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Infinitely many solutions for nonlinear Klein–Gordon–Maxwell system with general nonlinearity

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Abstract

This paper is concerned with the nonlinear Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta z + V(x)z - (2\omega + \phi)\phi z = g(x,z) & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)z^2 & x \in \mathbb{R}^3, \end{cases}$$

where the potential V and the primitive of g are both allowed to be sign-changing. Under more general superlinear assumptions on the nonlinearity, we obtain a new existence result of infinitely many high energy solutions by using variational methods. Some recent results in the literature are generalized and significantly improved.

MSC: 35J20; 35J60; 35Q40

Keywords: Klein–Gordon–Maxwell system; High energy solutions; Variational methods

1 Introduction and main result

It is well known that the nonlinear Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta z + [m_0^2 - (\omega + \phi)^2]z = f(z) & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)z^2 & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

arises in a very interesting physical context: as a model describing the nonlinear Klein–Gordon field interacting with the electromagnetic field. More specifically, it represents a standing wave $\psi = u(x)e^{i\omega t}$ in equilibrium with a purely electrostatic field $\mathbf{E} = -\nabla\phi(x)$. This system was first introduced in the pioneering work of Benci and Fortunato in [3], and in their subsequent paper [4] they considered $|\omega| < |m_0|$ and $f(z) = |z|^{q-2}z$, $4 < q < 2^* = 6$, and proved that system (1.1) has infinitely many radially symmetric solutions $(z,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$. Later, D'Aprile and Mugnai [8] extended the interval of definition of the power in the nonlinearity for the case $2 < q \leq 4$. A nonexistence result has been established by the same authors (e.g., see [9] and the references therein).

Since then, a lot of works have been devoted to system (1.1), and we cite a couple of them. For example, Azzollini and Pomponio [1] proved the existence of a ground state

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solution for the very special power nonlinearity $f(z) = |z|^{q-2}z$ with $3 \le q < 5$ and $m_0 > \omega$, or $1 < p < 3$ and $m_0\sqrt{p-1} > \omega\sqrt{5-p}$. Georgiev and Visciglia [11] also introduced a system like (1.1) with potentials; however, they considered a small external Coulomb potential in the corresponding Lagrangian density. Chen and Tang [7] obtained the existence of two solutions in the radically symmetric function space by using the mountain pass theorem and Ekeland's variational principle for the nonhomogeneous case. For the critical growth case, that is, $f(z) = \mu|z|^{p-2}z + |z|^4 z$, Cassani [6] showed that the above system has at least a radially symmetric solution when $4 < p < 6$ or $p = 4$ provided that $\mu > 0$ is sufficiently large. Soon after that, Carrião et al. [5] also studied the existence of a radially symmetric solution for $2 < p < 6$, which extended and generalized the results in [1] and [6], respectively.

Recently, He [12] considered the following nonlinear Klein–Gordon–Maxwell system with non-constant external potential:

$$\begin{cases} -\Delta z + V(x)z - (2\omega + \phi)\phi z = g(x,z) & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)z^2 & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

Moreover, the author proved the existence of infinitely many solutions by using variant fountain theorem under the following assumptions:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > 0$;

(V_2) there exists a constant $r > 0$ such that

$$\lim_{|y|\to+\infty} \operatorname{meas}\bigl(\bigl\{x \in \mathbb{R}^3 : |x-y| \le r, V(x) \le M\bigr\}\bigr) = 0, \quad \forall M > 0;$$

(h_0) $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $g(x,t)t \ge 0$, and $\lim_{|t|\to 0} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$;

(h_1) there exists $c > 0$ such that $|g(x,t)| \le c(1 + |t|^{p-1})$ for $2 < p < 6$;

(h_2) there exists $\mu > 4$ such that $\mu G(x,t) \le g(x,t)t$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$, where $G(x,t) = \int_0^t g(x,s)\,ds$;

(h_3) there exists $4 < \alpha < 6$ such that $\liminf_{|t|\to\infty} \frac{G(x,t)}{|t|^\alpha} > 0$ uniformly in $x \in \mathbb{R}^3$;

(h_4) $\lim_{|t|\to\infty} \frac{g(x,t)}{t^3} = \infty$ uniformly in $x \in \mathbb{R}^3$;

(h_5) $\widetilde{G}(x,t) = \frac{1}{4}g(x,t)t - G(x,t) \to \infty$ as $|t| \to \infty$ uniformly in $x \in \mathbb{R}^3$.

Here condition (V) was introduced in [2] to guarantee the compactness of Sobolev embedding. As we know, conditions (h_2) and (h_5) are a very strong technical hypothesis, it is convenient to achieve a mountain pass geometry structure of the energy functional and show the boundedness of the Palais–Smale sequences. Subsequently, under condition (V), Li and Tang [13] studied the subject for superlinear and sublinear nonlinearities case, and two multiplicity results were obtained by using symmetric mountain pass theorem and fountain theorem. It should be noted that the above mentioned works [12, 13] always require the potential V to be positive in the sense that the quadratic form of the energy functional is positive definite.

Very recently, Ding and Li [10] considered system (1.2) with sign-changing potential. In addition to conditions (V_2) and (h_1), the authors assumed

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > -\infty$;

(h_6) $\lim_{|t|\to\infty} \frac{G(x,t)}{t^4} = \infty$ uniformly in $x \in \mathbb{R}^3$, and $g(t)t \ge 0$;

(h_7) there exists $r > 0$ such that

$$G(x,t) \le \frac{1}{4}g(x,t)t, \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}, |t| \ge r.$$

Condition (V_1) implies that the potential V may be sign-changing. Because V is sign-changing, the function

$$z \mapsto \left(\int_{\mathbb{R}^3} |\nabla z|^2 + V(x) z^2 \, dx \right)^{1/2}$$

does not have a norm on the working space in general. This will bring some problems for verifying the boundedness of Palais–Smale sequences or Cerami sequences. To overcome this difficulty, they chose a constant $V_0' > 0$ such that $\tilde{V}(x) = V(x) + V_0' > 0$ and considered the equivalent system

$$\begin{cases} -\Delta z + \tilde{V}(x) z - (2\omega + \phi)\phi z = \tilde{g}(x,z) & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi) z^2 & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $\tilde{g}(x,z) = g(x,z) + V_0' z$. Unfortunately, after a careful calculus, we could not deduce

$$\frac{1}{4}\tilde{g}(x,z)z - \tilde{G}(x,z) = \frac{1}{4} g(x,z) z - G(x,z) - \frac{1}{4} V_0' z^2 \geq 0,$$

which implies that $\tilde{g}(x,z)$ does not satisfy condition (h_7), where $\tilde{G}(x,z)$ is primitive of $\tilde{g}(x,z)$. Hence, their result can only be valid for the case that the potential V is positive.

Motivated by [10, 12, 13], in the present paper we will further study system (1.2) with non-constant external potential and general superlinear growth conditions. More specifically, we are interested in the case where the potential V and the primitive of g are both sign-changing, which is called a double sign-changing case and prevents us from applying a standard variational argument directly. For the above reasons, few papers dealt with such a double sign-changing case as regards system (1.2) until now. The main purpose of this paper is to establish a new result about the existence of infinitely many high energy solutions under some weaker conditions. Before stating our main result, we make the following assumptions for the nonlinearity g:

(g_0) $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and there exist constants $c_1, c_2 > 0$ and $p \in (2, 6)$ such that

$$\left| g(x,t) \right| \leq c_1 |t| + c_2 |t|^{p-1}, \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R};$$

(g_1) $\lim_{|t| \to \infty} \frac{G(x,t)}{t^4} = \infty$ uniformly in x, and there exists $r_0 \geq 0$ such that $G(x,t) \geq 0$, $\forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}$, $|t| \geq r_0$;

(g_2) there exist constants β, r_1 such that

$$G(x,t) \leq \frac{1}{4} g(x,t) t + \beta t^2, \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}, |t| \geq r_1;$$

(g_3) $g(x,t) = -g(x,-t)$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

Now, we are ready to state the main result of this paper as follows.

Theorem 1.1 *Assume that* (V_1)–(V_2), (g_0)–(g_2), *and* (g_3) *are satisfied. Then system* (1.2) *possesses infinitely many nontrivial solutions* $\{(z_n, \phi_n)\}$ *such that*

$$\frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla z_n|^2 + V(x) |z_n|^2 - \omega \phi_n z_n^2 \right) dx - \int_{\mathbb{R}^3} G(x, z_n) \, dx \to \infty \quad \text{as } n \to \infty.$$

Remark 1.2 Theorem 1.1 improves and generalizes the results in [10, 12, 13]. To show this, on the one hand, condition (V_1) implies that the potential $V(x)$ is allowed to be sign-changing, which is weaker than condition (V). On the other hand, our conditions on the nonlinearity seem more general than conditions of [10, 12], even weaker. Indeed, it follows from (h_0) and (h_1) that, for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|g(x,t)\right| \leq \varepsilon|t| + C_\varepsilon|t|^{p-1} \leq (\varepsilon + C_\varepsilon)\left(|t| + |t|^{p-1}\right) \quad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

which implies that (g_0) holds. However, we do not need the usual "superlinear condition" at the origin in (h_0), which is very important in checking the geometry structure of the corresponding energy functional. Furthermore, it is clear that (h_2), (h_3), (h_5), (h_6), and (h_7) imply that (g_1) and (g_2) hold. Additionally, from $g(x,t)t \geq 0$ in (h_0) and (h_6), we can deduce that $G(x,t) \geq 0$, but it follows from (g_1) that $G(x,t)$ is allowed to be sign-changing.

The remainder of the paper is organized as follows. In Sect. 2, we formulate the variational setting for system (1.2) and introduce some useful preliminaries. We prove Theorem 1.1 in Sect. 3.

2 Variational setting and preliminary results

Below by $\|\cdot\|_q$ we denote the usual L^q-norm for $1 \leq p < +\infty$, c_i, C, C_i stand for different positive constants. First, we observe that, in view of (V_1), the potential $V(x)$ is sign-changing in $\mathbb{R}^3$. In this case, the corresponding energy functional to system (1.2) is rather complicated, because the quadratic form

$$B(z,z) := \int_{\mathbb{R}^3} |\nabla z|^2 + V(x)z^2 \, dx$$

appearing in the energy functional is indefinite. In order to overcome the indefiniteness of the quadratic form, we do not handle system (1.2) directly, but instead we handle an equivalent system to (1.2). In indeed, it follows from (V_1) that there exists a constant $V_0 > 0$ such that $\tilde{V}(x) := V(x) + V_0 > 0$ for all $x \in \mathbb{R}^3$, and the quadratic form

$$\tilde{B}(z,z) := \int_{\mathbb{R}^3} |\nabla z|^2 + \tilde{V}(x)z^2 \, dx$$

is positive definite. Therefore, let $\tilde{g}(x,z) := g(x,z) + V_0 g$, we consider the following new system:

$$\begin{cases} -\Delta z + \tilde{V}(x)z - (2\omega + \phi)\phi z = \tilde{g}(x,z) & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)z^2 & x \in \mathbb{R}^3. \end{cases} \tag{2.1}$$

Clearly, system (1.2) is equivalent to system (2.1). Moreover, conditions (V), (V_2) and (g_0)–(g_3) still hold for $\tilde{V}$ and $\tilde{g}$ provided that those hold for V and g. Therefore, in what follows, we just need to study system (2.1). Throughout this section, we make the following assumption instead of (V_1):

$(\tilde{V}_1)$ $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > 0$.

Let $\mathcal{D}^{1,2}(\mathbb{R}^3) = \{z \in L^6(\mathbb{R}^3) : \nabla z \in L^2(\mathbb{R}^3)\}$ with the norm

$$\|z\|_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla z|^2\,dx\right)^{\frac{1}{2}}.$$

Under assumption $(\tilde{V}_1)$, we define the working space

$$E := \left\{z \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)z^2\,dx < \infty\right\}$$

with the inner product and norm

$$(z,w) = \int_{\mathbb{R}^3} \big(\nabla z \cdot \nabla w + V(x)zw\big)\,dx \quad \text{and} \quad \|z\| = (z,z)^{\frac{1}{2}}.$$

Evidently, E is a Hilbert space. According to [2], under assumptions $(\tilde{V}_1)$ and (V_2), the embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is continuous for $2 \le p \le 6$, $E \hookrightarrow L^p(\mathbb{R}^3)$ is compact for $2 \le p < 6$. Moreover, the embedding inequality

$$\|z\|_p \le \tau_p \|z\|, \quad \forall z \in E, p \in [2,6],$$

holds for some $c_p > 0$.

Due to the variational characteristic of system (2.1), its weak solutions $(z,\phi) \in E \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ are critical points of the functional given by

$$\mathcal{J}(z,\phi) = \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla z|^2 + V(x)z^2 - |\nabla\phi|^2 - (2\omega+\phi)\phi z^2\big)\,dx - \int_{\mathbb{R}^3} G(x,z)\,dx. \tag{2.2}$$

It is clear that the functional $\mathcal{J}$ is strongly indefinite, i.e., unbounded from below and from above on infinite dimensional spaces. In this sense the functional $\mathcal{J}$ possesses complicated geometry structure. To avoid this difficulty, we reduce the study of (2.2) to the study of a functional in the only variable z, as it has been done by the aforementioned authors.

In order to reduce functional (2.2), we need the following technical result (see [12]).

Lemma 2.1 *For any fixed $z \in H^1(\mathbb{R}^3)$, there exists unique $\phi = \phi_z \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, which solves the equation*

$$-\Delta\phi + z^2\phi = -\omega z^2.$$

Moreover, the map $\Phi : z \in H^1(\mathbb{R}^3) \to \Phi[z] := \phi_z \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and

(i) $-\omega \le \phi_z \le 0$ *on the set* $\{x | z(x) \ne 0\}$;

(ii) $\|\phi_z\|_{\mathcal{D}^{1,2}} \le C_0\|z\|^2$, *and* $\int_{\mathbb{R}^3} |\phi_z| z^2\,dx \le C_0\|z\|_{12/5}^4 \le C_0\|z\|^4$.

In virtue of Lemma 2.1, we can rewrite $\mathcal{J}$ as the C^1-reduced functional $\Phi : E \to \mathbb{R}$ given by

$$\Phi(z) = \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla z|^2 + V(x)z^2 - \omega\phi_z z^2\big)\,dx - \int_{\mathbb{R}^3} G(x,z)\,dx, \tag{2.3}$$

while for any $\varphi \in E$, we have

$$\langle \Phi'(z), \varphi \rangle = \int_{\mathbb{R}^3} \bigl(\nabla z \cdot \nabla \varphi + V(x) z\varphi - (2\omega + \phi_z)\phi_z z\varphi \bigr)\, dx - \int_{\mathbb{R}^3} g(x,z)\varphi \, dx. \tag{2.4}$$

Now the functional Φ obtained is not strongly indefinite anymore, and we will look for its critical points since if the pair $(z,\phi) \in E \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a critical point for $\mathcal{J}$, then z is a critical point for Φ with $\phi = \phi_z$. Recall that a sequence $\{z_n\} \subset E$ is said to be a Cerami sequence ($(C)_c$-sequence in short) if $\Phi(z_n) \to c$ and $(1 + \|z_n\|)\|\Phi'(z_n)\| \to 0$, Φ is said to satisfy the Cerami condition ($(C)_c$-condition in short) if any $(C)_c$-sequence has a convergent subsequence.

In order to obtain the existence of high energy solutions, we will use the symmetric mountain pass theorem of Rabinowitz [14]. It should be noted that the symmetric mountain pass theorem is established under the Palais–Smale condition. Since the deformation lemma is still valid under the $(C)_c$-condition, we see that the symmetric mountain pass theorem also holds under the $(C)_c$-condition.

Proposition 2.2 ([14]) *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $\varphi \in C^1(X,\mathbb{R})$ satisfies the Cerami condition, and*

(I_1) *$\varphi(0) = 0$, $\varphi(-u) = \varphi(u)$ for all $u \in X$;*

(I_2) *there exist constants $\rho, \alpha > 0$ such that $\varphi|_{\partial B_\rho \cap Z} \geq \alpha$;*

(I_3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $\varphi(u) \leq 0$ on $\tilde{X} \setminus B_R$;*

then φ possesses an unbounded sequence of critical values $c_j \to +\infty$.

Next we check if the energy functional Φ satisfies the mountain pass geometry structure of Proposition 2.2. First, we give a direct sum decomposition of the working space E. Let $\{\phi_j\}$ be a total orthonormal basis of E, we define $X_j = \mathbb{R}\phi_j$,

$$Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k+1}^{\infty} X_j}, \quad k \in \mathbb{Z}.$$

It is clear that Y_k is a finite dimensional space, and $E = Z_k \oplus Y_k$ for all $k \in \mathbb{Z}$. In addition, based on the fact that $E \hookrightarrow L^p(\mathbb{R}^3)$ is compact for $2 \leq p < 6$, we have the following result by the argument of Lemma 3.8 in [15] (see also [16, 17]).

Lemma 2.3 *Assume that $(\tilde{V}_1)$ and (V_2) hold, for $2 \leq p < 6$,*

$$\beta_k(p) := \sup_{z \in Z_k, \|z\|=1} \|z\|_p \to 0, \quad k \to \infty.$$

It follows from Lemma 2.3 that we may take an integer $m \geq 1$ such that

$$\|z\|_2^2 \leq \frac{1}{2c_1}\|z\|^2, \qquad \|z\|_p^p \leq \frac{p}{4c_2}\|z\|^p, \quad \forall\, z \in Z_m. \tag{2.5}$$

Lemma 2.4 *Assume that $(\tilde{V}_1)$, (V_2), and (g_0) hold, there exists a positive constant ρ such that $\Phi|_{\partial B_\rho \cap Z_m} > 0$.*

Proof Observe that, for any $z \in Z_m$, by Lemma 2.1, (2.5), and (g_0) we have

$$\begin{aligned}\Phi(z) &= \frac{1}{2}\|z\|^2 - \frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_z z^2\,dx - \int_{\mathbb{R}^3} G(x,z)\,dx \\ &\geq \frac{1}{2}\|z\|^2 - \frac{c_1}{2}\|z\|_2^2 - \frac{c_2}{p}\|z\|_p^p \\ &\geq \frac{1}{4}\big(\|z\|^2 - \|z\|^p\big).\end{aligned}$$

Since $p > 2$, choosing suitable $\rho > 0$, we see that the desired conclusion holds. □

Lemma 2.5 *Assume that* $(\tilde{V}_1)$, (V_2), (g_0), *and* (g_1) *hold; for any finite dimensional subspace* $\tilde{E} \subset E$, *there is* $R = R(\tilde{E}) > 0$ *such that* $\Phi(z) \leq 0$ *for any* $z \in \tilde{E}\backslash B_R$.

Proof Let $\tilde{E}$ be a finite dimensional subspace of E, then there is a positive integral number l such that $\tilde{E} \subset E_l$. Since all norms are equivalent in a finite dimensional space, there is a constant $c_3 > 0$ such that

$$\|z\|_4 \geq c_3\|z\|, \quad \forall E_l. \tag{2.6}$$

By (g_0) and (g_1), for any $M > \frac{\omega C_0}{2c_3^4}$, there is a constant $C_M > 0$ such that

$$G(x,z) \geq M|z|^4 - C_M|z|^2, \quad \forall (x,z) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.7}$$

It follows from Lemma 2.1, (2.7), and (2.6) that

$$\begin{aligned}\Phi(z) &\leq \frac{1}{2}\|z\|^2 + \frac{\omega C_0}{2}\|z\|^4 - M\|z\|_4^4 + C_M\|z\|_2^2 \\ &\leq \frac{1}{2}\|z\|^2 - \left(Mc_3^4 - \frac{\omega C_0}{2}\right)\|z\|^4 + C_M C_2^2\|z\|^2\end{aligned}$$

for all $z \in E_l$. Consequently, there exists large $R = R(\tilde{E}) > 0$ such that $\Phi(z) \leq 0$ on $\tilde{E}\backslash B_R$. The proof is completed. □

Now we discuss the property of the $(C)_c$-sequence, we have the following lemma.

Lemma 2.6 *Assume that* $(\tilde{V}_1)$, (V_2), (g_0)–(g_2) *hold. Then any* $(C)_c$*-sequence of* Φ *is bounded.*

Proof Let $\{z_n\} \subset E$ be a $(C)_c$-sequence of Φ, then

$$\Phi(z_n) \to c \quad \text{and} \quad \big(1 + \|z_n\|\big)\Phi'(z_n) \to 0, \tag{2.8}$$

and there exists a constant $C > 0$ such that

$$\Phi(z_n) - \frac{1}{4}\Phi'(z_n)z_n \leq C. \tag{2.9}$$

Suppose to the contrary that $\|z_n\| \to \infty$. Setting $w_n := z_n/\|z_n\|$, then $\|w_n\| = 1$. After passing to a subsequence, we may assume that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^p(\mathbb{R}^3)$ for $2 \le p < 6$ and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^3$. Then there are two possibilities: $w = 0$ or $w \neq 0$.

If $w = 0$. Observe that it follows from (g_0) that there exist constants r_1, c such that

$$\left|\frac{1}{4}g\big(|x|,z\big)z - G\big(|x|,z\big)\right| \le c|z|^2 \quad \text{for all } |z| < r_1.$$

Therefore, by (g_2), (2.3), (2.4), and (2.9), we have

$$\begin{aligned}
\frac{C}{\|z_n\|^2} &\ge \frac{1}{\|z_n\|^2}\left(\Phi(z_n) - \frac{1}{4}\big\langle \Phi'(z_n), z_n\big\rangle\right) \\
&= \frac{1}{4} + \frac{1}{\|z_n\|^2}\int_{\mathbb{R}^3}\left(\frac{1}{4}\phi_{z_n}^2 z_n^2 + \left(\frac{1}{4}g(x,z_n)z_n - G(x,z_n)\right)\right)dx \\
&\ge \frac{1}{4} - c\int_{|z_n|<r_1} w_n^2\,dx - \beta\int_{|z_n|\ge r_1} w_n^2\,dx \\
&\ge \frac{1}{4} - (c+\beta)\int_{\mathbb{R}^3} w_n^2\,dx.
\end{aligned}$$

Taking limit on both sides, we obtain $\frac{1}{4} \le 0$. Clearly, this is a contradiction.

If $w \neq 0$. For $0 \le a < b$, let $\Omega_n(a,b) = \{x \in \mathbb{R}^3 : a \le |z_n(x)| < b\}$. Setting $\Sigma := \{x \in \mathbb{R}^3 : w(x) \neq 0\}$. Obviously, $\operatorname{meas}(\Sigma) > 0$. For $x \in \Sigma$, $|z_n(x)| \to \infty$ as $n \to \infty$. Hence $x \in \Omega_n(r_0,\infty)$ for large $n \in \mathbb{N}$, which implies that $\chi_{\Omega_n(r_0,\infty)}(x) = 1$ for large n, where χ_Ω denotes the characteristic function on Ω, r_0 is given in (g_1). Since $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^3$, we have $\chi_{\Omega_n(r_0,\infty)}(x)w_n(x) \to w(x)$ a.e. in Σ. It follows from (g_1) and (2.5) and Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty}\frac{c+o(1)}{\|z_n\|^4} = \lim_{n\to\infty}\frac{\Phi(z_n)}{\|z_n\|^4} \\
&= \lim_{n\to\infty}\frac{1}{\|z_n\|^4}\left(\frac{1}{2}\|z_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_{z_n}z_n^2 - \int_{\mathbb{R}^3}G(x,z_n)\,dx\right) \\
&= \lim_{n\to\infty}\frac{1}{\|z_n\|^4}\left(\frac{1}{2}\|z_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_{z_n}z_n^2 - \int_{\Omega_n(0,r_0)}G(x,z_n)\,dx - \int_{\Omega_n(r_0,\infty)}G(x,z_n)\,dx\right) \\
&\le \frac{C_0}{2} + \limsup_{n\to\infty}\left[\left(\frac{c_4}{2} + \frac{c_5}{p}r_0^{p-2}\right)\frac{1}{\|z_n\|^2}\int_{\mathbb{R}^3}|w_n|^2\,dx\right] - \liminf_{n\to\infty}\int_{\Omega_n(r_0,\infty)}\frac{G(x,z_n)}{z_n^4}w_n^4\,dx \\
&\le c_6 + \frac{C_0}{2} - \liminf_{n\to\infty}\int_{\mathbb{R}^3}\frac{G(x,z_n)}{z_n^4}\big[\chi_{\Omega_n(r_0,\infty)}(x)\big]w_n^4\,dx \\
&\le c_6 + \frac{C_0}{2} - \int_\Sigma \liminf_{n\to\infty}\frac{G(x,z_n)}{z_n^4}\big[\chi_{\Omega_n(r_0,\infty)}(x)\big]w_n^4\,dx = +\infty,
\end{aligned}$$

which implies a contradiction. Thus $\{z_n\}$ is bounded in E. □

Lemma 2.7 *Assume that $(\tilde{V}_1)$, (V_2), and (g_0)–(g_2) hold. Then Φ satisfies the $(C)_c$-condition.*

Proof Let $\{z_n\} \subset E$ be a $(C)_c$-sequence of Φ. It follows from Lemma 2.6 that $\{z_n\}$ is bounded in E. Passing to a subsequence, we can assume that $z_n \rightharpoonup z$ in E, then $z_n \to z$

in $L^p(\mathbb{R}^3)$ for $2 \le p < 6$, hence it is easy to check that

$$\int_{\mathbb{R}^3} \bigl|g(x,z_n) - g(x,z)\bigr||z_n - z|\,dx \to 0, \quad \text{as } n \to \infty. \tag{2.10}$$

By view of (2.3) and (2.4), we get

$$\begin{aligned} \|z_n - z\|^2 &= \bigl\langle \Phi'(z_n) - \Phi'(z), z_n - z\bigr\rangle + \int_{\mathbb{R}^3} \bigl(g(x,z_n) - g(x,z)\bigr)(z_n - z)\,dx \\ &\quad + \int_{\mathbb{R}^3} \bigl((2\omega + \phi_{z_n})\phi_{z_n} z_n - (2\omega + \phi_z)\phi_z z\bigr)(z_n - z)\,dx. \end{aligned}$$

On the one hand, by Hölder's inequality and Sobolev's embedding inequality, we get

$$\begin{aligned} \left|\int_{\mathbb{R}^3} (2\omega + \phi_{z_n})\phi_{z_n} z_n (z_n - z)\,dx\right| &\le \left|\int_{\mathbb{R}^3} 2\omega\phi_{z_n} z_n (z_n - z)\,dx\right| + \left|\int_{\mathbb{R}^3} \phi_{z_n}^2 z_n (z_n - u)\,dx\right| \\ &\le 2\omega\|\phi_{z_n} z_n\|_2\|z_n - z\|_2 + \bigl\|\phi_{z_n}^2 z_n\bigr\|_2\|z_n - z\|_2 \\ &\le 2\omega\|\phi_{z_n}\|_6\|z_n\|_3\|z_n - z\|_2 + \|\phi_{z_n}\|_6^2\|z_n\|_6\|z_n - z\|_2 \\ &\le c_4\bigl(\|\phi_{z_n}\|_{\mathcal{D}^{1,2}}\|z_n\|_3 + \|\phi_{z_n}\|_{\mathcal{D}^{1,2}}^2\|z_n\|_6\bigr)\|z_n - z\|_2 \\ &\le c_5\bigl(\|z_n\|_{12/5}^2\|z_n\|_3 + \|z_n\|_{12/5}^4\|z_n\|_6\bigr)\|z_n - z\|_2 \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Moreover, using some similar arguments, we can show that

$$\left|\int_{\mathbb{R}^3} (2\omega + \phi_z)\phi_z z(z_n - z)\,dx\right| \to 0, \quad \text{as } n \to \infty. \tag{2.11}$$

On the other hand, by view of the definition of weak convergence, we have

$$\bigl\langle \Phi'(z_n) - \Phi'(z), z_n - z\bigr\rangle \to 0, \quad \text{as } n \to \infty. \tag{2.12}$$

It follows from (2.10)–(2.12) that $\|z_n - z\| \to 0$ as $n \to \infty$. The proof is completed. □

3 Proof of the theorem

In this section, we give the proof of Theorem 1.1.

Proof of Theorem 1.1 Obviously, (g_3) implies that $\Phi(0) = 0$ and Φ is even. Lemmas 2.4 and 2.5 imply that Φ satisfies the geometry structure of Proposition 2.2. Lemmas 2.6 and 2.7 show that Φ satisfies the $(C)_c$-condition. Thus, by Proposition 2.2, system (2.1) possesses a sequence of infinitely many nontrivial solutions $\{z_n\}$ such that $\Phi(z_n) \to \infty$ as $n \to \infty$. Moreover, system (1.2) also possesses a sequence of infinitely many nontrivial solutions $\{z_n\}$ such that $\Phi(z_n) \to \infty$ as $n \to \infty$. □

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