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A quasi-reversibility regularization method for a Cauchy problem of the modified Helmholtz-type equation

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Abstract

The Cauchy problem of the modified Helmholtz-type equation is severely ill-posed, i.e., the solution does not depend continuously on the given Cauchy data. Thus the regularization methods are required to recover the numerical stability. In this paper, we propose a quasi-reversibility regularization method to deal with this ill-posed problem. Convergence estimates are obtained under a-priori bound assumptions for the exact solution and the selection of regularization parameter. Some numerical results are given to show that this method is stable and feasible.

Keywords: Modified Helmholtz-type equation; Cauchy problem; Quasi-reversibility regularization method; Convergence estimates

1 Introduction

In this study, a Cauchy problem of the Helmholtz-type equation is considered as follows:

$$\begin{cases} w_{xx} + w_{yy} - k^2 w = 0, & 0 < x < \pi, 0 < y < T, \\ w(x,0) = \varphi(x), & 0 \le x \le \pi, \\ w_y(x,0) = \psi(x), & 0 \le x \le \pi, \\ w(0,y) = w(\pi,y) = 0, & 0 \le y \le T. \end{cases} \tag{1.1}$$

By solving equations (1.2) and (1.3) as follows, respectively, the solution to equation (1.1) can be obtained.

$$\begin{cases} u_{xx} + u_{yy} - k^2 u = 0, & 0 < x < \pi, 0 < y < T, \\ u(x,0) = \varphi(x), & 0 \le x \le \pi, \\ u_y(x,0) = 0, & 0 \le x \le \pi, \\ u(0,y) = u(\pi,y) = 0, & 0 \le y \le T, \end{cases} \tag{1.2}$$

and

$$\begin{cases} v_{xx} + v_{yy} - k^2 v = 0, & 0 < x < \pi, 0 < y < T, \\ v(x,0) = 0, & 0 \le x \le \pi, \\ v_y(x,0) = \psi(x), & 0 \le x \le \pi, \\ v(0,y) = v(\pi,y) = 0, & 0 \le y \le T, \end{cases} \tag{1.3}$$

where $u(x,y) : [0,\pi] \times [0,T] \to R, v(x,y) : [0,\pi] \times [0,T] \to R$ and $w(x,y) : [0,\pi] \times [0,T] \to R$ are all second-order continuous differentiable functions.

This problem appears in many applications [1] such as in Debye–Huckel theory, implicit marching strategies of the heat equation, the linearization of the Poisson–Boltzmann equation [2–4], and so on. The direct problems of the Helmholtz-type equation have been studied widely in the past century [5, 6]. In recent years, some new methods have been proposed for the Helmholtz problems, such as fast solution of three-dimensional modified Helmholtz equations by the method of fundamental solutions [7], a new radial basis function for Helmholtz problems [8], a new investigation into regularization techniques for the method of fundamental solutions [9], the blow-up of radial solutions to a cubic non-linear system equation in dimension 2 [10], and a modified and simple algorithm for fractional modelling arising in unidirectional propagation of long wave in dispersive media by using the fractional homotopy analysis transform method [11]. However, the noisy data can be obtained only on a part of the boundary or at some interior points in some practical problems giving rise to an inverse problem [12]. Problem (1.1) is well known to be a highly ill-posed problem, which means the solution does not depend continuously on the given Cauchy data, i.e., any small change in the given data may cause large error to the solution [13, 14]. In recent years, the Cauchy problems associated with the Helmholtz-type equation have been studied by using different numerical methods such as the conjugate gradient method [15], the Landweber method with boundary element method [16], Tikhonov-type regularization method [17], the method of fundamental solutions [18–20], quasi-reversibility and truncation methods [21], and so on. In paper [22], a non-local boundary value problem method is used to solve a Cauchy problem for elliptic equations in a cylindrical domain. Recently this method has been used to solve the backward heat conduction problem [23–26] and the Cauchy problem for hyper-parabolic partial differential equations [27].

In this study, a quasi-reversibility regularization method will be considered to construct stable approximate solutions to problems (1.2) and (1.3). Our method has a little difference with the one in [21]. There are two ways to propose quasi-reversibility methods: by modifying the disturbance equation or by modifying the initial-boundary value condition. In [21], the main strategy is to modify the disturbance equation. In our paper, the initial-boundary value condition is modified. Here the initial conditions $u(x,0) = \varphi(x)$ in (1.2) and $v_y(x,0) = \psi(x)$ in (1.3) are replaced with

$$u(x,0) + \alpha \frac{\partial^p u(x,T)}{\partial y^p} = \varphi(x), \tag{1.4}$$

$$v_y(x,0) + \alpha \frac{\partial^p v(x,T)}{\partial y^p} = \psi(x), \tag{1.5}$$

respectively, where $p \geq 1$ is an integer and $\alpha > 0$ is the regularization parameter. In order to overcome the ill-posedness of problems (1.2) and (1.3), the perturbation conditions (1.4) and (1.5) will be adopted. For compatibility of physical dimension, here we make the regularization parameter α include some coefficients of thermodynamics.

The remainder of this paper is organized as follows. In Sect. 2, a quasi-reversibility regularization method and error estimates are given. In Sect. 3, numerical results are shown. Some conclusions are given in Sect. 4.

2 Regularization method and error estimates

Firstly, as for equation (1.2), the solution to the following perturbation equation will be adopted to approach the solution to equation (1.2):

$$\begin{cases} (u_\alpha^\delta)_{xx} + (u_\alpha^\delta)_{yy} - k^2(u_\alpha^\delta) = 0, \quad 0 < x < \pi, 0 < y < T, & (2.1a) \\ u_\alpha^\delta(x,0) + \alpha \dfrac{\partial^p u_\alpha^\delta(x,T)}{\partial^p y} = \varphi^\delta(x), \quad 0 \leq x \leq \pi, & (2.1b) \\ (u_\alpha^\delta)_y(x,0) = 0, \quad 0 \leq x \leq \pi, & (2.1c) \\ u_\alpha^\delta(0,y) = u_\alpha^\delta(\pi,y) = 0, \quad 0 \leq y \leq T. & (2.1d) \end{cases}$$

where $p \geq 1$ is an integer, $\alpha > 0$ is a regularization parameter, and the measured data $\varphi^\delta \in L^2(0,\pi)$ satisfies

$$\|\varphi^\delta - \varphi\| \leq \delta, \tag{2.2}$$

in which $\|\cdot\|$ denotes the L^2-norm and the constant $\delta > 0$ is called an error level.

By the technique of separation of variables, we can obtain a solution to equation (1.2) as follows:

$$u(x,y) = \sum_{n=1}^{\infty} \varphi_n \sin(nx) \cosh\bigl(\sqrt{k^2+n^2}y\bigr), \tag{2.3}$$

where

$$\varphi_n = \frac{2}{\pi} \int_0^\pi \varphi(x) \sin(nx)\, dx. \tag{2.4}$$

Similarly, the solution to problem (2.1a)–(2.1d) is

$$u_\alpha^\delta(x,y) = \begin{cases} \sum_{n=1}^\infty \frac{\cosh(\sqrt{k^2+n^2}y)/\sinh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^p + 1/\sinh(\sqrt{k^2+n^2}T)} \varphi_n^\delta \sin(nx) & p \text{ is odd}, \\ \sum_{n=1}^\infty \frac{\cosh(\sqrt{k^2+n^2}y)/\cosh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^p + 1/\cosh(\sqrt{k^2+n^2}T)} \varphi_n^\delta \sin(nx) & p \text{ is even}, \end{cases} \tag{2.5}$$

where

$$\varphi_n^\delta = \frac{2}{\pi} \int_0^\pi \varphi^\delta(x) \sin(nx)\, dx. \tag{2.6}$$

Next, the deduction of (2.5) will be given. By the technique of separation of variables, let $u_\alpha^\delta(x,y) = X(x)T(y)$, and plug that into equation (2.1a). We can obtain

$$X''(x)T(y) + X(x)T''(y) - k^2\bigl(X(x)T(y)\bigr) = 0.$$

By separation of variables, we have

$$\frac{X''(x)}{X(x)} = \frac{k^2 T(y) - T''(y)}{T(y)}. \tag{2.5a}$$

Since the left-hand side is independent of t and the right-hand side is independent of x in equation (2.5a), we can let equation (2.5a) equal $-\lambda$(constant).

Hence, we can obtain two second-order linear ordinary differential equations as follows:

$$X''(x) + \lambda X(x) = 0, \tag{2.5b}$$

$$T''(y) - \left(k^2 + \lambda\right) T(y) = 0. \tag{2.5c}$$

Now, plug $u_\alpha^\delta(x,y) = X(x)T(y)$ into equation (2.1d), and we have

$$X(0)T(y) = X(\pi)T(y) = 0.$$

Apparently, $T(y) \not\equiv 0$, we have

$$X(0) = X(\pi) = 0. \tag{2.5d}$$

So, the Sturm–Liouville eigenvalue problems of equations (2.5b) and (2.5c) can be obtained, and we can obtain all eigenvalues $\lambda_n = n^2, n = 1, 2, 3, \ldots$, and eigenfunctions

$$X_n = \sin(nx), \quad n = 1, 2, 3, \ldots. \tag{2.5e}$$

For any $\lambda_n = n^2, n = 1, 2, 3, \ldots$, from equation (2.5b) we have

$$T_n(y) = C_n e^{\sqrt{k^2+n^2}y} + D_n e^{-\sqrt{k^2+n^2}y}. \tag{2.5f}$$

From equations (2.5e) and (2.5f), we can obtain

$$u_\alpha^\delta(x,y) = \sum_{n=1}^{\infty} \left(C_n e^{\sqrt{k^2+n^2}y} + D_n e^{-\sqrt{k^2+n^2}y}\right) \sin(nx). \tag{2.5g}$$

We plug equation (2.5g) into equation (2.1c), and we have

$$\sum_{n=1}^{\infty} \left(C_n \sqrt{k^2+n^2} - D_n \sqrt{k^2+n^2}\right) \sin(nx) = 0,$$

$$\left(C_n \sqrt{k^2+n^2} - D_n \sqrt{k^2+n^2}\right) = \frac{2}{\pi} \int_0^\pi 0 \cdot \sin(nx)\, dx = 0.$$

Hence, we can obtain $C_n = D_n$, $T_n(y) = 2C_n \cosh(\sqrt{k^2+n^2}y)$, and

$$u_\alpha^\delta(x,y) = \left(2C_n \cosh\left(\sqrt{k^2+n^2}y\right)\right) \sin(nx). \tag{2.5h}$$

From equations (2.5f) and (2.1b), we can obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big[2C_n+\alpha C_n\big(e^{\sqrt{k^2+n^2}T}+(-1)^p e^{-\sqrt{k^2+n^2}T}\big)\big(\sqrt{k^2+n^2}\big)^p\Big]\sin(nx)=\varphi^\delta(x),\\
&\sum_{n=1}^{\infty}2C_n\left[1+\frac{e^{\sqrt{k^2+n^2}T}+(-1)^p e^{-\sqrt{k^2+n^2}T}}{2}\alpha\big(\sqrt{k^2+n^2}\big)^p\right]\sin(nx)=\varphi^\delta(x),\\
&2C_n\left[1+\frac{e^{\sqrt{k^2+n^2}T}+(-1)^p e^{-\sqrt{k^2+n^2}T}}{2}\alpha\big(\sqrt{k^2+n^2}\big)^p\right]=\frac{2}{\pi}\int_0^\pi\varphi(x)\cdot\sin(nx)\,dx\triangleq\varphi_n^\delta.
\end{aligned}\tag{2.5i}$$

From equation (2.5i), we have

$$2C_n=\frac{\varphi_n^\delta}{1+\frac{e^{\sqrt{k^2+n^2}T}+(-1)^p e^{-\sqrt{k^2+n^2}T}}{2}\alpha(\sqrt{k^2+n^2})^p}.\tag{2.5j}$$

Therefore, from equations (2.5g) and (2.5j), we can obtain equation (2.5).

In the following Theorem 2.1, we will prove that solution (2.5) depends continuously on the Cauchy data φ^δ.

Theorem 2.1 *Suppose that $u_{\alpha1}^\delta$ is the solutions to equation* (2.1a)–(2.1d) *corresponding to the data φ_1^δ, and $u_{\alpha2}^\delta$ is the solutions to equation* (2.1a)–(2.1d) *corresponding to the data φ_2^δ, then, for $\alpha < T$, we obtain*

$$\big\|u_{\alpha1}^\delta(\cdot,y)-u_{\alpha2}^\delta(\cdot,y)\big\|\le\left(\frac{2}{1-e^{-2T}}\right)\left(\frac{T}{\alpha(1+\ln(T/\alpha))}\right)\big\|\varphi_1^\delta-\varphi_2^\delta\big\|.\tag{2.7}$$

Proof The case that p is even will be considered first. From (2.5), we can obtain

$$u_{\alpha1}^\delta=\sum_{n=1}^{\infty}\frac{\cosh(\sqrt{k^2+n^2}y)/\cosh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^p+1/\cosh(\sqrt{k^2+n^2}T)}\varphi_{1,n}^\delta\sin(nx),\tag{2.8}$$

$$u_{\alpha2}^\delta=\sum_{n=1}^{\infty}\frac{\cosh(\sqrt{k^2+n^2}y)/\cosh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^p+1/\cosh(\sqrt{k^2+n^2}T)}\varphi_{2,n}^\delta\sin(nx),\tag{2.9}$$

where $\varphi_{i,n}^\delta=\frac{2}{\pi}\int_0^\pi\varphi_i^\delta(x)\sin(nx)\,dx$ for $i=1,2$.

For $x>0$, we define the function

$$h(x)=\frac{1}{\alpha x+e^{-xT}}.\tag{2.10}$$

It is easy to prove that $h(x)$ has a unique maximizer x_0 as $\alpha < T$ such that

$$h(x)\le h(x_0)=h\left(\frac{\ln(T/\alpha)}{T}\right)=\frac{T}{\alpha(1+\ln(T/\alpha))}.\tag{2.11}$$

Then, from Parseval equality, equation (2.11), and Bessel inequality, we have

$$\big\|u_{\alpha1}^\delta(\cdot,y)-u_{\alpha2}^\delta(\cdot,y)\big\|^2$$

$$
\begin{aligned}
&= \left\| \sum_{n=1}^{\infty} \frac{\cosh(\sqrt{k^2+n^2}y)/\cosh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^p + 1/\cosh(\sqrt{k^2+n^2}T)} \left(\varphi_{1,n}^{\delta} - \varphi_{2,n}^{\delta}\right) \sin(nx) \right\|^2 \\
&= \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{\cosh(\sqrt{k^2+n^2}y)/\cosh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^p + 1/\cosh(\sqrt{k^2+n^2}T)} \right)^2 \left(\varphi_{1,n}^{\delta} - \varphi_{2,n}^{\delta}\right)^2 \\
&\le \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{\cosh(\sqrt{k^2+n^2}T)/\cosh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2}) + 1/\cosh(\sqrt{k^2+n^2}T)} \right)^2 \left(\varphi_{1,n}^{\delta} - \varphi_{2,n}^{\delta}\right)^2 \\
&= \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{1}{\alpha(\sqrt{k^2+n^2}) + 1/\cosh(\sqrt{k^2+n^2}T)} \right)^2 \left(\varphi_{1,n}^{\delta} - \varphi_{2,n}^{\delta}\right)^2 \\
&\le \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{1}{\alpha(\sqrt{k^2+n^2}) + e^{-\sqrt{k^2+n^2}T}} \right)^2 \left(\varphi_{1,n}^{\delta} - \varphi_{2,n}^{\delta}\right)^2 \\
&\le \left(\frac{T}{\alpha(1+\ln(T/\alpha))} \right)^2 \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\varphi_{1,n}^{\delta} - \varphi_{2,n}^{\delta}\right)^2 \\
&= \left(\frac{T}{\alpha(1+\ln(T/\alpha))} \right)^2 \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} \left(\varphi_1^{\delta} - \varphi_2^{\delta}\right) \sin(nx)\, dx \right)^2 \\
&\le \left(\frac{T}{\alpha(1+\ln(T/\alpha))} \right)^2 \frac{\pi}{2} \cdot \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\pi} \left(\varphi_1^{\delta} - \varphi_2^{\delta}\right) \left(\frac{2}{\pi} \sin(nx) \right) dx \right)^2 \\
&\le \left(\frac{T}{\alpha(1+\ln(T/\alpha))} \right)^2 \left\| \varphi_1^{\delta} - \varphi_2^{\delta} \right\|^2.
\end{aligned} \tag{2.12}
$$

Next, the case that p is odd will be discussed. From (2.5), using the inequality $\frac{\cosh(\sqrt{k^2+n^2}T)}{\sinh(\sqrt{k^2+n^2}T)} \le \frac{2}{1-e^{-2(\sqrt{k^2+n^2}T)}} \le \frac{2}{1-e^{-2T}}$, we can obtain

$$
\begin{aligned}
&\left\| u_{\alpha 1}^{\delta}(\cdot,y) - u_{\alpha 2}^{\delta}(\cdot,y) \right\|^2 \\
&\quad = \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{\cosh(\sqrt{k^2+n^2}y)/\sinh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^p + 1/\sinh(\sqrt{k^2+n^2}T)} \right)^2 \left(\varphi_{1,n}^{\delta} - \varphi_{2,n}^{\delta}\right)^2 \\
&\quad \le \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{\cosh(\sqrt{k^2+n^2}T)/\sinh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2}) + e^{-(\sqrt{k^2+n^2})T}} \right)^2 \left(\varphi_{1,n}^{\delta} - \varphi_{2,n}^{\delta}\right)^2 \\
&\quad \le \left(\frac{2}{1-e^{-2T}} \right)^2 \left(\frac{T}{\alpha(1+\ln(T/\alpha))} \right)^2 \left\| \varphi_1^{\delta} - \varphi_2^{\delta} \right\|^2.
\end{aligned} \tag{2.13}
$$

By (2.12), (2.13), we have (2.7). □

In Theorem 2.2 below, we will verify that a stable approximation to the exact solution u given by (2.3) is the regularized solution u_{α}^{δ} given by (2.5).

Theorem 2.2 *Let u be the solution to equation* (1.2) *and u_{α}^{δ} be the solution to equation* (2.1a)–(2.1d). *Suppose that the measured data φ^{δ} satisfies $\|\varphi^{\delta} - \varphi\| \le \delta$ and the exact solution u satisfies $\|\frac{\partial^p u}{\partial y^p}(\cdot,T)\| \le E$ with $p \ge 1$. We choose the regularization parameter*

$$
\alpha = \delta. \tag{2.14}
$$

Then, for fixed $0 < y \le T$ *and* $\delta < T$, *we can obtain the following error estimate*:

$$\left\| u_\alpha^\delta(\cdot,y) - u(\cdot,y) \right\| \le C\left(1 + \ln\left(\frac{T}{\delta}\right)\right)^{-1}, \tag{2.15}$$

where $C = \frac{2}{1-e^{-2T}} T(1+E)$.

Proof Denote by u_α the solution of equation (2.1a)–(2.1d) corresponding to the exact data φ. We have

$$\left\| u_\alpha^\delta - u \right\| \le \left\| u_\alpha^\delta - u_\alpha \right\| + \|u_\alpha - u\|. \tag{2.16}$$

When p is even, from Theorem 2.1, we get

$$\left\| u_\alpha^\delta(\cdot,y) - u_\alpha(\cdot,y) \right\|^2 \le \left(\frac{T}{\alpha(1+\ln(T/\alpha))}\right)^2 \left\| \varphi^\delta - \varphi \right\|^2.$$

From (2.2), (2.3), (2.5), (2.11), we can obtain

$$\begin{aligned}
&\left\| u_\alpha(\cdot,y) - u(\cdot,y) \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \left(\frac{\cosh(\sqrt{k^2+n^2}y)/\cosh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^p + 1/\cosh(\sqrt{k^2+n^2}T)} - \cosh\left(\sqrt{k^2+n^2}y\right)\right)\varphi_n \sin(nx) \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \frac{\alpha(\sqrt{k^2+n^2})^p \cosh(\sqrt{k^2+n^2}y)}{\alpha(\sqrt{k^2+n^2})^p + 1/\cosh(\sqrt{k^2+n^2}T)} \varphi_n \sin(nx) \right\|^2 \\
&= \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha(\sqrt{k^2+n^2})^p \cosh(\sqrt{k^2+n^2}y)}{\alpha(\sqrt{k^2+n^2})^p + 1/\cosh(\sqrt{k^2+n^2}T)} \right)^2 \varphi_n^2 \\
&\le \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha}{\alpha(\sqrt{k^2+n^2}) + e^{-(\sqrt{k^2+n^2})T}} \right)^2 \left(\sqrt{k^2+n^2}\right)^{2p} \cosh^2\left(\sqrt{k^2+n^2}T\right)\varphi_n^2 \\
&\le \left(\frac{T}{1+\ln(T/\alpha)}\right)^2 \left\| \frac{\partial^p u}{\partial y^p}(\cdot,T) \right\|^2.
\end{aligned}$$

From (2.16) and the above two estimates, we have

$$\left\| u_\alpha^\delta(\cdot,y) - u(\cdot,y) \right\| \le \left(1 + \ln\left(\frac{T}{\delta}\right)\right)^{-1} T(1+E). \tag{2.17}$$

In the following equation, the case that p is odd is considered. From Theorem 2.1 and the inequality $\frac{\cosh(\sqrt{k^2+n^2}T)}{\sinh(\sqrt{k^2+n^2}T)} \le \frac{2}{1-e^{-2(\sqrt{k^2+n^2}T)}} \le \frac{2}{1-e^{-2T}}$, we have

$$\left\| u_\alpha^\delta(\cdot,y) - u_\alpha(\cdot,y) \right\|^2 \le \left(\frac{2}{1-e^{-2T}}\right)^2 \left(\frac{T}{\alpha(1+\ln(T/\alpha))}\right)^2 \left\| \varphi^\delta - \varphi \right\|^2, \tag{2.18}$$

$$\begin{aligned}
&\left\| u_\alpha(\cdot,y) - u(\cdot,y) \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \left(\frac{\cosh(\sqrt{k^2+n^2}y)/\sinh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^p + 1/\sinh(\sqrt{k^2+n^2}T)} - \cosh\left(\sqrt{k^2+n^2}y\right)\right)\varphi_n \sin(nx) \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_{n=1}^{\infty} \frac{\alpha(\sqrt{k^2+n^2})^p \cosh(\sqrt{k^2+n^2}y)}{\alpha(\sqrt{k^2+n^2})^p + 1/\sinh(\sqrt{k^2+n^2}T)} \varphi_n \sin(nx) \right\|^2 \\
&\leq \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha(\sqrt{k^2+n^2})^p \cosh(\sqrt{k^2+n^2}y)}{\alpha(\sqrt{k^2+n^2}) + e^{-(\sqrt{k^2+n^2}T)}} \right)^2 \varphi_n^2 \\
&\leq \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha}{\alpha(\sqrt{k^2+n^2}) + e^{-(\sqrt{k^2+n^2}T)}} \right)^2 \\
&\quad \times \left(\frac{\cosh(\sqrt{k^2+n^2}y)}{\sinh(\sqrt{k^2+n^2}T)} \right)^2 \left(\sqrt{k^2+n^2}\right)^{2p} \sinh^2\left(\sqrt{k^2+n^2}T\right) \varphi_n^2 \\
&\leq \left(\frac{T}{1+\ln(T/\alpha)} \right)^2 \left(\frac{2}{1-e^{-2T}} \right)^2 \left\| \frac{\partial^p u}{\partial y^p}(\cdot, T) \right\|^2.
\end{aligned} \tag{2.19}
$$

From (2.14), (2.18), (2.19), we get

$$
\begin{aligned}
\left\| u_\alpha^\delta(\cdot, y) - u(\cdot, y) \right\| &\leq \left(1 + \ln\left(\frac{T}{\delta} \right) \right)^{-1} \frac{2T}{1-e^{-2T}} \left(1 + \left\| \frac{\partial^p u}{\partial y^p}(\cdot, T) \right\| \right) \\
&\leq C \left(1 + \ln\left(\frac{T}{\delta} \right) \right)^{-1}.
\end{aligned} \tag{2.20}
$$

By (2.17), (2.20), the estimate form of (2.15) can be obtained. □

Secondly, as for equation (1.3), the following perturbation equation is considered:

$$
\begin{cases}
(v_\alpha^\delta)_{xx} + (v_\alpha^\delta)_{yy} - k^2(v_\alpha^\delta) = 0, & 0 < x < \pi, 0 < y < T, \\
v_\alpha^\delta(x, 0) = 0, & 0 \leq x \leq \pi, \\
(v_\alpha^\delta)_y(x, 0) + \alpha \frac{\partial^p v_\alpha^\delta(x,T)}{\partial y^p} = \psi^\delta(x), & 0 \leq x \leq \pi, \\
v_\alpha^\delta(0, y) = v_\alpha^\delta(\pi, y) = 0, & 0 \leq y \leq T,
\end{cases} \tag{2.21}
$$

where $p \geq 1$ is an integer, α is a regularization parameter, and the measured data $\psi^\delta \in L^2(0, \pi)$ satisfies

$$
\left\| \psi^\delta - \psi \right\| \leq \delta, \tag{2.22}
$$

the $\| \cdot \|$ denotes L^2-norm and the constant $\delta > 0$ is an error level.

By the technique of separation of variables, we get a solution to equation (1.3) as follows:

$$
v(x, y) = \sum_{n=1}^{\infty} \frac{\psi_n}{\sqrt{k^2+n^2}} \sin(nx) \sinh\left(\sqrt{k^2+n^2}y\right), \tag{2.23}
$$

where

$$
\psi_n = \frac{2}{\pi} \int_0^\pi \psi(x) \sin(nx)\, dx. \tag{2.24}
$$

In a similar way, we get that the solution to equation (2.21) is

$$v_\alpha^\delta(x,y)=\begin{cases}\sum_{n=1}^{\infty}\frac{\sinh(\sqrt{k^2+n^2})y)/\cosh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^{p-1}+1/\cosh(\sqrt{k^2+n^2}T)}\frac{\psi_n^\delta}{\sqrt{k^2+n^2}}\sin(nx) & p \text{ is odd},\\ \sum_{n=1}^{\infty}\frac{\sinh(\sqrt{k^2+n^2}y)/\sinh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^{p-1}+1/\sinh(\sqrt{k^2+n^2}T)}\frac{\psi_n^\delta}{\sqrt{k^2+n^2}}\sin(nx) & p \text{ is even},\end{cases} \tag{2.25}$$

where

$$\psi_n^\delta=\frac{2}{\pi}\int_0^\pi\psi^\delta(x)\sin(nx)\,dx. \tag{2.26}$$

Lemma 2.3 *Suppose* $0<y<T$, *then for* $\alpha<1$ *we get*

$$\sup_{n>0}\frac{e^{ny}}{n(1+\alpha e^{nT})}\le\frac{T}{\ln(1/\alpha)}\alpha^{-\frac{y}{T}}. \tag{2.27}$$

Lemma 2.3 is required in the following proof, and its proof can be found in [28].

Theorem 2.4 *Let* v *be the solution to equation* (1.3) *and* v_α^δ *be the solution to equation* (2.21). *Suppose that the measured data* ψ^δ *satisfies* $\|\psi^\delta-\psi\|\le\delta$ *and the exact solution* v *satisfies* $\|\frac{\partial^p v}{\partial y^p}(\cdot,T)\|\le E$ *with* $p\ge1$. *We choose the regularization parameter*

$$\alpha=\delta. \tag{2.28}$$

Then, for fixed $0<y\le T$ *and* $\delta<2$, *we get the following error estimate*:

$$\left\|v_\alpha^\delta(\cdot,y)-v(\cdot,y)\right\|\le 2^{\frac{y}{T}}\left(1-e^{-2T}\right)^{-\frac{y}{T}}\delta^{1-\frac{y}{T}}\frac{T}{\ln(2/\delta(1-e^{-2T}))}(1+E). \tag{2.29}$$

Proof Firstly, the case that p is odd will be proved. From the condition $\|\psi^\delta-\psi\|\le\delta$ we derive

$$\frac{\pi}{2}\sum_{n=1}^{\infty}\left(\psi_n^\delta-\psi_n\right)^2\le\delta^2. \tag{2.30}$$

Then, from (2.23), (2.25), (2.30), note that $n\ge1$, we get

$$\begin{aligned}&\left\|v_\alpha^\delta(\cdot,y)-v_\alpha(\cdot,y)\right\|^2\\ &\quad=\left\|\sum_{n=1}^{\infty}\frac{\sinh(\sqrt{k^2+n^2}y)/\cosh(\sqrt{k^2+n^2}T)}{\sqrt{k^2+n^2}(\alpha(\sqrt{k^2+n^2})^{p-1}+1/\cosh(\sqrt{k^2+n^2}T))}\left(\psi_n^\delta-\psi_n\right)\sin(nx)\right\|^2\\ &\quad=\frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\sinh(\sqrt{k^2+n^2}y)/\cosh(\sqrt{k^2+n^2}T)}{\sqrt{k^2+n^2}(\alpha(\sqrt{k^2+n^2})^{p-1}+1/\cosh(\sqrt{k^2+n^2}T))}\right)^2\left(\psi_n^\delta-\psi_n\right)^2\\ &\quad\le\frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\sinh(\sqrt{k^2+n^2}y)}{\sqrt{k^2+n^2}(1+\alpha\cosh(\sqrt{k^2+n^2}T))}\right)^2\left(\psi_n^\delta-\psi_n\right)^2\\ &\quad\le\frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{e^{\sqrt{k^2+n^2}y}}{\sqrt{k^2+n^2}(1+\frac{\alpha}{2}e^{\sqrt{k^2+n^2}T})}\right)^2\left(\psi_n^\delta-\psi_n\right)^2.\end{aligned}$$

From (2.27) in Lemma 2.3, for $\delta < 2$, we have

$$\left\| v_\alpha^\delta(\cdot,y) - v_\alpha(\cdot,y)\right\| \le 2^{\frac{y}{T}}\delta^{1-\frac{y}{T}}\frac{T}{\ln(2/\delta)}. \tag{2.31}$$

And

$$\begin{aligned}
&\left\| v_\alpha(\cdot,y) - v(\cdot,y)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty}\left(\frac{\sinh(\sqrt{k^2+n^2}y)/\cosh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^{p-1} + 1/\cosh(\sqrt{k^2+n^2}T)} - \sinh\left(\sqrt{k^2+n^2}y\right)\right)\right. \\
&\quad \left. \times \frac{\psi_n}{\sqrt{k^2+n^2}}\sin(nx)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty}\frac{\alpha(\sqrt{k^2+n^2})^{p-1}\sinh(\sqrt{k^2+n^2}y)}{\alpha(\sqrt{k^2+n^2})^{p-1} + 1/\cosh(\sqrt{k^2+n^2}T)}\frac{\psi_n}{\sqrt{k^2+n^2}}\sin(nx)\right\|^2 \\
&= \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha(\sqrt{k^2+n^2})^{p-1}\sinh(\sqrt{k^2+n^2}y)\cosh(\sqrt{k^2+n^2}T)}{\sqrt{k^2+n^2}(1+\alpha(\sqrt{k^2+n^2})^{p-1}\cosh(\sqrt{k^2+n^2}T))}\right)^2\psi_n^2 \\
&\le \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha(\sqrt{k^2+n^2})^{p-1}\sinh(\sqrt{k^2+n^2}y)\cosh(\sqrt{k^2+n^2}T)}{\sqrt{k^2+n^2}(1+\alpha\cosh(\sqrt{k^2+n^2}T))}\right)^2\psi_n^2 \\
&\le \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha e^{\sqrt{k^2+n^2}y}}{\sqrt{k^2+n^2}(1+\frac{\alpha}{2}e^{\sqrt{k^2+n^2}T})}\right)^2\left(\sqrt{k^2+n^2}\right)^{2(p-1)}\cosh^2\left(\sqrt{k^2+n^2}T\right)\psi_n^2,
\end{aligned}$$

thus

$$\left\| v_\alpha(\cdot,y) - v(\cdot,y)\right\| \le 2^{\frac{y}{T}}\delta^{1-\frac{y}{T}}\frac{T}{\ln(2/\delta)}\left\|\frac{\partial^p v}{\partial y^p}(\cdot,T)\right\|. \tag{2.32}$$

From (2.31), (2.32), we get

$$\left\| v_\alpha^\delta(\cdot,y) - v(\cdot,y)\right\| \le 2^{\frac{y}{T}}\delta^{1-\frac{y}{T}}\frac{T}{\ln(2/\delta)}(1+E). \tag{2.33}$$

To even p, note that $n \ge 1$, $\sinh(\sqrt{k^2+n^2}T) \ge 1/2e^{\sqrt{k^2+n^2}T}(1-e^{-2T})$, we have

$$\begin{aligned}
&\left\| v_\alpha^\delta(\cdot,y) - v_\alpha(\cdot,y)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty}\frac{\sinh(\sqrt{k^2+n^2}y)/\sinh(\sqrt{k^2+n^2}T)}{\sqrt{k^2+n^2}(\alpha(\sqrt{k^2+n^2})^{p-1} + 1/\sinh(\sqrt{k^2+n^2}T))}\left(\psi_n^\delta - \psi_n\right)\sin(nx)\right\|^2 \\
&= \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\sinh(\sqrt{k^2+n^2}y)/\sinh(\sqrt{k^2+n^2}T)}{\sqrt{k^2+n^2}(\alpha(\sqrt{k^2+n^2})^{p-1} + 1/\sinh(\sqrt{k^2+n^2}T))}\right)^2\left(\psi_n^\delta - \psi_n\right)^2 \\
&\le \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\sinh(\sqrt{k^2+n^2}y)}{\sqrt{k^2+n^2}(1+\alpha\sinh(\sqrt{k^2+n^2}T))}\right)^2\left(\psi_n^\delta - \psi_n\right)^2 \\
&\le \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{e^{\sqrt{k^2+n^2}y}}{\sqrt{k^2+n^2}(1+\frac{\alpha(1-e^{-2T})}{2}e^{\sqrt{k^2+n^2}T})}\right)^2\left(\psi_n^\delta - \psi_n\right)^2.
\end{aligned}$$

From (2.27) in Lemma 2.3, for $\delta < 2(1-e^{-2T})^{-1}$, we can obtain

$$\left\|v_\alpha^\delta(\cdot,y)-v_\alpha(\cdot,y)\right\| \le 2^{\frac{y}{T}}\left(1-e^{-2T}\right)^{-\frac{y}{T}}\delta^{1-\frac{y}{T}}\frac{T}{\ln(2/(\delta(1-e^{-2T})))}. \tag{2.34}$$

And

$$\begin{aligned}
&\left\|v_\alpha(\cdot,y)-v(\cdot,y)\right\|^2 \\
&= \left\|\sum_{n=1}^{\infty}\left(\frac{\sinh(\sqrt{k^2+n^2}y)/\sinh(\sqrt{k^2+n^2}T)}{\alpha(\sqrt{k^2+n^2})^{p-1}+1/\sinh(\sqrt{k^2+n^2}T)}-\sinh\left(\sqrt{k^2+n^2}y\right)\right)\right. \\
&\quad \left.\times\frac{\psi_n}{\sqrt{k^2+n^2}}\sin(nx)\right\|^2 \\
&= \left\|\sum_{n=1}^{\infty}\frac{\alpha(\sqrt{k^2+n^2})^{p-1}\sinh(\sqrt{k^2+n^2}y)}{\alpha(\sqrt{k^2+n^2})^{p-1}+1/\sinh(\sqrt{k^2+n^2}T)}\frac{\psi_n}{\sqrt{k^2+n^2}}\sin(nx)\right\|^2 \\
&= \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha(\sqrt{k^2+n^2})^{p-1}\sinh(\sqrt{k^2+n^2}y)\sinh(\sqrt{k^2+n^2}T)}{\sqrt{k^2+n^2}(1+\alpha(\sqrt{k^2+n^2})^{p-1}\sinh(\sqrt{k^2+n^2}T))}\right)^2\psi_n^2 \\
&\le \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha(\sqrt{k^2+n^2})^{p-1}\sinh(\sqrt{k^2+n^2}y)\sinh(\sqrt{k^2+n^2}T)}{\sqrt{k^2+n^2}(1+\alpha\sinh(\sqrt{k^2+n^2}T))}\right)^2\psi_n^2 \\
&\le \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha e^{\sqrt{k^2+n^2}y}}{\sqrt{k^2+n^2}(1+\frac{\alpha(1-e^{-2T})}{2}e^{\sqrt{k^2+n^2}T})}\right)^2\left(\sqrt{k^2+n^2}\right)^{2(p-1)}\sinh^2\left(\sqrt{k^2+n^2}T\right)\psi_n^2.
\end{aligned}$$

Then, from (2.27) in Lemma 2.3, we have

$$\left\|v_\alpha(\cdot,y)-v(\cdot,y)\right\| \le 2^{\frac{y}{T}}\left(1-e^{-2T}\right)^{-\frac{y}{T}}\delta^{1-\frac{y}{T}}\frac{T}{\ln(2/(\delta(1-e^{-2T})))}\left\|\frac{\partial^p v}{\partial y^p}(\cdot,T)\right\|. \tag{2.35}$$

Using (2.34), (2.35), we can obtain the error estimate (2.29). □

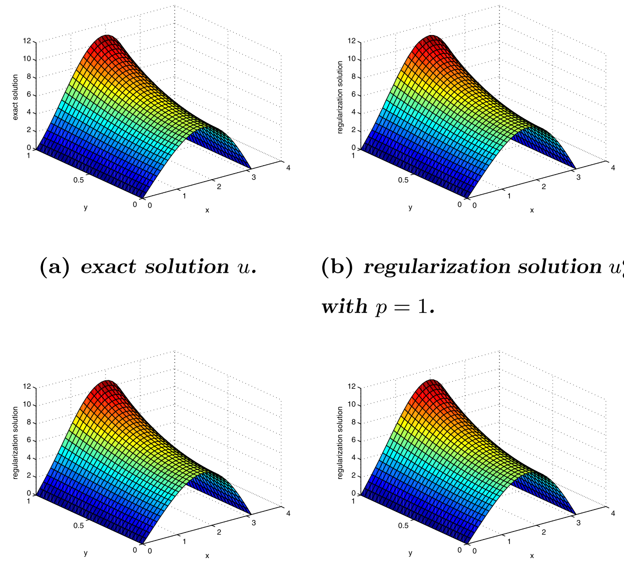
3 Numerical experiments

In order to verify the accuracy and efficiency of the proposed regularization method, two numerical examples are performed.

Example 1 The following direct problem for the modified Helmholtz equation is considered:

$$\begin{cases}
u_{xx}+u_{yy}-k^2u=0, & 0<x<\pi, 0<y<1, \\
u_y(x,0)=0, & 0\le x\le\pi, \\
u(x,1)=x(\pi-x)(1+x), & 0\le x\le\pi, \\
u(0,y)=u(\pi,y)=0, & 0\le y\le 1,
\end{cases} \tag{3.1}$$

where we take $T=1$.



(a) *exact solution* u.

(b) *regularization solution* u_α^δ *with* $p = 1$.

(c) *regularization solution* u_α^δ *with* $p = 2$.

(d) *regularization solution* u_α^δ *with* $p = 3$.

Figure 1 $\varepsilon = 1 \times 10^{-3}$, $k = 0.5$

By the technique of separation of variables, we can obtain the solution to the direct problem (3.1) as follows:

$$u(x,y) = \sum_{n=1}^{\infty} \varphi_n \sin(nx) \cosh\bigl(\sqrt{k^2 + n^2}y\bigr), \tag{3.2}$$

where $\varphi_n = \frac{2}{\pi \cosh(k^2+n^2)} d_n$, $d_n = \int_0^\pi x(\pi - x)(1 + x)\sin(nx)\,dx$, which can be computed by employing the Simpson formulation.

Next, the initial data $\varphi(x)$ is chosen as follows:

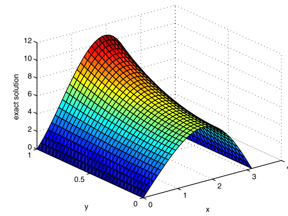
$$\varphi(x) = u(x,0) \approx \sum_{n=1}^{25} \varphi_n \sin(nx). \tag{3.3}$$

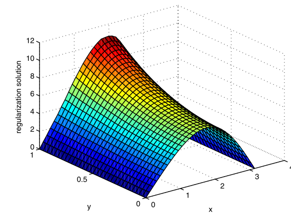
We give the measured data $\varphi^\delta(x_i) = \varphi(x_i) + \varepsilon\,\mathrm{rand}(i)$, where ε is an error level and

$$\delta := \|\varphi_\delta - \varphi\|_{l_2} = \left(\frac{1}{N_1} \sum_{i=1}^{N_1} \bigl|\varphi_\delta(x_i) - \varphi(x_i)\bigr|^2 \right)^{1/2}. \tag{3.4}$$

The function rand(·) denotes a random number uniformly distributed in the interval $[0,1]$. The relative root mean square error between the exact and regularization solution is given by

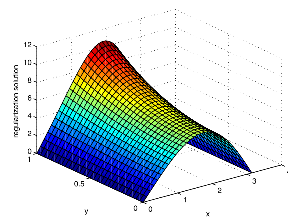
$$\epsilon(u) = \frac{\sqrt{\frac{1}{N_1 \times N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (u(x_i,y_j) - u_\alpha^\delta(x_i,y_j))^2}}{\sqrt{\frac{1}{N_1 \times N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (u(x_i,y_j))^2}}, \tag{3.5}$$

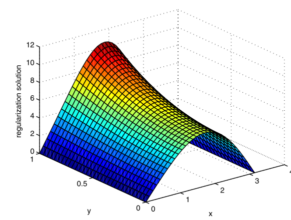




(a) ***exact solution*** u.

(b) ***regularization solution*** u_α^δ ***with*** $p = 1$.

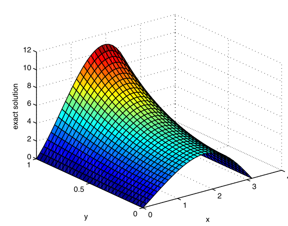


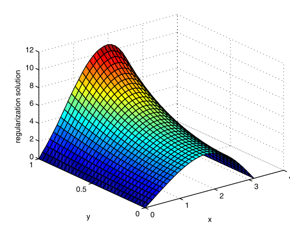


(c) ***regularization solution*** u_α^δ ***with*** $p = 2$.

(d) ***regularization solution*** u_α^δ ***with*** $p = 3$.

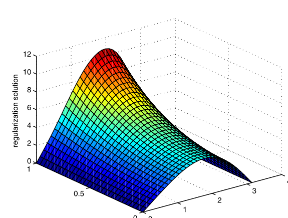
Figure 2 $\varepsilon = 1 \times 10^{-2}$, $k = 0.5$

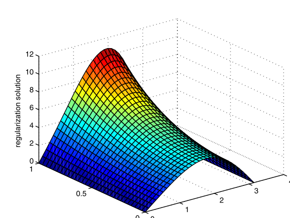




(a) ***exact solution*** u.

(b) ***regularization solution*** u_α^δ ***with*** $p = 1$.





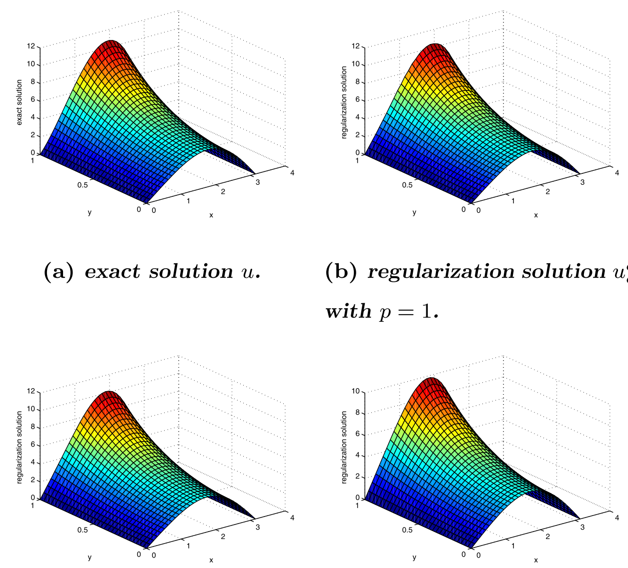
(c) ***regularization solution*** u_α^δ ***with*** $p = 2$.

(d) ***regularization solution*** u_α^δ ***with*** $p = 3$.

Figure 3 $\varepsilon = 1 \times 10^{-3}$, $k = 1.2$

where

$$x_i = \frac{(i-1)}{N_1 - 1}\pi, \qquad y_j = \frac{(j-1)}{N_2 - 1}, \quad i = 1, 2, \dots, N_1, j = 1, 2, \dots, N_2. \tag{3.6}$$



(a) ***exact solution*** u.

(b) ***regularization solution*** u_α^δ ***with*** $p = 1$.

(c) ***regularization solution*** u_α^δ ***with*** $p = 2$.

(d) ***regularization solution*** u_α^δ ***with*** $p = 3$.

Figure 4 $\varepsilon = 1 \times 10^{-2}$, $k = 1.2$

Table 1 $k = 0.5$, the relative root mean square errors $\epsilon(u)$ for various noise levels

ε	0.0001	0.001	0.005	0.01	0.05
$p = 1$	0.0014	0.0037	0.0117	0.0270	0.0814
$p = 2$	0.0025	0.0073	0.0205	0.0344	0.1184
$p = 3$	0.0046	0.0129	0.0311	0.0492	0.1323

Table 2 $k = 1.2$, the relative root mean square errors $\epsilon(u)$ for various noise levels

ε	0.0001	0.001	0.005	0.01	0.05
$p = 1$	0.0018	0.0056	0.0212	0.0394	0.1581
$p = 2$	0.0032	0.0113	0.0394	0.9705	0.2479
$p = 3$	0.0060	0.0203	0.0637	0.1075	0.3240

In the numerical computations, we only consider the cases when $p = 1, 2, 3$, and always take $N_1 = N_2 = 31$. We choose the regularization parameter α by (2.14).

We have shown the numerical results in Figs. 1–4 and Tables 1–2. The numerical results for $u(\cdot, y)$ and $u_\alpha^\delta(\cdot, y)$ with $k = 0.5$ and $\varepsilon = 0.001, 0.01$ are respectively shown in Fig. 1 and Fig. 2. The numerical results for $u(\cdot, y)$ and $u_\alpha^\delta(\cdot, y)$ with $k = 1.2$ and $\varepsilon = 0.001, 0.01$ are respectively shown in Fig. 3 and Fig. 4. The relative root mean square errors for the computed solution versus the error levels ε are respectively shown in Table 1 ($k = 0.5$) and Table 2 ($k = 1.2$).

Example 2 The following direct problem for the modified Helmholtz equation is considered:

$$\begin{cases} v_{xx} + v_{yy} - k^2 v = 0, \quad 0 < x < \pi, 0 < y < 1, \\ v(x,0) = 0, \quad 0 \le x \le \pi, \\ v_y(x,1) = x(\pi - x), \quad 0 \le x \le \pi, \\ v(0,y) = v(\pi,y) = 0, \quad 0 \le y \le 1, \end{cases} \tag{3.7}$$

where $T = 1$.

By the technique of separation of variables, we get the solution to the direct problem (3.7) as follows:

$$v(x,y) = \sum_{n=1}^{\infty} \frac{2}{\pi \sinh(\sqrt{k^2+n^2})} \int_0^{\pi} x(\pi - x)\sin(nx)\,dx \sin(nx)\sinh\bigl(\sqrt{k^2+n^2}y\bigr). \tag{3.8}$$

Then

$$v_y(x,y) = \sum_{n=1}^{\infty} \psi_n \sin(nx)\cosh\bigl(\sqrt{k^2+n^2}y\bigr), \tag{3.9}$$

where $\psi_n = \frac{2n}{\pi \sinh(k^2+n^2)} e_n$, $e_n = \int_0^{\pi} x(\pi - x)\sin(nx)\,dx$.

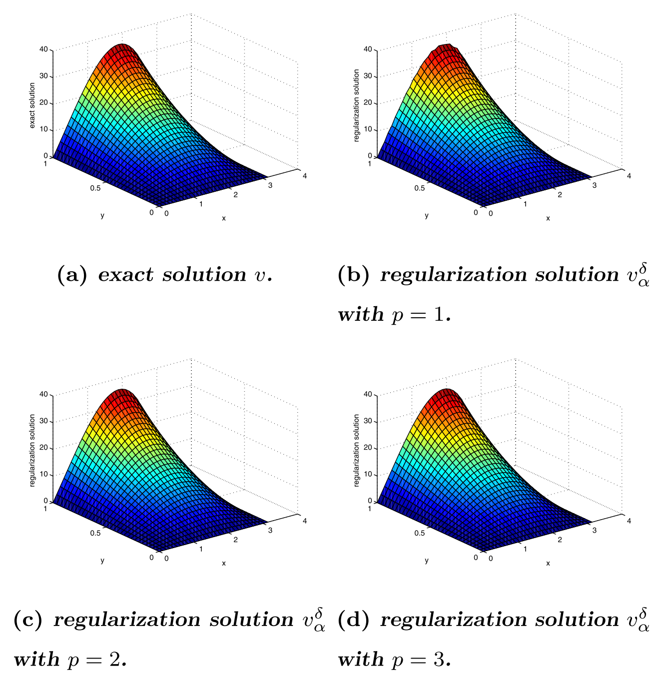
We give the initial data

$$\psi(x) = v_y(x,0) \approx \sum_{n=1}^{20} \psi_n \sin(nx) \tag{3.10}$$

and the measured data $\psi^{\delta}(x_i) = \psi(x_i) + \varepsilon\,\mathrm{rand}(i)$, where ε is an error level.

We have shown the numerical results in Figs. 5–8 and Tables 3–4. The numerical results for $v(\cdot,y)$ and $v_{\alpha}^{\delta}(\cdot,y)$ with $k = 0.5$ and $\varepsilon = 0.001, 0.01$ are respectively shown in Fig. 5 and Fig. 6. The numerical results for $v(\cdot,y)$ and $v_{\alpha}^{\delta}(\cdot,y)$ with $k = 1.2$ and $\varepsilon = 0.001, 0.01$ are respectively shown in Fig. 7 and Fig. 8. The relative root mean square errors for the computed solution versus the error levels ε are respectively shown in Table 3 ($k = 0.5$) and Table 4 ($k = 1.2$).

By Figs. 1–8 and Tables 1–4, we observe that our proposed method is effective and stable. From Tables 1–2 and 3–4, we note that the smaller ε is, the better the calculation effect is, which means that our proposed regularization method is convergent with respect to decreasing the noise level ε. In addition, from Tables 1 to 4, we can see the relative root mean square errors $\epsilon(u) = 0.0014$ and $\epsilon(v) = 0.0313$ for various noise levels when $k = 0.5, \varepsilon = 0.0001, p = 1$, and the relative root mean square errors $\epsilon(u) = 0.0018$ and $\epsilon(v) = 0.0204$ for various noise levels when $k = 1.2, \varepsilon = 0.0001, p = 1$. Since $\epsilon(u) = 0.0014$ is less than $\epsilon(v) = 0.0313$ and $\epsilon(u) = 0.0018$ is less than $\epsilon(v) = 0.0204$, our proposed method is more effective to problem (1.2) than to (1.3) when $p = 1$. These results show that our proposed method is applicable in dealing with Cauchy problem (1.2) and (1.3).



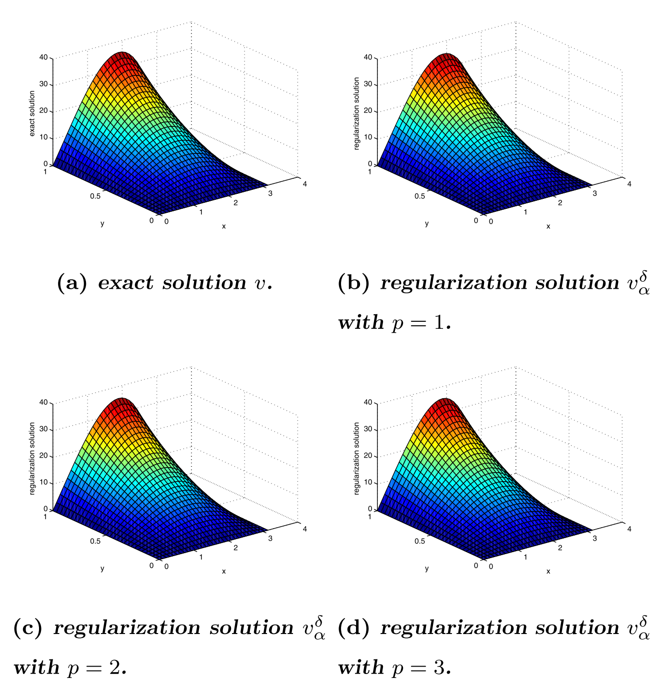
(a) ***exact solution*** v.

(b) ***regularization solution*** v_α^δ ***with*** $p = 1$.

(c) ***regularization solution*** v_α^δ ***with*** $p = 2$.

(d) ***regularization solution*** v_α^δ ***with*** $p = 3$.

Figure 5 $\varepsilon = 1 \times 10^{-3}$, $k = 0.5$



(a) ***exact solution*** v.

(b) ***regularization solution*** v_α^δ ***with*** $p = 1$.

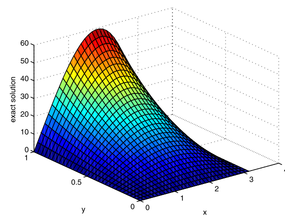
(c) ***regularization solution*** v_α^δ ***with*** $p = 2$.

(d) ***regularization solution*** v_α^δ ***with*** $p = 3$.

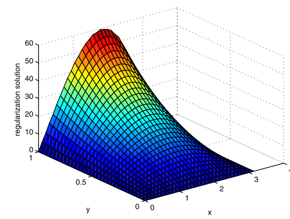
Figure 6 $\varepsilon = 1 \times 10^{-2}$, $k = 0.5$

4 Conclusions

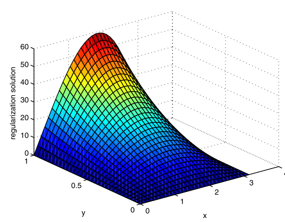
In this study, we propose a quasi-reversibility regularization method to solve a Cauchy problem for the modified Helmholtz-type equation. The error and stability estimates for



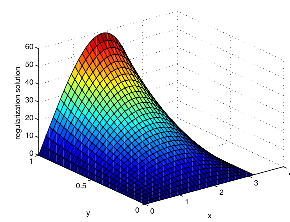
(a) ***exact solution*** v.



(b) ***regularization solution*** v_α^δ ***with*** $p = 1$.

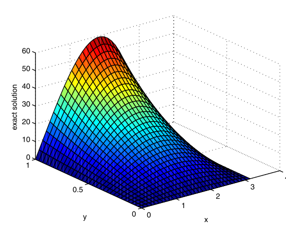


(c) ***regularization solution*** v_α^δ ***with*** $p = 2$.

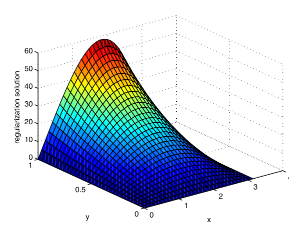


(d) ***regularization solution*** v_α^δ ***with*** $p = 3$.

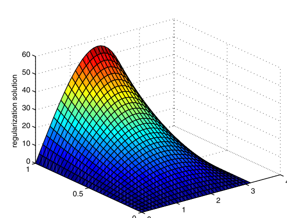
Figure 7 $\varepsilon = 1 \times 10^{-3}$, $k = 1.2$



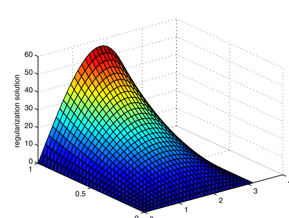
(a) ***exact solution*** v.



(b) ***regularization solution*** v_α^δ ***with*** $p = 1$.



(c) ***regularization solution*** v_α^δ ***with*** $p = 2$.



(d) ***regularization solution*** v_α^δ ***with*** $p = 3$.

Figure 8 $\varepsilon = 1 \times 10^{-2}$, $k = 1.2$

$0 < y \le T$ have been obtained under a-priori bound assumptions for the exact solution. Some numerical results show that our proposed regularization method is effective and stable. In addition, our proposed method is easy to be extended to the three-dimensional

Table 3 $k = 0.5$, the relative root mean square errors $\epsilon(v)$ for various noise levels

ε	0.0001	0.001	0.005	0.01	0.05
$p = 1$	0.0313	0.0040	0.0096	0.0183	0.0815
$p = 2$	0.0019	0.0043	0.0129	0.0223	0.0850
$p = 3$	0.0029	0.0084	0.0226	0.0367	0.1206

Table 4 $k = 1.2$, the relative root mean square errors $\epsilon(v)$ for various noise levels

ε	0.0001	0.001	0.005	0.01	0.05
$p = 1$	0.0204	0.0034	0.0129	0.0251	0.1127
$p = 2$	0.0014	0.0048	0.0199	0.0378	0.1573
$p = 3$	0.0223	0.0093	0.0355	0.0651	0.2426

case and the proofs are similar. It should be mentioned that the method of separation of variables is used to give the expression of solution, so the proposed method in this paper can be extended to solve the Cauchy problems of Helmholtz-type equation in a cylindrical domain. But it cannot be applied in more general geometries, which is a limit of the non-local boundary value problem method.

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Authors' contributions
HY and QYY completed the main study together. HY wrote the manuscript, QYY checked the proofs process and verified the calculation. Moreover, all the authors read and approved the last version of the manuscript.

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