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On the $\mathcal{PC}$ -mild solutions of abstract fractional evolution equations with non-instantaneous impulses via the measure of noncompactness

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Abstract

In this paper, we deal with the existence results for mild solutions of abstract fractional evolution equations with non-instantaneous impulses on an unbounded interval. We also establish the existence of $\mathcal{S}$ -asymptotically ω -periodic mild solutions. The applied techniques are supported by the concept of measure of noncompactness in conjunction with the well-known Darbo–Sadovskii and Tichonov fixed-point theorems. Furthermore, an example to the fractional initial/boundary value Cauchy problem is concerned to illustrate our main results.

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1 Introduction

Throughout this work, inspired by the problem stated by Hernández and O'Regan [22], we consider the following fractional evolution equations with non-instantaneous impulses:

$$\begin{cases} {}^C D_{0,t}^q x(t) = Ax(t) + f(t, x(t)), & t \in [s_i, t_{i+1}], i \in \mathbb{N}, \\ x(t) = g_i(t, N_i(t)(x)), & t \in (t_i, s_i], i \in \mathbb{N}, \\ x(0) = x_0, \end{cases} \quad (1.1)$$

where ${}^C D_{0,t}^q$ denotes the Caputo fractional derivative of order $0 < q < 1$ with the lower limit 0, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operator $\{T(t)\}_{t \geq 0}$ defined on Banach space $(E, \|\cdot\|)$, $x_0 \in E$ and $0 = t_0 = s_0 < t_1 < s_1 < \dots < t_i < s_i < t_{i+1} < \dots$ are pre-fixed real numbers. Moreover, $N_i(t) : C([t_i, s_i]; E) \rightarrow E$ are continuous maps for $t \in [t_i, s_i]$, the function $t \rightarrow N_i(t)(x)$ is continuous for each $x \in C([t_i, s_i]; E)$, $g_i \in C([t_i, s_i] \times E; E)$ for all $i \in \mathbb{N}$ and $f : \mathbb{R}_+ \times E \rightarrow E$ is a given function which will be specified later.

The concept of fractional calculus appeared in 1695 in the letter between de L'Hôpital and Leibniz. Since then, further development in this area has been explored by many math-

ematicians, and we recommend to read the study of Riemann, Liouville, Caputo, and other famous mathematicians. Fractional calculus plays an important role in various fields such as electricity, biology, economics, signal and image processing. From decades ago, so many researchers studied in this area and obtained theoretical results (see, for example, [4–6, 23, 51] and the references therein). Fractional calculus is also important in the regularity theory of solutions to partial differential equations. For example, in 2018, Scapellato [43] studied the second-order divergence-form operators $\mathcal{L}$ with coefficients satisfying the vanishing mean oscillation property, and then presented some regularity results concerning with the divergence form elliptic equation $\mathcal{L}u = \operatorname{div} f$ and applying the fractional integral operators (see also [19, 38, 49]).

To know more information about the pioneers in fractional differential operators, Lacroix [25] observed that $\frac{d^m}{dx^m} x^n = \frac{n!}{(n-m)!} x^{n-m}$ for $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $n \geq m$. He write the latter derivative in terms of the Γ -function in the form $\frac{d^m}{dx^m} x^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m}$ and then set $m = \frac{1}{2}$. About 50 years later, Grünwald [18] defined the differentiation $\frac{d^q}{[d(x-a)]^q}$ based on the infinite series where q is arbitrary. The fractional integral of an arbitrary order is a generalization of the ordinary n th order integral ($n \in \mathbb{N}$). One of the most fundamental definitions of fractional integral of arbitrary order is the Riemann–Liouville fractional derivative operator which will be defined further on. This operator has novel applications in the modeling and study the neural networks [50], electrical conductivity and temperature control [44], etc., see also [8, 16, 40, 45].

In general, the classical instantaneous impulses cannot describe certain dynamics of evolution processes. For example, when we consider the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes. In fact, the above situation can be characterized by a new case of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval. It is remarkable that Hernández and O'Regan [22] and Pierri et al. [36] introduced some initial value problems for a new class of non-instantaneous impulsive differential equations to describe some certain dynamic change of evolution processes in the pharmacotherapy (as therapy using pharmaceutical drugs). Very recently, Pierri et al. [35] studied the existence of global solutions for a class of impulsive abstract differential equations with non-instantaneous impulses. As a part of their investigation, the existence of mild solutions on $\mathbb{R}_+$ and the existence of $\mathcal{S}$ -asymptotically ω -periodic mild solutions based on the Hausdorff measure of noncompactness have been established. We remark that the measure of noncompactness has been recently utilized in several papers (for example, see [2, 27]). Both integer- or fractional-order differential equations with impulses have been studied previously. One can see the monographs [1, 3, 7, 10, 11, 14, 15, 24, 26, 41, 42, 47, 48], and the references therein.

In the next section, we introduce some helpful preliminaries. In Sect. 3, we establish an existence result of mild solutions for problem (1.1) by considering an integral equation which is given in terms of probability density and semigroup. The methods of the functional analysis concerning a C_0 -semigroup of operators and some fixed point theorems are applied effectively. At the end of this section we give also an example to illustrate the application of the abstract result. Finally, in Sect. 4, we focus on the existence of $\mathcal{S}$ -asymptotically ω -periodic mild solutions.

2 Notations and auxiliary facts

This section is devoted to collecting a few auxiliary facts concerning mainly measures of noncompactness which are used throughout this paper (cf. [9]). Denote by $B(x; r)$ the closed ball centered at x and with radius r . We will write B_r to denote the ball $B(\theta; r)$ where θ is the zero element of given real Banach space $(E; \|\cdot\|)$. Let $L^p([a, b]; E)$ be the space of E -valued Bochner functions on $[a, b]$ with the norm $\|x\|_{[a, b]} = (\int_a^b \|x(s)\|^p ds)^{\frac{1}{p}}$. Suppose $C([a, b]; E)$ stands for the space of continuous functions from $[a, b]$ into E .

If X is a subset of E then symbols $\overline{X}$ and $\text{Conv } X$ denote the closure and convex closure of X , respectively. The family of all nonempty and bounded subsets of E will be indicated by $\mathfrak{M}_E$ while its subfamily consisting of all relatively compact sets is denoted by $\mathfrak{N}_E$. Following [9], we accept the following definition of a regular measure of noncompactness.

Definition 2.1 ([9]) A mapping $\mu: \mathfrak{M}_E \rightarrow \mathbb{R}^+$ is said to be a *regular measure of noncompactness* in E if it satisfies the following conditions:

- (i) $\mu(X) = 0 \iff X \in \mathfrak{N}_E$.
- (ii) $X \subset Y \implies \mu(X) \leq \mu(Y)$.
- (iii) $\mu(\text{Conv } X) = \mu(X)$.
- (iv) For all $\lambda \in [0, 1]$ and $X, Y \in \mathfrak{M}_E$,

$$\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y).$$

- (v) $\mu(\lambda X) = |\lambda| \mu(X)$ for $\lambda \in \mathbb{R}$.
- (vi) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
- (vii) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.
- (viii) If $(X_n)_{n \in \mathbb{N}}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that

$$X_{n+1} \subset X_n \quad \text{for all } n = 1, 2, \dots \text{ and } \lim_{n \rightarrow \infty} \mu(X_n) = 0,$$

then the intersection set

$$X_\infty = \bigcap_{n=1}^{\infty} X_n \quad \text{is nonempty.}$$

For a bounded, closed, and convex subset $C \subseteq E$, the mapping $T: C \subseteq E \rightarrow E$ is said to be a μ -contraction map, if there exists a positive constant $k < 1$ such that $\mu(T(W)) \leq k\mu(W)$, and is said to be μ -condensing map if $\mu(T(W)) < \mu(W)$ for any bounded closed subset $W \subseteq C$.

For our purposes, we will need the following theorem which was established by Darbo [13] in 1955 for μ -contractions, and by Sadovskii [39] in 1967 for μ -condensing mappings.

Theorem 2.2 (Darbo–Sadovskii) *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let the continuous mapping $T: C \subseteq E \rightarrow E$ be a μ -contraction map. Then T has at least one fixed point in C .*

One of the most frequently used regular measures are the so-called Hausdorff and Kuratowski measures of noncompactness (see [9]).

In what follows, we will work in the space $C([a, b]; E)$ and consider the measure of noncompactness on it similar to that for $C(\mathbb{R}^+; E)$ introduced in [30]. In order to formulate it, for $I = [a, b]$ with $a \geq 0$ let $r : I \rightarrow (0, \infty)$ be a given function and

$$\mathfrak{M}_r = \{X \subset C(I; E) : X \neq \emptyset, \|x(t)\| \leq r(t) \text{ for } x \in X, t \in I\}.$$

Denote by $\mathfrak{N}_r$ the family of all relatively compact members of $\mathfrak{M}_r$.

Fix $X \in \mathfrak{M}_r$, then for $x \in X$ and $\epsilon > 0$, denote by $\omega^I(x, \epsilon)$ the modulus of continuity of the function x on the interval I as follows:

$$\omega^I(x, \epsilon) = \{\|x(t) - x(s)\| : t, s \in I, |t - s| \leq \epsilon\}.$$

Further, we define

$$\omega^I(X, \epsilon) = \sup\{\omega^I(x, \epsilon) : x \in X\}, \quad \omega_0^I(X) = \lim_{\epsilon \rightarrow 0^+} \omega^I(X, \epsilon).$$

We observe that functions from the set $X \in \mathfrak{M}_r$ are equicontinuous on any compact interval of I if and only if $\omega_0^I(X) = 0$ for arbitrary compact interval I (see also [9]).

Suppose μ is the regular measure of noncompactness in E and let us put $\overline{\mu}^I(X) = \sup\{\mu(X(t)) : t \in I\}$. We define the function γ_R on the family $\mathfrak{M}_r$ by

$$\gamma_R(X) = \sup\left\{\frac{1}{R(b-a)}(\omega_0^I(X) + \overline{\mu}^I(X)) : \text{for any interval } I = [a, b]\right\}, \quad (2.1)$$

where $R : \mathbb{R}^+ \rightarrow (0, \infty)$ is a given function such that $r(t) \leq R(t)$ for $t \geq 0$.

Olszowy et al. [32] proved the following basic properties of the quantity γ_R :

- (i) The family $\ker \gamma_R := \{X \in \mathfrak{M}_r : \gamma_R(X) = 0\} = \mathfrak{N}_r$.
- (ii) $\gamma_R(\text{Conv}(X)) = \gamma_R(X)$.
- (iii) If $(X_n)_{n \in \mathbb{N}}$ is a sequence of closed sets from $\mathfrak{M}_r$ such that

$$X_{n+1} \subset X_n \quad \text{for all } n = 1, 2, \dots \text{ and } \lim_{n \rightarrow \infty} \gamma_R(X_n) = 0,$$

then the intersection set

$$X_\infty = \bigcap_{n=1}^{\infty} X_n \quad \text{is nonempty.}$$

For $X \in \mathfrak{M}_r$, let us denote $\int_0^t X(\tau) d\tau = \{\int_0^t x(\tau) d\tau, x \in X\}$.

Lemma 2.3 ([31]) *If all functions belonging to X are equicontinuous on any compact subset of $\mathbb{R}^+$ then*

$$\mu\left(\int_0^t X(\tau) d\tau\right) \leq \int_0^t \mu(X(\tau)) d\tau.$$

Lemma 2.4 ([9]) *If μ is a regular measure of noncompactness then*

$$|\mu(X) - \mu(Y)| \leq \mu(B(\mathbf{0}, 1)) d_H(X, Y)$$

for any bounded subsets $X, Y \in E$, where d_H is the Hausdorff distance between X and Y .

Lemma 2.5 ([9]) *Suppose that $x \geq 1$, then*

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi x} \left(1 + \frac{1}{12x}\right) < \Gamma(1+x) < \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \left(1 + \frac{1}{12x-0.5}\right).$$

Similar to Cauchy's formula, we have the following lemma which can be easily proved by changing the integral order and some calculations.

Lemma 2.6 *If $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function and $q > 0$, then*

$$\begin{aligned} & \int_0^t (t-s_1)^{q-1} \int_0^{s_1} (s_1-s_2)^{q-1} \cdots \int_0^{s_n} (s_n-s_{n+1})^{q-1} f(s_{n+1}) ds_{n+1} ds_n \cdots ds_1 \\ &= \frac{\Gamma^{n+1}(q)}{\Gamma((n+1)q)} \int_0^t (t-s)^{(n+1)q-1} f(s) ds, \quad \text{for } t \geq 0. \end{aligned}$$

Theorem 2.7 ([12], Tikhonov fixed-point theorem) *Let V be a locally convex topological vector space. For any nonempty compact convex X in V , if the function $F: X \rightarrow X$ is continuous, then F has a fixed point in X .*

For the convenience of the reader, we recall the following generalized forms of classic concepts from [23].

Definition 2.8 ([48]) The Caputo derivative of order q for a function $f: [a, b] \rightarrow \mathbb{R}$ can be written as

$${}^C D_{a,t}^q f(t) = {}^L D_{a,t}^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(a) \right), \quad t > a, n-1 < q < n.$$

Here, the function f can be discontinuous and ${}^L D_{a,t}^q$ is understood by the following definition:

Definition 2.9 ([48]) For a function f given on the interval $[a, b]$, the q th Riemann–Liouville fractional order derivative of f is defined by

$${}^L (D_{a,t}^q f)(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-q-1} f(s) ds,$$

where Γ is the gamma function, $n = [q] + 1$ and $[q]$ denotes the integer part of q .

Definition 2.10 ([48]) The fractional order integral of the function $f \in L^1([a, b], \mathbb{R})$ with order $q \in \mathbb{R}^+$ is defined by

$$I_{a,t}^q f(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > a.$$

We also remark that if f is an abstract function with values in E , then integrals which appear in the previous definitions are taken in Bochner's sense.

3 Existence of mild solutions

In this section, we deal with establishing the existence results via the measure of noncompactness as introduced before. To treat with the impulsive action, we consider the vector space $\mathcal{PC}(E)$ which is formed by all functions $x: [0, \infty) \rightarrow E$ such that $x(\cdot)$ is continuous at $t \neq t_i$, $x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exists for all $i \in \mathbb{N}$. For $x \in \mathcal{PC}(E)$ and $i \in \mathbb{N}_0$, we denote by $\tilde{x}_i$ the function $\tilde{x}_i \in C([t_i, t_{i+1}]; E)$ given by

$$\tilde{x}_i = \begin{cases} x(t), & t \in (t_i, t_{i+1}], \\ x(t_i^+), & t = t_i. \end{cases}$$

Inspired by the result of Zhou et al. [51], we adopt the following definition of mild solutions of problem (1.1).

Definition 3.1 A function $x \in \mathcal{PC}(E)$ is called a *mild solution* of problem (1.1) if $x(0) = x_0$, $x(t) = g_i(t, N_i(t)(y_i))$, for all $t \in (t_i, s_i]$ and each $i \in \mathbb{N}$, and x satisfies

$$\begin{aligned} x(t) &= \mathfrak{S}(t)x_0 + \int_0^t (t-s)^{q-1} \mathfrak{T}(t-s) f(s, x(s)) ds, \quad t \in [0, t_1], \\ x(t) &= \mathfrak{S}(t-s_i)x(s_i) + \int_{s_i}^t (t-s)^{q-1} \mathfrak{T}(t-s) f(s, x(s)) ds, \quad t \in [s_i, t_{i+1}], i \in \mathbb{N}, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{S}(t) &= \int_0^\infty \xi_q(\theta) T(t^q \theta) d\theta, \\ \mathfrak{T}(t) &= q \int_0^\infty \theta \xi_q(\theta) T(t^q \theta) d\theta, \\ \xi_q(\theta) &= \frac{1}{q} \theta^{-1-\frac{1}{q}} \Psi_q(\theta^{\frac{1}{q}}), \\ \Psi_q(\theta) &= \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in \mathbb{R}^+. \end{aligned} \tag{3.1}$$

To show the validity of Definition 3.1, we remark that problem (1.1) can be formulated in the equivalent integral equation as form

$$x(t) = x(s_i) + \int_{s_i}^t (t-s)^{q-1} [Ax(s) + f(s, x(s))] ds, \quad t \in [s_i, t_{i+1}], i \in \mathbb{N} \cup \{0\}.$$

Following the technique of Laplace transform applied in [51], we easily see that

$$\begin{aligned} v(\lambda) &= \frac{x(s_i)}{\lambda} + \frac{1}{\lambda^q} Av(\lambda) + \frac{1}{\lambda^q} w(\lambda) \\ &= \lambda^{q-1} (\lambda^q I - A)^{-1} x(s_i) + (\lambda^q I - A)^{-1} w(\lambda) \\ &= \lambda^{q-1} \int_0^\infty e^{-s\lambda^q} T(s) x(s_i) ds + \int_0^\infty e^{-s\lambda^q} T(s) w(\lambda) ds, \end{aligned} \tag{3.2}$$

where I is the identity operator defined on E and ν, w are the Laplace transforms given by

$$\nu(\lambda) = \int_0^\infty e^{-\lambda s} x(s) ds, \quad w(\lambda) = \int_0^\infty e^{-\lambda s} f(s, x(s)) ds.$$

Now, since the Laplace transform of one-sided stable probability density Ψ_q given by (3.1) is equal to $e^{-\lambda^q}$ for $q \in (0, 1)$ (see [28, 51]), we find

$$\begin{aligned} \nu(\lambda) &= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \Psi_q(\theta) T\left(\frac{(t-s_i)^q}{\theta^q}\right) x(s_i) d\theta \right. \\ &\quad \left. + q \int_{s_i}^t \int_0^\infty \Psi_q(\theta) T\left(\frac{(t-s)^q}{\theta^q}\right) f(s, x(s)) \frac{(t-s)^{q-1}}{\theta^q} d\theta ds \right] dt, \end{aligned}$$

which for the inverse Laplace transform implies that

$$\begin{aligned} x(t) &= \int_0^\infty \Psi_q(\theta) T\left(\frac{(t-s_i)^q}{\theta^q}\right) x(s_i) d\theta \\ &\quad + q \int_{s_i}^t \int_0^\infty \Psi_q(\theta) T\left(\frac{(t-s)^q}{\theta^q}\right) f(s, x(s)) \frac{(t-s)^{q-1}}{\theta^q} d\theta ds \\ &= \int_0^\infty \xi_q(\theta) T(t-s_i)^q x(s_i) d\theta \\ &\quad + q \int_{s_i}^t \int_0^\infty \theta(t-s)^{q-1} \xi_q(\theta) T((t-s)^q) f(s, x(s)) d\theta ds \\ &= \mathfrak{S}(t-s_i)x(s_i) + \int_{s_i}^t (t-s)^{q-1} \mathfrak{T}(t-s)f(s, x(s)) ds, \end{aligned}$$

where ξ_q , $\mathfrak{S}$ and $\mathfrak{T}$ are given in Definition 3.1. In the following, we use the symbol y_i defined by

$$y_i(t) = \begin{cases} x(t), & t \in (t_i, s_i], \\ x(t_i^+), & t = t_i. \end{cases}$$

To state and prove our main results for the existence of mild solutions of problem (1.1), we need the hypotheses as below. In what follows we use the notations $J = \bigcup_{i=0}^\infty [s_i, t_{i+1}]$ and $J' = \bigcup_{i=1}^\infty [t_i, s_i]$.

(H0) Suppose that the C_0 -semigroup $\{T(t)\}_{t \geq 0}$ generated by A is compact and there exists a constant $M > 0$ such that $M = \sup\{\|T(t)\|; t \in \mathbb{R}^+\} < \infty$.

Remark 3.2 Obviously, one can see that under condition (H0) for any fixed $t \geq 0$, $\mathfrak{S}(t)$ and $\mathfrak{T}(t)$ defined as above are linear and bounded operators (see also [46]), i.e., for any $x \in E$,

$$\|\mathfrak{S}(t)x\| \leq M\|x\|, \quad \|\mathfrak{T}(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|.$$

Moreover, using the compactness of the semigroup $\{T(t)\}_{t \geq 0}$, we obtain the fact that such operators are continuous in the uniform operator topology for $t > 0$.

- (H1) There exist constants L_{g_i} such that $\|g_i(t, x) - g_i(t, y)\| \leq L_{g_i} \|x - y\|$ for all $x, y \in E$, $t \in [t_i, s_i]$ and each $i \in \mathbb{N}$.
- (H2) The function $f : \mathbb{R}^+ \times E \rightarrow E$ satisfies the Carathéodory type conditions, i.e., $f(t, \cdot) : E \rightarrow E$ is continuous for a.e. $t \in J$ and $f(\cdot, x) : J \rightarrow E$ is strongly measurable for each $x \in E$.
- (H3) There exist locally $L^{\frac{1}{p}}$ -integrable function h and function m both from J into $\mathbb{R}^+$ ($0 < p < q$) and a non-decreasing function $\Phi_i \in C(\mathbb{R}_+, \mathbb{R}^+)$, $i \in \mathbb{N}$, such that $m\Phi_i$ is locally $L^{\frac{1}{p}}$ -integrable and

$$\|f(t, x)\| \leq m(t)\Phi_i(\|x\|) + h(t)$$

for all $x \in E$ and a.e. $t \in [s_i, t_{i+1}]$.

- (H4) $k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a measurable and essentially bounded function on the compact intervals of $\mathbb{R}^+$ such that

$$\mu(f(t, X)) \leq k(t)\mu(X)$$

for a.e. $t \in J$ and bounded subsets X of E , where μ is a regular measure of noncompactness on E .

- (H5) There exist constants $\eta_i > 0$, $i \in \mathbb{N}$ such that

$$\gamma(\{N_i(\cdot)(v) : v \in W_i\}) \leq \eta_i \gamma(W_i)$$

for every bounded set $W_i \subseteq C([t_i, s_i]; E)$ where γ denotes the regular measure of noncompactness on $C([a, b]; E)$.

From the recent condition we observe that the mappings $N_i(\cdot) : C([t_i, s_i]; E) \rightarrow C([t_i, s_i]; E)$, $i \in \mathbb{N}$, given by $(N_i(\cdot)x)(t) = N_i(t)x$ are uniformly bounded on bounded sets. For such reason, from now on, we use the notation

$$\chi_{i,r} = \sup\{\|N_i(t)(x)\| : t \in [t_i, s_i], x \in C([t_i, s_i]; E), \|x\|_\infty \leq r\}.$$

Remark 3.3 If condition (H5) holds, then the maps $\overline{N}_i(\cdot) : C([t_i, s_i]; E) \rightarrow C([t_i, s_i]; E)$, $i \in \mathbb{N}$, defined by $\overline{N}_i(y)(t) = N_i(t)(y)$, are continuous. Indeed, if $(y_n)_n$ is a sequence convergent to y in $C([t_i, s_i]; E)$, then the set $Y = \{y_n : n \in \mathbb{N}\}$ is relatively compact in $C([t_i, s_i]; E)$ which yields $\{N_i(\cdot)y_n : n \in \mathbb{N}\}$ is so. Finally, there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ where $N_i(\cdot)(y_{n_k})$ tends into $N_i(\cdot)(y)$ as $k \rightarrow \infty$ in $C([t_i, s_i]; E)$. Now, from the fact that this property is independent from the sequence $(y_{n_k})_k$, we infer that $N_i(\cdot)(y_n)$ goes to $N_i(\cdot)(y)$ as $n \rightarrow \infty$.

Now we can formulate our result of the section as follows.

Theorem 3.4 Assume that hypotheses (H0)–(H5) hold and $L_{g_i}\eta_i < 1$ for $i \in \mathbb{N}_0$. Also suppose that there exist a function $r_i(t)$ and constant $\tilde{c}_i$ such that

$$MC_i + \frac{M}{\Gamma(q)} \left(\frac{1-p}{q-p} \right)^{1-p} (t-s_i)^{q-p} \left(\|m\Phi_i(\|r_i\|)\|_{L^{\frac{1}{p}}[s_i,t]} + \|h\|_{L^{\frac{1}{p}}[s_i,t]} \right) \leq r_i(t) \quad (3.3)$$

for $t \in [s_i, t_{i+1}]$ and

$$L_{g_i} \chi_i \tilde{c}_i + \sup_{t \in [t_i, s_i]} \|g_i(t, 0)\| \leq \tilde{c}_i. \quad (3.4)$$

Then problem (1.1) has at least one mild solution $x \in \mathcal{PC}(E)$.

Proof We divide the proof into solving of two fixed point problems as follows. First, let us define

$$F_i : C([t_i, s_i]; E) \rightarrow C([t_i, s_i]; E), \quad \text{for } i \in \mathbb{N} \quad (3.5)$$

given by $F_i(y)(t) = g_i(t, N_i(t)(y))$, $t_i \leq t \leq s_i$. It follows from our main assumptions and Remark 3.3 that F_i is a continuous. Furthermore, if $y \in C([t_i, s_i]; E)$ with $\|y\|_\infty = \sup_{t \in [t_i, s_i]} \|y(t)\| \leq \tilde{c}_i$ then

$$\begin{aligned} \|F_i(y)(t)\| &\leq \|g_i(t, N_i(t)(y)) - g_i(t, 0)\| + \|g_i(t, 0)\| \\ &\leq L_{g_i} \chi_i \tilde{c}_i + \sup_{t \in [t_i, s_i]} \|g_i(t, 0)\|, \end{aligned}$$

which shows that $F_i(B_{\tilde{c}_i}(0)) \subseteq B_{\tilde{c}_i}(0)$ followed by (3.4). Moreover, for every bounded set $W \subseteq C([t_i, s_i]; E)$ with $\gamma(W) > 0$, if $\tilde{W} = \{N_i(\cdot)(v) : v \in W\}$ then, using (H1) and (H5), we obtain

$$\gamma(F_i(W)) \leq L_{g_i} \gamma(\tilde{W}) \leq L_{g_i} \eta_i \gamma(W).$$

Accordingly, F_i is a γ -contraction map, and, applying Theorem 2.2, we obtain that there is a fixed point y_i of F_i .

As the second part of the proof, we define the operator $\mathcal{F}_i : C([s_i, t_{i+1}]; E) \rightarrow C([s_i, t_{i+1}]; E)$, $i \in \mathbb{N}_0$, by

$$\begin{aligned} (\mathcal{F}_i x)(t) &= \mathfrak{G}(t - s_i) y_i(s_i) \\ &\quad + \int_{s_i}^t (t - s)^{q-1} \mathfrak{T}(t - s) f(s, x(s)) ds, \quad t \in [s_i, t_{i+1}], x \in C([s_i, t_{i+1}]; E), \end{aligned} \quad (3.6)$$

where $y_0(s_0) = x_0$. Using (H2), since the function $s \rightarrow f(s, x(s))$ is integrable on $[s_i, t_{i+1}]$, $\mathcal{F}_i$ is well-defined. We shall show that there exists a function $r_i : [s_i, t_{i+1}] \rightarrow \mathbb{R}$ such that if $x \in C([s_i, t_{i+1}]; E)$ and $\|x(t)\| \leq r_i(t)$ for $t \in [s_i, t_{i+1}]$, then

$$\|(\mathcal{F}_i x)(t)\| \leq r_i(t). \quad (3.7)$$

In fact, if we choose $r_i(t)$ as a solution of inequality (3.3), then from the hypotheses we have

$$\|(\mathcal{F}_i x)(t)\| \leq \|\mathfrak{G}(t - s_i) y_i(s_i)\| + \left\| \int_{s_i}^t (t - s)^{q-1} \mathfrak{T}(t - s) f(s, x(s)) ds \right\|$$

$$\begin{aligned}
&= \left\| \int_0^\infty \xi_q(\theta) T((t-s_i)^q \theta) y_i(s_i) d\theta \right\| \\
&\quad + \left\| q \int_{s_i}^t (t-s)^{q-1} \int_0^\infty \theta \xi_q(\theta) T((t-s)^q \theta) d\theta f(s, x(s)) ds \right\|,
\end{aligned}$$

which, together with the fact that $\xi_q(\theta)$ is the probability density function defined on $\mathbb{R}^+$, implies that

$$\begin{aligned}
\|(\mathcal{F}_i x)(t)\| &\leq M \|y_i(s_i)\| + \frac{M}{\Gamma(q)} \left(\int_{s_i}^t (t-s)^{\frac{q-1}{1-p}} ds \right)^{1-p} (\|m\Phi_i(\|r_i\|)\|_{L^{\frac{1}{p}}[s_i, t]} + \|h\|_{L^{\frac{1}{p}}[s_i, t]}) \\
&= M \|y_i(s_i)\| + \frac{M}{\Gamma(q)} \left(\frac{1-p}{q-p} \right)^{1-p} (t-s_i)^{q-p} (\|m\Phi_i(\|r_i\|)\|_{L^{\frac{1}{p}}[s_i, t]} + \|h\|_{L^{\frac{1}{p}}[s_i, t]}) \\
&\leq r_i(t)
\end{aligned}$$

for $t \in [s_i, t_{i+1}]$. Let us fix $x \in C([s_i, t_{i+1}], E)$ such that $x(t) \leq r_i(t)$. We will estimate the modulus of continuity of the function $\mathcal{F}_i x$. Fix arbitrary $T_i \geq s_i$ and $\epsilon > 0$ and take $t_{i1}, t_{i2} \in [s_i, T_i]$ such that $|t_{i1} - t_{i2}| \leq \epsilon$. Without loss of generality, we assume that $t_{i2} \geq t_{i1}$, then

$$\begin{aligned}
&\|(\mathcal{F}_i x)(t_{i2}) - (\mathcal{F}_i x)(t_{i1})\| \\
&= \left\| \mathfrak{S}(t_{i2} - s_i) y_i(s_i) + \int_{s_i}^{t_{i2}} (t_{i2} - s)^{q-1} \mathfrak{T}(t_{i2} - s) f(s, x(s)) ds \right. \\
&\quad \left. - \mathfrak{S}(t_{i1} - s_i) y_i(s_i) - \int_{s_i}^{t_{i1}} (t_{i1} - s)^{q-1} \mathfrak{T}(t_{i1} - s) f(s, x(s)) ds \right\| \\
&\leq \left\| (\mathfrak{S}(t_{i2} - s_i) - \mathfrak{S}(t_{i1} - s_i)) y_i(s_i) \right\| + \left\| \int_{t_{i1}}^{t_{i2}} (t_{i2} - s)^{q-1} \mathfrak{T}(t_{i2} - s) f(s, x(s)) ds \right\| \\
&\quad + \left\| \int_{s_i}^{t_{i1}} [(t_{i2} - s)^{q-1} - (t_{i1} - s)^{q-1}] \mathfrak{T}(t_{i2} - s) f(s, x(s)) ds \right\| \\
&\quad + \left\| \int_{s_i}^{t_{i1}} (t_{i1} - s)^{q-1} [\mathfrak{T}(t_{i2} - s) - \mathfrak{T}(t_{i1} - s)] f(s, x(s)) ds \right\| \\
&\leq \omega^{T_i}(\mathfrak{S}, \epsilon) \|y_i(s_i)\| + \frac{M}{\Gamma(q)} \int_{t_{i1}}^{t_{i2}} (t_{i2} - s)^{q-1} (m(s) \Phi_i(\|x(s)\|) + h(s)) ds \\
&\quad + \frac{M}{\Gamma(q)} \int_{s_i}^{t_{i1}} [(t_{i1} - s)^{q-1} - (t_{i2} - s)^{q-1}] (m(s) \Phi_i(\|x(s)\|) + h(s)) ds \\
&\quad + \int_{s_i}^{t_{i1}} (t_{i1} - s)^{q-1} \|\mathfrak{T}(t_{i2} - s) - \mathfrak{T}(t_{i1} - s)\| (m(s) \Phi_i(\|x(s)\|) + h(s)) ds.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\|(\mathcal{F}_i x)(t_{i2}) - (\mathcal{F}_i x)(t_{i1})\| \\
&\leq \omega^{T_i}(\mathfrak{S}, \epsilon) \|y_i(s_i)\| + \frac{M}{\Gamma(q)} \left(\frac{1-p}{q-p} \right)^{1-p} \epsilon^{q-p} (\|m\Phi_i(\|r_i\|)\|_{L^{\frac{1}{p}}[t_{i1}, t_{i2}]} + \|h\|_{L^{\frac{1}{p}}[t_{i1}, t_{i2}]}) \\
&\quad + \frac{M}{\Gamma(q)} \left(\int_{s_i}^{t_{i1}} [(t_{i1} - s)^{q-1} - (t_{i2} - s)^{q-1}]^{\frac{1}{1-p}} ds \right)^{1-p}
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(\|m\Phi_i(\|r_i\|)\|_{L^{\frac{1}{p}}[s_i, t_{i_1}]} + \|h\|_{L^{\frac{1}{p}}[s_i, t_{i_1}]} \right) \\
& + v^{T_i}(\mathfrak{T}, \epsilon) \int_{s_i}^{t_{i_1}} (t_{i_1} - s)^{q-1} (m(s)\Phi_i(\|x(s)\|) + h(s)) ds \\
& \leq \omega^{T_i}(\mathfrak{S}, \epsilon) \|y_i(s_i)\| + \frac{M\Omega_i}{\Gamma(q)} \left(\frac{1-p}{q-p} \right)^{1-p} \epsilon^{q-p} \\
& + \frac{M}{\Gamma(q)} \left(\frac{1-p}{q-p} \right)^{1-p} \epsilon^{q-p} \left(\|m\Phi_i(\|r_i\|)\|_{L^{\frac{1}{p}}[s_i, T_i]} + \|h\|_{L^{\frac{1}{p}}[s_i, T_i]} \right) \\
& + v^{T_i}(\mathfrak{T}, \epsilon) \left(\frac{1-p}{q-p} \right)^{1-p} (T_i - s_i)^{q-p} \left(\|m\Phi_i(\|r_i\|)\|_{L^{\frac{1}{p}}[s_i, T_i]} + \|h\|_{L^{\frac{1}{p}}[s_i, T_i]} \right) \\
& := \Lambda(T_i, \epsilon),
\end{aligned}$$

where

$$\begin{aligned}
\Omega_i &= \sup \left\{ \left(\|m\Phi_i(\|r_i\|)\|_{L^{\frac{1}{p}}[t_{i_1}, t_{i_2}]} + \|h\|_{L^{\frac{1}{p}}[t_{i_1}, t_{i_2}]} \right) : s_i \leq t_{i_1} \leq t_{i_2} \leq T_i \right\}, \\
\omega^{T_i}(\mathfrak{S}, \epsilon) &= \sup \left\{ \left\| \mathfrak{S}(t_{i_2}) - \mathfrak{S}(t_{i_1}) \right\| : t_{i_1}, t_{i_2} \in [0, T_i - s_i], |t_{i_1} - t_{i_2}| \leq \epsilon \right\}, \\
v^{T_i}(\mathfrak{T}, \epsilon) &= \sup \left\{ \left\| \mathfrak{T}(t_{i_2}) - \mathfrak{T}(t_{i_1}) \right\| : t_{i_1}, t_{i_2} \in [0, T_i - s_i], |t_{i_1} - t_{i_2}| \leq \epsilon \right\}.
\end{aligned}$$

Thus, we obtain

$$\|(\mathcal{F}_i x)(t_{i_2}) - (\mathcal{F}_i x)(t_{i_1})\| \leq \Lambda(T_i, \epsilon) \quad (3.8)$$

for x such that $x(t) \leq r_i(t)$. From Remark 3.2, we have

$$\lim_{\epsilon \rightarrow 0^+} \Lambda(T_i, \epsilon) = 0, \quad \text{for } T_i \geq s_i. \quad (3.9)$$

Now define the subset B_i of $C([s_i, t_{i+1}]; E)$ as follows:

$$B_i = \{x \in C([s_i, t_{i+1}]; E) : \|x(t)\| \leq r_i(t), \omega^{T_i}(x, \epsilon) \leq \Lambda(T_i, \epsilon) \text{ for } t, T_i \geq s_i \text{ and } \epsilon \geq 0\}.$$

In view of $x(t) \equiv My_i(s_i) \in B_i$, we see that B_i is nonempty. Moreover, B_i is a closed and convex subset of $C([s_i, t_{i+1}]; E)$. From (3.9) and using the definition of $\omega_0^{T_i}$ as mentioned in the previous section, we find that the set B_i is the family consisting of functions equicontinuous on compact intervals $[s_i, t_{i+1}]$. By (3.7) we observe that $\mathcal{F}_i$ maps B_i into itself. Next, we will show that $\mathcal{F}_i : B_i \rightarrow B_i$ is continuous. For $x, \hat{x}_n \in B_i$ such that $\lim_{n \rightarrow \infty} \hat{x}_n = x$ in $C([s_i, t_{i+1}]; E)$, we have

$$\lim_{n \rightarrow \infty} \sup_{s_i \leq t \leq T_i} \|\hat{x}_n(t) - x(t)\| = 0, \quad T_i \geq s_i.$$

Now fix $T_i \geq s_i$. Then we get

$$\begin{aligned}
\sup_{s_i \leq t \leq T_i} \|(\mathcal{F}_i \hat{x}_n)(t) - (\mathcal{F}_i x)(t)\| &\leq \sup_{s_i \leq t \leq T_i} \int_{s_i}^t (t-s)^{q-1} \|\mathfrak{T}(t-s)[f(s, \hat{x}_n(s)) - f(s, x(s))]\| ds \\
&\leq \frac{M}{\Gamma(q)} \sup_{s_i \leq t \leq T_i} \int_{s_i}^t (t-s)^{q-1} \|f(s, \hat{x}_n(s)) - f(s, x(s))\| ds.
\end{aligned}$$

This means that $\lim_{n \rightarrow \infty} \mathcal{F} \hat{x}_n = \mathcal{F} x$ in $C([s_i, t_{i+1}]; E)$, which is implied by the Lebesgue dominated convergence theorem together with (H2).

Let us consider $B_{i_0} = B_i$, $B_{i_n} = \text{Conv } \mathcal{F}(B_{i_{n-1}})$ for $n = 1, 2, \dots$. Then all sets of this sequence are nonempty, closed, and convex. Moreover, $B_{i_{n+1}} \subseteq B_{i_n}$ for $n = 0, 1, \dots$. By the equicontinuity of the set B_i on compact intervals, we have

$$\omega_0^{T_i}(B_{i_n}) = 0 \quad \text{for } n = 0, 1, \dots, \text{ and } T_i \geq s_i.$$

Put $a_{i_n}(t) = \mu(B_{i_n}(t))$. From Lemma 2.4 and (3.8), we get

$$|a_{i_n}(t) - a_{i_n}(s)| \leq \mu(B(\mathbf{0}, 1)) \Lambda(T_i, |t - s|),$$

which, together with (3.9), yields the continuity of $a_{i_n}(t)$ on $[s_i, t_{i+1}]$. Focusing on the properties of μ and Lemma 2.3, together with assumption (H4), we obtain

$$\begin{aligned} a_{i_{n+1}}(t) &= \mu((\text{Conv } \mathcal{F} B_{i_n})(t)) \\ &= \mu\left(\int_{s_i}^t (t-s)^{q-1} \mathfrak{T}(t-s) f(s, B_{i_n}(s)) ds\right) \\ &= M \int_{s_i}^t (t-s)^{q-1} k(s) a_{i_n}(s) ds \\ &\leq M k_i(t) \int_{s_i}^t (t-s)^{q-1} a_{i_n}(s) ds, \end{aligned}$$

where $k_i(t) = \text{ess sup}\{k(s) : s_i \leq s \leq t\}$; clearly, $k_i(t)$ is nondecreasing. Using the mathematical induction and Lemma 2.6, we have

$$\begin{aligned} a_{i_{n+1}}(t) &\leq M k_i(t) \int_{s_i}^t (t-s)^{q-1} a_{i_n}(s) ds \\ &\leq M^2 k_i^2(t) \int_{s_i}^t (t-r_1)^{q-1} \int_{s_i}^{r_1} (r_1-r_2)^{q-1} a_{i_{n-1}}(r_2) dr_2 dr_1 \\ &\leq M^{n+1} k_i^{n+1}(t) \int_{s_i}^t (t-r_1)^{q-1} \int_{s_i}^{r_1} (r_1-r_2)^{q-1} \times \dots \\ &\quad \times \int_{s_i}^{r_n} (r_n-r_{n+1})^{q-1} a_{i_0}(r_{n+1}) dr_{n+1} dr_n \dots dr_1 \\ &\leq M^{n+1} k_i^{n+1}(t) \frac{\Gamma^{n+1}(q)}{\Gamma((n+1)q)} \int_{s_i}^t (t-s)^{(n+1)q-1} a_{i_0}(s) ds. \end{aligned}$$

Thus for $n \geq \frac{1-q}{q}$ we have

$$a_{i_{n+1}}(t) \leq M^{n+1} k_i^{n+1}(t) \frac{\Gamma^{n+1}(q)}{\Gamma((n+1)q)} t^{(n+1)q-1} \int_{s_i}^t a_{i_0}(s) ds. \quad (3.10)$$

Now we utilize the measure of noncompactness γ_R defined in $C([s_i, t_{i+1}]; E)$ by formula (2.1), where

$$R_i(t) = r_i(t) \left(1 + (M\Gamma(q)k_i(t))^{\frac{1}{q}}\right) \left(1 + \int_{s_i}^t a_{i_0}(s) ds\right) e^{(M\Gamma(q)k_i(t))^{\frac{1}{q}} t}.$$

Clearly, $r_i(t) \leq R_i(t)$ and, using inequality (3.10), for $n \geq \frac{1-q}{q}$ we get

$$\bar{\mu}^{T_i}(B_{i_{n+1}}) = \sup_{s_i \leq t \leq T_i} a_{i_{n+1}}(t) \leq M^{n+1} k_i^{n+1}(T_i) \frac{\Gamma^{n+1}(q)}{\Gamma((n+1)q)} T_i^{(n+1)q-1} \int_{s_i}^{T_i} a_{i_0}(s) ds$$

and also

$$\begin{aligned} \frac{\bar{\mu}^{T_i}(B_{i_{n+1}})}{R_i(T_i)} &= \frac{(M\Gamma(q)k_i(T_i))^{n+1-\frac{1}{q}} T_i^{(n+1)q-1}}{r_i(T_i)\Gamma((n+1)q)e^{(M\Gamma(q)k_i(T_i))^{\frac{1}{q}} T_i}} \\ &= \frac{((M\Gamma(q)k_i(T_i))^{\frac{1}{q}} T_i)^{(n+1)q-1}}{r_i(T_i)\Gamma((n+1)q)e^{(M\Gamma(q)k_i(T_i))^{\frac{1}{q}} T_i}}. \end{aligned}$$

On the other hand, from the estimate $\sup_{a \geq 0} \{\frac{a^n}{e^a}\} \leq \frac{n^n}{e^n}$ we infer

$$\frac{\bar{\mu}^{T_i}(B_{i_{n+1}})}{R_i(T_i)} \leq \frac{(n+1)q-1}{r_i(T_i)\Gamma((n+1)q)e^{(n+1)q-1}}.$$

Then from Lemma 2.5, for $n > \max\{1, \frac{1-q}{q}\}$ we infer

$$\frac{\bar{\mu}^{T_i}(B_{i_{n+1}})}{R_i(T_i)} < \frac{1}{r_i(T_i)\sqrt{2\pi((n+1)q-1)}}.$$

Therefore, we get

$$\lim_{n \rightarrow \infty} \gamma_{R_i}(B_{i_{n+1}}) = \lim_{n \rightarrow \infty} \sup_{s_i \leq T_i \leq t_{i+1}} \left\{ \frac{1}{R_i(T_i)} (\omega_0^{T_i}(B_{i_{n+1}}), \bar{\mu}^{T_i}(B_{i_{n+1}})) \right\} = 0.$$

In view of the properties of γ_{R_i} , we get $B_{i_\infty} = \bigcap_{n=0}^\infty B_{i_n} \neq \emptyset$. Since $0 \leq \gamma_{R_i}(B_{i_\infty}) \leq \lim_{n \rightarrow \infty} \gamma_{R_i}(B_{i_n})$, we have $\gamma_{R_i}(B_{i_\infty}) = 0$, which yields that B_{i_∞} is a compact subset in $C([s_i, t_{i+1}]; E)$.

Consider $\mathcal{F}_i : B_{i_\infty} \rightarrow B_{i_\infty}$. From the above arguments, we see that all the conditions of the Tikhonov fixed-point theorem are satisfied. Therefore $\mathcal{F}_i$ has at least one fixed point x_i in B_{i_∞} , which is the mild solution of Eq. (3.6). Now using this inductive procedure, we are led to define

$$x(t) = \begin{cases} x_i(t), & t \in [s_i, t_{i+1}), i \in \mathbb{N}_0, \\ y_i(t), & t \in [t_i, s_i], i \in \mathbb{N}. \end{cases} \quad (3.11)$$

It is easily seen that $x \in \mathcal{PC}(E)$ is a mild solution of problem (1.1). $\square$

Now, we give a simple example to illustrate the feasibility of the assumptions made before.

Example 3.5 Let $B^n \subseteq \mathbb{R}^n$ be an n -ball bounded by S^{n-1} as $(n-1)$ -sphere in n -dimensional Euclidean space $\mathbb{R}^n$. Consider a fractional initial/boundary value Cauchy problem of the

form

$$\begin{cases} {}^C D_{0,t}^q u(t, z) = u_{zz}(t, z) + f(t, u(t, z)), & (t, z) \in J \times B^n, \\ u(t, z) = p_i(t, q_i(t, u(\cdot, z))), & (t, z) \in I_i \times B^n, \\ u(t, z) = 0, & (t, z) \in J \times S^{n-1}, \\ u(0, z) = u_0, & z \in B^n, \end{cases} \quad (3.12)$$

where $I_i = (t_i, s_i]$, $i \in \mathbb{N}$, $J = \bigcup_{i=0}^{\infty} [s_i, t_{i+1}]$, ${}^C D_{0,t}^q$ is the Caputo fractional partial derivative of order $0 < q < 1$, and f is a given function.

Assume $E = L^2(B^n)$, we define an operator $Au = \frac{\partial^2 u}{\partial z^2}$ on E with the domain

$$D(A) = \left\{ u \in E \mid u, \frac{\partial u}{\partial z} \text{ are absolutely continuous, } \frac{\partial^2 u}{\partial z^2} \in E, \text{ and } u \equiv 0 \text{ on } S^{n-1} \right\}.$$

Then, operator A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ which is compact and analytic with $\|T(t)\| \leq e^{-t}$ for all $t \geq 0$. Therefore, system (3.12) can be reformulated in E as follows:

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + f(t, x(t)), & t \in J, \\ x(t) = p_i(t, q_i(t, x(t))), & t \in I_i, \\ x(t) = 0, & t \in J, \\ x(0) = u_0, \end{cases}$$

where $x(t) = u(t, \cdot)$, that is, $x(t) = u(t, z)$ for all $z \in B^n$. Let us choose $q = \frac{1}{2}$, $f(t, x(t)) = \sqrt[3]{t} \sin(\arctan |x(t)|) + \sqrt[3]{\ln(t+1)}$ and $g_i(t, x(t)) = \frac{e^t \sin tx(t)}{2^i + t + \cos tx(t)}$. Firstly, following the argument as above, we see that (H0) is satisfied. Also, for mapping g_i we have

$$\begin{aligned} \|g_i(t, x) - g_i(t, y)\| &= \left\| \frac{e^t \sin tx}{2^i + t + \cos tx} - \frac{e^t \sin ty}{2^i + t + \cos ty} \right\| \\ &= e^t \left\| \frac{\sin t(x-y) + (2^i + t)(\sin tx - \sin ty)}{(2^i + t + \cos tx)(2^i + t + \cos ty)} \right\| \\ &\leq \frac{e^{s_i}}{(2^i + t_i - 1)^2} (s_i \|x - y\| + s_i (2^i + s_i) \|x - y\|) \\ &:= L_{g_i} \|x - y\| \end{aligned}$$

for all $t \in [t_i, s_i]$, $i \in \mathbb{N}$ and $x, y \in E$ where

$$L_{g_i} = \frac{s_i e^{s_i} (2^i + s_i + 1)}{(2^i + t_i - 1)^2}.$$

This shows that (H1) holds. On the other hand, considering the given function f , we claim that it satisfies the Carathéodory type conditions and

$$\begin{aligned} \left\| \sqrt[3]{t} \sin(\arctan |x(t)|) + \sqrt[3]{\ln(t+1)} \right\| &\leq \sqrt[3]{t} \arctan(\|x\|) + \sqrt[3]{\ln(t+1)} \\ &:= m(t) \Phi_i(\|x\|) + h(t) \end{aligned}$$

for all $x \in E$ and a.e. $t \in [s_i, t_{i+1}]$. This means that (H2) and (H3) are satisfied. Moreover, from the fact

$$\begin{aligned} \|f(t, x(t)) - f(t, y(t))\| &\leq \sqrt[3]{t} \|\arctan|x(t)| - \arctan|y(t)|\| \\ &\leq \sqrt[3]{t} \|x - y\|_\infty, \end{aligned}$$

we conclude that (H4) is also valid. Since N_i defined in (1.1) is considered as the trivial mapping, then hypothesis (H5) is also fulfilled, and our above result can be used to derive the existence of $\mathcal{PC}$ -mild solutions to problem (3.12) under the considerations as mentioned.

4 $\mathcal{S}$ -Asymptotically ω -periodic solutions

Throughout this section we investigate the existence of $\mathcal{S}$ -asymptotically ω -periodic mild solutions for (1.1). Concentrating on the theory of $\mathcal{S}$ -asymptotically ω -periodic functions, we invite the reader to see the recent papers [20, 21, 29, 34–37]. Next, we need to adapt the concept of $\mathcal{S}$ -asymptotically ω -periodic function introduced in the cited works to include piecewise continuous functions. Initially, we recall the concept of $\mathcal{S}$ -asymptotically ω -periodic function and gather some related definitions (see also [35]).

From now on, by $C_b([0, \infty); E)$ and $\mathcal{PC}_b$ we denote all bounded continuous functions from $\mathbb{R}_+$ into E and the subspace of $\mathcal{PC}(E)$ consisting of all bounded functions endowed with the norm of uniform convergence symbolized by $\|\cdot\|_{\mathcal{PC}}$, respectively. It is well-known that $\mathcal{PC}_b(E)$ is a Banach space.

Definition 4.1 A function $x \in C_b([0, \infty); E)$ is said to be $\mathcal{S}$ -asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t \rightarrow \infty} [x(t + \omega) - x(t)] = 0$. In this case, we say that $x(\cdot)$ is an $\mathcal{S}$ -asymptotically ω -periodic function.

In what follows, $\text{SAP}_\omega(X)$ stands for the space including all E -valued $\mathcal{S}$ -asymptotically ω -periodic functions provided with the norm $\|\cdot\|_{C_b([0, \infty); E)}$.

Definition 4.2 We say that a function $x \in \mathcal{PC}_b$ is $\mathcal{IS}$ -asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t \rightarrow \infty} [x(t + \omega) - x(t)] = 0$. In this case, we say that ω is an asymptotic period of $x(\cdot)$ and that $x(\cdot)$ is an $\mathcal{IS}$ -asymptotically ω -periodic function.

We next use the notation $\text{ISAP}_\omega(E)$ for the space formed by all E -valued $\mathcal{S}$ -asymptotically ω -periodic functions provided with the norm $\|\cdot\|_{\mathcal{PC}(E)}$. It is not difficult to see that $\text{ISAP}_\omega(E)$ is a Banach space,

Definition 4.3 A continuous function $\varphi : [0, \infty) \times E \rightarrow E$ is said to be uniformly $\mathcal{S}$ -asymptotically ω -periodic on bounded sets if for every bounded subset K of E , the set $\{\varphi(t, x) : t \geq 0, x \in K\}$ is bounded and $\lim_{t \rightarrow \infty} [\varphi(t, x) - \varphi(t + \omega, x)] = 0$ uniformly for $x \in K$.

In the remainder of this section, we always assume that there is a $k \in \mathbb{N}$ such that the impulsive points s_i, t_i satisfy that $s_i - t_i = t_{i+1} - s_i = \omega 2^{-k}$ for all $i \in \mathbb{N}_0$. Motivated by the previous concept, we give the next definitions which are also needed some notations to simplify the text. Let us define $g : [0, \infty) \times E \rightarrow E$ as $g(t, x) = g_i(t, x)$ for $t \in [t_i, s_i]$, $g_0(0, x) =$

x_0 , and

$$g(t, x) = \frac{t_{i+1} - t}{t_{i+1} - s_i} g_i(s_i, x) + \frac{t - s_i}{t_{i+1} - s_i} g_{i+1}(t_{i+1}, x)$$

for all $t \in [s_i, t_{i+1}]$ and $i \in \mathbb{N}_0$. It is clear that g is continuous. Suppose that $x \in \mathcal{PC}(E)$ and $i \in \mathbb{N}$. We denote by $x_i \in C([t_i, s_i]; E)$ the function given by $x_i(t) = x(t)$ for $t \in (t_i, s_i]$ and $x_i(t_i) = \lim_{t \rightarrow t_i^+} x(t)$. We define $N(t) : \mathcal{PC}(E) \rightarrow E$ by $N(t)(x) = N_i(t)(x_i)$ for $t \in [t_i, s_i]$, and

$$N(t)(x) = \frac{t_{i+1} - t}{t_{i+1} - s_i} N_i(s_i)(x) + \frac{t - s_i}{t_{i+1} - s_i} N_{i+1}(t_{i+1})(x)$$

for all $t \in [s_i, t_{i+1}]$ and $i \in \mathbb{N}_0$. Here, we set $N(0) = 0$.

Definition 4.4 We say that the family of functions $(g_i)_{i \in \mathbb{N}}$ is *uniformly $\mathcal{IS}$ -asymptotically ω -periodic* on bounded sets if g is uniformly $\mathcal{S}$ -asymptotically ω -periodic on bounded sets.

Finally, we also should consider the following concept.

Definition 4.5 The family $(N_i)_{i \in \mathbb{N}}$ is said to be *$\mathcal{IS}$ -asymptotically ω -periodic* if the set $\{N(t)(x) : t \geq 0\}$ is bounded and $[N(t + \omega)(x) - N(t)(x)] \rightarrow 0$ as $t \rightarrow \infty$ for each $x \in \text{ISAP}_\omega(E)$.

In our next results we consider the following Lipschitz conditions:

(H6) For the bounded linear operator T generated by the infinitesimal generator A , there exist constants $M \geq 1$ and $\sigma \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\sigma t}$ for all $t \geq 0$ and

$$\eta_{ij} = \sup_{t \in [s_i, t_{i+1}]} \int_0^\infty \theta^j |\xi_q(\theta)| \exp(\sigma(t - s_i)^q \theta) d\theta < \infty \quad \text{for every } i \in \mathbb{N}_0 \text{ and } j = 0, 1.$$

(H7) There is a function $L_f \in L^1_{\text{loc}}([0, \infty); \mathbb{R}^+)$ such that $\|f(t, x) - f(t, y)\| \leq L_f(t)\|x - y\|$ for all $x, y \in E$ and every $t \geq 0$.

(H8) There are constants $a_i \geq 0$ such that

$$\|N_i(t)(v_2) - N_i(t)(v_1)\| \leq a_i \|v_2 - v_1\|_\infty, \quad t \in [t_i, s_i], v_2, v_1 \in C([t_i, s_i]; E),$$

(H9)

$$\xi = \sup_{i \in \mathbb{N}} \sup_{t_i \leq t \leq s_i} [L_{g_i} \|N_i(t, 0)\| + \|g_i(t, 0)\|] < \infty,$$

(H10)

$$l = \sup_{i \in \mathbb{N}_0} \sup_{s_i \leq t \leq t_{i+1}} \int_{s_i}^{t_{i+1}} (t - s)^{q-1} L_f(s) ds < \infty.$$

Remark 4.6 We notice that condition (H6) is tangibly weaker than condition (H0) (for more details on semigroup theory, we refer the reader to [33]). Moreover, setting $\eta^{(j)} = \sup_{i \in \mathbb{N}_0} \eta_{ij}$, we conclude that

$$\eta^{(j)} = \begin{cases} \int_0^\infty \theta^j |\xi_q(\theta)| d\theta, & \sigma \leq 0, \\ \int_0^\infty \theta^j |\xi_q(\theta)| \exp(\frac{\omega^q \sigma \theta}{2kq}) d\theta, & \sigma > 0, \end{cases}$$

for $j = 0, 1$.

Theorem 4.7 Suppose that f is continuous and conditions (H1) and (H6)–(H10) are satisfied. Let $f(\cdot)$ be uniformly $\mathcal{S}$ -asymptotically ω -periodic on bounded sets, the family $(g_i)_{i \in \mathbb{N}}$ be uniformly $\mathcal{IS}$ -asymptotically ω -periodic on bounded sets, and the family $(N_i)_{i \in \mathbb{N}}$ be $\mathcal{IS}$ -asymptotically ω -periodic. If $\Delta = \sup_{i \in \mathbb{N}} L_{g_i}$ and $\eta^{(j)}$ are finite and $\tau = M\eta^{(0)} \sup_{i \in \mathbb{N}} L_{g_i} a_i < 1$, then there exists a unique $\mathcal{IS}$ -asymptotically ω -periodic mild solution of problem (1.1).

Proof In order to prove the claim, let us define $\mathcal{F}$ on $\mathcal{PS}^0(E) = \{x \in \mathcal{PS}(E) : x(0) = x_0\}$ by $\mathcal{F}x(t) = F_i x(t)$, for $t \in (t_i, s_i]$, $i \in \mathbb{N}$, and $\mathcal{F}x(t) = \mathcal{F}_i x(t)$, for $t \in [s_i, t_{i+1}]$, $i \in \mathbb{N}_0$, where the mappings F_i are defined by (3.5) and

$$(\mathcal{F}_i x)(t) = \mathfrak{S}(t - s_i) F_i(x)(s_i) + \int_{s_i}^t (t - s)^{q-1} \mathfrak{T}(t - s) f(s, x(s)) ds, \quad \text{for } t \in [s_i, t_{i+1}].$$

It is obvious that $\mathcal{F} : \mathcal{PS}^0(E) \rightarrow \mathcal{PS}^0(E)$. We divide the rest of proof into several steps:

Step 1. In what follows we will show that $\mathcal{F}$ takes bounded functions to bounded ones. Suppose that $x \in \mathcal{PS}^0(E)$ is a bounded function. For $t \in (t_i, s_i]$, $i \in \mathbb{N}$, from the hypotheses on τ and Δ , and applying (H7), we get

$$\begin{aligned} \|g_i(t, N_i(t)(x))\| &\leq \|g_i(t, N_i(t)(x)) - g_i(t, 0)\| + \|g_i(t, 0)\| \\ &\leq L_{g_i} \|N_i(t)(x)\| + \|g_i(t, 0)\| \\ &\leq L_{g_i} a_i \|x\|_\infty + L_{g_i} \|N_i(t)(0)\| + \|g_i(t, 0)\| \\ &\leq \tau \|x\|_\infty + \xi, \end{aligned}$$

which yields that $\{\mathcal{F}x(t) : t \in J'\}$ is a bounded set. Similarly, for $[s_i, t_{i+1}]$, $i \in \mathbb{N}_0$, we infer

$$\begin{aligned} \|\mathcal{F}x(t)\| &= \left\| \mathfrak{S}(t - s_i) F_i(x)(s_i) + \int_{s_i}^t (t - s)^{q-1} \mathfrak{T}(t - s) f(s, x(s)) ds \right\| \\ &\leq M\eta^{(0)} \|x(s_i)\| + q \int_0^\infty \int_{s_i}^t \theta |\xi_q(\theta)| \exp(\sigma(t - s)^q \theta) (t - s)^{q-1} \|f(s, x(s))\| ds d\theta \\ &\leq M\eta^{(0)} \|x(s_i)\| + q\eta^{(1)} \int_{s_i}^t (t - s)^{q-1} L_f(s) \|x(s)\| ds \\ &\quad + q\eta^{(1)} \int_{s_i}^t (t - s)^{q-1} \|f(s, 0)\| ds \end{aligned}$$

and, using (H9) and the fact that $f(t, 0)$ is bounded on $\mathbb{R}_+$, we have that $\{\mathcal{F}x(t) : t \in J\}$ is a bounded set. Therefore, we can consider $\mathcal{F} : \mathcal{PS}_b^0(E) \rightarrow \mathcal{PS}_b^0(E)$.

Step 2. In this step we show that $\mathcal{F}$ is Lipschitz continuous on $\mathcal{PS}_b^0(E)$, and that there exists $n \in \mathbb{N}$ such that $\mathcal{F}^n$ is a contraction. As $M\eta^{(0)} \geq 1$, so $\gamma := \frac{\tau}{M\eta^{(0)}} < 1$, then for $x, y \in \mathcal{PS}_b^0(E)$ if $t \in [t_i, s_i]$, $i \in \mathbb{N}$ we obtain

$$\|\mathcal{F}x(t) - \mathcal{F}y(t)\| \leq L_{g_i} a_i \sup_{t_i \leq t \leq s_i} \|x(t) - y(t)\| \leq \gamma \|x - y\|_\infty.$$

Instantly, we get

$$\|\mathcal{F}^n x - \mathcal{F}^n y\|_\infty \leq \gamma^n \|x - y\|_\infty, \quad \text{for all } n \in \mathbb{N}.$$

Also, if $t \in [s_i, t_{i+1}]$, $i \in \mathbb{N}_0$, then

$$\begin{aligned} & \|\mathcal{F}x(t) - \mathcal{F}y(t)\| \\ & \leq M\eta_{i0} \|F_i(x)(s_i) - F_i(y)(s_i)\| + \int_{s_i}^t (t-s)^{q-1} \mathfrak{T}(t-s) L_f(s) \|x(s) - y(s)\| ds \\ & \leq M\gamma \eta^{(0)} \|x - y\|_\infty \\ & \quad + q \int_0^\infty \int_{s_i}^{t_{i+1}} \theta |\xi_q(\theta)| \exp(\sigma(t-s)^q \theta) (t-s)^{q-1} L_f(s) \|x(s) - y(s)\| ds d\theta \\ & \leq M\gamma \eta^{(0)} \|x - y\|_\infty + q\eta^{(1)} \left(\int_{s_i}^{t_{i+1}} (t-s)^{q-1} L_f(s) ds \right) \|x - y\|_\infty. \end{aligned}$$

By iterating this process, one can easily obtain

$$\|\mathcal{F}^n x - \mathcal{F}^n y\|_\infty \leq \tau^n \left(\sum_{k=1}^n \frac{(\frac{L}{\tau})^k}{k!} \right) \|x - y\|_\infty,$$

where

$$\sup_{i \in \mathbb{N}_0} \sup_{s_i \leq t \leq t_{i+1}} \int_{s_i}^{t_{i+1}} (t-s)^{q-1} L_f(s) ds < \infty.$$

This, combining with the fact that $\sum_{k=1}^\infty \frac{(\frac{L}{\tau})^k}{k!} < \infty$ and hypothesis $\tau < 1$, explains that $\mathcal{F}^n$ is a contraction for n sufficiently large.

Step 3. Following the last step, in order to establish that there is an $\mathcal{IS}$ -asymptotically ω -periodic mild solution, one only needs to prove that $\text{ISAP}_\omega^0(E)$ is $\mathcal{F}$ -invariant where $\text{ISAP}_\omega^0(E) = \{x \in \text{ISAP}_\omega(E) : x(0) = x_0\}$. To show this fact, we choose $x \in \text{ISAP}_\omega(E)$ and $t \geq 0$. We consider the following two cases:

If $t \in [t_i, s_i]$, $i \in \mathbb{N}$, then, using the fact that $t + \omega \in [t_{i+1}, s_{i+1}]$, we have

$$\begin{aligned} \mathcal{F}x(t + \omega) - \mathcal{F}x(t) &= g_{i+1}(t + \omega, N_{i+1}(t + \omega)(x)) - g_i(t, N_i(t)(x)) \\ &= g(t + \omega, N(t + \omega)(x)) - g(t, N(t + \omega)(x)) \\ &\quad + g_i(t, N(t + \omega)(x)) - g_i(t, N(t)(x)). \end{aligned}$$

Now, since $\{N(t)(x) : t \geq 0\}$ is a bounded set, $g(t + \omega, N(t + \omega)(x)) - g(t, N(t + \omega)(x))$ vanishes as t tends to infinity. On the other hand,

$$\|g_i(t, N(t + \omega)(x)) - g_i(t, N(t)(x))\| \leq \sup_{i \in \mathbb{N}} L_{g_i} \|N(t + \omega)(x) - N(t)(x)\| \rightarrow 0, \quad t \rightarrow \infty.$$

Hence, we get

$$\mathcal{F}x(t + \omega) - \mathcal{F}x(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty, t \in [t_i, s_i], i \in \mathbb{N}.$$

For the case $t \in [s_i, t_{i+1}]$, since $t + \omega \in [s_{i+1}, t_{i+2}]$, we obtain

$$\begin{aligned} & \mathcal{F}x(t + \omega) - \mathcal{F}x(t) \\ &= \mathfrak{G}(t + \omega - s_{i+1})F_{i+1}(x)(s_{i+1}) - \mathfrak{G}(t - s_i)F_i(x)(s_i) \\ & \quad + \int_{s_{i+1}}^{t+\omega} (t + \omega - s)^{q-1} \mathfrak{T}(t + \omega - s) f(s, x(s)) ds - \int_{s_i}^t (t - s)^{q-1} \mathfrak{T}(t - s) f(s, x(s)) ds \\ &= \mathfrak{G}(t - s_i)(F_{i+1}(x)(s_{i+1}) - F_i(x)(s_i)) \\ & \quad + \int_{s_i}^t (t - s)^{q-1} \mathfrak{T}(t - s) [f(s + \omega, x(s + \omega)) - f(s, x(s))] ds, \end{aligned}$$

which yields

$$\begin{aligned} & \|\mathcal{F}x(t + \omega) - \mathcal{F}x(t)\| \\ & \leq M\eta^{(0)} \|F_{i+1}(x)(s_{i+1}) - F_i(x)(s_i)\| \\ & \quad + q \int_0^\infty \int_{s_i}^t \theta |\xi_q(\theta)| \exp(\sigma(t - s)^q \theta) (t - s)^{q-1} \\ & \quad \times \|f(s + \omega, x(s + \omega)) - f(s, x(s + \omega))\| ds d\theta \\ & \quad + q \int_0^\infty \int_{s_i}^t \theta |\xi_q(\theta)| \exp(\sigma(t - s)^q \theta) (t - s)^{q-1} \|f(s, x(s + \omega)) - f(s, x(s))\| ds d\theta \\ & \leq M\eta^{(0)} \|F_{i+1}(x)(s_{i+1}) - F_i(x)(s_i)\| \\ & \quad + \frac{\eta^{(1)} \omega^q}{2^{kq}} \sup_{s_i \leq s \leq t_{i+1}} \|f(s + \omega, x) - f(s, x)\| \\ & \quad + q\eta^{(1)} \left(\int_{s_i}^t (t - s)^{q-1} L_f(s) ds \right) \sup_{s_i \leq s \leq t_{i+1}} \|x(s + \omega) - x(s)\|. \end{aligned}$$

Applying the conclusion of the first case, we see that the first term on the right-hand side of the latter inequality vanishes as $t \rightarrow \infty$. Moreover, using the hypotheses, we infer that the other two terms also converge to zero as t tends to infinity. Now, following the steps as above, we easily observe that $\mathcal{F}^n$ is a contraction on $\text{ISAP}_\omega^0(E)$, which implies that there exists a unique $\mathcal{IS}$ -asymptotically ω -periodic mild solution of (1.1). $\square$

Definition 4.8 ([35]) We say that the family of functions $(g_i)_{i \in \mathbb{N}}$ vanishes uniformly at infinity on bounded sets if for every bounded set $K \subseteq E$, $g(t, x) \rightarrow 0$ as $t \rightarrow \infty$ uniformly for $x \in K$.

Inspired by the proof of the previous theorem and modifying some conditions around g_i and N_i , we can restate Theorem 4.7 as follows.

Theorem 4.9 *Suppose that f is continuous and conditions (H1), (H7) and (H10) are satisfied. Let $\|T(t)\| \leq Me^{\sigma t}$ for all $t \geq 0$ and $\sigma \in \mathbb{R}$. Suppose that $f(\cdot)$ is uniformly $\mathcal{S}$ -asymptotically ω -periodic on bounded sets such that $lq\eta^{(1)} < 1$ and there exists nonnegative number $r_0 \geq \|x_0\|$ such that*

$$L_g \chi_{i+1, r_0} + \sup_{t_{i+1} \leq t \leq s_{i+1}} \|g_{i+1}(t, 0)\| \leq r_0 \quad (4.1)$$

for each $i \in \mathbb{N}_0$. Moreover, suppose the family $(g_i)_{i \in \mathbb{N}}$ vanishes uniformly at infinity on bounded sets, and the mappings N_i , $i \in \mathbb{N}$ are uniformly bounded on bounded sets. If the mappings $N_i : C([t_i, s_i]; E) \rightarrow C([t_i, s_i]; E)$, $i \in \mathbb{N}$, are completely continuous, then there exists an $\mathcal{TS}$ -asymptotically ω -periodic mild solution of (1.1).

Proof Let us introduce the space Z of all bounded continuous functions $x : J' \rightarrow E$ equipped with the topology of uniform convergence. We define $\mathcal{F}_2$ on Z by

$$\mathcal{F}_2 x(t) = g_i(t, N(t)(x)), \quad t \in [t_i, s_i], i \in \mathbb{N}.$$

We remark that, as a consequence of (H1), continuity of N_i , and vanishing of the family $(g_i)_{i \in \mathbb{N}}$ uniformly at infinity on bounded sets, $\mathcal{F}_2$ is a continuous mapping from Z into Z . Moreover, mixing (H1) with the property that N_i are completely continuous, we deduce that the mappings F_i for $i \in \mathbb{N}$ are completely continuous, too.

Applying again that the family $(g_i)_{i \in \mathbb{N}}$ vanishes uniformly at infinity on bounded sets and using the compactness criterion (which states that for any $W \subseteq \mathcal{PC}_b(E)$ if $\tilde{W}_i = \{\tilde{x}_i : x \in W\}$ is relatively compact in $C([t_i, t_{i+1}]; E)$ for all $i \in \mathbb{N}_0$ and $x(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly for $x \in W$, then W is relatively compact in $\mathcal{PC}_b(E)$), we can approve that $\mathcal{F}_2$ is completely continuous. Recall that compact operators on Banach spaces are always completely continuous. In addition, using (4.1) we can confirm that there exists a constant $r_0 > 0$ such that $B_{r_0}(0, Z)$ is $\mathcal{F}_2$ -invariant.

Now applying the well-known Schauder–Tychonoff theorem (see [17]), we are allowed to conclude the existence of a function $\bar{x} \in Z$ such that $\mathcal{F}_2 \bar{x} = \bar{x}$. From $\bar{x}(t) = g_i(t, N(t)(\bar{x}))$ for $t \in [t_i, s_i]$, we obtain that $\bar{x}(t) \rightarrow 0$ as $t \in J'$, $t \rightarrow \infty$. We now introduce $\mathcal{F}_1$ on $\mathcal{PC}_b(E)$ by

$$\mathcal{F}_1 x(t) = \begin{cases} \mathfrak{S}(t)x_0 + \int_0^t (t-s)^{q-1} \mathfrak{T}(t-s) f(s, x(s)) ds, & t \in [0, t_1], \\ \bar{x}(t), & t \in (t_i, s_i], i \in \mathbb{N}, \\ \mathfrak{S}(t-s_i)\bar{x}(s_i) + \int_{s_i}^t (t-s)^{q-1} \mathfrak{T}(t-s) f(s, x(s)) ds, & t \in (s_i, t_{i+1}], i \in \mathbb{N}. \end{cases} \quad (4.2)$$

Now, following the argument as in the proof of Theorem 4.7, we obtain that $\mathcal{F}_1$ is a map from $\mathcal{PC}_b(E)$ into itself. Furthermore, proceeding as in the third step of its proof, we can

establish that $\text{ISAP}_\omega(E)$ is $\mathcal{F}_1$ -invariant. Indeed, if $t \in [s_i, t_{i+1}]$, then $t + \omega \in [s_{i+1}, t_{i+2}]$, and

$$\begin{aligned} & \mathcal{F}_1 x(t + \omega) - \mathcal{F}_1 x(t) \\ &= \mathfrak{G}(t + \omega - s_{i+1})\bar{x}(s_{i+1}) - \mathfrak{G}(t - s_i)\bar{x}(s_i) \\ & \quad + \int_{s_{i+1}}^{t+\omega} (t + \omega - s)^{q-1} \mathfrak{T}(t + \omega - s) f(s, x(s)) ds - \int_{s_i}^t (t - s)^{q-1} \mathfrak{T}(t - s) f(s, x(s)) ds \\ &= \mathfrak{G}(t - s_i)(\bar{x}(s_i + \omega) - \bar{x}(s_i)) + \int_{s_i}^t (t - s)^{q-1} \mathfrak{T}(t - s) [f(s + \omega, x(s + \omega)) - f(s, x(s))] ds. \end{aligned}$$

Using the assumption that f is uniformly $\mathcal{S}$ -asymptotically ω -periodic on bounded sets, together with the fact that $\bar{x}(s_i) \rightarrow 0$ as $i \rightarrow \infty$, $i \in \mathbb{N}$, we can conclude that $\mathcal{F}_1 x(t + \omega) - \mathcal{F}_1 x(t) \rightarrow 0$ as $i \rightarrow \infty$. Moreover, by the virtue of proof of Theorem 4.7, together with (4.2), we obtain

$$\|\mathcal{F}_1 x_2(t) - \mathcal{F}_1 x_1(t)\| \leq lq\eta^{(1)} \|x_2 - x_1\|_\infty,$$

which means $\mathcal{F}_1$ is a contraction. Mixing these assertions, we infer that there is an $x \in \text{ISAP}_\omega(E)$ such that $\mathcal{F}_1 x = x$. This yields that $x(t) = \bar{x}(t)$ for $t \in (t_i, s_i]$, $i \in \mathbb{N}$, and

$$x(t) = \mathfrak{G}(t - s_i)x(s_i) + \int_{s_i}^t (t - s)^{q-1} \mathfrak{T}(t - s) f(s, x(s)) ds,$$

for $t \in (s_i, t_{i+1}]$. Therefore, $x(\cdot)$ is an $\mathcal{IS}$ -asymptotically ω -periodic mild solution of Eq. (1.1). $\square$

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