Qing and Zhang *Boundary Value Problems* (2019) 2019:35
https://doi.org/10.1186/s13661-019-1149-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Sharp criteria of blow-up solutions for the cubic nonlinear beam equation

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Abstract
In this paper, we obtain the precisely sharp criteria of blow-up and global existence for the cubic nonlinear beam equation in the H^2 energy-critical and H^2 sub-critical cases, respectively.

MSC: 35G25; 35B44

Keywords: Cubic nonlinear beam equation; H^2 energy-critical; Sharp criteria; Blow-up

1 Introduction

In this paper, we consider the Cauchy problem of the nonlinear beam equation

$$\begin{cases} \frac{\partial^2}{\partial t^2}u + \Delta^2 u + mu - |u|^2 u = 0, & t \geq 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0, & \frac{\partial}{\partial t}u(0,x) = u_1, \end{cases} \tag{1.1}$$

where $u = u(t,x)$: $\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and d is the space dimension; $\Delta^2 = \Delta\Delta$ and $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian. The parameter $m > 0$. In fact, Eq. (1.1) is a class of fourth-order partial differential equations having different physical settings (see [19]). In particular, when $d = 1$, Eq. (1.1) is called Bretherton's type equation, which appears in studying of weak interactions of dispersive waves (see [2]). When $d = 2$, Eq. (1.1) models the motion of the clamped plate and beams (see [15]). During the last two decades, Eq. (1.1) has been widely studied (see [18] for a review).

The local well-posedness of Cauchy problem (1.1) was established by Levandosky in [12, 13]. The stability of traveling waves and standing waves was investigated by Levandosky in [12, 13]. The asymptotic behavior and scattering properties of global solutions were widely studied in [3, 14, 17]. On the other hand, for the blow-up solutions, a sufficient condition for the existence of blow-up solutions was given by Hebey and Pausader in [9]. In the L^2-critical case: $m = 1$, $d = 4$, the sharp criteria and limiting profile of blow-up solutions were investigated by Zheng and Leng in [22].

This motivates us to further study the blow-up solutions of Eq. (1.1) in the following sense: Under what conditions will the waves become unstable to collapse (blow-up)? Under what conditions will the waves be stable for all time (global existence)? In other words, what are the sharp criteria of blow-up and global existence?

In the present paper, we focus on the two typical cases: H^2 energy-critical (i.e., $d = 8$) and H^2 sub-critical (i.e., $d = 6$). Indeed, we remark that Eq. (1.1) has the scaling symmetry $v^\lambda := \lambda^2 u(\lambda^4 t, \lambda x)$ provided $m = 0$. More precisely, when $d = 8$, v^λ is a solution of Eq. (1.1) provided u solves Eq. (1.1) and $\|v^\lambda\|_{\dot{H}^2} = \|u\|_{\dot{H}^2}$. This case is called the H^2 energy-critical (see [9]). And when $d = 6$, this case is called the H^2 sub-critical due to $\|v^\lambda\|_{\dot{H}^s} = \|u\|_{\dot{H}^s}$ being invariant, where $s < 2$.

Our main arguments are from the studies of the nonlinear Schrödinger equations. In fact, Kenig and Merle in [11] and Holmer and Roudenko in [10] injected the sharp Gagliardo–Nirenberg inequality into the energy functional to obtain the sharp threshold of blow-up and global existence for the nonlinear Schrödinger equations, in which the scaling invariance and conservation of L^2-norm play a crucial role. For the cubic nonlinear Schrödinger equations with defect, Goubet and Hamraoui in [7] investigated both numerically and theoretically the influence of a defect on the blow-up of radial solutions to a cubic NLS equation in dimension two.

But for Eq. (1.1), the scaling invariance and conservation of L^2-norm fail, which is the main difficulty. For the H^2 energy-critical case: $d = 8$, by injecting some new estimates into the best constant of the Sobolev embedding inequality, we find two invariant sets and obtain the precisely sharp criteria of blow-up and global existence for Eq. (1.1). For the H^2 sub-critical case: and $d = 6$, the best constant of the Sobolev inequality is not determined, and the method in [11] for the H^1 energy-critical Schrödinger equation and the method in [10] for the L^2 sup-critical nonlinear Schrödinger equation cannot be directly applied. By exploring the convex properties of the energy functional, we can find two invariant sets and obtain precisely sharp criteria of blow-up and global existence for Eq. (1.1).

2 Notations and preliminaries

In this paper, we abbreviate $L^q(\mathbb{R}^d)$, $\|\cdot\|_{L^q(\mathbb{R}^d)}$, $H^2(\mathbb{R}^d)$, $\dot{H}^2(\mathbb{R}^d)$, and $\int_{\mathbb{R}^d} \cdot\, dx$ by L^q, $\|\cdot\|_q$, H^2, $\dot{H}^2$, and $\int \cdot\, dx$. The various positive constants will be simply denoted by C.

The work space is defined by

$$H^2 := \left\{ v \in L^2 \,\middle|\, \int \left(|v|^2 + |\nabla v|^2 + |\Delta v|^2\right) dx < +\infty \right\}.$$

The norm is denoted by $\|v\|_{H^2} = (\|v\|_2^2 + \|\nabla v\|_2^2 + \|\Delta v\|_2^2)^{\frac{1}{2}}$, which is equivalent to $(\|v\|_2^2 + \|\Delta v\|_2^2)^{\frac{1}{2}}$. For Cauchy problem (1.1), we define two functionals in $H^2 \times L^2$ by

$$E\left(\left(v(t), \frac{\partial}{\partial t} v(t)\right)\right) := \int \left[\frac{1}{2}\left|\frac{\partial}{\partial t} v(t)\right|^2 + \frac{1}{2}\left|\Delta v(t)\right|^2 + \frac{1}{2}\left|v(t)\right|^2 - \frac{1}{4}\left|v(t)\right|^4\right] dx,$$

$$H\big(v(t)\big) := \int \left[\frac{1}{2}\left|v(t)\right|^2 dx - \frac{1}{4}\left|v(t)\right|^4\right] dx.$$

The functionals E and H are well-defined by the Sobolev embedding theorem (see [9]). The local well-posedness of Cauchy problem (1.1) is established by Hebey and Pausader in the energy space $H^2 \times L^2$ (see [9]) as follows.

Proposition 2.1 *Let $m = 1$, $1 \le d \le 8$, and $(u_0, u_1) \in H^2 \times L^2$. There exists a unique solution $u(t,x)$ of Cauchy problem (1.1) on $[0,T)$ such that $u(t,x) \in C([0,T); H^2 \times L^2)$. Moreover,* *either* $T = +\infty$ (*global existence*) *or* $0 < T < +\infty$ *and* $\lim_{t\to T} \|u(t,x)\|_{H^2} = +\infty$ (*blow-up*). *Furthermore, for all* $t \in [0,T)$, $u(t,x)$ *satisfies the following conservation law*:

$$E\left(\left(u(t), \frac{\partial}{\partial t}u(t)\right)\right) = E\big((u_0,u_1)\big). \tag{2.1}$$

Remark 2.2 It follows from Hebey and Pausader's result in [9] that the local well-posedness of Cauchy problem (1.1) is also true in $H^2 \times L^2$. Moreover, let $u(t,x) \in C([0,T); H^2 \times L^2)$ be the solution of Cauchy problem (1.1). If $0 < T < +\infty$, then $\lim_{t\to T} \|u(t,x)\|_{H^2} = +\infty$ or $\limsup_{t\to T} \|u(t,x)\|_{L_t^q([0,T);L_x^r)} = +\infty$ (blow-up), where $(q,r) = (\frac{2(d+4)}{d-4}, \frac{2d(d+4)}{(d-4)(d+2)})$ is B-admissible.

Next, we introduce two important sharp inequalities, which are used to describe the sharp criteria of blow-up and global existence for Cauchy problem (1.1).

Lemma 2.3 *Let* $v \in H^2$. *Then*

$$\|v\|_4^4 \le \frac{2}{\|Q\|_2^2} \|v\|_2^{\frac{8-d}{2}} \|\Delta v\|_2^{\frac{d}{2}}, \tag{2.2}$$

where Q *is a ground state of*

$$\frac{d}{4}\Delta^2 Q + \frac{8-d}{4} Q - |Q|^2 Q = 0, \quad Q \in H^2. \tag{2.3}$$

Lemma 2.4 *Let* $d = 8$ *and* $W(x) = \frac{8\sqrt{30}}{(1+|x|^2)^2}$ *solve*

$$\Delta^2 W - |W|^2 W = 0, \quad W \in \dot{H}^2. \tag{2.4}$$

Then the best constant $C_* > 0$ *of the Sobolev inequality*

$$\|v\|_4 \le C_* \|\Delta v\|_2, \quad v \in \dot{H}^2 \tag{2.5}$$

is given by $C_* = \|\Delta W\|_2^{-\frac{1}{2}}$, *where* W *is the solution of* (2.4).

Remark 2.5 The sharp Gagliardo–Nirenberg inequality (2.2) was obtained by Fibich, Ilan, and Papanicolaou in [6], and the existence of the ground state solution of Eq. (2.3) was proved by Zhu, Zhang, and Yang in [23]. For the sharp generalized Gagliardo–Nirenberg inequalities (2.2) with fractional order derivatives, the readers can refer to [4, 5, 8, 21, 24, 25]. The sharp Sobolev inequality (2.5) was established in [1, 16, 20].

3 Sharp criteria of blow-up and global existence

In this section, we first consider the H^2 energy-critical case: $p = 3$ and $d = 8$. By constructing the invariants, we can find the sharp criteria of blow-up and global existence for Cauchy problem (1.1).

Theorem 3.1 *Let* $m = 1$, $p = 3$, $d = 8$, *and* W *be the solution of* (2.4). *Assume that* $(u_0,u_1) \in H^2 \times L^2$ *and*

$$E\big((u_0,u_1)\big) < \frac{1}{4}\|\Delta W\|_2^2. \tag{3.1}$$

Then

(i) *if* $\|\Delta u_0\|_2^2 + \|u_0\|_2^2 < \|\Delta W\|_2^2$, *then the solution* $u(t,x)$ *of Cauchy problem* (1.1) *exists globally, and* $u(t,x)$ *satisfies that, for all time* t,

$$\left\|\Delta u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2 < \|\Delta W\|_2^2, \tag{3.2}$$

(ii) *if* $\|\Delta u_0\|_2^2 + \|u_0\|_2^2 > \|\Delta W\|_2^2$, *then the solution* $u(t,x)$ *of Cauchy problem* (1.1) *blows up in finite time* $0 < T < +\infty$.

Proof Firstly, applying the best constant of Sobolev inequality (2.5) to the energy functional E, for all $t \in I$ (maximal existence interval), we get

$$\begin{aligned} E\left(\left(u(t), \frac{\partial}{\partial t}u(t)\right)\right) &\geq \frac{1}{2}\left(\left\|\Delta u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2\right) - \frac{C_*^4}{4}\left(\left\|\Delta u(t)\right\|_2^2\right)^2 \\ &\geq \frac{1}{2}\left(\left\|\Delta u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2\right) - \frac{C_*^4}{4}\left(\left\|\Delta u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2\right)^2, \end{aligned} \tag{3.3}$$

where $C_* = \|\Delta W\|_2^{-\frac{1}{2}}$ and W is the solution of (2.4). Now, define a function on the interval $[0, +\infty)$ by

$$f(y) := \frac{1}{2}y^2 - \frac{C_*^4}{4}y^4.$$

Then, we see that $f'(y) = y - C_*^4 y^3 = y(1 - C_*^4 y^2)$, and there are two roots for the equation $f'(y) = 0$: $y_1 = 0$, $y_2 = \frac{1}{C_*^2} = \|\Delta W\|_2$. Hence, y_1 is the minimizer of $f(y)$ and y_2 is the maximum of $f(y)$. Meanwhile, $f(y)$ is increasing on $[y_1, y_2)$ and decreasing on $[y_2, +\infty)$. Note that $f(y_1) = 0$ and $f(y_2) = \frac{\|\Delta W\|_2^2}{4}$. It follows from (2.1) and (3.1) that, for all $t \in I$,

$$f\left(\sqrt{\left\|\Delta u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2}\right) \leq E\left(\left(u(t), \frac{\partial}{\partial t}u(t)\right)\right) = E\left((u_0, u_1)\right) < f(y_2). \tag{3.4}$$

Secondly, using the convexity and monotony of $f(y)$ and the conservation of energy, we construct two invariant evolution flows generated by Cauchy problem (1.1) as follows:

$$\begin{aligned} K_1 &:= \left\{u(t) \in H^2 \setminus \{0\} \,\middle|\, \left\|\Delta u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2 < \|\Delta W\|_2^2, 0 < E < \frac{\|\Delta W\|_2^2}{4}\right\}, \\ K_2 &:= \left\{u(t) \in H^2 \setminus \{0\} \,\middle|\, \left\|\Delta u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2 > \|\Delta W\|_2^2, 0 < E < \frac{\|\Delta W\|_2^2}{4}\right\}, \end{aligned}$$

where $E := E((u(t), \frac{\partial}{\partial t}u(t)))$. Indeed, if $u_0 \in K_1$, i.e., $\|\Delta u_0\|_2^2 + \|u_0\|_2^2 < \|\Delta W\|_2^2$, then $\sqrt{\|\Delta u_0\|_2^2 + \|u_0\|_2^2} < y_2$. Since $f(y)$ is increasing on $[0, y_2)$ and for all $t \in I$,

$$f\left(\sqrt{\left\|\Delta u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2}\right) \leq E\left((u_0, u_1)\right) < \frac{\|\Delta W\|_2^2}{4} = f_{\max} = f(y_2). \tag{3.5}$$

According to the bootstrap and continuity argument, we can show that the corresponding solution $u(t,x)$ satisfies that, for all $t \in I$,

$$\sqrt{\left\|\Delta u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2} < y_2, \tag{3.6}$$

which implies that K_1 is invariant. In fact, if (3.6) is not true, then there exists $t_1 \in I$ such that $\sqrt{\|\Delta u(t_1)\|_2^2 + \|u(t_1)\|_2^2} \geq y_2$. Since the corresponding solution $u(t,x) \in C([0,T); H^2 \times L^2)$ of Cauchy problem (1.1) is continuous with respect to t, there exists $0 < t_0 \leq t_1$ such that

$$\sqrt{\|\Delta u(t_0)\|_2^2 + \|u(t_0)\|_2^2} = y_2.$$

Thus, injecting this into (3.5) with $t = t_0$,

$$f(y_2) = f\left(\sqrt{\|\Delta u(t_0)\|_2^2 + \|u(t_0)\|_2^2}\right) \leq E\bigl((u_0, u_1)\bigr) < \frac{\|\Delta W\|_2^2}{4} = f_{\max} = f(y_2),$$

which is a contradiction. Hence, (3.6) is true. On the other hand, if $u_0 \in K_2$, i.e., $\|\Delta u_0\|_2^2 + \|u_0\|_2^2 > \|\Delta W\|_2^2$, then $\sqrt{\|\Delta u_0\|_2^2 + \|u_0\|_2^2} > y_2$. Since $f(y)$ is decreasing on $[y_2, +\infty)$ and for all $t \in I$,

$$f\left(\sqrt{\|\Delta u(t)\|_2^2 + \|u(t)\|_2^2}\right) \leq E\bigl((u_0, u_1)\bigr) < \frac{\|\Delta W\|_2^2}{4} = f_{\max} = f(y_2).$$

According to the bootstrap and continuity argument (see the proof of the invariance of K_1), the corresponding solution $u(t,x)$ satisfies that, for all $t \in I$,

$$\sqrt{\|\Delta u(t)\|_2^2 + \|u(t)\|_2^2} > y_2, \tag{3.7}$$

which implies that K_2 is invariant.

Finally, we return to the proof of Theorem 3.1. By (3.1) and $\|\Delta u_0\|_2^2 + \|u_0\|_2^2 < \|\Delta W\|_2^2$, we get $u_0 \in K_1$. Applying the invariant of K_1, (3.2) is true and the solution $u(t,x)$ of Cauchy problem (1.1) exists globally by the local well-posedness (see Proposition 2.1). This completes part (i) of the proof.

On the other hand, (3.1) and $\|\Delta u_0\|_2^2 + \|u_0\|_2^2 > \|\Delta W\|_2^2$ imply $u_0 \in K_2$. Applying the invariant of K_2, (3.7) is true. According to the conservation of energy and assumption (3.1), we deduce that, for all $t \in I$,

$$\begin{aligned}\|u(t)\|_4^4 &= -4E\bigl((u_0, u_1)\bigr) + 2\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 + 2\|\Delta u(t)\|_2^2 + 2\|u(t)\|_2^2 \\ &> -\|\Delta W\|_2^2 + 2\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 + 2\|\Delta u(t)\|_2^2 + 2\|u(t)\|_2^2.\end{aligned} \tag{3.8}$$

Letting $J(t) := \int |u(t,x)|^2\,dx$ and computing the derivatives of $J(t)$, we see that, for all $t \in I$, $J'(t) = 2\int u(t)\frac{\partial}{\partial t}u(t)\,dx$ and

$$J''(t) = 2\int \left(\left|\frac{\partial}{\partial t}u(t)\right|^2 + |u(t)|^4 - |\Delta u(t)|^2 - |u(t)|^2\right)dx. \tag{3.9}$$

Inject (3.8) into (3.9). For all $t \in I$,

$$J''(t) \geq 6\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 - 2\|\Delta W\|_2^2 + 2\|\Delta u(t)\|_2^2 + 2\|u(t)\|_2^2 > 6\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2, \tag{3.10}$$

which implies that $J''(t)$ is positive and has a lower bound for all $t \in I$. Applying the Hölder inequality to $J'(t)$, we get

$$\big(J'(t)\big)^2 \le 4\big\|u(t)\big\|_2^2\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2$$

for all $t \in I$. Multiplying (3.10) with $J(t)$, we deduce that, for all $t \in I$,

$$J(t)J''(t) > 6\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2\big\|u(t)\big\|_2^2 > \frac{3}{2}\big(J'(t)\big)^2, \tag{3.11}$$

and there exists $t_0 > 0$ such that $J'(t) > 0$ for all $t > t_0$. Therefore, as the proof of Theorem 3.1, the solution $u(t,x)$ of Cauchy problem (1.1) blows up in finite time $0 < T < +\infty$. □

Now, we investigate the sharp criteria of blow-up and global existence for Cauchy problem (1.1) in the H^2 sub-critical case: $p = 3$ and $d = 6$.

Theorem 3.2 *Let* $m = 1$, $p = 3$, *and* $d = 6$. *Assume that* $(u_0,u_1) \in H^2 \times L^2$ *and*

$$E\big((u_0,u_1)\big) < \frac{\sqrt{3}}{6}\|Q\|_2^2. \tag{3.12}$$

Then

(i) *if* $\|\Delta u_0\|_2 < (\frac{4}{3})^{\frac{1}{4}}\|Q\|_2$, *then the solution* $u(t,x)$ *of Cauchy problem* (1.1) *exists globally and* $u(t,x)$ *satisfies that, for all time* t,

$$\big\|\Delta u(t)\big\|_2 < \left(\frac{4}{3}\right)^{\frac{1}{4}}\|Q\|_2, \tag{3.13}$$

(ii) *if* $\|\Delta u_0\|_2 > (\frac{4}{3})^{\frac{1}{4}}\|Q\|_2$, *then the solution* $u(t,x)$ *of Cauchy problem* (1.1) *blows up in finite time* $0 < T < +\infty$,

where Q *is the ground state of* (2.3).

Proof It follows from the proof of Theorem 3.1 that we give the main schedule of the proof of Theorem 3.2 in the following. From the sharp generalized Gagliardo–Nirenberg inequality (2.2) with $d = 6$, we see that, for all $t \in I$ (maximal existence interval),

$$\begin{aligned} E\left(\left(u(t),\frac{\partial}{\partial t}u(t)\right)\right) &\ge \frac{1}{2}\big\|\Delta u(t)\big\|_2^2 + \frac{1}{2}\big\|u(t)\big\|_2^2 - \frac{1}{2\|Q\|_2^2}\big\|u(t)\big\|_2\big\|\Delta u(t)\big\|_2^3 \\ &\ge \frac{1}{2}\big\|\Delta u(t)\big\|_2^2 + \frac{1}{2}\big\|u(t)\big\|_2^2 - \frac{1}{2}\big\|u(t)\big\|_2^2 - \frac{1}{8\|Q\|_2^4}\big\|\Delta u(t)\big\|_2^6 \\ &\ge \frac{1}{2}\big\|\Delta u(t)\big\|_2^2 - \frac{1}{8\|Q\|_2^4}\big\|\Delta u(t)\big\|_2^6. \end{aligned} \tag{3.14}$$

Define a function $f(y)$ on the interval $[0,+\infty)$ by

$$f(y) = \frac{1}{2}y^2 - \frac{1}{8\|Q\|_2^4}y^6.$$

We see that $y_1 = 0$ is the minimizer of $f(y)$ and $y_2 = (\frac{4}{3})^{\frac{1}{4}}\|Q\|_2$ is the maximum of $f(y)$. Meanwhile, $f(y)$ is increasing on $[y_1, y_2)$ and decreasing on $[y_2, +\infty)$. Note that $f(y_1) = 0$ and $f(y_2) = \frac{2\sqrt{3}}{9}\|Q\|_2^2$. By (2.1) and (3.12), we see that, for all $t \in I$,

$$f(\|\Delta u(t)\|_2) \leq E\left(\left(u(t), \frac{\partial}{\partial t}u(t)\right)\right) = E((u_0, u_1)) < \frac{\sqrt{3}}{6}\|Q\|_2^2 < f(y_2). \tag{3.15}$$

Then, as in the proof of Theorem 3.1, we can construct two invariant evolution flows generated by Cauchy problem (1.1) as follows:

$$K_1 := \left\{u(t) \in H^2 \setminus \{0\} \,\middle|\, \|\Delta u(t)\|_2 < \left(\frac{4}{3}\right)^{\frac{1}{4}}\|Q\|_2, 0 < E < \frac{\sqrt{3}}{6}\|Q\|_2^2\right\},$$

$$K_2 := \left\{u(t) \in H^2 \setminus \{0\} \,\middle|\, \|\Delta u(t)\|_2 > \left(\frac{4}{3}\right)^{\frac{1}{4}}\|Q\|_2, 0 < E < \frac{\sqrt{3}}{6}\|Q\|_2^2\right\},$$

where $E := E((u(t), \frac{\partial}{\partial t}u(t)))$. Finally, using the invariances of K_1 and K_2, we can complete the proof of Theorem 3.2. (3.12) and $\|\Delta u_0\|_2 < (\frac{4}{3})^{\frac{1}{4}}\|Q\|_2$ imply that $u_0 \in K_1$. Applying the invariant of K_1, (3.13) is true and the solution $u(t,x)$ of Cauchy problem (1.1) exists globally by Proposition 2.1. This completes the proof of part (i).

(3.12) and $\|\Delta u_0\|_2 > (\frac{4}{3})^{\frac{1}{4}}\|Q\|_2$ imply that $u_0 \in K_2$. Applying the invariant of K_2, (2.1), and (3.12), we deduce that, for all $t \in I$,

$$\begin{aligned} 2\|u(t)\|_4^4 &= -8E((u_0, u_1)) + 4\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 + 4\|\Delta u(t)\|_2^2 + 4\|u(t)\|_2^2 \\ &> -\frac{4\sqrt{3}}{3}\|Q\|_2^2 + 4\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 + 4\|\Delta u(t)\|_2^2 + 4\|u(t)\|_2^2. \end{aligned} \tag{3.16}$$

Letting $J(t) := \int |u(t,x)|^2\,dx$, by some basic computations, we deduce that, for all $t \in I$, $J'(t) = 2\int u(t)\frac{\partial}{\partial t}u(t)\,dx$ and

$$J''(t) = 2\int \left(\left|\frac{\partial}{\partial t}u(t)\right|^2 + |u(t)|^4 - |\Delta u(t)|^2 - |u(t)|^2\right)dx. \tag{3.17}$$

It follows from (3.16) and (3.17) that, for all $t \in I$,

$$J''(t) > 6\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 - \frac{4\sqrt{3}}{3}\|Q\|_2^2 + 2\|\Delta u(t)\|_2^2 + 2\|u(t)\|_2^2 \geq 6\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2, \tag{3.18}$$

which implies that $J''(t)$ is positive and has a lower bound for all $t \in I$. Applying the Hölder inequality, we get $J'(t)^2 \leq 4\|u(t)\|_2^2\|\frac{\partial}{\partial t}u(t)\|_2^2$. Multiplying (3.18) with $J(t)$, one deduces that, for all $t \in I$,

$$J(t)J''(t) > 6\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2\|u(t)\|_2^2 > \frac{3}{2}J'(t)^2. \tag{3.19}$$

Thus, as the proof of Theorem 3.1, the solution $u(t,x)$ of Cauchy problem (1.1) blows up in finite time $0 < T < +\infty$. □

Acknowledgements
The authors would like to thank the reviewers for their helpful comments and suggestions.

Funding
This work is supported by the National Natural Science Foundation of China (No. 11871138).

Abbreviations
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JQ and CYZ have the same contribution to this work. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 5 December 2018 Accepted: 4 February 2019 Published online: 12 February 2019

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