotes the usual pressure which can be recovered from the first and the third equation in (1.3) by taking the divergence and then inverting the Laplacian operator. The scalar function $\theta=\theta(x, y, t)$ quantifies the temperature variation in a gravity field. The Bénard system describes the Rayleigh-Bénard convective motion in a heated 2D inviscid incompressible fluid under thermal effects (see, e.g., [1, 8, 9, 19, 22, 25]). Especially, Ma and Zhang [18] wrote the velocity equation of the Bénard system in its two components and considered the global weak solution of the resulting 2D Bénard system with partial dissipation.
However, the questions of global regularity or finite time singularity of the weak solutions of 3D magnetic Bénard system, MHD system, and Bénard system are still challenging open problems. Many attempts have been made, but there are no satisfactory results. It is of interest to study the regularity of weak solutions under additional critical growth conditions on the velocity, the magnetic, the temperature, or the pressure. There are numerous papers related to the regularity criteria of 3D MHD system (see [4, 10-12] and the references therein).
The 2D flow generates a large family of 3D flow with vorticity stretching [20]. We refer to these as $2 \frac{1}{2} \mathrm{D}$ flows because the flow in the $z$ direction is predetermined by the underlying 2D flows. Recently, Cheng and Du [5] proved the global regularity (in time) for 2D magnetic Bénard system with mixed partial viscosity. Ma and Zhang [18] studied the global existence of weak solutions and regularity criteria for the 2D Bénard system with partial dissipation. Later, Ma [16] generalized the results in [18] and established the global wellposedness and conditional regularity for the 2D incompressible magnetic Bénard fluid
system with mixed partial viscosity. One natural question is "Can we extend these results related to the 2D magnetic Bénard system to the $2 \frac{1}{2} \mathrm{D}$ magnetic Bénard system?"
In this paper, motivated by the works [5, 14, 16-18, 23], we will firstly study the global regularity of classical solution to a $2 \frac{1}{2} \mathrm{D}$ (i.e., $u, b, \theta$, and $p$ are independent of $z$ ) magnetic Bénard system with horizontal dissipation, horizontal magnetic diffusion, and horizontal thermal diffusivity. Next, we will consider the global regularity of $2 \frac{1}{2} \mathrm{D}$ magnetic Bénard system with vertical dissipation, vertical magnetic diffusion, and vertical thermal diffusivity.

Let

$$
\begin{aligned}
& u=\left(u_{1}, u_{2}, u_{3}\right)=\left(\tilde{u}, u_{3}\right), \quad b=\left(b_{1}, b_{2}, b_{3}\right)=\left(\tilde{b}, b_{3}\right), \\
& \tilde{\nabla}=\left(\partial_{x}, \partial_{y}\right), \quad \tilde{\Delta}=\partial_{x x}+\partial_{y y},
\end{aligned}
$$

then we can formulate the $2 \frac{1}{2} \mathrm{D}$ magnetic Bénard system as follows:

$$
\left\{\begin{array}{l}
\partial_{t} \tilde{u}+(\tilde{u} \cdot \tilde{\nabla}) \tilde{u}+\tilde{\nabla} p=\mu_{1} \partial_{x x} \tilde{u}+\mu_{2} \partial_{y y} \tilde{u}+(\tilde{b} \cdot \tilde{\nabla}) \tilde{b},  \tag{1.4}\\
\partial_{t} u_{3}+(\tilde{u} \cdot \tilde{\nabla}) u_{3}=\mu_{1} \partial_{x x} u_{3}+\mu_{2} \partial_{y y} u_{3}+(\tilde{b} \cdot \tilde{\nabla}) b_{3}+\theta, \\
\partial_{t} \tilde{b}+(\tilde{u} \cdot \tilde{\nabla}) \tilde{b}=v_{1} \partial_{x x} \tilde{b}+v_{2} \partial_{y y} \tilde{b}+(\tilde{b} \cdot \tilde{\nabla}) \tilde{u}, \\
\partial_{t} b_{3}+(\tilde{u} \cdot \tilde{\nabla}) b_{3}=v_{1} \partial_{x x} b_{3}+v_{2} \partial_{y y} b_{3}+(\tilde{b} \cdot \tilde{\nabla}) u_{3} \\
\partial_{t} \theta+(\tilde{u} \cdot \tilde{\nabla}) \theta=\kappa_{1} \partial_{x x} \theta+\kappa_{2} \partial_{y y} \theta+u_{3} \\
\tilde{\nabla} \cdot \tilde{u}=\tilde{\nabla} \cdot \tilde{b}=0, \\
u(x, y, 0)=u_{0}(x, y), \quad b(x, y, 0)=b_{0}(x, y), \quad \theta(x, y, 0)=\theta_{0}(x, y),
\end{array}\right.
$$

where $u: \mathbb{R}^{2} \times[0, \infty) \rightarrow \mathbb{R}^{3}$ denotes the fluid velocity field, $b: \mathbb{R}^{2} \times[0, \infty) \rightarrow \mathbb{R}^{3}$ magnetic field, $\theta: \mathbb{R}^{2} \times[0, \infty) \rightarrow \mathbb{R}$ temperature, and $p: \mathbb{R}^{2} \times[0, \infty) \rightarrow \mathbb{R}$ pressure.
It is well known that the vorticity $\omega=\nabla \times u=\left(\partial_{y} u_{3},-\partial_{x} u_{3}, \partial_{x} u_{2}-\partial_{y} u_{1}\right)=\left(\omega_{1}, \omega_{2}, \omega_{3}\right)=$ $\left(\tilde{\omega}, \omega_{3}\right)$ and the current density $j=\nabla \times b=\left(\partial_{y} b_{3},-\partial_{x} b_{3}, \partial_{x} b_{2}-\partial_{y} b_{1}\right)=\left(j_{1}, j_{2}, j_{3}\right)=\left(\tilde{j}, j_{3}\right)$ play an important role in establishing the regularity for a magnetic Bénard system. Taking the curl operator for the first four equations and applying $\tilde{\nabla}$ to both sides of the fifth equation in (1.4), system (1.4) can be written as follows:

$$
\left\{\begin{array}{l}
\partial_{t} \omega_{1}+(\tilde{u} \cdot \tilde{\nabla}) \omega_{1}=(\tilde{\omega} \cdot \tilde{\nabla}) u_{1}+(\tilde{b} \cdot \tilde{\nabla}) j_{1}-(\tilde{j} \cdot \tilde{\nabla}) b_{1}+\mu_{1} \partial_{x x} \omega_{1}+\mu_{2} \partial_{y y} \omega_{1}+\partial_{y} \theta,  \tag{1.5}\\
\partial_{t} \omega_{2}+(\tilde{u} \cdot \tilde{\nabla}) \omega_{2}=(\tilde{\omega} \cdot \tilde{\nabla}) u_{2}+(\tilde{b} \cdot \tilde{\nabla}) j_{2}-(\tilde{j} \cdot \tilde{\nabla}) b_{2}+\mu_{1} \partial_{x x} \omega_{2}+\mu_{2} \partial_{y y} \omega_{2}-\partial_{x} \theta, \\
\partial_{t} \omega_{3}+(\tilde{u} \cdot \tilde{\nabla}) \omega_{3}=(\tilde{b} \cdot \tilde{\nabla}) j_{3}+\mu_{1} \partial_{x x} \omega_{3}+\mu_{2} \partial_{y y} \omega_{3}, \\
\partial_{t} j_{1}+(\tilde{u} \cdot \tilde{\nabla}) j_{1}=(\tilde{j} \cdot \tilde{\nabla}) u_{1}+(\tilde{b} \cdot \tilde{\nabla}) \omega_{1}-(\tilde{\omega} \cdot \tilde{\nabla}) b_{1}+\nu_{1} \partial_{x x} j_{1}+\nu_{2} \partial_{y y} j_{1} \\
\partial_{t} j_{2}+(\tilde{u} \cdot \tilde{\nabla}) j_{2}=(\tilde{j} \cdot \tilde{\nabla}) u_{2}+(\tilde{b} \cdot \tilde{\nabla}) \omega_{2}-(\tilde{\omega} \cdot \tilde{\nabla}) b_{2}+v_{1} \partial_{x x} j_{2}+\nu_{2} \partial_{y y} j_{2}, \\
\partial_{t} j_{3}+(\tilde{u} \cdot \tilde{\nabla}) j_{3} \\
\quad=(\tilde{b} \cdot \tilde{\nabla}) \omega_{3}+v_{1} \partial_{x x} j_{3}+v_{2} \partial_{y y} j_{3}+2 \partial_{x} b_{1}\left(\partial_{y} u_{1}+\partial_{x} u_{2}\right)-2 \partial_{x} u_{1}\left(\partial_{x} b_{2}+\partial_{y} b_{1}\right), \\
\partial_{t} \tilde{\nabla} \theta+\tilde{\nabla}[(\tilde{u} \cdot \tilde{\nabla}) \theta]=\kappa_{1} \tilde{\nabla} \partial_{x x} \theta+\kappa_{2} \tilde{\nabla} \partial_{y y} \theta+\tilde{\nabla} u_{3} .
\end{array}\right.
$$

Now let us state our main results as follows.

Our first goal is to consider the $2 \frac{1}{2} \mathrm{D}$ magnetic Bénard system with horizontal dissipation, horizontal magnetic diffusion, and horizontal thermal diffusivity

$$
\left\{\begin{array}{l}
\partial_{t} \tilde{u}+(\tilde{u} \cdot \tilde{\nabla}) \tilde{u}+\tilde{\nabla} p=\mu_{1} \partial_{x x} \tilde{u}+(\tilde{b} \cdot \tilde{\nabla}) \tilde{b},  \tag{1.6}\\
\partial_{t} u_{3}+(\tilde{u} \cdot \tilde{\nabla}) u_{3}=\mu_{1} \partial_{x x} u_{3}+(\tilde{b} \cdot \tilde{\nabla}) b_{3}+\theta, \\
\partial_{t} \tilde{b}+\left(\tilde{u} \cdot \tilde{\nabla} \tilde{b}=v_{1} \partial_{x x} \tilde{b}+(\tilde{b} \cdot \tilde{\nabla}) \tilde{u},\right. \\
\partial_{t} b_{3}+(\tilde{u} \cdot \tilde{\nabla}) b_{3}=v_{1} \partial_{x x} b_{3}+(\tilde{b} \cdot \tilde{\nabla}) u_{3}, \\
\partial_{t} \theta+(\tilde{u} \cdot \tilde{\nabla}) \theta=\kappa_{1} \partial_{x x} \theta+u_{3}, \\
\tilde{\nabla} \cdot \tilde{u}=\tilde{\nabla} \cdot \tilde{b}=0, \\
u(x, y, 0)=u_{0}(x, y), \quad b(x, y, 0)=b_{0}(x, y), \quad \theta(x, y, 0)=\theta_{0}(x, y) .
\end{array}\right.
$$

Theorem 1.1 Given a positive time $T \in(0, \infty)$. Suppose that $\left(u_{0}, b_{0}, \theta_{0}\right) \in H^{2}\left(\mathbb{R}^{2}\right)$ and $\nabla$. $u_{0}=\nabla \cdot b_{0}=0$. Let $(u, b, \theta)$ be a corresponding local smooth solution of (1.6) at the interval $(0, T)$. If one of the following three conditions holds

$$
\begin{align*}
& \int_{0}^{T}\left\|\left(u_{1}, b_{1}\right)\right\|_{L^{\infty}}^{2} d t<\infty  \tag{1.7}\\
& \int_{0}^{T}\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2} d t<\infty  \tag{1.8}\\
& \int_{0}^{T}\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2} d t<\infty \tag{1.9}
\end{align*}
$$

then the solution $(u, b, \theta)$ can be extended beyond $T$.

Our second goal is to consider the $2 \frac{1}{2} \mathrm{D}$ magnetic Bénard system with vertical dissipation, vertical magnetic diffusion, and vertical thermal diffusivity

$$
\left\{\begin{array}{l}
\partial_{t} \tilde{u}+(\tilde{u} \cdot \tilde{\nabla}) \tilde{u}+\tilde{\nabla} p=\mu_{2} \partial_{y y} \tilde{u}+(\tilde{b} \cdot \tilde{\nabla}) \tilde{b},  \tag{1.10}\\
\partial_{t} u_{3}+(\tilde{u} \cdot \tilde{\nabla}) u_{3}=\mu_{2} \partial_{y y} u_{3}+(\tilde{b} \cdot \tilde{\nabla}) b_{3}+\theta, \\
\partial_{t} \tilde{b}+(\tilde{u} \cdot \tilde{\nabla}) \tilde{b}=\nu_{2} \partial_{y y} \tilde{b}+(\tilde{b} \cdot \tilde{\nabla}) \tilde{u}, \\
\partial_{t} b_{3}+(\tilde{u} \cdot \tilde{\nabla}) b_{3}=v_{2} \partial_{y y} b_{3}+(\tilde{b} \cdot \tilde{\nabla}) u_{3}, \\
\partial_{t} \theta+(\tilde{u} \cdot \tilde{\nabla}) \theta=\kappa_{2} \partial_{y y} \theta+u_{3}, \\
\tilde{\nabla} \cdot \tilde{u}=\tilde{\nabla} \cdot \tilde{b}=0, \\
u(x, y, 0)=u_{0}(x, y), \quad b(x, y, 0)=b_{0}(x, y), \quad \theta(x, y, 0)=\theta_{0}(x, y),
\end{array}\right.
$$

Theorem 1.2 Given a positive time $T \in(0, \infty)$. Suppose that $\left(u_{0}, b_{0}, \theta_{0}\right) \in H^{2}\left(\mathbb{R}^{2}\right)$ and $\nabla$. $u_{0}=\nabla \cdot b_{0}=0$. Let $(u, b, \theta)$ be a corresponding local smooth solution of $(1.10)$ at the interval $(0, T)$. If one of the following three conditions holds

$$
\begin{align*}
& \int_{0}^{T}\left\|\left(u_{2}, b_{2}\right)\right\|_{L^{\infty}}^{2} d t<\infty  \tag{1.11}\\
& \int_{0}^{T}\left\|\partial_{x} u_{2}\right\|_{L^{2}}^{2} d t<\infty \tag{1.12}
\end{align*}
$$

$$
\begin{equation*}
\int_{0}^{T}\left\|\partial_{x} b_{2}\right\|_{L^{2}}^{2} d t<\infty \tag{1.13}
\end{equation*}
$$

then the solution $(u, b, \theta)$ can be extended beyond $T$.

Remark 1.1 It is clear that Theorem 1.1 and Theorem 1.2 here improve the results of Case 6 and Case 9 in Theorem 3 in [5] on the global regularity to the 2D magnetic Bénard system. More precisely, the conditional global regularity they established for the horizontal dissipation, horizontal diffusion, and horizontal thermal diffusivity is $\left\|\partial_{y} u_{1}\right\|_{L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{2}\right)\right)}<$ $\infty$ or $\left\|\partial_{y} b_{1}\right\|_{L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{2}\right)\right)}<\infty$, and the conditional global regularity for the vertical dissipation, vertical diffusion, and vertical thermal diffusivity is only $\left\|\partial_{x} u_{2}\right\|_{L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{2}\right)\right)}<\infty$.

Remark 1.2 Our main results generalize the recent work by Regmi [23], in which they study the global regularity of classical solution to a $2 \frac{1}{2} \mathrm{D}$ magnetohydrodynamic system with horizontal dissipation and horizontal magnetic diffusion, and with vertical dissipation and vertical magnetic diffusion, but they do not consider the thermal effects. Our main results also extend the recent works [16, 18], where they study the global regularity for the 2D Bénard and magnetic Bénard systems.

Remark 1.3 Our methods are similar to the 2D magnetic Bénard system with horizontal dissipation, horizontal magnetic diffusion, and horizontal thermal diffusivity [5]. However, in the presence of the vortex stretching term, the mathematical analysis for $2 \frac{1}{2} \mathrm{D}$ is harder than for 2D case.

Remark 1.4 Our results also show that any possible finite time blow-up can be controlled by the $L^{\infty}$-norm of the vertical components of the velocity field and magnetic field (see (1.7) in Theorem 1.1 and (1.11) in Theorem 1.2).

## 2 The proof of Theorem 1.1

In this paper, all constants will be denoted by $C$ that is a generic constant depending only on the quantities specified in the context.
This section is devoted to the proof of Theorem 1.1. Now we explain the main process involved in proving Theorem 1.1 and the methods used here. The general approach to establish the global existence and regularity results consists of two main steps. The first step assesses the local (in time) well-posedness, while the second extends the local solution into a global one by obtaining global (in time) a priori bounds. For the systems of equations concerned here, the local well-posedness follows from a standard approach and shall be skipped here. Our main efforts are devoted to proving the necessary global a priori bounds. More precisely, we show that, for some $T>0$ and $t \leq T$,

$$
\begin{equation*}
\|(u, b, \theta)\|_{H^{2}\left(\mathbb{R}^{2}\right)} \leq C, \tag{2.1}
\end{equation*}
$$

where $C$ denotes a bound that depends on $T$ and the initial data.
To begin with, the following anisotropic Sobolev inequality which bounds a tripleproduct in terms of the Lebesgue norms of the functions and their directional derivatives is needed, please see for example [2].

Lemma 2.1 Let $f, g, h, \partial_{x} g, \partial_{y} h \in L^{2}\left(\mathbb{R}^{2}\right)$. Then there exists an absolute constant $C$ such that

$$
\iint_{\mathbb{R}^{2}}|f g h| d x d y \leq C\|f\|_{L^{2}}\|g\|_{L^{2}}^{\frac{1}{2}}\left\|\partial_{x} g\right\|_{L^{2}}^{\frac{1}{2}}\|h\|_{L^{2}}^{\frac{1}{2}}\left\|\partial_{y} h\right\|_{L^{2}}^{\frac{1}{2}}
$$

Now let us move to proving the main theorem.

### 2.1 Case i

Under condition (1.7).
We first state the global $L^{2}$-bound.
Lemma 2.2 Assume that $\left(u_{0}, b_{0}, \theta_{0}\right) \in H^{2}\left(\mathbb{R}^{2}\right)$. Let $(u, b, \theta)$ be a corresponding solution of (1.6). Then $(u, b, \theta)$ obeys the following global $L^{2}$-bound:

$$
\begin{align*}
& \|u\|_{L^{2}}^{2}+\|b\|_{L^{2}}^{2}+\|\theta\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{t}\left\|\partial_{x} u\right\|_{L^{2}}^{2} d \tau+v_{1} \int_{0}^{t}\left\|\partial_{x} b\right\|_{L^{2}}^{2} d \tau \\
& \quad+\kappa_{1} \int_{0}^{t}\left\|\partial_{x} \theta\right\|_{L^{2}}^{2} d \tau \leq C\left\|\left(u_{0}, b_{0}, \theta_{0}\right)\right\|_{L^{2}}^{2} \tag{2.2}
\end{align*}
$$

for any $t \geq 0$.
Proof Taking the $L^{2}$-inner product of $\left(\tilde{u}, u_{3}, \tilde{b}, b_{3}, \theta\right)$ with (1.6), respectively, yields

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\|\tilde{u}\|_{L^{2}}^{2}+\mu_{1}\left\|\partial_{x} \tilde{u}\right\|_{L^{2}}^{2}=\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) \tilde{b} \cdot \tilde{u} d x d y \\
& \frac{1}{2} \frac{d}{d t}\left\|u_{3}\right\|_{L^{2}}^{2}+\mu_{1}\left\|\partial_{x} u_{3}\right\|_{L^{2}}^{2}=\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) b_{3} \cdot u_{3} d x d y+\iint_{\mathbb{R}^{2}} \theta \cdot u_{3} d x d y \\
& \frac{1}{2} \frac{d}{d t}\|\tilde{b}\|_{L^{2}}^{2}+v_{1}\left\|\partial_{x} \tilde{b}\right\|_{L^{2}}^{2}=\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) \tilde{u} \cdot \tilde{b} d x d y \\
& \frac{1}{2} \frac{d}{d t}\left\|b_{3}\right\|_{L^{2}}^{2}+v_{1}\left\|\partial_{x} b_{3}\right\|_{L^{2}}^{2}=\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) u_{3} \cdot b_{3} d x d y \\
& \frac{1}{2} \frac{d}{d t}\|\theta\|_{L^{2}}^{2}+\kappa_{1}\left\|\partial_{x} \theta\right\|_{L^{2}}^{2}=\iint_{\mathbb{R}^{2}} u_{3} \cdot \theta d x d y
\end{aligned}
$$

Adding them up, we have

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left(\|\tilde{u}\|_{L^{2}}^{2}+\left\|u_{3}\right\|_{L^{2}}^{2}+\|\tilde{b}\|_{L^{2}}^{2}+\left\|b_{3}\right\|_{L^{2}}^{2}+\|\theta\|_{L^{2}}^{2}\right)+\mu_{1}\left\|\left(\partial_{x} \tilde{u}, \partial_{x} u_{3}\right)\right\|_{L^{2}}^{2} \\
& \quad+v_{1}\left\|\left(\partial_{x} \tilde{b}, \partial_{x} b_{3}\right)\right\|_{L^{2}}^{2}+\kappa_{1}\left\|\partial_{x} \theta\right\|_{L^{2}}^{2} \\
& \quad=2 \iint_{\mathbb{R}^{2}} u_{3} \cdot \theta d x d y
\end{aligned}
$$

where we have used the identities

$$
\begin{aligned}
& \iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) \tilde{b} \cdot \tilde{u} d x d y+\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) \tilde{u} \cdot \tilde{b} d x d y=0 \\
& \iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) b_{3} \cdot u_{3} d x d y+\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) u_{3} \cdot b_{3} d x d y=0
\end{aligned}
$$

Applying Hölder's inequality yields

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left(\|\tilde{u}\|_{L^{2}}^{2}+\left\|u_{3}\right\|_{L^{2}}^{2}+\|\tilde{b}\|_{L^{2}}^{2}+\left\|b_{3}\right\|_{L^{2}}^{2}+\|\theta\|_{L^{2}}^{2}\right)+\mu_{1}\left\|\left(\partial_{x} \tilde{u}, \partial_{x} u_{3}\right)\right\|_{L^{2}}^{2} \\
& \quad+v_{1}\left\|\left(\partial_{x} \tilde{b}, \partial_{x} b_{3}\right)\right\|_{L^{2}}^{2}+\kappa_{1}\left\|\partial_{x} \theta\right\|_{L^{2}}^{2} \leq C\left(\left\|u_{3}\right\|_{L^{2}}^{2}+\|\theta\|_{L^{2}}^{2}\right) .
\end{aligned}
$$

Gronwall's inequality then implies

$$
\begin{aligned}
& \|\tilde{u}\|_{L^{2}}^{2}+\left\|u_{3}\right\|_{L^{2}}^{2}+\|\tilde{b}\|_{L^{2}}^{2}+\left\|b_{3}\right\|_{L^{2}}^{2}+\|\theta\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{t}\left\|\left(\partial_{x} \tilde{u}, \partial_{x} u_{3}\right)\right\|_{L^{2}}^{2} d \tau \\
& \quad+v_{1} \int_{0}^{t}\left\|\left(\partial_{x} \tilde{b}, \partial_{x} b_{3}\right)\right\|_{L^{2}}^{2} d \tau+\kappa_{1} \int_{0}^{t}\left\|\partial_{x} \theta\right\|_{L^{2}}^{2} d \tau \leq C
\end{aligned}
$$

for any $0<t \leq T$, where $C$ depends only on the initial data.

We next prove the global $H^{1}$-bound for $u, b$, and $\theta$.

Proposition 2.3 Assume that $\left(u_{0}, b_{0}, \theta_{0}\right)$ satisfies the condition stated in Theorem 1.1. Let $(u, b, \theta)$ be the corresponding solution of (1.6). Then $(u, b, \theta)$ satisfies, for any $T>0$ and $t \leq T$,

$$
\begin{equation*}
\|(u, b, \theta)\|_{H^{1}\left(\mathbb{R}^{2}\right)} \leq C_{1} e^{C_{2} \int_{0}^{t}\left(\left\|u_{1}\right\|_{L^{\infty}}^{2}+\left\|b_{1}\right\|_{L^{\infty}}^{2}\right) d \tau} \tag{2.3}
\end{equation*}
$$

where $C_{1}$ is a constant depending on $T$ and the initial data and $C_{2}$ is a pure constant.

Proof Proposition 2.3 is an easy consequence of Lemmas 2.4 and 2.5.

Lemma 2.4 Consider (1.5) with $\mu_{1}>0, \mu_{2}=0, \nu_{1}>0, \nu_{2}=0, \kappa_{1}>0, \kappa_{2}=0$. Assume that $\left(u_{0}, b_{0}, \theta_{0}\right)$ satisfies the condition stated in Theorem 1.1. Let $(u, b, \theta)$ be the corresponding solution of (1.6). Then $\omega_{3}$ and $j_{3}$ satisfy

$$
\begin{equation*}
\left\|\omega_{3}\right\|_{L^{2}}^{2}+\left\|j_{3}\right\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{T}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2} d t+v_{1} \int_{0}^{T}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2} d t \leq C \tag{2.4}
\end{equation*}
$$

provided $\int_{0}^{T}\left\|\left(u_{1}, b_{1}\right)\right\|_{L^{\infty}}^{2} d t<\infty$ for some $T>0$.
Proof Set

$$
F_{1}(t) \triangleq\left(\left\|\omega_{3}(t)\right\|_{L^{2}}^{2}+\left\|j_{3}(t)\right\|_{L^{2}}^{2}\right)^{\frac{1}{2}}
$$

Taking the inner product of the third and sixth equations in (1.5) with $\omega_{3}$ and $j_{3}$, respectively, then integrating by parts in $\mathbb{R}^{2}$, we get

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} F_{1}^{2}(t)+\mu_{1}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+v_{1}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2} \\
& \quad=\sum_{i=1}^{7} I_{i}
\end{aligned}
$$

$$
\begin{align*}
\leq & \frac{\mu_{1}}{2}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\frac{\nu_{1}}{2}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2} \\
& +C\left(\left\|u_{1}\right\|_{L^{\infty}}^{2}+\left\|b_{1}\right\|_{L^{\infty}}^{2}+\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{x} b\right\|_{L^{2}}^{2}\right) F_{1}^{2}(t) \tag{2.5}
\end{align*}
$$

where

$$
\begin{array}{ll}
I_{1}=-2 \iint_{\mathbb{R}^{2}} b_{1} \partial_{x y} u_{1} j_{3} d x d y, & I_{2}=-2 \iint_{\mathbb{R}^{2}} b_{1} \partial_{y} u_{1} \partial_{x} j_{3} d x d y \\
I_{3}=2 \iint_{\mathbb{R}^{2}} \partial_{x} b_{1} \partial_{x} u_{2} j_{3} d x d y, & I_{4}=-2 \iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{x} b_{2} j_{3} d x d y \\
I_{5}=2 \iint_{\mathbb{R}^{2}} b_{1} \partial_{x y} u_{1} \partial_{x} b_{2} d x d y, & I_{6}=2 \iint_{\mathbb{R}^{2}} b_{1} \partial_{x} u_{1} \partial_{x y} b_{2} d x d y, \\
I_{7}=-4 \iint_{\mathbb{R}^{2}} u_{1} \partial_{y} b_{1} \partial_{x y} b_{1} d x d y, &
\end{array}
$$

and we have used Lemma 2.1, Cauchy-Schwarz's inequality, and Young's inequality.
Taking advantage of Gronwall's inequality, together with Lemma 2.2, we complete the proof of this lemma.

Lemma 2.5 Consider (1.5) with $\mu_{1}>0, \mu_{2}=0, v_{1}>0, \nu_{2}=0, \kappa_{1}>0, \kappa_{2}=0$. Assume that $\left(u_{0}, b_{0}, \theta_{0}\right)$ satisfies the condition stated in Theorem 1.1. Let $(u, b, \theta)$ be the corresponding solution of (1.6). Then $\tilde{\omega}, \tilde{j}$, and $\tilde{\nabla} \theta$ satisfy

$$
\begin{align*}
& \|\tilde{\omega}\|_{L^{2}}^{2}+\|\tilde{j}\|_{L^{2}}^{2}+\|\tilde{\nabla} \theta\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{T}\left\|\partial_{x} \tilde{\omega}\right\|_{L^{2}}^{2} d t+v_{1} \int_{0}^{T}\left\|\partial_{x} \tilde{j}\right\|_{L^{2}}^{2} d t \\
& \quad+\kappa_{1} \int_{0}^{T}\left\|\tilde{\nabla} \partial_{x} \theta\right\|_{L^{2}}^{2} d t \leq C \tag{2.6}
\end{align*}
$$

provided $\int_{0}^{T}\left\|\left(u_{1}, b_{1}\right)\right\|_{L^{\infty}}^{2} d t<\infty$ for some $T>0$.
Proof Set

$$
G_{1}(t) \triangleq\left(\|\tilde{\omega}(t)\|_{L^{2}}^{2}+\|\tilde{j}(t)\|_{L^{2}}^{2}+\|\tilde{\nabla} \theta(t)\|_{L^{2}}^{2}\right)^{\frac{1}{2}}
$$

Multiplying the first, second, forth, fifth, and seventh equations of (1.5) with $\omega_{1}, \omega_{2}, j_{1}, j_{2}$, and $\tilde{\nabla} \theta$ respectively, integrating them in space domain and adding the resulting equations together, we have

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} G_{1}^{2}(t)+\mu_{1}\left\|\left(\partial_{x} \omega_{1}, \partial_{x} \omega_{2}\right)\right\|_{L^{2}}^{2}+v_{1}\left\|\left(\partial_{x} j_{1}, \partial_{x} j_{2}\right)\right\|_{L^{2}}^{2}+\kappa_{1}\left\|\tilde{\nabla} \partial_{x} \theta\right\|_{L^{2}}^{2} \\
& \quad=\sum_{i=1}^{28} J_{i} \leq \frac{\mu_{1}}{2}\left\|\partial_{x} \tilde{\omega}\right\|_{L^{2}}^{2}+\frac{\nu_{1}}{2}\left\|\partial_{x} \tilde{j}\right\|_{L^{2}}^{2}+\frac{\kappa_{1}}{2}\left\|\tilde{\nabla} \partial_{x} \theta\right\|_{L^{2}}^{2}+C A_{1}(t) G_{1}^{2}(t) \tag{2.7}
\end{align*}
$$

where

$$
\begin{aligned}
& J_{1}=-2 \iint_{\mathbb{R}^{2}} u_{1} \omega_{1} \partial_{x} \omega_{1} d x d y, \quad J_{2}=\iint_{\mathbb{R}^{2}} \omega_{1} \partial_{y} u_{1} \omega_{2} d x d y \\
& J_{3}=\iint_{\mathbb{R}^{2}} b_{1} \partial_{x} j_{1} \omega_{1} d x d y, \quad J_{4}=\iint_{\mathbb{R}^{2}} b_{1} j_{1} \partial_{x} \omega_{1} d x d y
\end{aligned}
$$

$$
\begin{array}{ll}
J_{5}=-\iint_{\mathbb{R}^{2}} j_{2} \partial_{y} b_{1} \omega_{1} d x d y, & J_{6}=\iint_{\mathbb{R}^{2}} \partial_{y} \theta \omega_{1} d x d y, \\
J_{7}=\iint_{\mathbb{R}^{2}} \omega_{1} \partial_{x} u_{2} \omega_{2} d x d y, & J_{8}=2 \iint_{\mathbb{R}^{2}} u_{1} \omega_{2} \partial_{x} \omega_{2} d x d y, \\
J_{9}=-\iint_{\mathbb{R}^{2}} j_{1} \partial_{x} b_{2} \omega_{2} d x d y, & J_{10}=-\iint_{\mathbb{R}^{2}} b_{1} \partial_{x} j_{2} \omega_{2} d x d y, \\
J_{11}=-\iint_{\mathbb{R}^{2}} b_{1} j_{2} \partial_{x} \omega_{2} d x d y, & J_{12}=-\iint_{\mathbb{R}^{2}} \partial_{x} \theta \omega_{2} d x d y, \\
J_{13}=-2 \iint_{\mathbb{R}^{2}} u_{1} j_{1} \partial_{x} j_{1} d x d y, & J_{14}=\iint_{\mathbb{R}^{2}} j_{1} \partial_{y} u_{1} j_{2} d x d y, \\
J_{15}=\iint_{\mathbb{R}^{2}} b_{1} \partial_{x} \omega_{1} j_{1} d x d y, & J_{16}=\iint_{\mathbb{R}^{2}} b_{1} \omega_{1} \partial_{x} j_{1} d x d y, \\
J_{17}=-\iint_{\mathbb{R}^{2}} \omega_{2} \partial_{y} b_{1} j_{1} d x d y, & J_{18}=\iint_{\mathbb{R}^{2}} j_{1} \partial_{x} u_{2} j_{2} d x d y, \\
J_{19}=2 \iint_{\mathbb{R}^{2}} u_{1} j_{2} \partial_{x} j_{2} d x d y, & J_{20}=-\iint_{\mathbb{R}^{2}} \omega_{1} \partial_{x} b_{2} j_{2} d x d y, \\
J_{21}=-\iint_{\mathbb{R}^{2}} b_{1} \partial_{x} \omega_{2} j_{2} d x d y, & J_{22}=-\iint_{\mathbb{R}^{2}} b_{1} \omega_{2} \partial_{x} j_{2} d x d y, \\
J_{23}=2 \iint_{\mathbb{R}^{2}} u_{1} \partial_{x} \theta \partial_{x x} \theta d x d y, & J_{24}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{2} \partial_{x} \theta \partial_{y} \theta d x d y, \\
J_{25}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{1} \partial_{x} \theta \partial_{y} \theta d x d y, & J_{26}=-2 \iint_{\mathbb{R}^{2}} u_{1} \partial_{y} \theta \partial_{x y} \theta d x d y, \\
J_{27}=\iint_{\mathbb{R}^{2}} \partial_{x} u_{3} \partial_{x} \theta d x d y, & J_{28}=\iint_{\mathbb{R}^{2}} \partial_{y} u_{3} \partial_{y} \theta d x d y,
\end{array}
$$

and we have applied Lemma 2.1, Young's inequality, and the simple facts

$$
\begin{array}{ll}
\partial_{y} \omega_{2}=-\partial_{x} \omega_{1}, & \partial_{y} j_{2}=-\partial_{x} j_{1}, \\
\partial_{y} u_{1}=\partial_{x} u_{2}-\omega_{3} & \Rightarrow \int_{0}^{T}\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2} d t<\infty, \\
\partial_{y} b_{1}=\partial_{x} b_{2}-j_{3} \quad \Rightarrow \quad \int_{0}^{T}\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2} d t<\infty,
\end{array}
$$

and $A_{1}(t)=\left(\left\|u_{1}\right\|_{L^{\infty}}^{2}+\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}+\left\|b_{1}\right\|_{L^{\infty}}^{2}+\left\|\partial_{x} b\right\|_{L^{2}}^{2}+\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2}+1\right)$.
Applying Gronwall's inequality, together with $A_{1}(t)$ is an integrable function over $(0, T)$, we obtain

$$
\begin{aligned}
& G_{1}^{2}(t)+\mu_{1} \int_{0}^{T}\left\|\left(\partial_{x} \omega_{1}, \partial_{x} \omega_{2}\right)\right\|_{L^{2}}^{2} d t+\nu_{1} \int_{0}^{T}\left\|\left(\partial_{x} j_{1}, \partial_{x} j_{2}\right)\right\|_{L^{2}}^{2} d t \\
& \quad+\kappa_{1} \int_{0}^{T}\left\|\tilde{\nabla} \partial_{x} \theta\right\|_{L^{2}}^{2} d t \leq C
\end{aligned}
$$

for any $t \leq T$, where $C$ depends on $T$ and the initial $H^{1}$-norm. Especially, (2.6) is proven. This completes the proof of Lemma 2.5.

Finally, we prove Theorem 1.1 by establishing the global $H^{2}$-bound for the solution.

Proof of Theorem 1.1. As we explained before, it suffices to establish the global $H^{2}$-bound in order to prove Theorem 1.1. The rest of this proof we devote to establishing the global $H^{2}$-bound.

Set

$$
F_{2}(t) \triangleq\left(\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{3}\right\|_{L^{2}}^{2}\right)^{\frac{1}{2}}
$$

Consider (1.5) with $\mu_{1}>0, \mu_{2}=0, v_{1}>0, \nu_{2}=0, \kappa_{1}>0, \kappa_{2}=0$. Dotting the third and sixth equations in (1.5) by $\tilde{\Delta} \omega_{3}$ and $\tilde{\Delta} j_{3}$, respectively, and integrating them over spatial domain, we have

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} F_{2}^{2}(t)+\mu_{1}\left\|\tilde{\nabla} \partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\nu_{1}\left\|\tilde{\nabla} \partial_{x} j_{3}\right\|_{L^{2}}^{2} \\
& \quad=\sum_{i=1}^{19} K_{i} \leq \frac{\mu_{1}}{2}\left\|\tilde{\nabla} \partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\frac{\nu_{1}}{2}\left\|\tilde{\nabla} \partial_{x} j_{3}\right\|_{L^{2}}^{2}+C A_{2}(t) F_{2}^{2}(t) \tag{2.8}
\end{align*}
$$

where

$$
\begin{array}{ll}
K_{1}=\iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{x} \omega_{3} \partial_{x} \omega_{3} d x d y, & K_{2}=\iint_{\mathbb{R}^{2}} \partial_{x} u_{2} \partial_{x} \omega_{3} \partial_{y} \omega_{3} d x d y \\
K_{3}=\iint_{\mathbb{R}^{2}} \partial_{y} u_{1} \partial_{x} \omega_{3} \partial_{y} \omega_{3} d x d y, & K_{4}=\iint_{\mathbb{R}^{2}} \partial_{y} u_{2} \partial_{y} \omega_{3} \partial_{y} \omega_{3} d x d y \\
K_{5}=2 \iint_{\mathbb{R}^{2}} \partial_{x} \omega_{3} \partial_{x} b_{1} \partial_{x} j_{3} d x d y, & K_{6}=2 \iint_{\mathbb{R}^{2}} \partial_{x} \omega_{3} \partial_{x} b_{2} \partial_{y} j_{3} d x d y \\
K_{7}=2 \iint_{\mathbb{R}^{2}} \partial_{y} \omega_{3} \partial_{y} b_{1} \partial_{x} j_{3} d x d y, & K_{8}=2 \iint_{\mathbb{R}^{2}} b_{1} \partial_{x y} \omega_{3} \partial_{y} j_{3} d x d y \\
K_{9}=2 \iint_{\mathbb{R}^{2}} b_{1} \partial_{y} \omega_{3} \partial_{x y} j_{3} d x d y, \quad K_{10}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{x} j_{3} \partial_{x} j_{3} d x d y \\
K_{11}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{2} \partial_{x} j_{3} \partial_{y} j_{3} d x d y, \quad K_{12}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{1} \partial_{x} j_{3} \partial_{y} j_{3} d x d y \\
K_{13}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{2} \partial_{y} j_{3} \partial_{y} j_{3} d x d y, \quad K_{14}=-2 \iint_{\mathbb{R}^{2}} \partial_{x} b_{1}\left(\partial_{y} u_{1}+\partial_{x} u_{2}\right) \partial_{x x} j_{3} d x d y, \\
K_{15}=2 \iint_{\mathbb{R}^{2}} \partial_{x y} b_{1}\left(\partial_{y} u_{1}+\partial_{x} u_{2}\right) \partial_{y} j_{3} d x d y, \\
K_{16}=2 \iint_{\mathbb{R}^{2}} \partial_{x} b_{1} \partial_{y}\left(\partial_{y} u_{1}+\partial_{x} u_{2}\right) \partial_{y} j_{3} d x d y, \\
K_{17}=2 \iint_{\mathbb{R}^{2}} \partial_{x} u_{1}\left(\partial_{x} b_{2}+\partial_{y} b_{1}\right) \partial_{x x} j_{3} d x d y, \\
K_{18}=-2 \iint_{\mathbb{R}^{2}} \partial_{x y} u_{1}\left(\partial_{x} b_{2}+\partial_{y} b_{1}\right) \partial_{y} j_{3} d x d y, \\
K_{19}=-2 \iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{y}\left(\partial_{x} b_{2}+\partial_{y} b_{1}\right) \partial_{y} j_{3} d x d y,
\end{array}
$$

and $A_{2}(t)=\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}+\left\|b_{1}\right\|_{L^{\infty}}^{2}+\left\|\partial_{x} b\right\|_{L^{2}}^{2}+\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2}+\left\|\omega_{3}\right\|_{L^{2}}^{2}+\left\|j_{3}\right\|_{L^{2}}^{2}+\left(\left\|\partial_{x} u\right\|_{L^{2}}^{\frac{2}{3}}+\right.\right.$ $\left.\left.\left\|j_{3}\right\|_{L^{2}}^{\frac{2}{3}}\right)\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}+\left(\left\|\omega_{3}\right\|_{L^{2}}^{2}+\left\|\partial_{x} b\right\|_{L^{2}}^{2}\right)\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}+1\right)$ is also an integrable function over $(0, T)$.

It thus follows from Gronwall's inequality that

$$
\begin{equation*}
\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{3}\right\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{T}\left\|\tilde{\nabla} \partial_{x} \omega_{3}\right\|_{L^{2}}^{2} d t+\nu_{1} \int_{0}^{T}\left\|\tilde{\nabla} \partial_{x} j_{3}\right\|_{L^{2}}^{2} d t \leq C \tag{2.9}
\end{equation*}
$$

where $C$ depends on $T$ and the initial $H^{2}$-norm. Therefore, we obtain the desired $H^{2}$ bound for $u_{3}$ and $b_{3}$.
At last, we establish the global $H^{2}$-bound for $u_{1}, u_{2}, b_{1}, b_{2}$, and $\theta$. Similarly, consider (1.5) with $\mu_{1}>0, \mu_{2}=0, v_{1}>0, \nu_{2}=0, \kappa_{1}>0, \kappa_{2}=0$.

Set

$$
G_{2}(t) \triangleq\left(\|\tilde{\nabla} \tilde{\omega}\|_{L^{2}}^{2}+\|\tilde{\nabla} \tilde{j}\|_{L^{2}}^{2}+\|\tilde{\Delta} \theta\|_{L^{2}}^{2}\right)^{\frac{1}{2}}
$$

Taking the inner product of the first, second, forth, and fifth equations in (1.5) with $\tilde{\Delta} \omega_{1}$, $\tilde{\Delta} \omega_{2}, \tilde{\Delta} j_{1}$, and $\tilde{\Delta} j_{2}$, and integrating by parts, we obtain

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left(\left\|\tilde{\nabla} \omega_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} \omega_{2}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{2}\right\|_{L^{2}}^{2}\right)+\mu_{1}\left\|\left(\tilde{\nabla} \partial_{x} \omega_{1}, \tilde{\nabla} \partial_{x} \omega_{2}\right)\right\|_{L^{2}}^{2} \\
& \quad+v_{1}\left\|\left(\tilde{\nabla} \partial_{x} j_{1}, \tilde{\nabla} \partial_{x} j_{2}\right)\right\|_{L^{2}}^{2}=\sum_{i=1}^{18} L_{i}, \tag{2.10}
\end{align*}
$$

where

$$
\begin{array}{ll}
L_{1}=-\iint_{\mathbb{R}^{2}} \tilde{\nabla} \omega_{1} \cdot \tilde{\nabla} u \cdot \tilde{\nabla} \omega_{1} d x d y, & L_{2}=-\iint_{\mathbb{R}^{2}}(\tilde{\omega} \cdot \tilde{\nabla}) u_{1} \cdot \tilde{\Delta} \omega_{1} d x d y, \\
L_{3}=-\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) j_{1} \cdot \tilde{\Delta} \omega_{1} d x d y, & L_{4}=\iint_{\mathbb{R}^{2}}(\tilde{j} \cdot \tilde{\nabla}) b_{1} \cdot \tilde{\Delta} \omega_{1} d x d y, \\
L_{5}=-\iint_{\mathbb{R}^{2}} \tilde{\nabla} \omega_{2} \cdot \tilde{\nabla} u \cdot \tilde{\nabla} \omega_{2} d x d y, & L_{6}=-\iint_{\mathbb{R}^{2}}(\tilde{\omega} \cdot \tilde{\nabla}) u_{2} \cdot \tilde{\Delta} \omega_{2} d x d y, \\
L_{7}=-\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) j_{2} \cdot \tilde{\Delta} \omega_{2} d x d y, & L_{8}=\iint_{\mathbb{R}^{2}}(\tilde{j} \cdot \tilde{\nabla}) b_{2} \cdot \tilde{\Delta} \omega_{2} d x d y, \\
L_{9}=-\iint_{\mathbb{R}^{2}} \tilde{\nabla} j_{1} \cdot \tilde{\nabla} u \cdot \tilde{\nabla} j_{1} d x d y, & L_{10}=-\iint_{\mathbb{R}^{2}}(\tilde{j} \cdot \tilde{\nabla}) u_{1} \cdot \tilde{\Delta} j_{1} d x d y, \\
L_{11}=-\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) \omega_{1} \cdot \tilde{\Delta} j_{1} d x d y, & L_{12}=\iint_{\mathbb{R}^{2}}(\tilde{\omega} \cdot \tilde{\nabla}) b_{1} \cdot \tilde{\Delta} j_{1} d x d y, \\
L_{13}=-\iint_{\mathbb{R}^{2}} \tilde{\nabla} j_{2} \cdot \tilde{\nabla} u \cdot \tilde{\nabla} j_{2} d x d y, & L_{14}=-\iint_{\mathbb{R}^{2}}(\tilde{j} \cdot \tilde{\nabla}) u_{2} \cdot \tilde{\Delta} j_{2} d x d y, \\
L_{15}=-\iint_{\mathbb{R}^{2}}(\tilde{b} \cdot \tilde{\nabla}) \omega_{2} \cdot \tilde{\Delta} j_{2} d x d y, & L_{16}=\iint_{\mathbb{R}^{2}}(\tilde{\omega} \cdot \tilde{\nabla}) b_{2} \cdot \tilde{\Delta} j_{2} d x d y, \\
L_{17}=-\iint_{\mathbb{R}^{2}} \partial_{y} \theta \cdot \tilde{\Delta} \omega_{1} d x d y, & L_{18}=\iint_{\mathbb{R}^{2}} \partial_{x} \theta \cdot \tilde{\Delta} \omega_{2} d x d y .
\end{array}
$$

Applying $\tilde{\nabla}$ to the seventh equation in (1.5) and taking the $L^{2}$-inner product with $\tilde{\Delta} \theta$, and integrating by parts, we obtain

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\|\tilde{\Delta} \theta\|_{L^{2}}^{2}+\kappa_{1}\left\|\tilde{\Delta} \partial_{x} \theta\right\|_{L^{2}}^{2} \\
& \quad=-\iint_{\mathbb{R}^{2}} \tilde{\Delta}[(\tilde{u} \cdot \tilde{\nabla}) \theta] \cdot \tilde{\Delta} \theta d x d y+\iint_{\mathbb{R}^{2}} \tilde{\Delta} u_{3} \cdot \tilde{\Delta} \theta d x d y=L_{19}+L_{20} \tag{2.11}
\end{align*}
$$

Adding (2.10) and (2.11) yields

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} G_{2}^{2}(t)+\mu_{1}\left\|\left(\tilde{\nabla} \partial_{x} \omega_{1}, \tilde{\nabla} \partial_{x} \omega_{2}\right)\right\|_{L^{2}}^{2}+\nu_{1}\left\|\left(\tilde{\nabla} \partial_{x} j_{1}, \tilde{\nabla} \partial_{x} j_{2}\right)\right\|_{L^{2}}^{2}+\kappa_{1}\left\|\tilde{\Delta} \partial_{x} \theta\right\|_{L^{2}}^{2} \\
& \quad=\sum_{i=1}^{20} L_{i} \tag{2.12}
\end{align*}
$$

We now estimate $L_{1}$ through $L_{20}$ as follows:

$$
L_{1}=\sum_{i=1}^{4} L_{1 i} \leq \frac{\mu_{1}}{10}\left\|\tilde{\nabla} \partial_{x} \omega_{1}\right\|_{L^{2}}^{2}+C\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}+\left\|\partial_{x} u\right\|_{L^{2}}^{\frac{2}{3}}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}\right)\left\|\tilde{\nabla} \omega_{1}\right\|_{L^{2}}^{2},
$$

where

$$
\begin{aligned}
& L_{11}=- \iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{x} \omega_{1} \partial_{x} \omega_{1} d x d y, \quad L_{12}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{2} \partial_{x} \omega_{1} \partial_{y} \omega_{1} d x d y \\
& L_{13}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{1} \partial_{x} \omega_{1} \partial_{y} \omega_{1} d x d y, \quad L_{14}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{2} \partial_{y} \omega_{1} \partial_{y} \omega_{1} d x d y \\
& L_{2}+L_{4}= \sum_{i=1}^{8}\left(L_{2 i}+L_{8 i}\right) \\
& \quad \leq \frac{2 \mu_{1}}{5}\left\|\tilde{\nabla} \partial_{x} \omega_{1}\right\|_{L^{2}}^{2}+C\left(\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}+1\right)\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\left(\left\|\partial_{x} b\right\|_{L^{2}}^{2}\right.\right. \\
&\left.+\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2}+\left\|j_{2}\right\|_{L^{2}}^{2}+1\right)\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}+\left\|\partial_{x} u\right\|_{L^{2}}^{\frac{2}{3}}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{\frac{2}{3}}\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}} \\
&\left.+\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{2}+\left(\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2}+1\right)\left\|\tilde{\nabla} j_{3}\right\|_{L^{2}}^{2}+1\right)\left(\left\|\tilde{\nabla} \omega_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} \omega_{2}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{1}\right\|_{L^{2}}^{2}\right) \\
&+C\left(\left\|\omega_{1}\right\|_{L^{2}}^{2}+\left\|\omega_{2}\right\|_{L^{2}}^{2}+\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}+\left\|j_{1}\right\|_{L^{2}}^{2}+\left\|j_{2}\right\|_{L^{2}}^{2}\right),
\end{aligned}
$$

where

$$
\begin{array}{ll}
L_{21}=-\iint_{\mathbb{R}^{2}} \partial_{x y} u_{3} \partial_{y} u_{2} \partial_{x} \omega_{1} d x d y, & L_{22}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{3} \partial_{x y} u_{2} \partial_{x} \omega_{1} d x d y, \\
L_{23}=-\iint_{\mathbb{R}^{2}} \partial_{x x} u_{3} \partial_{y} u_{1} \partial_{x} \omega_{1} d x d y, & L_{24}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{3} \partial_{x y} u_{1} \partial_{x} \omega_{1} d x d y, \\
L_{25}=\iint_{\mathbb{R}^{2}} \partial_{x y} b_{3} \partial_{y} b_{2} \partial_{x} \omega_{1} d x d y, & L_{26}=\iint_{\mathbb{R}^{2}} \partial_{y} b_{3} \partial_{x y} b_{2} \partial_{x} \omega_{1} d x d y, \\
L_{27}=\iint_{\mathbb{R}^{2}} \partial_{x x} b_{3} \partial_{y} b_{1} \partial_{x} \omega_{1} d x d y, & L_{28}=\iint_{\mathbb{R}^{2}} \partial_{x} b_{3} \partial_{x y} b_{1} \partial_{x} \omega_{1} d x d y, \\
L_{41}=\iint_{\mathbb{R}^{2}} \partial_{y y} u_{3} \partial_{x} u_{1} \partial_{y} \omega_{1} d x d y, & L_{42}=\iint_{\mathbb{R}^{2}} \partial_{y} u_{3} \partial_{x y} u_{1} \partial_{y} \omega_{1} d x d y, \\
L_{43}=-\iint_{\mathbb{R}^{2}} \partial_{x y} u_{3} \partial_{y} u_{1} \partial_{y} \omega_{1} d x d y, & L_{44}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{3} \partial_{y y} u_{1} \partial_{y} \omega_{1} d x d y, \\
L_{45}=-\iint_{\mathbb{R}^{2}} \partial_{y y} b_{3} \partial_{x} b_{1} \partial_{y} \omega_{1} d x d y, & L_{46}=-\iint_{\mathbb{R}^{2}} \partial_{y} b_{3} \partial_{x y} b_{1} \partial_{y} \omega_{1} d x d y, \\
L_{47}=\iint_{\mathbb{R}^{2}} \partial_{x y} b_{3} \partial_{y} b_{1} \partial_{y} \omega_{1} d x d y, & L_{48}=\iint_{\mathbb{R}^{2}} \partial_{x} b_{3} \partial_{y y} b_{1} \partial_{y} \omega_{1} d x d y,
\end{array}
$$

$$
L_{3}+L_{11}=\sum_{i=1}^{4} \sharp_{i} \leq \frac{\nu_{1}}{12}\left\|\tilde{\nabla} \partial_{x} j_{1}\right\|_{L^{2}}^{2}+C\left(1+\left\|\partial_{x} b\right\|_{L^{2}}^{2}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}\right)\left(\left\|\tilde{\nabla} \omega_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{1}\right\|_{L^{2}}^{2}\right),
$$

where

$$
\begin{aligned}
& \sharp_{1}=2 \iint_{\mathbb{R}^{2}} \partial_{x} \omega_{1} \partial_{x} b_{1} \partial_{x} j_{1} d x d y, \quad \sharp_{2}=2 \iint_{\mathbb{R}^{2}} \partial_{x} \omega_{1} \partial_{x} b_{2} \partial_{y} j_{1} d x d y, \\
& \sharp_{3}=2 \iint_{\mathbb{R}^{2}} \partial_{y} \omega_{1} \partial_{y} b_{1} \partial_{x} j_{1} d x d y, \quad \sharp_{4}=2 \iint_{\mathbb{R}^{2}} \partial_{y} \omega_{1} \partial_{y} b_{2} \partial_{y} j_{1} d x d y, \\
& L_{5}=\sum_{i=1}^{4} L_{5 i} \leq \frac{\mu_{1}}{10}\left\|\tilde{\nabla} \partial_{x} \omega_{2}\right\|_{L^{2}}^{2}+C\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}+\left\|\partial_{x} u\right\|_{L^{2}}^{\frac{2}{3}}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}\right)\left\|\tilde{\nabla} \omega_{2}\right\|_{L^{2}}^{2},
\end{aligned}
$$

where

$$
\begin{aligned}
& L_{51}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{x} \omega_{2} \partial_{x} \omega_{2} d x d y, \quad L_{52}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{2} \partial_{x} \omega_{2} \partial_{y} \omega_{2} d x d y \\
& L_{53}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{1} \partial_{x} \omega_{2} \partial_{y} \omega_{2} d x d y, \quad L_{54}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{2} \partial_{y} \omega_{2} \partial_{y} \omega_{2} d x d y \\
& L_{6}+L_{8}= \\
& \quad \sum_{i=1}^{8}\left(L_{6 i}+L_{8 i}\right) \\
& \quad \leq \frac{2 \mu_{1}}{5}\left\|\tilde{\nabla} \partial_{x} \omega_{2}\right\|_{L^{2}}^{2}+C\left(\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+1\right)\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\left(\left\|\partial_{x} b\right\|_{L^{2}}^{2}+\left\|j_{3}\right\|_{L^{2}}^{2}\right.\right. \\
& \left.\quad+1)\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}+1\right)\left(\left\|\tilde{\nabla} \omega_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} \omega_{2}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{2}\right\|_{L^{2}}^{2}\right) \\
& \quad+C\left(\left\|\omega_{2}\right\|_{L^{2}}^{2}+\left\|j_{1}\right\|_{L^{2}}^{2}+\left\|j_{2}\right\|_{L^{2}}^{2}\right),
\end{aligned}
$$

where

$$
\begin{array}{ll}
L_{61}=\iint_{\mathbb{R}^{2}} \partial_{x y} u_{3} \partial_{x} u_{2} \partial_{x} \omega_{2} d x d y, & L_{62}=\iint_{\mathbb{R}^{2}} \partial_{y} u_{3} \partial_{x x} u_{2} \partial_{x} \omega_{2} d x d y, \\
L_{63}=-\iint_{\mathbb{R}^{2}} \partial_{x x} u_{3} \partial_{y} u_{2} \partial_{x} \omega_{2} d x d y, & L_{64}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{3} \partial_{x y} u_{2} \partial_{x} \omega_{2} d x d y, \\
L_{65}=-\iint_{\mathbb{R}^{2}} \partial_{x y} b_{3} \partial_{x} b_{2} \partial_{x} \omega_{2} d x d y, & L_{66}=-\iint_{\mathbb{R}^{2}} \partial_{x x} b_{2} \partial_{y} b_{3} \partial_{x} \omega_{2} d x d y, \\
L_{67}=\iint_{\mathbb{R}^{2}} \partial_{x x} b_{3} \partial_{y} b_{2} \partial_{x} \omega_{2} d x d y, & L_{68}=\iint_{\mathbb{R}^{2}} \partial_{x} b_{3} \partial_{x y} b_{2} \partial_{x} \omega_{2} d x d y, \\
L_{81}=\iint_{\mathbb{R}^{2}} \partial_{y y} u_{3} \partial_{x} u_{2} \partial_{y} \omega_{2} d x d y, & L_{82}=\iint_{\mathbb{R}^{2}} \partial_{y} u_{3} \partial_{x y} u_{2} \partial_{y} \omega_{2} d x d y, \\
L_{83}=\iint_{\mathbb{R}^{2}} \partial_{x y} u_{3} \partial_{x} u_{1} \partial_{y} \omega_{2} d x d y, & L_{84}=\iint_{\mathbb{R}^{2}} \partial_{x} u_{3} \partial_{x y} u_{1} \partial_{y} \omega_{2} d x d y, \\
L_{85}=-\iint_{\mathbb{R}^{2}} \partial_{y y} b_{3} \partial_{x} b_{2} \partial_{y} \omega_{2} d x d y, & L_{86}=-\iint_{\mathbb{R}^{2}} \partial_{y} b_{3} \partial_{x y} b_{2} \partial_{y} \omega_{2} d x d y, \\
L_{87}=-\iint_{\mathbb{R}^{2}} \partial_{x y} b_{3} \partial_{x} b_{1} \partial_{y} \omega_{2} d x d y, & L_{88}=-\iint_{\mathbb{R}^{2}} \partial_{x} b_{3} \partial_{x y} b_{1} \partial_{y} \omega_{2} d x d y, \\
L_{7}+L_{15}=\sum_{i=1}^{4} \diamond_{i} \leq \frac{\nu_{1}}{12}\left\|\tilde{\nabla} \partial_{x} j_{2}\right\|_{L^{2}}^{2}+C\left(1+\left\|\partial_{x} b\right\|_{L^{2}}^{2}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}\right)\left(\left\|\tilde{\nabla} \omega_{2}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{2}\right\|_{L^{2}}^{2}\right),
\end{array}
$$

where

$$
\begin{aligned}
& \diamond_{1}=2 \iint_{\mathbb{R}^{2}} \partial_{x} \omega_{2} \partial_{x} b_{1} \partial_{x} j_{2} d x d y, \quad \diamond_{2}=2 \iint_{\mathbb{R}^{2}} \partial_{x} \omega_{2} \partial_{x} b_{2} \partial_{y} j_{2} d x d y \\
& \diamond_{3}=2 \iint_{\mathbb{R}^{2}} \partial_{y} \omega_{2} \partial_{y} b_{1} \partial_{x} j_{2} d x d y, \quad \diamond_{4}=2 \iint_{\mathbb{R}^{2}} \partial_{y} \omega_{2} \partial_{y} b_{2} \partial_{y} j_{2} d x d y \\
& L_{9}=\sum_{i=1}^{4} L_{9 i} \leq \frac{\nu_{1}}{12}\left\|\tilde{\nabla} \partial_{x} j_{1}\right\|_{L^{2}}^{2}+C\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}+\left\|\partial_{x} u\right\|_{L^{2}}^{\frac{2}{3}}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}\right)\left\|\tilde{\nabla} j_{1}\right\|_{L^{2}}^{2},
\end{aligned}
$$

where

$$
\begin{aligned}
& L_{91}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{x} j_{1} \partial_{x} j_{1} d x d y, \quad L_{92}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{2} \partial_{x} j_{1} \partial_{y} j_{1} d x d y \\
& L_{93}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{1} \partial_{x} j_{1} \partial_{y} j_{1} d x d y, \quad L_{94}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{2} \partial_{y} j_{1} \partial_{y} j_{1} d x d y \\
& L_{10}+L_{12}= \\
& \quad \sum_{i=1}^{8}\left(L_{10 i}+L_{12 i}\right) \\
& \leq \frac{5 \mu_{1}}{12}\left\|\tilde{\nabla} \partial_{x} j_{1}\right\|_{L^{2}}^{2}+C\left(\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}+1\right)\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\left(\left\|\partial_{x} b\right\|_{L^{2}}^{2}\right.\right. \\
& \left.\quad+\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2}+1\right)\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}+\left\|\partial_{x} u\right\|_{L^{2}}^{\frac{2}{3}}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{\frac{2}{3}}\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}} \\
& \\
& \left.\quad+\left(\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2}+1\right)\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{3}\right\|_{L^{2}}^{2}+1\right)\left(\left\|\tilde{\nabla} \omega_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} \omega_{2}\right\|_{L^{2}}^{2}\right. \\
& \\
& \left.\quad+\left\|\tilde{\nabla} j_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{2}\right\|_{L^{2}}^{2}\right)+C\left(\left\|\omega_{1}\right\|_{L^{2}}^{2}+\left\|\omega_{2}\right\|_{L^{2}}^{2}+\left\|j_{1}\right\|_{L^{2}}^{2}+\left\|j_{2}\right\|_{L^{2}}^{2}\right)
\end{aligned}
$$

where

$$
\begin{array}{ll}
L_{101}=-\iint_{\mathbb{R}^{2}} \partial_{x y} b_{3} \partial_{y} u_{2} \partial_{x} j_{1} d x d y, & L_{102}=-\iint_{\mathbb{R}^{2}} \partial_{y} b_{3} \partial_{x y} u_{2} \partial_{x} j_{1} d x d y \\
L_{103}=-\iint_{\mathbb{R}^{2}} \partial_{x x} b_{3} \partial_{y} u_{1} \partial_{x} j_{1} d x d y, & L_{104}=-\iint_{\mathbb{R}^{2}} \partial_{x} b_{3} \partial_{x y} u_{1} \partial_{x} j_{1} d x d y, \\
L_{105}=\iint_{\mathbb{R}^{2}} \partial_{x y} u_{3} \partial_{y} b_{2} \partial_{x} j_{1} d x d y, & L_{106}=\iint_{\mathbb{R}^{2}} \partial_{y} u_{3} \partial_{x y} b_{2} \partial_{x} j_{1} d x d y, \\
L_{107}=\iint_{\mathbb{R}^{2}} \partial_{x x} u_{3} \partial_{y} b_{1} \partial_{x} j_{1} d x d y, & L_{108}=\iint_{\mathbb{R}^{2}} \partial_{x} u_{3} \partial_{x y} b_{1} \partial_{x} j_{1} d x d y, \\
L_{121}=\iint_{\mathbb{R}^{2}} \partial_{y y} b_{3} \partial_{x} u_{1} \partial_{y} j_{1} d x d y, & L_{122}=\iint_{\mathbb{R}^{2}} \partial_{y} b_{3} \partial_{x y} u_{1} \partial_{y} j_{1} d x d y, \\
L_{123}=-\iint_{\mathbb{R}^{2}} \partial_{x y} b_{3} \partial_{y} u_{1} \partial_{y} j_{1} d x d y, & L_{124}=-\iint_{\mathbb{R}^{2}} \partial_{x} b_{3} \partial_{y y} u_{1} \partial_{y} j_{1} d x d y, \\
L_{125}=\iint_{\mathbb{R}^{2}} \partial_{y y} u_{3} \partial_{x} b_{1} \partial_{y} j_{1} d x d y, & L_{126}=\iint_{\mathbb{R}^{2}} \partial_{y} u_{3} \partial_{x y} b_{1} \partial_{y} j_{1} d x d y, \\
L_{127}=\iint_{\mathbb{R}^{2}} \partial_{x y} u_{3} \partial_{y} b_{1} \partial_{y} j_{1} d x d y, & L_{128}=\iint_{\mathbb{R}^{2}} \partial_{x} u_{3} \partial_{y y} b_{1} \partial_{y} j_{1} d x d y, \\
L_{13}=\sum_{i=1}^{4} L_{13 i} \leq \frac{\mu_{1}}{12}\left\|\tilde{\nabla} \partial_{x} j_{2}\right\|_{L^{2}}^{2}+C\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}+\left\|\partial_{x} u\right\|_{L^{2}}^{\frac{2}{3}}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}\right)\left\|\tilde{\nabla} j_{2}\right\|_{L^{2}}^{2},
\end{array}
$$

where

$$
\begin{aligned}
& L_{131}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{x} j_{2} \partial_{x} j_{2} d x d y, \quad L_{132}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{2} \partial_{x} j_{2} \partial_{y} j_{2} d x d y \\
& L_{133}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{1} \partial_{x} j_{2} \partial_{y} j_{2} d x d y, \quad L_{134}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{2} \partial_{y} j_{2} \partial_{y} j_{2} d x d y \\
& L_{14}+L_{16}= \\
& \quad \sum_{i=1}^{8}\left(L_{14 i}+L_{16 i}\right) \\
& \quad \leq \frac{5 \mu_{1}}{12}\left\|\tilde{\nabla} \partial_{x} j_{2}\right\|_{L^{2}}^{2}+C\left(\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+1\right)\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\left(\left\|\partial_{x} b\right\|_{L^{2}}^{2}+1\right)\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}\right. \\
& \left.\quad+\left\|\partial_{x} u\right\|_{L^{2}}^{\frac{3}{3}}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}+1\right)\left(\left\|\tilde{\nabla} \omega_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} \omega_{2}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} j_{2}\right\|_{L^{2}}^{2}\right. \\
& \left.\quad+\left\|\tilde{\nabla} j_{2}\right\|_{L^{2}}^{2}\right)+C\left(\left\|\omega_{1}\right\|_{L^{2}}^{2}+\left\|\omega_{2}\right\|_{L^{2}}^{2}+\left\|j_{2}\right\|_{L^{2}}^{2}+\left\|j_{2}\right\|_{L^{2}}^{2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
& L_{141}=\iint_{\mathbb{R}^{2}} \partial_{x y} b_{3} \partial_{x} u_{2} \partial_{x} j_{2} d x d y, \quad L_{142}=\iint_{\mathbb{R}^{2}} \partial_{y} b_{3} \partial_{x x} u_{2} \partial_{x} j_{2} d x d y, \\
& L_{143}=-\iint_{\mathbb{R}^{2}} \partial_{x x} b_{3} \partial_{y} u_{2} \partial_{x} j_{2} d x d y, \quad L_{144}=-\iint_{\mathbb{R}^{2}} \partial_{x} b_{3} \partial_{x y} u_{2} \partial_{x} j_{2} d x d y, \\
& L_{145}=-\iint_{\mathbb{R}^{2}} \partial_{x y} u_{3} \partial_{x} b_{2} \partial_{x} j_{2} d x d y, \quad L_{146}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{3} \partial_{x x} b_{2} \partial_{x} j_{2} d x d y, \\
& L_{147}=\iint_{\mathbb{R}^{2}} \partial_{x x} u_{3} \partial_{y} b_{2} \partial_{x} j_{2} d x d y, \quad L_{148}=\iint_{\mathbb{R}^{2}} \partial_{x} u_{3} \partial_{x y} b_{2} \partial_{x} j_{2} d x d y, \\
& L_{161}=\iint_{\mathbb{R}^{2}} \partial_{y y} b_{3} \partial_{x} u_{2} \partial_{y} j_{2} d x d y, \quad L_{162}=\iint_{\mathbb{R}^{2}} \partial_{y} b_{3} \partial_{x y} u_{2} \partial_{y} j_{2} d x d y, \\
& L_{163}=\iint_{\mathbb{R}^{2}} \partial_{x y} b_{3} \partial_{x} u_{1} \partial_{y} j_{2} d x d y, \quad L_{164}=\iint_{\mathbb{R}^{2}} \partial_{x} b_{3} \partial_{x y} u_{1} \partial_{y} j_{2} d x d y, \\
& L_{165}=-\iint_{\mathbb{R}^{2}} \partial_{y y} u_{3} \partial_{x} b_{2} \partial_{y} j_{2} d x d y, L_{166}=-\iint_{\mathbb{R}^{2}} \partial_{y} u_{3} \partial_{x y} b_{2} \partial_{y} j_{2} d x d y, \\
& L_{167}=-\iint_{\mathbb{R}^{2}} \partial_{x y} u_{3} \partial_{x} b_{1} \partial_{y} j_{2} d x d y, L_{168}=-\iint_{\mathbb{R}^{2}} \partial_{x} u_{3} \partial_{x y} b_{1} \partial_{y} j_{2} d x d y, \\
& L_{17}+ L_{18} \leq\left\|\tilde{\nabla} \omega_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} \omega_{2}\right\|_{L^{2}}^{2}+\|\tilde{\Delta} \theta\|_{L^{2}}^{2}, \\
& L_{19} \leq C\left\|\tilde{\nabla} \partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+C\|\tilde{\nabla} \theta\|_{L^{2}}^{\frac{2}{3}\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}\|\tilde{\Delta} \theta\|_{L^{2}}^{2}+\frac{\kappa_{1}}{4}\left\|\tilde{\Delta} \partial_{x} \theta\right\| 2_{L^{2}}} \\
&+C\left\|\omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}\|\tilde{\Delta} \theta\|_{L^{2}}^{2},
\end{aligned}
$$

After combining all inequalities, we have

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} G_{2}^{2}(t)+\mu_{1}\left\|\left(\tilde{\nabla} \partial_{x} \omega_{1}, \tilde{\nabla} \partial_{x} \omega_{2}\right)\right\|_{L^{2}}^{2}+v_{1}\left\|\left(\tilde{\nabla} \partial_{x} j_{1}, \tilde{\nabla} \partial_{x} j_{2}\right)\right\|_{L^{2}}^{2}+\kappa_{1}\left\|\tilde{\Delta} \partial_{x} \theta\right\|_{L^{2}}^{2} \\
& \quad \leq \frac{\mu_{1}}{2}\left\|\left(\tilde{\nabla} \partial_{x} \omega_{1}, \tilde{\nabla} \partial_{x} \omega_{2}\right)\right\|_{L^{2}}^{2}+\frac{\nu_{1}}{2}\left\|\left(\tilde{\nabla} \partial_{x} j_{1}, \tilde{\nabla} \partial_{x} j_{2}\right)\right\|_{L^{2}}^{2}+C A_{3}(t) G_{2}^{2}(t)+C A_{4}(t) \tag{2.13}
\end{align*}
$$

where $A_{3}(t)=\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}+1\right)\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\left\|\partial_{x} u\right\|_{L^{2}}^{\frac{2}{3}}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}+\right.$ $\left(\partial_{y} b_{1} \|_{L^{2}}^{2}+1\right)\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{2}+\left(\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{\frac{2}{3}}+\left\|\omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}\right)\left\|\tilde{\nabla} \omega_{3}\right\|_{L^{2}}^{\frac{2}{3}}+\left(\left\|\partial_{x} b\right\|_{L^{2}}^{2}+\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2}+\left\|j_{2}\right\|_{L^{2}}^{2}+\left\|j_{3}\right\|_{L^{2}}^{2}+\right.$ 1) $\left.\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}+\left(\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2}+1\right)\left\|\tilde{\nabla} j_{3}\right\|_{L^{2}}^{2}\right)$ and $A_{4}(t)=\left(\left\|\omega_{1}\right\|_{L^{2}}^{2}+\left\|\omega_{2}\right\|_{L^{2}}^{2}+\left\|j_{1}\right\|_{L^{2}}^{2}+\left\|j_{2}\right\|_{L^{2}}^{2}+\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}\right)$ are integrable over $(0, T)$.
Thanks to Gronwall's inequality and the estimate for $\|(u, b, \theta)\|_{L^{2}}$ in (2.2), the estimate for $\left\|\left(\omega_{3}, j_{3}\right)\right\|_{L^{2}}$ in $(2.4)$, the bound for $\|(\tilde{\omega}, \tilde{j}, \tilde{\nabla} \theta)\|_{L^{2}}$ in (2.6), and the bound for $\left\|\left(\tilde{\nabla} \omega_{3}, \tilde{\nabla} j_{3}\right)\right\|_{L^{2}}$ in (2.9), we reach

$$
\begin{align*}
& \|\tilde{\nabla} \tilde{\omega}\|_{L^{2}}^{2}+\|\tilde{\nabla} \tilde{j}\|_{L^{2}}^{2}+\|\tilde{\Delta} \tilde{\theta}\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{T}\left\|\left(\tilde{\nabla} \partial_{x} \omega_{1}, \tilde{\nabla} \partial_{x} \omega_{2}\right)\right\|_{L^{2}}^{2} d t \\
& \quad+v_{1} \int_{0}^{T}\left\|\left(\tilde{\nabla} \partial_{x} j_{1}, \tilde{\nabla} \partial_{x} j_{2}\right)\right\|_{L^{2}}^{2} d t+\kappa_{1} \int_{0}^{T}\left\|\tilde{\Delta} \partial_{x} \theta\right\|_{L^{2}}^{2} d t \leq C \tag{2.14}
\end{align*}
$$

where $C$ depends on $T$ and the initial $H^{2}$-norm.
Thus taking the global $H^{1}$-bound for $\omega_{1}, \omega_{2}, \omega_{3}$ together with the global $H^{1}$-bound for $j_{1}, j_{2}, j_{3}$ and the global $H^{2}$-bound for $\theta$, we obtain the global $H^{2}$-bound for $(u, b, \theta)$ of $2 \frac{1}{2} \mathrm{D}$ magnetic Bénard system with horizontal dissipation, horizontal magnetic diffusion, and horizontal thermal diffusivity.

### 2.2 Case ii

Under conditions (1.8) and (1.9). For these two cases, the proof is much similar to the previous case. Since the higher-order estimates can be obtained as in case (1.7), provided that the uniform lower-order estimates have been done, it suffices to establish the lowerorder estimates of the solutions with the aid of regularity criterion.
2.2.1 Subcase \&: suppose that $\int_{0}^{T}\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2} d t<\infty$ for some $T>0$

Proposition 2.6 Assume that $\left(u_{0}, b_{0}, \theta_{0}\right)$ satisfies the condition stated in Theorem 1.1. Let ( $u, b, \theta$ ) be the corresponding solution of (1.6). Then $(u, b, \theta)$ satisfies, for any $T>0$ and $t \leq T$,

$$
\begin{equation*}
\|(u, b, \theta)\|_{H^{1}\left(\mathbb{R}^{2}\right)} \leq C_{3} e^{C_{4} \int_{0}^{t}\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2} d \tau} \tag{2.15}
\end{equation*}
$$

where $C_{3}$ is a constant depending on $T$ and the initial data and $C_{4}$ is a pure constant.

Proof Proposition 2.6 is an easy consequence of Lemmas 2.7 and 2.8.

Lemma 2.7 Consider (1.5) with $\mu_{1}>0, \mu_{2}=0, \nu_{1}>0, \nu_{2}=0, \kappa_{1}>0, \kappa_{2}=0$. Assume that $\left(u_{0}, b_{0}, \theta_{0}\right)$ satisfies the condition stated in Theorem 1.1. Let $(u, b, \theta)$ be the corresponding solution of (1.6). Then $\omega_{3}$ and $j_{3}$ satisfy

$$
\begin{equation*}
\left\|\omega_{3}\right\|_{L^{2}}^{2}+\left\|j_{3}\right\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{T}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2} d t+v_{1} \int_{0}^{T}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2} d t \leq C \tag{2.16}
\end{equation*}
$$

provided $\int_{0}^{T}\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2} d t<\infty$ for some $T>0$.

Proof The lower-order estimates of $u_{3}$ and $b_{3}$ follow from (2.5):

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} F_{1}^{2}(t)+\mu_{1}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+v_{1}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2} \\
& \quad=O_{1}+O_{2}+O_{3}+O_{4}+O_{5} \\
& \quad \leq \frac{v_{1}}{2}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2}+C\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left(\|u\|_{L^{2}}^{2}+1\right)\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2}\right)\left\|j_{3}\right\|_{L^{2}}^{2} \tag{2.17}
\end{align*}
$$

where

$$
\begin{array}{ll}
O_{1}=2 \iint_{\mathbb{R}^{2}} \partial_{x} b_{1} \partial_{y} u_{1} j_{3} d x d y, & O_{2}=2 \iint_{\mathbb{R}^{2}} \partial_{x} b_{1} \partial_{x} u_{2} j_{3} d x d y, \\
O_{3}=-2 \iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{x} b_{2} j_{3} d x d y, & O_{4}=2 \iint_{\mathbb{R}^{2}} u_{1} \partial_{x y} b_{1} j_{3} d x d y, \\
O_{5}=2 \iint_{\mathbb{R}^{2}} u_{1} \partial_{y} b_{1} \partial_{x} j_{3} d x d y .
\end{array}
$$

Thanks to Gronwall's inequality and the criterion $\int_{0}^{T}\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2} d t<\infty$, we have

$$
\begin{equation*}
\left\|\left(\omega_{3}, j_{3}\right)\right\|_{L^{\infty}\left(0, T ; L^{2}\left(\mathbb{R}^{2}\right)\right)}+\mu_{1}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{2}\right)\right)}+v_{1}\left\|\partial_{x} j_{3}\right\|_{L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{2}\right)\right)} \leq C . \tag{2.18}
\end{equation*}
$$

This completes the proof of Lemma 2.7.

Lemma 2.8 Consider (1.5) with $\mu_{1}>0, \mu_{2}=0, \nu_{1}>0, \nu_{2}=0, \kappa_{1}>0, \kappa_{2}=0$. Assume that $\left(u_{0}, b_{0}, \theta_{0}\right)$ satisfies the condition stated in Theorem 1.1. Let $(u, b, \theta)$ be the corresponding solution of (1.6). Then $\tilde{\omega}, \tilde{j}$, and $\tilde{\nabla} \theta$ satisfy

$$
\begin{align*}
& \|\tilde{\omega}\|_{L^{2}}^{2}+\|\tilde{j}\|_{L^{2}}^{2}+\|\tilde{\nabla} \theta\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{T}\left\|\partial_{x} \tilde{\omega}\right\|_{L^{2}}^{2} d t+v_{1} \int_{0}^{T}\left\|\partial_{x} \tilde{j}\right\|_{L^{2}}^{2} d t \\
& \quad+\kappa_{1} \int_{0}^{T}\left\|\tilde{\nabla} \partial_{x} \theta\right\|_{L^{2}}^{2} d t \leq C \tag{2.19}
\end{align*}
$$

provided $\int_{0}^{T}\left\|\partial_{y} u_{1}\right\|_{L^{2}}^{2} d t<\infty$ for some $T>0$.

Proof The proof of Lemma 2.8 is similar to [5]. So we omit the details.
2.2.2 Subcase $\boldsymbol{\oplus}$ : suppose that $\int_{0}^{T}\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2} d t<\infty$ for some $T>0$

Proposition 2.9 Assume that $\left(u_{0}, b_{0}, \theta_{0}\right)$ satisfies the condition stated in Theorem 1.1. Let $(u, b, \theta)$ be the corresponding solution of (1.6). Then $(u, b, \theta)$ satisfies, for any $T>0$ and $t \leq T$,

$$
\begin{equation*}
\|(u, b, \theta)\|_{H^{1}\left(\mathbb{R}^{2}\right)} \leq C_{5} e^{C_{6} \int_{0}^{t}\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2} d \tau} \tag{2.20}
\end{equation*}
$$

where $C_{5}$ is a constant depending on $T$ and the initial data and $C_{6}$ is a pure constant.

Proof Proposition 2.9 is an easy consequence of Lemmas 2.10 and 2.11.

Lemma 2.10 Consider (1.5) with $\mu_{1}>0, \mu_{2}=0, \nu_{1}>0, \nu_{2}=0, \kappa_{1}>0, \kappa_{2}=0$. Assume that $\left(u_{0}, b_{0}, \theta_{0}\right)$ satisfies the condition stated in Theorem 1.1. Let $(u, b, \theta)$ be the corresponding solution of (1.6). Then $\omega_{3}$ and $j_{3}$ satisfy

$$
\begin{equation*}
\left\|\omega_{3}\right\|_{L^{2}}^{2}+\left\|j_{3}\right\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{T}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2} d t+v_{1} \int_{0}^{T}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2} d t \leq C \tag{2.21}
\end{equation*}
$$

provided $\int_{0}^{T}\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2} d t<\infty$ for some $T>0$.
Proof Similar to the previous arguments, the lower-order estimates of $u_{3}$ and $b_{3}$ follow from (2.5):

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} F_{1}^{2}(t)+\mu_{1}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\nu_{1}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2} \\
& \quad=Q_{1}+Q_{2}+Q_{3}+Q_{4}+Q_{5} \\
& \quad \leq \frac{\mu_{1}}{2}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}}^{2}+\frac{\nu_{1}}{2}\left\|\partial_{x} j_{3}\right\|_{L^{2}}^{2} \\
& \quad+C\left(\left\|\partial_{x} u\right\|_{L^{2}}^{2}+\left(\|b\|_{L^{2}}^{2}+1\right)\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2}\right)\left(\left\|\omega_{3}\right\|_{L^{2}}^{2}+\left\|j_{3}\right\|_{L^{2}}^{2}\right) \tag{2.22}
\end{align*}
$$

where

$$
\begin{array}{ll}
Q_{1}=-2 \iint_{\mathbb{R}^{2}} b_{1} \partial_{x y} u_{1} j_{3} d x d y, & Q_{2}=-2 \iint_{\mathbb{R}^{2}} b_{1} \partial_{y} u_{1} \partial_{x} j_{3} d x d y \\
Q_{3}=2 \iint_{\mathbb{R}^{2}} \partial_{x} b_{1} \partial_{x} u_{2} j_{3} d x d y, & Q_{4}=-2 \iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{x} b_{2} j_{3} d x d y \\
Q_{5}=-2 \iint_{\mathbb{R}^{2}} \partial_{x} u_{1} \partial_{y} b_{1} j_{3} d x d y .
\end{array}
$$

Taking advantage of Gronwall's inequality and the criterion $\int_{0}^{T}\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2} d t<\infty$ gives

$$
\begin{equation*}
\left\|\left(\omega_{3}, j_{3}\right)\right\|_{L^{\infty}\left(0, T ; L^{2}\left(\mathbb{R}^{2}\right)\right)}+\mu_{1}\left\|\partial_{x} \omega_{3}\right\|_{L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{2}\right)\right)}+v_{1}\left\|\partial_{x} j_{3}\right\|_{L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{2}\right)\right)} \leq C . \tag{2.23}
\end{equation*}
$$

Thus we complete the proof of Lemma 2.10.

Lemma 2.11 Consider (1.5) with $\mu_{1}>0, \mu_{2}=0, \nu_{1}>0, \nu_{2}=0, \kappa_{1}>0, \kappa_{2}=0$. Assume that $\left(u_{0}, b_{0}, \theta_{0}\right)$ satisfies the condition stated in Theorem 1.1. Let $(u, b, \theta)$ be the corresponding solution of (1.6). Then $\tilde{\omega}, \tilde{j}$, and $\tilde{\nabla} \theta$ satisfy

$$
\begin{align*}
& \|\tilde{\omega}\|_{L^{2}}^{2}+\|\tilde{j}\|_{L^{2}}^{2}+\|\tilde{\nabla} \theta\|_{L^{2}}^{2}+\mu_{1} \int_{0}^{T}\left\|\partial_{x} \tilde{\omega}\right\|_{L^{2}}^{2} d t+v_{1} \int_{0}^{T}\left\|\partial_{x} \tilde{j}\right\|_{L^{2}}^{2} d t \\
& \quad+\kappa_{1} \int_{0}^{T}\left\|\tilde{\nabla} \partial_{x} \theta\right\|_{L^{2}}^{2} d t \leq C \tag{2.24}
\end{align*}
$$

provided $\int_{0}^{T}\left\|\partial_{y} b_{1}\right\|_{L^{2}}^{2} d t<\infty$ for some $T>0$.
Proof Using the similar arguments as in [5], we can easily obtain this lemma.

In view of the above arguments, we have completed the proof of Theorem 1.1. The proof of Theorem 1.2 is similar, so we leave it to the interested readers.

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## Authors' contributions

LM conceived and designed the work. LZ drafted the manuscript. LM revised the manuscript. LM and LZ approved the final version.

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