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Global existence and stability of a class of nonlinear evolution equations with hereditary memory and variable density

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Abstract

In this paper, we consider the initial boundary value problem of nonlinear evolution equation with hereditary memory, variable density, and external force term

$$\begin{cases} |u_t|^\rho u_{tt} - \alpha\Delta u - \Delta u_{tt} + \int_{-\infty}^t \mu(t-s)\Delta u(s)\,ds - \gamma\Delta u_t = f(u), \\ \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{cases}$$

Under suitable assumptions, we prove the existence of a global solution by means of the Galerkin method, establish the exponential stability result by using only one simple auxiliary functional, and give the polynomial stability result.

MSC: 35L05; 35L15; 35L70

Keywords: Hereditary memory; Variable density; Global existence; Exponential stability; Polynomial stability

1 Introduction

In this paper, we are concerned with the following problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \alpha\Delta u - \Delta u_{tt} + \int_{-\infty}^t \mu(t-s)\Delta u(s)\,ds - \gamma\Delta u_t = f(u), \\ \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$, ρ is a positive constant, and $\gamma \geq 0$. We prove the existence of a global solution by means of the Galerkin method and establish the exponential stability under suitable assumptions by using a simpler auxiliary functional than that in [1]. We also show the polynomial stability under suitable conditions.

Partial differential equations in viscoelastic materials have important physical background and important mathematical significance. The viscous effects are described and characterized by an integral term, and the integral term indicates a dissipative effect. For mathematical analysis on the motions of evolution equations with memory, we refer to [8, 32]. Problem (1.1) is related to the equations

$$f(u_t)u_{tt} - \Delta u - \Delta u_{tt} = 0, \tag{1.2}$$

which have several modeling features. If $f(u_t)$ is a constant, Eq. (1.2) has been used to model extensional vibrations of thin rods (see [27, Ch. 20]) and it differs from D'Alembert's wave equation because of Δu_{tt}, which is not a damping term. On the contrary, Δu_{tt} increases the energy functional. If $f(u_t)$ is not a constant, Eq. (1.2) shows that the density of materials depends on the velocity u_t.

In the past ten years, several authors studied the homogeneous Dirichlet boundary value problem for the following model with memory (*starting from the zero moment*) and variable density:

$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + F(u, u_t, u_{tt}) = 0$$

in a bounded domains $\Omega \subset \mathbb{R}^n$. Cavalcanti et al. [2] considered the model with integral dissipation and strong damping

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds - \gamma\Delta u_t = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+.$$

Assuming that $0 < \rho \leq \frac{2}{n-2}$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$ and that $g(t)$ decays exponentially, they obtained the global existence of a solution for $\gamma \geq 0$ and the uniform exponential decay of the energy for $\gamma > 0$. Cavalcanti et al. [3] considered this model and proved intrinsic decays for large classes of relaxation kernels described by the inequality $g' + H(g) \leq 0$ with convex function H. Han and Wang [11] considered the equation with integral dissipation and linear damping

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds + u_t = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+.$$

They proved the global existence and exponential decay when g is decaying exponentially by introducing two auxiliary functionals. Han and Wang [12] established the general decay of energy for the equation with integral dissipation and nonlinear damping

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds + |u_t|^m u_t = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+,$$

by introducing two auxiliary functionals. Messaoudi and Tatar [29, 30] considered the equation only with integral dissipation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+.$$

Under some assumptions on g, they obtained exponential and polynomial decay rates. Messaoudi and Tatar [28] studied the equation with external force term and only with integral dissipation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds = b|u|^{p-2}u, \quad (x,t) \in \Omega \times \mathbb{R}^+. \tag{1.3}$$

By introducing a new functional and using potential well method they showed that there exists an appropriate set S (called a stable set) such that if the initial datum is in S, then the solution continues to live there forever. They also showed that the solution goes to zero with an exponential or polynomial rate depending on the decay rate of the relaxation function g. Liu [26] considered (1.3) and proved that, for certain class of relaxation functions and certain initial data in the stable set, the decay rate of the solution energy is similar to that of the relaxation function. Conversely, for certain initial data in the unstable set, there are solutions that blow up in finite time.

Now, we list some important literature on the nonlinear evolution equation with *hereditary memory* and variable density. Araújo et al. [1] considered the equation with integral dissipation in infinite interval

$$|u_t|^\rho u_{tt} - \alpha\Delta u - \Delta u_{tt} + \int_{-\infty}^t \mu(t-s)\Delta u(s)\,ds - \gamma\Delta u_t + f(u) = h, \quad (x,t) \in \Omega \times \mathbb{R}^+,$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$. They established the uniqueness of the solution, exponential decay, and global attractors. However, the existence of a solution *is not given* in detail, *two auxiliary functionals* are introduced to prove the exponential decay result, and the polynomial decay result *is not given*. Conti et al. [5] established an existence, uniqueness, and continuous dependence result for weak solutions to the nonlinear viscoelastic equation with hereditary memory on a bounded three-dimensional domain

$$|\partial_t u|^\rho \partial_{tt} u - \Delta\partial_{tt} u + \gamma(-\Delta)^\theta \partial_t u - \alpha\Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\,ds + f(u) = h$$

with Dirichlet boundary conditions. In particular, the parameter ρ belongs to the interval $[0,4]$, the value 4 is critical for the Sobolev embeddings, whereas f can reach the critical polynomial order 5. Lately, Conti et al. [4] studied the nonlinear viscoelastic equation

$$|\partial_t u|^\rho \partial_{tt} u - \Delta\partial_{tt} u - \Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\,ds + f(u) = h$$

and showed that the sole weak dissipation given by the memory term is enough to ensure the existence and optimal regularity of the global attractor $\mathcal{A}_\rho$ for $\rho < 4$ and critical nonlinearity f.

In recent years, Fatori et al. [9] studied long-time behavior of a class of thermoelastic plates with nonlinear strain and long memory; the main result establishes the existence of global and exponential attractors for the strongly damped problem through a stabilizability inequality. In addition, for the weakly damped problem, they establish the exponential stability of its Galerkin semiflows. Li et al. [13–15] proved the existence uniqueness, uniform energy decay rates, and limit behavior of the solution to the nonlinear viscoelastic Marguerre–von Kármán shallow shells system. The global existence uniqueness and decay estimates for nonlinear viscoelastic equation with boundary dissipation were given in

[16, 17, 19, 22–25]. The authors in [10, 18, 20, 21] studied the blowup phenomenon for some evolution equations. Du and Li [6, 7] proved the integrability and regularity of the solution to some equations.

In this paper, we study the equation with hereditary memory ($u_0(x,t), t \leq 0$) and variable density

$$|u_t|^\rho u_{tt} - \alpha\Delta u - \Delta u_{tt} + \int_{-\infty}^{t} \mu(t-\tau)\Delta u(\tau)\,d\tau - \gamma\Delta u_t = f(u), \tag{1.4}$$

that is,

$$|u_t|^\rho u_{tt} - \alpha\Delta u - \Delta u_{tt} + \int_{0}^{\infty} \mu(\tau)\Delta u(t-\tau)\,d\tau - \gamma\Delta u_t = f(u),$$

which can be rewritten as

$$\begin{aligned} &|u_t|^\rho u_{tt} - \left(\alpha - \int_0^\infty \mu(\tau)\,d\tau\right)\Delta u - \Delta u_{tt} \\ &\quad - \int_0^\infty \mu(\tau)\Delta\bigl(u(x,t) - u(x,t-\tau)\bigr)\,d\tau - \gamma\Delta u_t = f(u). \end{aligned}$$

This equation inspires us to define

$$\eta := \eta(x,t,\tau) = u(x,t) - u(x,t-\tau), \quad (x,\tau) \in \Omega \times \mathbb{R}^+, t \geq 0,$$

which implies

$$\eta_t(x,t,\tau) = u_t(x,t) - u_t(x,t,\tau) = u_t(x,t) - \eta_\tau(x,t-\tau), \quad (x,\tau) \in \Omega \times \mathbb{R}^+, t \geq 0,$$

and

$$t = 0 : \eta(x,0,\tau) = u_0(x,0) - u_0(x,-\tau), \quad (x,\tau) \in \Omega \times \mathbb{R}^+.$$

Hence Eq. (1.4) can be rewritten as

$$|u_t|^\rho u_{tt} - \left(\alpha - \int_0^\infty \mu(\tau)\,d\tau\right)\Delta u - \Delta u_{tt} - \int_0^\infty \mu(\tau)\Delta\eta(\tau)\,d\tau - \gamma\Delta u_t = f(u).$$

Without loss of generality, we assume that $\alpha - \int_0^\infty \mu(\tau)\,d\tau = 1$. Then

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} - \int_0^\infty \mu(\tau)\Delta\eta\,d\tau - \gamma\Delta u_t = f(u), \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ \eta_t(x,t,\tau) = u_t(x,t) - \eta_\tau(x,t,\tau), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \\ \eta(x,t,0) = 0, \qquad \eta(x,0,\tau) = \eta_0(x,\tau), \\ u = 0 \qquad \partial\Omega \times \mathbb{R}^+, \qquad \eta = 0 \qquad \partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \end{cases} \tag{1.5}$$

where

$$\eta_0(x,\tau) = u_0(x,0) - u_0(x,-\tau), \quad (x,\tau) \in \Omega \times \mathbb{R}^+.$$

The main contribution of this paper are: (a) the equation with hereditary memory, variable density, and external force term is representative; (b) the detailed construction process of the energy functional is given by an integration method; (c) we give a detailed proof of the existence for the solution; (d) the proof of the exponential decay result is simplified by introducing only one auxiliary functional; (e) the polynomial decay result is established.

The outline of this paper is as follows. In Sect. 2, we present the preliminaries and important our results. In Sects. 3–5, we prove the main Theorems 2.1–2.3, respectively.

2 Assumptions and the main results

In this paper, we assume that the following conditions (A_1)–(A_3) hold:

(A_1)

$$0<\rho\le\frac{2}{n-2}\quad \textit{if } n\ge 3;\qquad \rho>0\quad \textit{if } n=1,2,$$

which implies that

$$H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega).$$

(A_2) $f:\mathbb{R}\to\mathbb{R}$ *and satisfies*

$$\bigl|f(u)-f(v)\bigr|\le c_0\bigl(1+|u|^p+|v|^p\bigr)|u-v|,\quad u,v\in\mathbb{R},$$

where $c_0>0$ *and*

$$0<p\le\frac{2}{n-2}\quad \textit{if } n\ge 3;\qquad \rho>0\quad \textit{if } n=1,2,$$

and

$$f(s)s\le F(s)\le 0,\quad \forall s\in\mathbb{R},$$

where $F(z)=\int_0^z f(\sigma)\,d\sigma$.

(A_3) μ *satisfies*

$$\mu\in C^1\bigl(\mathbb{R}^+\bigr)\cap L^1\bigl(\mathbb{R}^+\bigr),\qquad 0\le\mu(\tau)<\infty,\qquad \mu(0)>0,\qquad \mu(+\infty)=0$$

with

$$\int_0^\infty \mu(\tau)\,d\tau=:k_0>0,$$

and there exists a constant $k_1>0$ *satisfying*

$$\mu'(t)\le -k_1\mu^q(t),\quad \forall t\in\mathbb{R}^+, 1\le q<\frac{3}{2}.$$

To consider the relative displacement η as a new function, we introduce the weighted L^2-space

$$\mathcal{M}:=L^2_\mu\bigl(\mathbb{R}^+;H_0^1(\Omega)\bigr)=\biggl\{v:\mathbb{R}^+\to H_0^1(\Omega)\biggm|\int_0^\infty \mu(\tau)\bigl\|\nabla v(\tau)\bigr\|_2^2\,d\tau<\infty\biggr\},$$

which is a Hilbert space endowed with inner product

$$(v,w)_{\mathcal{M}} = \int_0^\infty \mu(\tau)\left(\int_\Omega \nabla v(\tau)\cdot\nabla w(\tau)\,dx\right)d\tau$$

and norm

$$\|v\|_{\mathcal{M}}^2 = \int_0^\infty \mu(\tau)\big\|\nabla v(\tau)\big\|_2^2\,d\tau.$$

We introduce the notation

$$\mathcal{H} = H_0^1(\Omega)\times H_0^1(\Omega)\times\mathcal{M}.$$

Remark 2.1

(1) $H_0^1(\Omega)\hookrightarrow L^r(\Omega)$ with

$$r: \begin{cases} 2\le r\le \frac{2n}{n-2}, & n\ge 3,\\ \ge 2, & n=1,2, \end{cases}$$

which implies

$$\|\varphi\|_r \le B\|\varphi\|_2, \quad \forall\varphi\in H_0^1(\Omega).$$

(2) From (A_2) we can easily get $f(0)=0$.

(3) The condition $q<\frac{3}{2}$ is imposed to ensure that $\int_0^\infty \mu^{2-q}(\tau)\,d\tau<\infty$. In fact, assumption (A_3) implies

$$\mu(t)\le \frac{C_1}{(1+t)^{\frac{1}{q-1}}}, \qquad \frac{2-q}{q-1}>1,$$

and therefore $\int_0^\infty \mu^{2-q}(\tau)\,d\tau<\infty$.

Give the initial data $(u_0,u_1,\eta_0)\in\mathcal{H}$, a function $\boldsymbol{z}=(u,u_t,\eta)\in C([0,T],\mathcal{H})$ is a weak solution of problem (1.5) if it satisfies the initial condition $\boldsymbol{z}(0)=(u_0,u_1,\eta_0)$ and

$$\begin{aligned}&\big(|u_t|^\rho u_{tt},w\big)+(\nabla u,\nabla w)+(\nabla u_{tt},\nabla w)+\gamma(\nabla u_t,\nabla w)+\int_0^\infty \mu(\tau)(\nabla\eta,\nabla w)\,d\tau\\ &\quad=\big(f(u),w\big),\\ &(\partial_t\eta+\partial_\tau\eta,v)_{\mathcal{M}}=(u_t,v)_{\mathcal{M}},\end{aligned}$$

for all $w\in H_0^1(\Omega)$, $v\in\mathcal{M}$, and a.e. $t\in[0,T]$.

Multiplying both sides of Eq. (1.5) by u_t, integrating the resulting equation over Ω, and using the Green formula, we have

$$\begin{aligned}&\int_\Omega |u_t|^\rho u_{tt}u_t\,dx+\int_\Omega \nabla u\cdot\nabla u_t\,dx+\int_\Omega \nabla u_{tt}\nabla u_t\,dx+\int_\Omega \nabla u_t\int_0^\infty \mu(\tau)\nabla\eta(\tau)\,d\tau\,dx\\ &\quad+\gamma\int_\Omega|\nabla u_t|^2\,dx=\int_\Omega f(u)u_t\,dx,\end{aligned}$$

that is,

$$\frac{d}{dt}\left[\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2}\big\|\nabla u_t(t)\big\|_2^2-\int_\Omega F(u)\,dx\right]$$
$$+\gamma\|\nabla u_t\|_2^2+\int_\Omega \nabla u_t\int_0^\infty \mu(\tau)\nabla\eta\,d\tau\,dx=0.$$

A direct computation and application of (1.5) show that

$$\begin{aligned}\int_\Omega \nabla u_t\int_0^\infty \mu\nabla\eta(\tau)\,d\tau\,dx&=\int_\Omega(\nabla\eta_t+\nabla\eta_\tau)\int_0^\infty \mu(\tau)\nabla\eta\,d\tau\,dx\\&=\int_\Omega \nabla\eta_t\int_0^\infty \mu(\tau)\nabla\eta\,d\tau\,dx+\int_\Omega \nabla\eta_\tau\int_0^\infty \mu(\tau)\nabla\eta\,d\tau\,dx\\&=\frac{1}{2}\frac{d}{dt}\|\nabla\eta\|_{\mathcal{M}}^2+(\nabla\eta_\tau,\nabla\eta)_{\mathcal{M}}.\end{aligned}$$

This computation inspires us to define an energy functional as follows:

$$E(t)=\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2}\|\nabla u_t\|_2^2+\frac{1}{2}\|\nabla\eta\|_{\mathcal{M}}^2-\int_\Omega F(u)\,dx \tag{2.1}$$

and

$$E'(t)=-\gamma\|\nabla u_t\|_2^2-(\eta_\tau,\eta)_{\mathcal{M}}.$$

Using (A_3) and (1.5), we have

$$(\eta_\tau,\eta)_{\mathcal{M}}=\frac{1}{2}\int_\Omega\left(\int_0^\infty \mu(\tau)\frac{\partial}{\partial\tau}|\nabla\eta|^2\,d\tau\right)dx=-\frac{1}{2}\int_\Omega\left(\int_0^\infty \mu'(\tau)|\nabla\eta|^2\,d\tau\right)dx.$$

Then

$$E'(t)=-\gamma\big\|\nabla u_t(t)\big\|_2^2+\frac{1}{2}\int_0^\infty \mu'(\tau)\big\|\nabla\eta(\tau)\big\|_2^2\,d\tau\le 0. \tag{2.2}$$

Theorem 2.1 *Assume that conditions (A_1)–(A_3) hold and $\gamma\ge 0$. If the initial data $(u_0,u_1,\eta_0)\in\mathcal{H}$, then for any $T>0$, problem (1.5) has a weak solution*

$$(u,u_t,\eta)\in C\big([0,T],\mathcal{H}\big)$$

satisfying

$$\begin{aligned}&u\in L^\infty\big(\mathbb{R}^+;H_0^1(\Omega)\big),\qquad u_t\in L^\infty\big(\mathbb{R}^+;H_0^1(\Omega)\big),\\&u_{tt}\in L^2\big([0,T];H_0^1(\Omega)\big),\qquad \eta\in L^2\big(\mathbb{R}^+;\mathcal{M}\big).\end{aligned}$$

Theorem 2.2 *Assume that conditions (A_1)–(A_3) hold and $\gamma>0$. If $q=1$, then*

$$E(t)\le Ke^{-\nu t},\quad t\ge 0,$$

where K and ν are positive constants.

Theorem 2.3 *Assume that conditions* (A_1)–(A_3) *hold and* $\gamma > 0$. *If* $1 < q < \frac{3}{2}$, *then*

$$E(t) \le K(1+t)^{-\frac{1}{2(q-1)}}, \quad t \ge 0,$$

where K *is a positive constant.*

3 Proof of Theorem 2.1

We study the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} - \int_0^\infty \mu(\tau)\Delta\eta \, d\tau - \gamma\Delta u_t = f(u).$$

Let $\{\omega_j\}_{j=1}^\infty$ be an orthogonal basis of H_0^1 with ω_j satisfying

$$\begin{cases} -\Delta\omega_j = \lambda_j\omega_j, & x \in \Omega, \\ \omega_j|_{\partial\Omega} = 0. \end{cases}$$

By normalization we have $\|\omega_j\|_2 = 1$ and write $V_k = \operatorname{span}\{\omega_1, \dots, \omega_k\}$. For any given integer k, we consider the approximate solution

$$u_k(t) = \sum_{j=1}^k c_k^j(t)\omega_j$$

that satisfies

$$\begin{cases} (|u_{kt}|^\rho u_{ktt}, \omega_j) + (\nabla u_k, \nabla\omega_j) + (\nabla u_{ktt}, \nabla\omega_j) + \gamma(\nabla u_{kt}, \nabla\omega_j) \\ \quad + \int_0^\infty \mu(\tau)(\nabla\eta_k, \nabla\omega_j)\, d\tau = (f(u_k), \omega_j), \\ u_k(0) = u_{0k}, \qquad u_{k,t}(0) = u_{1k}, \quad j = 1, 2, \dots, k, \end{cases} \tag{3.1}$$

and, as $k \to \infty$,

$$u_{0k} = \sum_{j=1}^k (u_0, \omega_j)\omega_j \to u_0 \quad \text{in } H_0^1 \quad \text{and} \quad u_{1k} = \sum_{j=1}^k (u_1, \omega_j)\omega_j \to u_1 \quad \text{in } H_0^1. \tag{3.2}$$

Here we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$. Then (3.1) can be reduced to the second-order ODE system

$$\begin{cases} (|\sum_{i=1}^k c_k^{i\,\prime}(t)\omega_i|^\rho \sum_{i=1}^k c_k^{i\,\prime\prime}(t)\omega_i, \omega_j) + \lambda_j c_k^j(t) + \lambda_j c_k^{j\,\prime\prime}(t) + \gamma\lambda_j c_k^{j\,\prime}(t) \\ \quad + \lambda_j \int_0^\infty \mu(\tau)(c_k^j(t) - c_k^j(t-\tau))\, d\tau = (f(\sum_{j=1}^k c_k^j(t)\omega_j), \omega_j), \\ c_k^j(0) = (u_0, \omega_j), \qquad c_k^{j\,\prime}(0) = (u_1, \omega_j), \quad j = 1, 2, \dots, k, \end{cases} \tag{3.3}$$

According to the standard existence theory for ordinary differential equations, we infer that system (3.3) admits a solution $c_k^j(t)$ in $[0, t_m)$, where $t_m > 0$. Then we can obtain an approximate solution $u_k(t)$ of (3.1) in V_k over $[0, t_m)$, and the solution can be extended to $[0, T]$ for any given $T > 0$.

Multiplying (3.3) by $c_k^{j\,\prime}(t)$ and summing with respect to j, we conclude that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{\rho+2}\|u_{kt}\|_{\rho+2}^{\rho+2}+\frac{1}{2}\|\nabla u_k\|_2^2+\frac{1}{2}\|\nabla u_{kt}\|_2^2\right)+\gamma\|\nabla u_{kt}\|_2^2+\int_0^\infty \mu(\tau)(\nabla\eta_k,\nabla u_{kt})\,d\tau\\&\quad=\bigl(f(u_k),u_{kt}'\bigr).\end{aligned}\tag{3.4}$$

Simple calculations yield

$$\begin{aligned}&\int_0^\infty \mu(\tau)(\nabla\eta_k,\nabla u_{kt})\,d\tau\\&\quad=\int_0^\infty \mu(\tau)(\nabla\eta_k,\nabla\eta_{kt}+\nabla\eta_{k\tau})\,d\tau=\frac{1}{2}\frac{d}{dt}\|\eta_k\|_{\mathcal{M}}^2+(\nabla\eta_\tau,\nabla\eta)_{\mathcal{M}},\end{aligned}\tag{3.5}$$

$$\begin{aligned}(\eta_{k\tau},\eta_k)_{\mathcal{M}}&=\frac{1}{2}\int_\Omega\left(\int_0^\infty \mu(\tau)\frac{\partial}{\partial\tau}|\nabla\eta_k|^2\,d\tau\right)dx\\&=-\frac{1}{2}\int_\Omega\left(\int_0^\infty \mu'(\tau)|\nabla\eta_k|^2\,d\tau\right)dx\\&=-\frac{1}{2}\int_0^\infty \mu'(\tau)\|\nabla\eta_k\|_2^2\,d\tau.\end{aligned}\tag{3.6}$$

Combining (3.4) and (3.6), we find

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{\rho+2}\|u_{kt}\|_{\rho+2}^{\rho+2}+\frac{1}{2}\|\nabla u_k\|_2^2+\frac{1}{2}\|\nabla u_{kt}\|_2^2+\frac{1}{2}\|\eta_k\|_{\mathcal{M}}^2-\int_\Omega F(u_k)\,dx\right)\\&\quad=-\gamma\|\nabla u_{kt}\|_2^2+\frac{1}{2}\int_0^\infty \mu'(\tau)\|\nabla_k\eta\|_2^2\,d\tau\le 0.\end{aligned}\tag{3.7}$$

Integrating (3.7) over $(0,t)$ and noting (3.2), we obtain

$$\begin{aligned}&\frac{1}{\rho+2}\|u_{kt}\|_{\rho+2}^{\rho+2}+\frac{1}{2}\|\nabla u_k\|_2^2+\frac{1}{2}\|\nabla u_{kt}\|_2^2+\frac{1}{2}\|\eta_k\|_{\mathcal{M}}^2-\int_\Omega F(u_k)\,dx\\&\quad\le\frac{1}{\rho+2}\bigl\|u_{kt}(0)\bigr\|_{\rho+2}^{\rho+2}+\frac{1}{2}\bigl\|\nabla u_k(0)\bigr\|_2^2+\frac{1}{2}\bigl\|\nabla u_{kt}(0)\bigr\|_2^2\\&\qquad+\frac{1}{2}\bigl\|\eta_k(0)\bigr\|_{\mathcal{M}}^2-\int_\Omega F\bigl(u_k(0)\bigr)\,dx\\&\quad\le K_1,\end{aligned}\tag{3.8}$$

where K_1 is a constant independent of k. It follows from (3.8) and the Poincaré inequality that

$$\begin{cases}\{u_k\}\text{ is bounded in }L^\infty(0,T;H_0^1),\\\{u_{kt}\}\text{ is bounded in }L^\infty(0,T;H_0^1),\\\{\eta_{kt}\}\text{ is bounded in }L^2(0,T;\mathcal{M}).\end{cases}\tag{3.9}$$

Multiplying (3.1) by $c_k^{j\,''}(t)$ and summing with respect to j, we obtain

$$\int_\Omega |u_{kt}|^\rho |u_{ktt}|^2\,dx + (\nabla u_k, \nabla u_{ktt}) + \|\nabla u_{ktt}\|_2^2 + \frac{\gamma}{2}\frac{d}{dt}\|\nabla u_{kt}\|_2^2 + \int_0^\infty \mu(\tau)(\nabla\eta_k, \nabla u_{ktt})\,d\tau = \bigl(f(u_k), u_{ktt}\bigr),$$

that is,

$$\begin{aligned}&\int_\Omega |u_{kt}|^\rho |u_{ktt}|^2\,dx + \|\nabla u_{ktt}\|_2^2 + \frac{\gamma}{2}\frac{d}{dt}\|\nabla u_{kt}\|_2^2 \\ &\quad= -(\nabla u_k, \nabla u_{ktt}) - \int_0^\infty \mu(\tau)(\nabla\eta_k, \nabla u_{ktt})\,d\tau + \bigl(f(u_k), u_{ktt}\bigr).\end{aligned} \tag{3.10}$$

The right-hand side of (3.10) can be estimated as follows:

$$\bigl|-(\nabla u_k, \nabla u_{ktt})\bigr| = \left|-\int_\Omega \nabla u_k \nabla u_{ktt}\,dx\right| \le \varepsilon\|\nabla u_{ktt}\|_2^2 + \frac{1}{4\varepsilon}\|\nabla u_k\|_2^2 \quad \forall\varepsilon > 0, \tag{3.11}$$

$$\begin{aligned}&\left|-\int_0^\infty \mu(\tau)(\nabla\eta_k, \nabla u_{ktt})\,d\tau\right| \\ &\quad\le \varepsilon\|\nabla u_{ktt}\|_2^2 + \frac{1}{4\varepsilon}\int_\Omega\left(\int_0^\infty \mu(\tau)|\nabla\eta_k|\,d\tau\right)^2 dx \\ &\quad\le \varepsilon\|\nabla u_{ktt}\|_2^2 + \frac{1}{4\varepsilon}\int_\Omega\left(\int_0^\infty \sqrt{\mu(\tau)}\sqrt{\mu(\tau)}|\nabla\eta_k|\,d\tau\right)^2 dx \\ &\quad\le \varepsilon\|\nabla u_{ktt}\|_2^2 + \frac{1}{4\varepsilon}\int_0^\infty \mu(\tau)\,d\tau\int_\Omega\int_0^\infty \mu(\tau)|\nabla\eta_k|^2\,d\tau\,dx \\ &\quad= \varepsilon\|\nabla u_{ktt}\|_2^2 + \frac{k_0}{4\varepsilon}\|\nabla\eta_k\|_{\mathcal{M}}^2\end{aligned} \tag{3.12}$$

with $k_0 = \int_0^\infty \mu(\tau)\,d\tau$. Using (A_2), the Sobolev embedding theorem, and the Poincaré inequality, we have

$$\begin{aligned}\bigl|\bigl(f(u_k), u_{ktt}\bigr)\bigr| &= \left|-\int_\Omega f(u_k)u_{ktt}\,dx\right| \\ &\le c_0\int_\Omega \bigl(1 + |u_k|^p\bigr)|u_k||u_{ktt}|\,dx \\ &\le C_\varepsilon\bigl(\|u_k\|_2^2 + \|u_k\|_{2(p+1)}^{2(p+1)}\bigr) + \varepsilon\|u_{ktt}\|_2^2 \\ &\le C^*\bigl(\|\nabla u_k\|_2^2 + \|\nabla u_k\|_2^{2(p+1)}\bigr) + \varepsilon C\|\nabla u_{ktt}\|_2^2.\end{aligned} \tag{3.13}$$

By (3.10)–(3.13) we have

$$\begin{aligned}&\int_\Omega |u_{kt}|^\rho |u_{ktt}|^2\,dx + \|\nabla u_{ktt}\|_2^2 + \frac{\gamma}{2}\frac{d}{dt}\|\nabla u_{kt}\|_2^2 \\ &\quad\le (2 + C)\varepsilon\|\nabla u_{ktt}\|_2^2 + \left(\frac{1}{4\varepsilon} + C^*\right)\|\nabla u_k\|_2^2 + C^*\|\nabla u_k\|_2^{2(p+1)} + \frac{k_0}{4\varepsilon}\|\nabla\eta_k\|_{\mathcal{M}}^2,\end{aligned}$$

that is,

$$\int_{\Omega} |u_{kt}|^{\rho} |u_{ktt}|^2 \, dx + \bigl[1 - (2 + C)\varepsilon\bigr] \|\nabla u_{ktt}\|_2^2 + \frac{\gamma}{2} \frac{d}{dt} \|\nabla u_{kt}\|_2^2 \\ \leq \left(\frac{1}{4\varepsilon} + C^* \right) \|\nabla u_k\|_2^2 + C^* \|\nabla u_k\|_2^{2(p+1)} + \frac{k_0}{4\varepsilon} \|\nabla \eta_k\|_{\mathcal{M}}^2. \tag{3.14}$$

From (3.14) and (3.18) we know that

$$\int_{\Omega} |u_{kt}|^{\rho} |u_{ktt}|^2 \, dx + \bigl[1 - (2 + C)\varepsilon\bigr] \|\nabla u_{ktt}\|_2^2 + \frac{\gamma}{2} \frac{d}{dt} \|\nabla u_{kt}\|_2^2 \leq K_2. \tag{3.15}$$

Integrating (3.15) over $(0, t)$ $(0 < t \leq T)$ and noting (3.2) yield

$$\int_0^t \int_{\Omega} |u_{ktt}|^{\rho} |u_{ktt}|^2 \, dx \, d\tau + \bigl[1 - (2 + C)\varepsilon\bigr] \int_0^t \|\nabla u_{ktt}\|_2^2 \, d\tau + \frac{\gamma}{2} \|\nabla u_{kt}\|_2^2 \leq C_T. \tag{3.16}$$

Taking ε suitably small in (3.16), we can obtain the second estimate

$$\int_0^t \|\nabla u_{ktt}\|_2^2 \, d\tau + \|\nabla u_{kt}\|_2^2 \leq C_T,$$

which implies that

$$\{u_{ktt}\} \text{ is uniformly bounded in } L^2\bigl(0, T; H_0^1\bigr). \tag{3.17}$$

According to estimates (3.9) and (3.17), we infer that there exists a subsequence in $\{u_m\}$ (denoted by the same symbol) such that

$$\begin{cases} u_k \overset{*}{\rightharpoonup} u & \text{weak-star in } L^{\infty}(0, T; H_0^1(\Omega)), \\ u_{kt} \overset{*}{\rightharpoonup} u_t & \text{weak-star in } L^{\infty}(0, T; H_0^1(\Omega)), \\ u_{ktt} \overset{*}{\rightharpoonup} u_{tt} & \text{weakly in } L^2(0, T; H_0^1(\Omega)), \\ \eta_k \overset{*}{\rightharpoonup} \eta & \text{weakly in } L^2(0, T; \mathcal{M}), \end{cases} \tag{3.18}$$

which, combined with the Aubin–Lions compactness lemma, implies

$$\begin{cases} u_k \to u & \text{strongly in } C([0, T]; H_0^1(\Omega)), \\ u_{kt} \to u_t & \text{strongly in } C([0, T]; H_0^1(\Omega)). \end{cases} \tag{3.19}$$

Using (A_2) and (3.19), we get

$$f(u_k) \to f(u).$$

From (3.19) we get $u_{kt} \to u_t$ a.e. in $\Omega \times (0, T)$. Hence

$$|u_{kt}|^{\rho} u_{kt} \to |u_t|^{\rho} u_t \quad \text{a.e. in } \Omega \times (0, T). \tag{3.20}$$

On the other hand, by the Sobolev embedding theorem and $\|\nabla u_{kt}\|_2^2 \le L_1$, we have

$$\begin{aligned}\big\||u_{kt}|^\rho u_{kt}\big\|_{L^2(0,T;L^2(\Omega))} &= \int_0^T \int_\Omega |u_{kt}|^{2(\rho+1)}\,dx\,dt \le B^{2(\rho+1)} \int_0^T \|\nabla u_{kt}\|_2^{2(\rho+1)}\,dt \\ &\le B^{2(\rho+1)} L_1^{(\rho+1)} T \le C_T. \end{aligned} \tag{3.21}$$

Thus, using (3.20), (3.21), and the Lions lemma, we derive

$$|u_{kt}|^\rho u_{kt} \rightharpoonup |u_t|^\rho u_t \quad \text{weakly in } L^2\big(0,T;L^2(\Omega)\big). \tag{3.22}$$

Let $\mathcal{D}(0,T)$ be the space of C^∞ functions with compact support in $(0,T)$. Multiplying (3.1) by $\theta(t) \in \mathcal{D}(0,T)$ and integrating over $(0,T)$, it follows that

$$\begin{aligned}&\int_0^T \big(|u_{kt}|^\rho u_{ktt}, \omega_j\big)\theta(t)\,dt + \int_0^T (\nabla u_k, \nabla\omega_j)\theta(t)\,dt + \int_0^T (\nabla u_{ktt}, \nabla\omega_j)\theta(t)\,dt \\ &\quad + \gamma \int_0^T (\nabla u_{kt}, \nabla\omega_j)\theta(t)\,dt + \int_0^T \int_0^\infty \mu(\tau)(\nabla\eta_k, \nabla\omega_j)\theta(t)\,dt\,d\tau \\ &= \int_0^T \big(f(u_k), \omega_j\big)\theta(t)\,dt. \end{aligned} \tag{3.23}$$

Noting that $\{\omega_j\}_{j=1}^\infty$ is a basis of H_0^1, via convergences (3.18), (3.19), and (3.22), we can get from (3.23) that

$$\begin{aligned}&\big(|u_t|^\rho u_{tt}, \omega_j\big) + (\nabla u, \nabla\omega_j) + (\nabla u_{tt}, \nabla\omega_j) + \gamma(\nabla u_t, \nabla\omega_j) + \int_0^\infty \mu(\tau)\big(\nabla\eta(\tau), \nabla\omega_j\big)\,d\tau \\ &= \big(f(u), \omega_j\big), \end{aligned}$$

and hence, for all $\omega \in H_0^1(\Omega)$,

$$\begin{aligned}&\big(|u_t|^\rho u_{tt}, \omega\big) + (\nabla u, \nabla\omega) + \big(\nabla u_{tt}(t), \nabla\omega\big) + \gamma(\nabla u_t, \nabla\omega) + \int_0^\infty \mu(\tau)\big(\nabla\eta(\tau), \nabla\omega\big)\,d\tau \\ &= \big(f(u), \omega\big). \end{aligned} \tag{3.24}$$

Using (3.2) and (3.19), we have

$$u(0) = u_0, \qquad u_t(0) = u_1. \tag{3.25}$$

On the other hand, by Pata and Zucchi [31] we have that

$$\eta \in C\big([0,T], \mathcal{M}\big). \tag{3.26}$$

Combining (3.20), (3.25), and (3.26), we complete the proof.

Remark 2.4 For the uniqueness of the weak solution, see [1].

4 Proof of Theorem 2.2

Define the functionals

$$\Phi(t) = \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx - \int_\Omega \Delta u_t u\,dx \tag{4.1}$$

and

$$\mathcal{L}(t) = ME(t) + \Phi(t), \tag{4.2}$$

where $M > 0$ will be fixed later. We recall that $E(t)$ is decreasing since $E'(t) \le 0$.

Lemma 4.1 *For $M > 0$ sufficiently large, there exist constants $\beta_1, \beta_2 > 0$ such that*

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t), \quad t \ge 0.$$

Proof By the Hölder and Cauchy inequalities we have

$$\begin{aligned}
\bigl|\Phi(t)\bigr| &\le \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+1}|u|\,dx + \left|\int_\Omega \nabla u_t \cdot \nabla u\,dx\right| \\
&\le \frac{1}{\rho+1}\left(\int_\Omega |u_t|^{2(\rho+1)}\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |u|^2\,dx\right)^{\frac{1}{2}} \\
&\quad + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \\
&= \frac{1}{\rho+1}\|u_t\|_{2(\rho+1)}^{\rho+1}\|u\|_2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2.
\end{aligned}$$

Since $E(t)$ is decreasing, from the Sobolev embedding theorem we have

$$\begin{aligned}
\frac{1}{\rho+1}\|u_t\|_{2(\rho+1)}^{\rho+1}\|u\|_2 &\le \frac{1}{2(\rho+1)}\|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \frac{1}{2(\rho+1)}\|u\|_2^2 \\
&\le C_1 E^\rho(0)\|\nabla u_t\|_2^2 + C_1\|\nabla u\|_2^2.
\end{aligned}$$

Therefore

$$\bigl|\Phi(t)\bigr| \le CE(t).$$

Then taking $M > C$, we complete the proof. □

Lemma 4.2 *There exist $C_2 > 0$ and $C_3 > 0$, dependent on the initial data, such that*

$$\begin{aligned}
\Phi'(t) &\le -E(t) - \left(\frac{1}{2} - \varepsilon\right)\bigl\|\nabla u(t)\bigr\|_2^2 + C_2\bigl\|\nabla u_t(t)\bigr\|_2^2 \\
&\quad - C_3\int_0^\infty \mu'(\tau)\bigl\|\nabla\eta(\tau)\bigr\|_2^2\,d\tau \quad \forall t \ge 0.
\end{aligned}$$

Proof From the definition of $\Psi(t)$ we get

$$\begin{aligned}\Phi'(t) &= \frac{1}{\rho+1}\int_\Omega \left(|u_t|^\rho u_{tt}u + |u_t|^\rho u_t u_t + \left(\left(u_t^2\right)^{\frac{\rho}{2}}\right)_t u_t u\right)dx - \int_\Omega (\Delta u_{tt}u + \Delta u_t u_t)\,dx \\ &= \frac{1}{\rho+1}\int_\Omega \left(\left|u_t(t)\right|^\rho u_{tt}u + |u_t|^\rho u_t u_t + \rho|u_t|^\rho u_{tt}u\right)dx - \int_\Omega \left(\Delta u_{tt}u + |\nabla u_t|^2\right)dx \\ &= \int_\Omega \left(|u_t|^\rho u_{tt} - \Delta u_{tt}\right)u\,dx + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2.\end{aligned}$$

Using (1.5), we easily see that

$$\begin{aligned}&\int_\Omega \left(|u_t|^\rho u_{tt} - \Delta u_{tt}\right)u\,dx \\ &\quad = -\|\nabla u\|_2^2 + \int_0^\infty \mu(\tau)\left(\int_\Omega \Delta\eta(\tau)u(t)\,dx\right)d\tau + \gamma\int_\Omega \Delta u_t u\,dx + \int_\Omega f(u)u\,dx.\end{aligned}$$

We now estimate the second and third terms in the right-hand side as follows. Using the Cauchy inequality with ε and the Hölder inequality, we have

$$\begin{aligned}&\left|\int_0^\infty \mu(\tau)\left(\int_\Omega \Delta\eta(\tau)u(t)\,dx\right)d\tau\right| \\ &\quad = \left|\int_\Omega \nabla u(t)\int_0^\infty \mu(\tau)\nabla\eta(\tau)\,d\tau\,dx\right| \\ &\quad \le \frac{1}{2}\varepsilon\|\nabla u\|_2^2 + \frac{1}{2\varepsilon}\int_\Omega\left(\int_0^\infty \mu(\tau)\nabla\eta(\tau)\,d\tau\right)^2 dx \\ &\quad \le \frac{1}{2}\varepsilon\|\nabla u\|_2^2 + \frac{k_0}{2\varepsilon}\|\eta\|_{\mathcal{M}}^2\end{aligned}$$

and

$$\left|\gamma\int_\Omega \Delta u_t u\,dx\right| \le \frac{1}{2}\varepsilon\|\nabla u\|_2^2 + \frac{\gamma^2}{2\varepsilon}\|\nabla u_t\|_2^2.$$

Therefore

$$\Phi'(t) \le -(1-\varepsilon)\|\nabla u\|_2^2 + \left(1 + \frac{\gamma^2}{2\varepsilon}\right)\|\nabla u_t\|_2^2 + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \frac{k_0}{2\varepsilon}\|\eta\|_{\mathcal{M}}^2.$$

Noting the definitions of $E(t)$ and (A_2), we obtain

$$\begin{aligned}\Phi'(t) \le{}& -E(t) - \left(\frac{1}{2} - \varepsilon\right)\|\nabla u\|_2^2 + \left(\frac{3}{2} + \frac{\gamma^2}{2\varepsilon}\right)\|\nabla u_t\|_2^2 + \frac{2}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} \\ &+ \left(\frac{1}{2}\frac{k_0}{2\varepsilon}\right)\|\eta\|_{\mathcal{M}}^2.\end{aligned} \tag{4.3}$$

By Sobolev embedding we have

$$\|u_t\|_{\rho+2}^{\rho+2} \le BE(0)^{\frac{\rho}{2}}\|\nabla u_t\|_2^2. \tag{4.4}$$

Using (A_3), we get

$$\|\eta\|_{\mathcal{M}}^2 \leq -\frac{1}{k_1}\int_0^\infty \mu'(\tau)\|\nabla\eta(\tau)\|_2^2\,d\tau. \tag{4.5}$$

Combining (4.3)–(4.5), we finish the proof of Lemma 4.2. □

Proof of Theorem 2.2 By Lemma 4.2 we have

$$\Phi'(t) \leq -E(t) + C_2\|\nabla u_t(t)\|_2^2 - C_3\int_0^\infty \mu'(\tau)\|\nabla\eta(\tau)\|_2^2\,d\tau.$$

Note that

$$E'(t) = -\gamma\|\nabla u_t(t)\|_2^2 + \frac{1}{2}\int_0^\infty \mu'(\tau)\|\nabla\eta(\tau)\|_2^2\,d\tau$$

and

$$\mathcal{L}(t) = ME(t) + \Phi(t).$$

Then taking $M = \max(2C_3, \frac{C_2}{\gamma})$, we have

$$\mathcal{L}'(t) \leq -E(t).$$

Using Lemma 4.1, we obtain

$$\mathcal{L}'(t) \leq -\frac{1}{\beta_2}\mathcal{L}(t).$$

By the Gronwall inequality we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\frac{1}{\beta_2}t}.$$

By Lemma 4.1 we have

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(0)e^{-\frac{1}{\beta_2}t},$$

that is,

$$E(t) \leq Ke^{-\nu t},$$

where $\nu = \frac{1}{\beta_2}$ and $K = \frac{\beta_2 E(0)}{\beta_1}$. □

5 Proof of Theorem 2.3

We define

$$\Phi(t) = \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx - \int_\Omega \Delta u_t u\,dx, \tag{5.1}$$

$$\Psi(t) = \int_\Omega \Delta u_t\left(\int_0^\infty \mu(\tau)\eta\,d\tau\right)dx - \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\left(\int_0^\infty \mu(\tau)\eta\,d\tau\right)dx \tag{5.2}$$

and set

$$\mathcal{L}(t) = ME(t) + \varepsilon\Psi(t) + \chi(t),$$

where M and ε will be fixed later.

Lemma 5.1 *For $M > 0$ sufficiently large, there exist constants $\beta_1, \beta_2 > 0$ such that*

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t), \quad t \ge 0,$$

for any $0 < \varepsilon \le 1$.

Proof Since

$$\begin{aligned} \left|\int_\Omega \Delta u_t \left(\int_0^\infty \mu(\tau)\eta\,d\tau\right)dx\right| &\le \int_0^\infty \mu(\tau)\left|\int_\Omega \Delta u_t \eta\,dx\right|d\tau \\ &= \int_0^\infty \mu(\tau)\left|\int_\Omega \nabla u_t \cdot \nabla\eta\,dx\right|d\tau \\ &\le \frac{k_0}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2, \end{aligned}$$

by the Hölder and Cauchy inequalities we have

$$\begin{aligned} \frac{1}{\rho+1}\left|\int_\Omega |u_t|^\rho u_t \eta\,dx\right| &\le \frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^{\rho+1}|\eta|\,dx \\ &\le \frac{1}{\rho+1}\left(\int_\Omega \left|u_t(t)\right|^{2(\rho+1)}dx\right)^{\frac{1}{2}}\left(\int_\Omega |\eta|^2\,dx\right)^{\frac{1}{2}} \\ &= \frac{1}{\rho+1}\|u_t\|_{2(\rho+1)}^{\rho+1}\|\eta\|_2. \end{aligned}$$

Since $E(t)$ is decreasing, from the Sobolev inequality we have

$$\begin{aligned} \frac{1}{\rho+1}\|u_t\|_{2(\rho+1)}^{\rho+1}\|\eta\|_2 &\le \frac{1}{2(\rho+1)}\|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \frac{1}{2(\rho+1)}\|\eta\|_2^2 \\ &\le C_1 E^\rho(0)\|\nabla u_t\|_2^2 + C_1\|\nabla\eta\|_2^2. \end{aligned}$$

Therefore

$$\left|-\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \left(\int_0^\infty \mu(\tau)\eta\,d\tau\right)dx\right| \le k_0 C_1 E^\rho(0)\|\nabla u_t\|_2^2 + C_1\|\nabla\eta\|_{\mathcal{M}}^2.$$

Consequently,

$$\left|\Psi(t)\right| \le CE(t).$$

Using Lemma 4.1 and taking M large enough, we complete the proof of Lemma 5.1. □

Lemma 5.2 *Under the conditions of Theorem 2.2, the functional*

$$\Phi(t) = \frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^\rho u_t(t)u(t)\,dx - \int_\Omega \Delta u_t(t)u(t)\,dx$$

satisfies

$$\begin{aligned}\Phi'(t) \le{}& -\left[1-(1+k_0)\delta_1\right]\|\nabla u\|_2^2 + \left(1+\frac{\gamma^2}{4\delta_1}\right)\|\nabla u_t\|_2^2 + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2}\\ &+ \frac{1}{4\delta_1}\int_0^\infty \mu^{2-q}(\tau)\,d\tau \int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau + \int_\Omega f(u)u\,dx\end{aligned}$$

for any $\delta_1 > 0$.

Proof As in the proof of Lemma 4.2, we get

$$\Phi'(t) = \int_\Omega \left(|u_t|^\rho u_{tt} - \Delta u_{tt}\right)u\,dx + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 \tag{5.3}$$

and

$$\begin{aligned}&\int_\Omega \left(|u_t|^\rho u_{tt} - \Delta u_{tt}\right)u\,dx\\ &\quad= -\|\nabla u\|_2^2 + \int_0^\infty \mu(\tau)\left(\int_\Omega \Delta\eta(\tau)u(t)\,dx\right)d\tau\\ &\qquad+ \gamma\int_\Omega \Delta u_t u\,dx + \int_\Omega f(u)u\,dx.\end{aligned} \tag{5.4}$$

By the Cauchy inequality the second and third terms can be estimated as follows:

$$\begin{aligned}&\int_0^\infty \mu(\tau)\left(\int_\Omega \Delta\eta u(t)\,dx\right)d\tau\\ &\quad\le \delta_1 k_0\|\nabla u\|_2^2 + \frac{1}{4\delta_1}\int_0^\infty \mu^{2-q}(\tau)\,d\tau \int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau,\end{aligned} \tag{5.5}$$

and

$$\left|\gamma\int_\Omega \Delta u_t u\,dx\right| = \left|-\gamma\int_\Omega \nabla u_t\cdot\nabla u\,dx\right| \le \delta_1\|\nabla u\|_2^2 + \frac{\gamma^2}{4\delta_1}\|\nabla u_t\|_2^2, \tag{5.6}$$

where $\delta_1 > 0$.

Combining (5.3)–(5.6), we establish Lemma 5.2. □

Lemma 5.3 *Under the conditions of Theorem 2.2, there exist constants* $C, C', C'' > 0$ *such that*

$$\Psi(t) = \int_\Omega \Delta u_t\left(\int_0^\infty \mu(\tau)\eta\,d\tau\right)dx - \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\left(\int_0^\infty \mu(\tau)\eta\,d\tau\right)dx$$

satisfies

$$\Psi'(t) \le (\delta_2 + \delta_2 C)\|\nabla u\|_2^2 + \big(\delta_2\gamma^2 - k_0 + 2\delta_2 C'\big)\|\nabla u_t\|_2^2 - \frac{k_0}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + C'\int_0^\infty \mu^{2-q}(\tau)\,ds \int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau - 2C''\int_0^\infty \mu'(\tau)\big\|\nabla\eta(\tau)\big\|_2^2\,d\tau$$

for any $\delta_2 > 0$.

Proof From definition of Ψ we have

$$\begin{aligned}
\Psi'(t) &= \int_\Omega \Delta u_{tt}\Big(\int_0^\infty \mu(\tau)\eta\,d\tau\Big)dx + \int_\Omega \Delta u_t\Big(\int_0^\infty \mu(\tau)\eta_t\,d\tau\Big)dx \\
&\quad - \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_{tt}\Big(\int_0^\infty \mu(\tau)\eta\,d\tau\Big)dx \\
&\quad - \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\Big(\int_0^\infty \mu(\tau)\eta_t\,d\tau\Big)dx \\
&\quad - \frac{1}{\rho+1}\int_\Omega \big((u_t^2)^{\frac{\rho}{2}}\big)_t u_t\Big(\int_0^\infty \mu(\tau)\eta\,d\tau\Big)dx \\
&= \int_\Omega \big(-|u_t|^\rho u_{tt} + \Delta u_{tt}\big)\Big(\int_0^\infty \mu(\tau)\eta\,d\tau\Big)dx \\
&\quad + \int_\Omega \Big(-\frac{|u_t|^\rho u_t}{\rho+1} + \Delta u_t\Big)\Big(\int_0^\infty \mu(\tau)\eta_t\,d\tau\Big)dx \\
&:= I_1 + I_2.
\end{aligned}$$

From (1.5) we see that

$$I_1 = \int_\Omega \Big(-\Delta u - \int_0^\infty \mu(\tau)\Delta\eta\,d\tau - \gamma\Delta u_t - f(u)\Big)\Big(\int_0^\infty \mu(\tau)\eta\,d\tau\Big)dx.$$

By the Green formula and the Cauchy and Hölder inequalities we have the following estimates:

$$\begin{aligned}
&\int_\Omega -\Delta u\Big(\int_0^\infty \mu(\tau)\eta\,d\tau\Big)dx \\
&\quad = \int_\Omega \nabla u\cdot\Big(\int_0^\infty \mu(\tau)\nabla\eta\,d\tau\Big)dx \\
&\quad \le \delta_2\|\nabla u\|_2^2 + \frac{1}{4\delta_2}\int_0^\infty \mu^{2-q}(\tau)\,d\tau\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau, \\
&-\gamma\int_\Omega \Delta u_t\Big(\int_0^\infty \mu(\tau)\eta(\tau)\,d\tau\Big)dx \\
&\quad = \gamma\int_\Omega \nabla u_t\cdot\Big(\int_0^\infty \mu(\tau)\nabla\eta\,d\tau\Big)dx \\
&\quad \le \delta_2\gamma^2\|\nabla u_t\|_2^2 + \frac{1}{4\delta_2}\int_0^\infty \mu^{2-q}(\tau)\,d\tau\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau,
\end{aligned}$$

and

$$-\int_\Omega\left(\int_0^\infty \mu(\tau)\Delta\eta\,d\tau\right)\left(\int_0^\infty \mu(\tau)\eta\,d\tau\right)dx = \int_\Omega\left(\int_0^\infty \mu^{\frac{2-q}{2}}\mu^{\frac{q}{2}}(\tau)\nabla\eta(\tau)\,d\tau\right)^2 dx$$
$$\leq \int_0^\infty \mu^{2-q}(\tau)\,d\tau\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau.$$

Using (A_2) and the Cauchy, Hölder, and Poincaré inequalities, we obtain

$$-\int_\Omega f(u)\left(\int_0^\infty \mu(\tau)\eta\,d\tau\right)dx \leq \delta_2 C\|\nabla u\|_2^2 + \frac{1}{4\delta_2}\int_0^\infty \mu^{2-q}(\tau)\,d\tau\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau.$$

Therefore

$$I_1 \leq (\delta_2+\delta_2 C)\|\nabla u\|_2^2 + \delta_2\gamma_2\|\nabla u_t\|_2^2 + C'\int_0^\infty \mu^{2-q}(\tau)\,d\tau\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau. \tag{5.7}$$

From (1.5) we easily obtain

$$\int_0^\infty \mu(\tau)\eta_t\,d\tau = -\int_0^\infty \mu(\tau)\eta_\tau\,d\tau + \int_0^\infty \mu(\tau)u_t(t)\,d\tau = \int_0^\infty \mu'(\tau)\eta\,d\tau + k_0 u_t(t).$$

Then

$$I_2 = -k_0\left\|\nabla u_t(t)\right\|_2^2 - \frac{k_0}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \int_0^\infty \mu'(\tau)\left(\int_\Omega \Delta u_t(t)\eta\,dx\right)d\tau$$
$$+ \frac{1}{\rho+1}\int_0^\infty \mu'(\tau)\left(\int_\Omega -\left|u_t(t)\right|^\rho u_t(t)\eta\,dx\right)d\tau.$$

Using the Green formula and the Cauchy and Hölder inequalities, we have

$$\int_0^\infty \mu'(\tau)\left(\int_\Omega \Delta u_t(t)\eta\,dx\right)d\tau = -\int_0^\infty \mu'(\tau)\left(\int_\Omega \nabla u_t(t)\cdot\nabla\eta\,dx\right)d\tau$$
$$\leq -\int_0^\infty \mu'(\tau)\|\nabla u_t\|_2\|\nabla\eta\|_2\,d\tau$$
$$\leq \delta_2 C'\|\nabla u_t\|_2^2 - C''\int_0^\infty \mu'(\tau)\|\nabla\eta\|_2^2\,d\tau.$$

Using the method similar to that in the proof of Lemma 4.1, we get

$$\frac{1}{\rho+1}\int_0^\infty \mu'(\tau)\left(\int_\Omega -\left|u_t(t)\right|^\rho u_t(t)\eta\,dx\right)d\tau \leq \delta_2 C'\|\nabla u_t\|_2^2 - C''\int_0^\infty \mu'(\tau)\|\nabla\eta\|_2^2\,d\tau.$$

Then

$$I_2 \leq \left(-k_0 + 2\delta_2 C'\right)\|\nabla u_t\|_2^2 - \frac{k_0}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - 2C''\int_0^\infty \mu'(\tau)\|\nabla\eta\|_2^2\,d\tau. \tag{5.8}$$

Considering (5.7) and (5.8), we arrive at the conclusion. □

Proof of Theorem 2.3 Using

$$\mathcal{L}(t) = ME(t) + \varepsilon\Phi(t) + \Psi(t),$$
$$E'(t) = -\gamma\|\nabla u_t(t)\|_2^2 + \frac{1}{2}\int_0^\infty \mu'(\tau)\|\nabla\eta(\tau)\|_2^2\,d\tau \le 0,$$

and Lemmas 5.2–5.3, we get

$$\begin{aligned}
\mathcal{L}'(t) &= ME'(t) + \varepsilon\Phi'(t) + \Psi'(t)\\
&\le \left[-\gamma M + \left(1 + \frac{\gamma^2}{4\delta_1}\right)\varepsilon + \gamma^2\delta_2 - k_0 + 2\delta_2 C'\right]\|\nabla u_t\|_2^2\\
&\quad + \left[-\left(1-(k_0+1)\delta_1\right)\varepsilon + \delta_2 + \delta_2 C\right]\|\nabla u\|_2^2 - \left[\frac{k_0}{\rho+1} - \frac{\varepsilon}{\rho+1}\right]\|u_t\|_{\rho+2}^{\rho+2}\\
&\quad + \left[\frac{M}{2} - 2C''\right]\int_0^\infty \mu'(\tau)\|\nabla\eta\|_2^2\,d\tau\\
&\quad + \left[\frac{\varepsilon}{4\delta_1} + C'\right]\int_0^\infty \mu^{2-q}(\tau)\,d\tau\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau + \varepsilon\int_\Omega f(u)u\,dx.
\end{aligned}$$

Taking $M > 0$ sufficiently large and suitable $\varepsilon, \delta_1, \delta_2 > 0$ such that

$$\frac{k_0}{\rho+1} - \frac{\varepsilon}{\rho+1} > 0, \qquad -(1-2\delta_1)\varepsilon + \delta_2 + \delta_2 C < 0,$$
$$-\gamma M + \left(1 + \frac{\gamma^2}{4\delta_1}\right)\varepsilon + \gamma^2\delta_2 - k_0 + 2\delta_2 C' < 0, \qquad \frac{M}{2} - 2C'' > 0,$$

by using the inequality $\mu'(\tau) \le -k_1\mu^q(\tau)$ we have

$$\left(\frac{M}{2} - 2C''\right)\int_0^\infty \mu'(\tau)\|\nabla\eta\|_2^2\,d\tau \le -k_1\left(\frac{M}{2} - 2C''\right)\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau.$$

Therefore

$$\begin{aligned}
\mathcal{L}'(t) &\le \left[-\gamma M + \left(1 + \frac{\gamma^2}{4\delta_1}\right)\varepsilon + \gamma^2\delta_2 - k_0 + 2\delta_2 C'\right]\|\nabla u_t\|_2^2\\
&\quad + \left[-\left(1-(k_0+1)\delta_1\right)\varepsilon + \delta_2 + \delta_2 C\right]\|\nabla u\|_2^2 - \left[\frac{k_0}{\rho+1} - \frac{\varepsilon}{\rho+1}\right]\|u_t\|_{\rho+2}^{\rho+2}\\
&\quad - \left[k_1\left(\frac{M}{2} - 2C''\right) - \left(\frac{\varepsilon}{4\delta_1} + C'\right)\int_0^\infty \mu^{2-q}(\tau)\,d\tau\right]\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau\\
&\quad + \varepsilon\int_\Omega f(u)u\,dx.
\end{aligned}$$

By Remark 2.1 and (A_2), taking suitable $M > 0$, $\varepsilon, \delta_1, \delta_2 > 0$, we get

$$\begin{aligned}
\mathcal{L}'(t) &\le -C\left(\|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + \|u_t\|_{\rho+2}^{\rho+2} + \int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau - \int_\Omega f(u)u\,dx\right)\\
&\le -C\left(\|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + \|u_t\|_{\rho+2}^{\rho+2} + \int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau\right).
\end{aligned} \tag{5.9}$$

Using (A_3), we can easily show that $\int_0^\infty \mu^{1-\theta}(\tau)\,d\tau < \infty$ for any $\theta < 2-q$. Then

$$\begin{aligned} \Lambda &:= \int_0^\infty \mu^{1-\theta}(\tau)\int_\Omega |\nabla\eta|^2\,dx\,d\tau \\ &\le 2\int_0^\infty \mu^{1-\theta}(\tau)\int_\Omega \left(\left|\nabla u(t)\right|^2 + \left|\nabla u(t-\tau)\right|^2\right)dx\,d\tau \\ &\le C\int_0^\infty \mu^{1-\theta}(\tau)\,d\tau < L \end{aligned} \tag{5.10}$$

with positive constant $L > 1$.

Using the conditions of Theorem 2.3, the Hölder inequality, and (5.10), we see that

$$\begin{aligned} \|\eta\|_{\mathcal{M}}^2 &= \int_0^\infty \mu(\tau)\int_\Omega |\nabla\eta|^2\,dx\,d\tau = \int_0^\infty \mu(\tau)\|\nabla\eta\|_2^2\,d\tau \\ &= \int_0^\infty \mu^{\frac{(q-1)(1-\theta)}{q-1+\theta}}(\tau)\|\nabla\eta\|_2^{2\frac{(q-1)}{q-1+\theta}}\cdot\mu^{\frac{\theta q}{q-1+\theta}}(\tau)\|\nabla\eta\|_2^{2\frac{\theta}{q-1+\theta}}\,d\tau \\ &\le \left[\int_0^\infty \mu^{1-\theta}(\tau)\|\nabla\eta\|_2^2\,d\tau\right]^{\frac{q-1}{q-1+\theta}}\cdot\left[\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau\right]^{\frac{\theta}{q-1+\theta}} \\ &\le L\left[\int_0^\infty \mu^q(\tau)|\nabla\eta|_2^2\,d\tau\right]^{\frac{\theta}{q-1+\theta}}. \end{aligned} \tag{5.11}$$

Therefore we get, for $\sigma > 1$,

$$\begin{aligned} E^\sigma(t) &\le CE^{\sigma-1}(0)\left\{\|\nabla u_t\|_2^2 + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 - \int_\Omega F(u)\,dx\right\} \\ &\quad + C\left(\int_0^\infty \mu(\tau)\|\nabla\eta\|_2^2\,dx\,d\tau\right)^\sigma \\ &\le CE^{\sigma-1}(0)\left\{\|\nabla u_t\|_2^2 + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 - \int_\Omega F(u)\,dx\right\} \\ &\quad + C\left\{\int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau\right\}^{\frac{\sigma\theta}{q-1+\theta}}. \end{aligned} \tag{5.12}$$

By choosing $\theta = \frac{1}{2}$ and $\sigma = 2q-1$ (hence $\frac{\sigma\theta}{q-1+\theta} = 1$) estimate (5.12) gives

$$E^\sigma(t) \le C\left\{\|\nabla u_t\|_2^2 + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 - \int_\Omega F(u)\,dx + \int_0^\infty \mu^q(\tau)\|\nabla\eta\|_2^2\,d\tau\right\}. \tag{5.13}$$

A combination of (5.9) and (5.13) then leads to

$$\mathcal{L}'(t) \le -CE^\sigma(t).$$

By Lemma 5.1 we have

$$\mathcal{L}'(t) \le -C\frac{1}{\beta_2^\sigma}\mathcal{L}^\sigma(t). \tag{5.14}$$

A simple integration of (5.14) over $(0,t)$ yields

$$\mathcal{L}(t) \le C_1(1+t)^{-\frac{1}{\sigma-1}}, \quad t \ge 0, \tag{5.15}$$

that is,

$$\mathcal{L}(t) \le C_1(1+t)^{-\frac{1}{2(q-1)}}, \quad t \ge 0. \qquad \square$$

6 Conclusions

In this paper, we consider the Dirichlet boundary value problem of nonlinear evolution equation with hereditary memory, variable density, and external force term. We prove the existence of a global solution by means of the Galerkin method, establish the exponential stability by using only one auxiliary functional (this method is simpler than that in [1]), and also show the polynomial stability under suitable conditions. Under suitable hypotheses on the external force term function f and integral kernel function μ with $\gamma \ge 0$ in the model, we can further consider the local existence and blowup phenomenon of the solution.

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The authors contributed equally to the writing of this paper. The authors read and approved the final manuscript.

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