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Positive solutions of fourth-order problems with dependence on all derivatives in nonlinearity under Stieltjes integral boundary conditions

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Abstract

In this article, we investigate the existence of positive solutions to fourth-order problems with dependence on all derivatives in nonlinearities subject to the Stieltjes integral boundary conditions

$$\begin{cases} u^{(4)}(t)=f(t,u(t),u'(t),u''(t),u'''(t)), & t\in[0,1], \\ u'(0)+\beta_1[u]=0, \quad u''(0)+\beta_2[u]=0, \quad u(1)=\beta_3[u], \quad u'''(1)=0, \end{cases}$$

and

$$\begin{cases} -u^{(4)}(t)=g(t,u(t),u'(t),u''(t),u'''(t)), & t\in[0,1], \\ u(0)=\alpha_1[u], \quad u'(0)=\alpha_2[u], \quad u''(0)=\alpha_3[u], \quad u'''(1)=0, \end{cases}$$

where $f:[0,1]\times\mathbb{R}_+\times\mathbb{R}_-^3\to\mathbb{R}_+$, $g:[0,1]\times\mathbb{R}_+^3\to\mathbb{R}_+$ are continuous and $\beta_i[u]$, $\alpha_i[u]$ ($i=1,2,3$) are linear functionals involving Stieltjes integrals of signed measures. Some growth conditions are posed on nonlinearities f, g, meanwhile the spectral radii of corresponding linear operators are restricted, which means the superlinear or sublinear conditions. On the cones in $C^3[0,1]$ we apply the theory of fixed point index, the existence of positive solutions is obtained. We also give some examples under mixed multi-point and integral boundary conditions with sign-changing coefficients.

MSC: Primary 34B18; 34B10; secondary 34B15

Keywords: Positive solution; Fixed point index; Cone; Spectral radius

1 Introduction and preliminaries

In the article, we investigate the existence of positive solutions to fourth-order boundary value problems (BVPs) with dependence on all derivatives in nonlinearities under the boundary conditions involving Stieltjes integrals

$$\begin{cases} u^{(4)}(t)=f(t,u(t),u'(t),u''(t),u'''(t)), & t\in[0,1], \\ u'(0)+\beta_1[u]=0, \quad u''(0)+\beta_2[u]=0, \quad u(1)=\beta_3[u], \quad u'''(1)=0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} -u^{(4)}(t) = g(t, u(t), u'(t), u''(t), u'''(t)), & t \in [0,1], \\ u(0) = \alpha_1[u], \quad u'(0) = \alpha_2[u], \quad u''(0) = \alpha_3[u], & u'''(1) = 0, \end{cases} \tag{1.2}$$

where $\beta_i[u] = \int_0^1 u(t)\,dB_i(t)$ and $\alpha_i[u] = \int_0^1 u(t)\,dA_i(t)$ are Stieltjes integrals with B_i, A_i of bounded variation ($i = 1,2,3$).

Webb, Infante, and Franco [1] were concerned with the existence of positive solutions for the fourth-order differential equation

$$u^{(4)}(t) = g(t)f\bigl(t, u(t)\bigr), \qquad \text{a.e. } t \in (0,1)$$

subject to several nonlocal boundary conditions such as

$$u(0) = 0, \qquad u'(0) = 0, \qquad u(1) = \alpha[u], \qquad u'(1) = 0$$

and

$$u(0) = 0, \qquad u''(0) = 0, \qquad u(1) = \alpha[u], \qquad u''(1) = 0,$$

etc. In these equations $\alpha[u]$ denotes a linear functional on $C[0,1]$ given by $\alpha[u] = \int_0^1 u(s)\,dA(s)$ involving a Stieltjes integral. Infante and Pietramala [2] proved the existence of positive solutions for the cantilever equation

$$\begin{cases} u^{(4)}(t) = g(t)f(t, u(t)), & t \in (0,1), \\ u(0) = u'(0) = u''(1) = 0, & u'''(1) + k_0 + B(\alpha[u]) = 0, \end{cases}$$

where k_0 is a nonnegative constant, B is a nonnegative continuous function, and $\alpha[u]$ is as the above. Their main ingredient is the classical fixed point index. By making use of the monotonically iterative technique, Yao [3] studied the positive solution for a nonlinear fourth-order two-point boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), & t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases}$$

Alves et al. [4] considered, also by using the monotone iteration method, the existence of positive solutions for the beam equation

$$u^{(4)}(t) = f\bigl(t, u(t), u'(t)\bigr)$$

subject to boundary conditions

$$u(0) = u'(0) = 0, \qquad u'''(1) = g\bigl(u(1)\bigr), \qquad u'(1) = 0 \quad \text{or} \quad u''(1) = 0,$$

where g is a continuous function. Li [5] and Ma [6] discussed the conditions ensuring the existence of positive solutions for the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), & t \in (0,1), \\ u(0) = u''(0) = u(1) = u''(1) = 0. \end{cases}$$

The proofs of main results are respectively based upon fixed point index theory on cones and global bifurcation techniques. Respectively, by Krasnosel'skii's fixed point theorem and convex functional fixed point theorem, Bai [7] and Guo et al. [8] explored the existence of positive solutions for the nonlocal fourth-order problems

$$u^{(4)}(t) + \beta u''(t) = \lambda f\big(t, u(t), u''(t)\big)$$

and

$$u^{(4)}(t) + \beta u''(t) = \lambda f\big(t, u(t), u'(t), u''(t), u'''(t)\big)$$

with the same boundary conditions

$$u(0) = u(1) = \int_0^1 p(s)u(s)\,ds, \qquad u''(0) = u''(1) = \int_0^1 q(s)u''(s)\,ds,$$

where $p, q \in L[0,1]$ are nonnegative. Recently in [9], Li obtained the existence of positive solutions for the local fully nonlinear problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where $f : [0,1] \times \mathbb{R}_+^3 \times \mathbb{R}_- \to \mathbb{R}_+$ is continuous and $f(t, x_1, x_2, x_3, x_4)$ may have superlinear or sublinear growth in x_1, x_2, x_3, x_4. We also refer to some other relevant articles, for example, [10–14].

By fixed point index on cones of completely continuous operators, Webb and Infante [15] put forward a unified method to establish the existence of positive solutions to local and nonlocal boundary problems if f does not depend on derivatives. They dealt with the boundary problems involving Stieltjes integrals with signed measures.

Motivated by the above-mentioned works, we consider BVPs (1.1) and (1.2) in which the nonlinearities depend on all derivatives and the boundary conditions include Stieltjes integrals of signed measures. Some growth conditions are posed on nonlinearities f, g, meanwhile the spectral radii of corresponding linear operators are restricted, which means the superlinear or sublinear conditions. On the cones in $C^3[0,1]$ we apply the theory of fixed point index, the existence of positive solutions to BVPs (1.1) and (1.2) is obtained. For the superlinear case, we require the Nagumo-type condition similar to [9]. In view of the above features, we treated them in a different way from those in the references earlier. It is worth noting that two cones are defined, the large one is reproducing and serves as the partial ordering, the small one is applied to compute fixed point index. Especially in the process of derivation, the partial ordering induced by cone and the natural ordering of functions in function space are combined to use. In order to illustrate the results in this paper, we give some examples under mixed multi-point and integral boundary conditions with sign-changing coefficients.

For the sake of proving the theorems, we state the following lemmas, see [16, 17]. Let X be a Banach space and P be a closed convex set in X, if $\lambda x \in P$ for any $\lambda > 0$, $x \in P$, and $x = 0$ (the zero element in X) provided $\pm x \in P$, then P is said to be a cone in X. A cone P in X is called reproducing if $X = P - P$.

Lemma 1.1 *Let Ω be a bounded open set with $0 \in \Omega$ in X and P be a cone. If $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator, and $\mu Au \neq u$ for $u \in P \cap \partial\Omega$, $\mu \in [0,1]$, then the fixed point index $i(A, P \cap \Omega, P) = 1$.*

Lemma 1.2 *Let Ω be a bounded open set in X and P be a cone. If $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator, and there exists $v_0 \in P \setminus \{0\}$ such that $u - Au \neq \nu v_0$ for $u \in P \cap \partial\Omega$ and $\nu \geq 0$, then the fixed point index $i(A, P \cap \Omega, P) = 0$.*

Lemma 1.3 (Krein–Rutman) *Let P be a reproducing cone in Banach space X and $L : X \to X$ be a completely continuous linear operator with $L(P) \subset P$. If the spectral radius $r(L) > 0$, then there exists $\varphi \in P \setminus \{0\}$ such that $L\varphi = r(L)\varphi$.*

Throughout this paper, $X = C^3[0,1]$ is the Banach space which consists of all third-order continuously differentiable functions on $[0,1]$, and its norm is $\|u\|_{C^3} = \max\{\|u\|_C, \|u'\|_C, \|u''\|_C, \|u'''\|_C\}$. For $r > 0$, denote the bounded open set $\Omega_r = \{u \in C^3[0,1] : \|u\|_{C^3} < r\}$.

2 Positive solutions of BVP (1.1)

For BVP (1.1) we make the following assumption:

(C_1) $f : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_-^3 \to \mathbb{R}_+$ is continuous, here $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty, 0]$.

As shown by Webb and Infante [15], there exists a solution to BVP (1.1) if and only if the integral equation

$$u(t) = \sum_{i=1}^{3} \beta_i[u]\gamma_i(t) + \int_0^1 k_0(t,s) f\bigl(s, u(s), u'(s), u''(s), u'''(s)\bigr)\, ds =: (Tu)(t), \tag{2.1}$$

where $\gamma_1(t) = 1 - t$, $\gamma_2(t) = \frac{1}{2}(1 - t^2)$, $\gamma_3(t) = 1$,

$$k_0(t,s) = \begin{cases} \frac{1}{2}s(1-s) + \frac{1}{6}(s^3 - t^3), & 0 \leq t \leq s \leq 1, \\ \frac{1}{2}s(1-s) + \frac{1}{2}ts(s-t), & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.2}$$

and $\beta_i[u] = \int_0^1 u(t)\, dB_i(t)$ $(i = 1,2,3)$, has a solution in $C^3[0,1]$. We set

$$(Bu)(t) =: \sum_{i=1}^{3} \beta_i[u]\gamma_i(t), \qquad (Fu)(t) =: \int_0^1 k_0(t,s) f\bigl(s, u(s), u'(s), u''(s), u'''(s)\bigr)\, ds,$$

so that $(Tu)(t) = (Bu)(t) + (Fu)(t)$.

We impose other hypotheses:

(C_2) B_i is of bounded variation, moreover

$$\mathcal{K}_i(s) := \int_0^1 k_0(t,s)\,dB_i(t) \ge 0, \quad \forall s \in [0,1]\ (i = 1,2,3);$$

(C_3) The 3×3 matrix $[B]$ is positive whose (i,j)th entry is $\beta_i[\gamma_j]$, i.e., it has nonnegative entries. Furthermore its spectrum radius $r([B]) < 1$.

Making use of the notations in [15] and writing $\langle \beta, \gamma \rangle = \sum_{i=1}^3 \beta_i \gamma_i$ for the inner product in $\mathbb{R}^3$, we define the operator S as

$$(Su)(t) = \bigl\langle \bigl(I - [B]\bigr)^{-1} \beta[Fu], \gamma(t) \bigr\rangle + (Fu)(t),$$

and thus S can be written as follows:

$$\begin{aligned}
&(Su)(t) \\
&\quad = \int_0^1 \bigl(\bigl\langle \bigl(I - [B]\bigr)^{-1} \mathcal{K}(s), \gamma(t) \bigr\rangle + k_0(t,s)\bigr) f\bigl(s, u(s), u'(s), u''(s), u'''(s)\bigr)\,ds \\
&\quad =: \int_0^1 k_S(t,s) f\bigl(s, u(s), u'(s), u''(s), u'''(s)\bigr)\,ds,
\end{aligned} \tag{2.3}$$

i.e.,

$$k_S(t,s) = \bigl\langle \bigl(I - [B]\bigr)^{-1} \mathcal{K}(s), \gamma(t) \bigr\rangle + k_0(t,s) = \sum_{i=1}^3 \kappa_i(s)\gamma_i(t) + k_0(t,s), \tag{2.4}$$

where $\kappa_i(s)$ is the ith component of $(I - [B])^{-1}\mathcal{K}(s)$.

Lemma 2.1 *If* (C_2) *and* (C_3) *hold, then* $\kappa_i(s) \ge 0$ $(i = 1,2,3)$ *and for* $t, s \in [0,1]$,

$$c_0(t)\Phi_0(s) \le k_S(t,s) \le \Phi_0(s), \tag{2.5}$$

where

$$\Phi_0(s) = \sum_{i=1}^3 \kappa_i(s) + \frac{1}{2}s(1-s) + \frac{1}{6}s^3, \qquad c_0(t) = \frac{1}{2}\bigl(1 - t^2\bigr),$$

and

$$c_1(t)\Phi_1(s) \le -\frac{\partial k_S(t,s)}{\partial t} \le \Phi_1(s), \qquad c_2(t)\Phi_2(s) \le -\frac{\partial^2 k_S(t,s)}{\partial t^2} \le \Phi_2(s), \tag{2.6}$$

where

$$\Phi_1(s) = \sum_{i=1}^2 \kappa_i(s) + \frac{1}{2}s(2-s), \qquad c_1(t) = t^2, \qquad \Phi_2(s) = \kappa_2(s) + s, \qquad c_2(t) = t.$$

Proof $\kappa_i(s) \geq 0$ $(i = 1,2,3)$ are due to [15]. (2.5) and (2.6) come directly from the inequalities

$$\begin{aligned}
&\frac{1}{2}\left(1-t^2\right)\sum_{i=1}^{3}\kappa_i(s) \leq \sum_{i=1}^{3}\kappa_i(s)\gamma_i(t) \leq \sum_{i=1}^{3}\kappa_i(s),\\
&\frac{1}{2}\left(1-t^2\right)\left(\frac{1}{2}s(1-s)+\frac{1}{6}s^3\right) \leq \left(1-t^2\right)\left(\frac{1}{2}s(1-s)+\frac{1}{6}s^3\right) \leq k_0(t,s) \leq \frac{1}{2}s(1-s)+\frac{1}{6}s^3
\end{aligned}$$

and

$$\begin{aligned}
&t^2\sum_{i=1}^{2}\kappa_i(s) \leq t\sum_{i=1}^{2}\kappa_i(s) \leq -\sum_{i=1}^{3}\kappa_i(s)\gamma_i'(t) \leq \sum_{i=1}^{2}\kappa_i(s),\\
&\frac{1}{2}t^2 s(2-s) \leq -\frac{\partial k_0(t,s)}{\partial t} \leq \frac{1}{2}s(2-s),\\
&t\kappa_2(s) \leq \kappa_2(s) = -\sum_{i=1}^{3}\kappa_i(s)\gamma_i''(t), \qquad ts \leq -\frac{\partial^2 k_0(t,s)}{\partial t^2} \leq s
\end{aligned}$$

for $t,s \in [0,1]$. □

In $C^3[0,1]$ we denote the subsets

$$P = \left\{u \in C^3[0,1] : u(t) \geq 0, u'(t) \leq 0, u''(t) \leq 0, u'''(t) \leq 0, \forall t \in [0,1]\right\}, \tag{2.7}$$

$$\begin{aligned}
K = \Big\{&u \in P : u(t) \geq c_0(t)\|u\|_C, -u'(t) \geq c_1(t)\left\|u'\right\|_C,\\
&-u''(t) \geq c_2(t)\left\|u''\right\|_C, \forall t \in [0,1];\\
&\beta_1[u] \geq 0, \beta_2[u] \geq 0, \beta_3[u] \geq 0, u'''(1) = 0\Big\}.
\end{aligned}\tag{2.8}$$

It is easy to verify that P and K are cones in $C^3[0,1]$ with $K \subset P$. Now define the following linear operators:

$$(L_i u)(t) = \int_0^1 k_S(t,s)\left(a_i u(s) - b_i u'(s) - c_i u''(s) - d_i u'''(s)\right) ds \quad (i = 1,2), \tag{2.9}$$

$$(L_3 u)(t) = a_1 \int_0^1 k_S(t,s) u(s)\, ds, \tag{2.10}$$

where a_i, b_i, c_i, d_i $(i = 1,2)$ are nonnegative constants.

Stipulate the partial ordering induced by P: $u \preceq v$, equivalently $v \succeq u$, if and only if $v - u \in P$. We know that if P is a solid cone, i.e., the interior point set $\mathring{P} \neq \emptyset$, then P is reproducing (refer to [16–18]).

By the routine method we can prove the following Lemma 2.2 via Lemma 2.1 (cf. [15]).

Lemma 2.2 *If* (C_1)–(C_3) *hold, then* $S : P \to K$ *and* $L_i : C^3[0,1] \to C^3[0,1]$ *are all completely continuous, and* $L_i(P) \subset K$ $(i = 1,2,3)$.

Lemma 2.3 ([15]) *If* (C_1)–(C_3) *hold, then* S *has the same fixed points in* K *as* T. *Furthermore, the positive solutions to BVP* (1.1) *are equivalent to the fixed points of* S *in* K.

Theorem 2.1 *Suppose that* (C_1)–(C_3) *hold and that*

(F_1) *there are constants* $a_1, b_1, c_1, d_1, C_0 \geq 0$ *such that*

$$f(t, x_1, x_2, x_3, x_4) \leq a_1 x_1 - b_1 x_2 - c_1 x_3 - d_1 x_4 + C_0 \tag{2.11}$$

for $(t, x_1, x_2, x_3, x_4) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}_-^3$, *the spectral radius* $r(L_1) < 1$;

(F_2) *there are constants* $a_2, b_2, c_2, d_2 \geq 0$, *and* $r > 0$ *such that*

$$f(t, x_1, x_2, x_3, x_4) \geq a_2 x_1 - b_2 x_2 - c_2 x_3 - d_2 x_4 \tag{2.12}$$

for $(t, x_1, x_2, x_3, x_4) \in [0,1] \times [0,r] \times [-r,0]^3$, *the spectral radius* $r(L_2) \geq 1$; *here* $L_i : C^3[0,1] \to C^3[0,1]$ $(i = 1,2)$ *are defined by* (2.9).

Then BVP (1.1) *has a positive solution in* K.

Proof Let $W = \{u \in K : u = \mu S u, \mu \in [0,1]\}$, here S and K are defined by (2.3) and (2.8), respectively.

First we prove that W is a bounded set. Actually, if $u \in W$, then $u = \mu S u$ for some $\mu \in [0,1]$. It follows from (2.9) and (2.11) that

$$\begin{aligned} u(t) = \mu(Su)(t) &= \mu \int_0^1 k_S(t,s) f\bigl(s, u(s), u'(s), u''(s), u'''(s)\bigr)\, ds \\ &\leq \int_0^1 k_S(t,s)\bigl[a_1 u(s) - b_1 u'(s) - c_1 u''(s) - d_1 u'''(s) + C_0\bigr]\, ds \\ &= (L_1 u)(t) + C_0 \int_0^1 k_S(t,s)\, ds \end{aligned}$$

and

$$\bigl((I - L_1)u\bigr)(t) \leq C_0 \int_0^1 k_S(t,s)\, ds =: v(t), \quad t \in [0,1].$$

Clearly $v \in P$, and we can easily see from (2.11) with $\frac{\partial k_S(t,s)}{\partial t}$, $\frac{\partial^2 k_S(t,s)}{\partial t^2}$ and $\frac{\partial^3 k_S(t,s)}{\partial t^3}$ nonpositive (see Lemma 2.1) that, for $t \in [0,1]$,

$$u'(t) \geq (L_1 u)'(t) + v'(t), \qquad u''(t) \geq (L_1 u)''(t) + v''(t), \qquad u'''(t) \geq (L_1 u)'''(t) + v'''(t),$$

thus $(I - L_1)u \preceq v$. Since the spectral radius $r(L_1) < 1$, the bounded inverse operator $(I - L_1)^{-1}$ exists and it can be written as

$$(I - L_1)^{-1} = I + L_1 + L_1^2 + \cdots + L_1^n + \cdots.$$

Because $L_1(P) \subset K \subset P$ by Lemma 2.2, we have $(I - L_1)^{-1}(P) \subset P$, and thus the inequality $u \preceq (I - L_1)^{-1} v$ holds. So, for $t \in [0,1]$,

$$\begin{aligned} &0 \leq u(t) \leq \bigl((I - L_1)^{-1} v\bigr)(t), \qquad 0 \geq u'(t) \geq \bigl((I - L_1)^{-1} v\bigr)'(t), \\ &0 \geq u''(t) \geq \bigl((I - L_1)^{-1} v\bigr)''(t), \qquad 0 \geq u'''(t) \geq \bigl((I - L_1)^{-1} v\bigr)'''(t) \end{aligned}$$

which imply that $\|u\|_{C^3} \leq \|(I - L_1)^{-1} v\|_{C^3}$, i.e., W is bounded.

Take $R > \max\{r, \sup W\}$, then $\mu Su \neq u$ for $u \in K \cap \partial\Omega_R$ and $\mu \in [0,1]$, we have from Lemma 1.1 that $i(S, K \cap \Omega_R, K) = 1$.

Since $L_2 : P \to K \subset P$ and $r(L_2) \geq 1$, by Lemma 1.3 there exists $\varphi_0 \in P \setminus \{0\}$ such that $L_2\varphi_0 = r(L_2)\varphi_0$. Furthermore, $\varphi_0 = (r(L_2))^{-1}L_2\varphi_0 \in K$.

Suppose that S has no fixed points in $K \cap \partial\Omega_r$, and we will show that $u - Su \neq \nu\varphi_0$ for $u \in K \cap \partial\Omega_r$ and $\nu \geq 0$.

If otherwise, there exist $u_0 \in K \cap \partial\Omega_r$ and $\nu_0 \geq 0$ such that $u_0 - Su_0 = \nu_0\varphi_0$, and clearly $\nu_0 > 0$. Since $u_0 \in K \cap \partial\Omega_r$, we have

$$0 \leq u_0(t) \leq r, \qquad -r \leq u_0'(t), \qquad u_0''(t), \qquad u'''(t) \leq 0, \quad \forall t \in [0,1].$$

From (2.4), (2.9), and (2.12) it follows that $\forall t \in [0,1]$,

$$\begin{aligned} &(Su_0)(t) \geq (L_2u_0)(t), \qquad (Su_0)'(t) \leq (L_2u_0)'(t), \\ &(Su_0)''(t) \leq (L_2u_0)''(t), \qquad (Su_0)'''(t) \leq (L_2u_0)'''(t), \end{aligned}$$

these imply that

$$u_0 = \nu_0\varphi_0 + Su_0 \succeq \nu_0\varphi_0 + L_2u_0 \succeq \nu_0\varphi_0. \tag{2.13}$$

Set $\nu^* = \sup\{\nu > 0 : u_0 \succeq \nu\varphi_0\}$, then $\nu_0 \leq \nu^* < +\infty$ and $u_0 \succeq \nu^*\varphi_0$. Hence from (2.13) it follows that

$$u_0 \succeq \nu_0\varphi_0 + L_2u_0 \succeq \nu_0\varphi_0 + \nu^*L_2\varphi_0 = \nu_0\varphi_0 + \nu^*r(L_2)\varphi_0.$$

However, $r(L_2) \geq 1$, so $u_0 \succeq (\nu_0 + \nu^*)\varphi_0$ contradicts the definition of ν^*. Therefore $u - Su \neq \nu\varphi_0$ for $u \in K \cap \partial\Omega_r$ and $\nu \geq 0$.

Therefore it follows from Lemma 1.2 that $i(S, K \cap \Omega_r, K) = 0$.

Using the properties of fixed point index, we have that

$$i\bigl(S, K \cap (\Omega_R \setminus \overline{\Omega}_r), K\bigr) = i(S, K \cap \Omega_R, K) - i(S, K \cap \Omega_r, K) = 1,$$

and hence S has a fixed point in K. Thereby BVP (1.1) has a positive solution by Lemma 2.3. □

Theorem 2.2 *Suppose that* (C_1)*–*(C_3) *hold and that*

(F_3) *there are constants* $a_2, b_2, c_2, d_2 \geq 0$, *and* $r > 0$ *such that*

$$f(t, x_1, x_2, x_3, x_4) \leq a_2x_1 - b_2x_2 - c_2x_3 - d_2x_4 \tag{2.14}$$

for $(t, x_1, x_2, x_3, x_4) \in [0,1] \times [0.r] \times [-r.0]^3$, *the spectral radius* $r(L_2) < 1$, *where* L_2 *is defined by* (2.9);

(F_4) *there are positive constants* a_1, b_1, c_1, C_0 *satisfying*

$$\min\left\{\frac{a_1}{4}\int_0^1 (1 - s^2)\Phi_0(s)\,ds, b_1\int_0^1 s^2\Phi_1(s)\,ds, c_1\int_0^1 s\Phi_2(s)\,ds\right\} > 1 \tag{2.15}$$

such that

$$f(t,x_1,x_2,x_3,x_4) \geq a_1x_1 - b_1x_2 - c_1x_3 - C_0 \tag{2.16}$$

for $(t,x_1,x_2,x_3,x_4) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}_-^3$.

If the condition of Nagumo type is fulfilled, i.e.,

(F_5) *for any* $M > 0$, *there exists a positive continuous function* $H_M(\rho)$ *on* $\mathbb{R}_+$ *which satisfies*

$$\int_0^{+\infty} \frac{\rho\,d\rho}{H_M(\rho)+1} = +\infty \tag{2.17}$$

such that $\forall (t,x_1,x_2,x_3,x_4) \in [0,1] \times [0,M] \times [-M,0]^2 \times \mathbb{R}_-$,

$$f(t,x_1,x_2,x_3,x_4) \leq H_M\bigl(|x_4|\bigr), \tag{2.18}$$

then BVP (1.1) *has a positive solution in* K.

Proof (i) In this step we will show that $\mu Su \neq u$ for $u \in K \cap \partial\Omega_r$, $\mu \in [0,1]$, which implies from Lemma 1.1 that $i(S, K \cap \Omega_r, K) = 1$.

If otherwise, there exist $u_1 \in K \cap \partial\Omega_r$ and $\mu_0 \in [0,1]$ such that $u_1 = \mu_0 Su_1$, then we have from

$$0 \leq u_1(t) \leq r, \quad 0 \leq -u_1'(t), -u_1''(t), -u_1'''(t) \leq r, \quad \forall t \in [0,1]$$

and (2.14) that, for $t \in [0,1]$,

$$\begin{aligned} &u_1(t) \leq (L_2u_1)(t), \qquad u_1'(t) \geq (L_2u_1)'(t), \\ &u_1''(t) \geq (L_2u_1)''(t), \qquad u_1'''(t) \geq (L_2u_1)'''(t), \end{aligned}$$

hence $(I-L_2)u_1 \preceq 0$. Because the spectral radius $r(L_2) < 1$, it follows that the bounded inverse operator $(I-L_2)^{-1} : P \to P$ exists and $u_1 \preceq (I-L_2)^{-1}0 = 0$, which is a contradiction to $u_1 \in K \cap \partial\Omega_r$.

(ii) Let M be

$$\max\left\{\frac{4C_0\int_0^1 \Phi_0(s)\,ds}{a_1\int_0^1(1-s^2)\Phi_0(s)\,ds - 4}, \frac{C_0\int_0^1 \Phi_1(s)\,ds}{b_1\int_0^1 s^2\Phi_1(s)\,ds - 1}, \frac{C_0\int_0^1 \Phi_2(s)\,ds}{c_1\int_0^1 s\Phi_2(s)\,ds - 1}\right\}. \tag{2.19}$$

Equation (2.15) tells us that $M > 0$. By (2.17) it can easily be seen that

$$\int_0^{+\infty} \frac{\rho\,d\rho}{H_M(\rho)+C_0} = +\infty,$$

and so there exists $M_1 > M$ such that

$$\int_0^{M_1} \frac{\rho\,d\rho}{H_M(\rho)+C_0} > M. \tag{2.20}$$

(iii) For $u \in K$, define the homotopy $H(\lambda, u) = Su + \lambda v$, where

$$v(t) = C_0 \int_0^1 k_S(t,s)\,ds,$$

then $v \in K$ and $H : [0,1] \times K \to K$ is completely continuous.

Let $R > \max\{r, M_1\}$, and we will prove that

$$H(\lambda, u) \neq u, \quad \forall u \in K \cap \partial\Omega_R, \lambda \in [0,1]. \tag{2.21}$$

If it is false, there exist $u_2 \in K \cap \partial\Omega_R$ and $\lambda_0 \in [0,1]$ such that

$$H(\lambda_0, u_2) = u_2, \tag{2.22}$$

then by (2.16) and Lemma 2.1 we have that

$$\begin{aligned}
\|u_2\|_C &= u_2(0) \\
&= \int_0^1 k_S(0,s) f\big(s, u_2(s), u_2'(s), u_2''(s), u_2'''(s)\big)\,ds + \lambda_0 C_0 \int_0^1 k_S(0,s)\,ds \\
&\geq \int_0^1 k_S(0,s)\big[a_1 u_2(s) - b_1 u_2'(s) - c_1 u_2''(s) - C_0 + \lambda_0 C_0\big]\,ds \\
&\geq \int_0^1 k_S(0,s)\big[a_1 u_2(s) - C_0\big]\,ds \geq \frac{a_1}{2}\int_0^1 \Phi_0(s) u_2(s)\,ds - C_0 \int_0^1 \Phi_0(s)\,ds \\
&\geq \frac{a_1 \|u_2\|_C}{4} \int_0^1 \big(1 - s^2\big)\Phi_0(s)\,ds - C_0 \int_0^1 \Phi_0(s)\,ds, \\
\big\|u_2'\big\|_C &= -u_2'(1) \\
&= -\int_0^1 \frac{\partial k_S(1,s)}{\partial t} f\big(s, u_2(s), u_2'(s), u_2''(s), u_2'''(s)\big)\,ds - \lambda_0 C_0 \int_0^1 \frac{\partial k_S(1,s)}{\partial t}\,ds \\
&\geq -\int_0^1 \frac{\partial k_S(1,s)}{\partial t}\big[a_1 u_2(s) - b_1 u_2'(s) - c_1 u_2''(s) - C_0 + \lambda_0 C_0\big]\,ds \\
&\geq -\int_0^1 \frac{\partial k_S(1,s)}{\partial t}\big[-b_1 u_2'(s) - C_0\big]\,ds \\
&\geq b_1 \int_0^1 \Phi_1(s)\big(-u_2'(s)\big)\,ds - C_0 \int_0^1 \Phi_1(s)\,ds \\
&\geq b_1 \big\|u_2'\big\|_C \int_0^1 s^2 \Phi_1(s)\,ds - C_0 \int_0^1 \Phi_1(s)\,ds, \\
\big\|u_2''\big\|_C &= -u_2''(1) \\
&= -\int_0^1 \frac{\partial^2 k_S(1,s)}{\partial t^2} f\big(s, u_2(s), u_2'(s), u_2''(s), u_2'''(s)\big)\,ds - \lambda_0 C_0 \int_0^1 \frac{\partial^2 k_S(1,s)}{\partial t^2}\,ds \\
&\geq -\int_0^1 \frac{\partial^2 k_S(1,s)}{\partial t^2}\big[a_1 u_2(s) - b_1 u_2'(s) - c_1 u_2''(s) - C_0 + \lambda_0 C_0\big]\,ds \\
&\geq -\int_0^1 \frac{\partial^2 k_S(1,s)}{\partial t^2}\big[-c_1 u_2''(s) - C_0\big]\,ds
\end{aligned}$$

$$\geq c_1 \int_0^1 \Phi_2(s)\bigl(-u_2''(s)\bigr)\,ds - C_0 \int_0^1 \Phi_2(s)\,ds$$
$$\geq c_1 \|u_2''\|_C \int_0^1 s\Phi_2(s)\,ds - C_0 \int_0^1 \Phi_2(s)\,ds.$$

These imply by (2.19) that

$$\|u_2\|_C \leq M, \qquad \|u_2'\|_C \leq M, \qquad \|u_2''\|_C \leq M. \tag{2.23}$$

From (2.18), (2.22), and (2.23) it follows that

$$\begin{aligned} u_2^{(4)}(t) &= f\bigl(t, u_2(t), u_2'(t), u_2''(t), u_2'''(t)\bigr) + \lambda_0 C_0 \\ &\leq f\bigl(t, u_2(t), u_2'(t), u_2''(t), u_2'''(t)\bigr) + C_0 \leq H_M\bigl(-u_2'''(t)\bigr) + C_0. \end{aligned} \tag{2.24}$$

Multiplying both sides of (2.24) by $-u_2'''(t) \geq 0$, we have that

$$\frac{-u_2'''(t)u_2^{(4)}(t)}{H_M(-u_2'''(t)) + C_0} \leq -u_2'''(t), \quad t \in [0,1]. \tag{2.25}$$

Then integrating (2.25) over $[0,1]$ and making the variable transformation such as $\rho = -u_2'''(t)$, we have from (2.23) that

$$\int_0^{\|u_2'''\|_C} \frac{\rho\,d\rho}{H_M(\rho) + C_0} = \int_{-u_2'''(1)}^{-u_2'''(0)} \frac{\rho\,d\rho}{H_M(\rho) + C_0} \leq u_2''(0) - u_2''(1) \leq \|u_2''\|_C \leq M$$

since $u_2'''(1) = 0$ and $u_2^{(4)}(t) \geq 0$ by (2.24). Hence by (2.20) we also have that $\|u_2'''\|_C \leq M_1$ and $\|u_2\|_{C^3} \leq M_1$, a contradiction to $\|u_2\|_{C^3} = R > M_1$.

By the homotopy invariance property, from (2.21) the fixed point index is

$$i(S, K \cap \Omega_R, K) = i\bigl(H(0,\cdot), K \cap \Omega_R, K\bigr) = i\bigl(H(1,\cdot), K \cap \Omega_R, K\bigr). \tag{2.26}$$

(iv) Let $h(t) = \frac{1}{2}(1 - t^2)$, we have from (2.10) and Lemma 2.1 that

$$\begin{aligned} (L_3 h)(t) &= a_1 \int_0^1 \frac{1}{2}\bigl(1 - s^2\bigr) k_S(t,s)\,ds \\ &\geq \frac{a_1}{2}\bigl(1 - t^2\bigr) \int_0^1 \frac{1}{2}\bigl(1 - s^2\bigr)\Phi_0(s)\,ds \\ &= \biggl(\frac{a_1}{2} \int_0^1 \bigl(1 - s^2\bigr)\Phi_0(s)\,ds\biggr) h(t), \end{aligned}$$

so by [18, p. 67, Theorem 2.5] and (2.15), there exist

$$\lambda_1 \geq \frac{a_1}{2} \int_0^1 \bigl(1 - s^2\bigr)\Phi_0(s)\,ds > 2$$

and $\varphi_0 \in C[0,1]\backslash\{0\}$ such that $\varphi_0(t) \geq 0$ and $\varphi_0 = \lambda_1^{-1} L_3 \varphi_0$ since L_3 is a completely continuous linear operator in $C[0,1]$. Obviously $\varphi_0 \in P$ and thus $\varphi_0 \in K$ by Lemma 2.2.

(v) Now we prove that $u - H(1,u) \neq \nu\varphi_0$ for $u \in K \cap \partial\Omega_R$ and $\nu \geq 0$, where φ_0 appears in step (iv), so then

$$i\big(H(1,\cdot), K \cap \Omega_R, K\big) = 0 \tag{2.27}$$

holds by Lemma 1.2.

If there exist $u_0 \in K \cap \partial\Omega_R$ and $\nu_0 \geq 0$ such that $u_0 - H(1,u_0) = \nu_0\varphi_0$. Clearly $\nu_0 > 0$ by (2.21) and thus

$$u_0(t) = \big(H(1,u_0)\big)(t) + \nu_0\varphi_0(t) \geq \nu_0\varphi_0(t)$$

for $t \in [0,1]$. Set

$$\nu^* = \sup\big\{\nu > 0 : u_0(t) \geq \nu\varphi_0(t), \forall t \in [0,1]\big\},$$

then $\nu_0 \leq \nu^* < +\infty$ and $u_0(t) \geq \nu^*\varphi_0(t)$ for $t \in [0,1]$. From (2.16) we have that, for $t \in [0,1]$,

$$\begin{aligned} u_0(t) &= \big(H(1,u_0)\big)(t) + \nu_0\varphi_0(t) \geq (L_3u_0)(t) + \nu_0\varphi_0(t) \\ &\geq \nu^*(L_3\varphi_0)(t) + \nu_0\varphi_0(t) = \lambda_1\nu^*\varphi_0(t) + \nu_0\varphi_0(t). \end{aligned}$$

From $\lambda_1 > 2$, we have that $\lambda_1\nu^* + \nu_0 > \nu^*$ contradicts the definition of ν^*.

(vi) Finally it follows from (2.26) and (2.27) that $i(S, K \cap \Omega_R, K) = 0$ and

$$i\big(S, K \cap (\Omega_R \setminus \overline{\Omega}_r), K\big) = i(S, K \cap \Omega_R, K) - i(S, K \cap \Omega_r, K) = -1.$$

Hence S has a fixed solution and BVP (1.1) has a positive solution by Lemma 2.3. □

In order to give some examples, consider the fourth-order boundary problem under mixed multi-point and integral boundary conditions with sign-changing coefficients

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & t \in [0,1], \\ u'(0) + \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4}) = 0, & u''(0) + \int_0^1 u(t)\cos(\pi t)\,dt = 0, \\ u(1) = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{4}u(\frac{3}{4}), & u'''(1) = 0, \end{cases} \tag{2.28}$$

thus $\beta_1[u] = \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4})$, $\beta_2[u] = \int_0^1 u(t)\cos(\pi t)\,dt$, $\beta_3[u] = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{4}u(\frac{3}{4})$. We estimate some coefficients, and Matlab is used to calculate in some places.

$$\begin{aligned} \mathcal{K}_1(s) &= \int_0^1 k_0(t,s)\,dB_1(t) = \frac{1}{4}k_0\left(\frac{1}{4}, s\right) - \frac{1}{12}k_0\left(\frac{3}{4}, s\right) \\ &= \begin{cases} -\frac{1}{12}s^2 + \frac{19}{192}s, & 0 \leq s \leq \frac{1}{4}, \\ \frac{1}{24}s^3 - \frac{11}{96}s^2 + \frac{41}{384}s - \frac{1}{1536}, & \frac{1}{4} < s \leq \frac{3}{4}, \\ \frac{1}{36}s^3 - \frac{1}{12}s^2 + \frac{1}{12}s + \frac{1}{192}, & \frac{3}{4} < s \leq 1, \end{cases} \end{aligned}$$

and hence $0 \leq \mathcal{K}_1(s) \leq \mathcal{K}_1(1) = \frac{19}{576} < 0.0330$;

$$\mathcal{K}_2(s) = \int_0^1 k_0(t,s)\cos(\pi t)\,dt = \frac{2s - s^2}{2\pi^2} + \frac{\cos\pi s}{\pi^4} - \frac{1}{\pi^4} \quad (0 \leq s \leq 1),$$

and hence $0 \le \mathcal{K}_2(s) \le \mathcal{K}_2(1) < 0.0302$;

$$\begin{aligned}
\mathcal{K}_3(s) &= \int_0^1 k_0(t,s)\,dB_3(t) = \frac{1}{2}k_0\left(\frac{1}{2},s\right) - \frac{1}{4}k_0\left(\frac{3}{4},s\right) \\
&= \begin{cases} -\frac{3}{32}s^2 + \frac{17}{128}s, & 0 \le s \le \frac{1}{2}, \\ \frac{1}{12}s^3 - \frac{7}{32}s^2 + \frac{25}{128}s - \frac{1}{96}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{1}{24}s^3 - \frac{1}{8}s^2 + \frac{1}{8}s + \frac{11}{1536}, & \frac{3}{4} < s \le 1, \end{cases}
\end{aligned}$$

and hence $0 \le \mathcal{K}_3(s) \le \mathcal{K}_3(1) = \frac{25}{512} < 0.0489$.

The 3×3 matrix

$$[B] = \begin{pmatrix} \beta_1[\gamma_1] & \beta_1[\gamma_2] & \beta_1[\gamma_3] \\ \beta_2[\gamma_1] & \beta_2[\gamma_2] & \beta_2[\gamma_3] \\ \beta_3[\gamma_1] & \beta_3[\gamma_2] & \beta_3[\gamma_3] \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{19}{192} & \frac{1}{6} \\ \frac{2}{\pi^2} & \frac{1}{\pi^2} & 0 \\ \frac{3}{16} & \frac{17}{128} & \frac{1}{4} \end{pmatrix}$$

and its spectrum radius $r([B]) \approx 0.4479 < 1$. Those mean that (C_2) and (C_3) are satisfied.

Now we take into account the constants in Theorem 2.1 and Theorem 2.2.

$$\left(I - [B]\right)^{-1} < \begin{pmatrix} 1.3112 & 0.1875 & 0.2915 \\ 0.2957 & 1.1551 & 0.0658 \\ 0.3802 & 0.2515 & 1.4179 \end{pmatrix}$$

and

$$\left(I - [B]\right)^{-1}\mathcal{K}(s) < \begin{pmatrix} 0.0633 \\ 0.0480 \\ 0.0896 \end{pmatrix},$$

thus $k_S(t,s) < 0.0633(1-t) + 0.0480 \times \frac{1}{2}(1-t^2) + 0.0896 + k_0(t,s) < 0.3437$. So, for $u \in C^3[0,1]$ and $t \in [0,1]$,

$$\begin{aligned}
\left|(L_i u)(t)\right| &\le 0.3437 \int_0^1 \left(a_i\left|u(s)\right| + b_i\left|u'(s)\right| + c_i\left|u''(s)\right| + d_i\left|u'''(s)\right|\right) ds \\
&\le 0.3437(a_i + b_i + c_i + d_i)\|u\|_{C^3} \quad (i = 1,2),
\end{aligned}$$

here L_i $(i = 1,2)$ are defined in (2.9). Since all the terms are nonpositive in the first, second, and third derivatives of $k_S(t,s)$ with respect to t, we also have that, for $u \in C^3[0,1]$ and $t \in [0,1]$,

$$\begin{aligned}
\left|(L_i u)'(t)\right| &\le 0.6114 \int_0^1 \left(a_i\left|u(s)\right| + b_i\left|u'(s)\right| + c_i\left|u''(s)\right| + d_i\left|u'''(s)\right|\right) ds \\
&\le 0.6114(a_i + b_i + c_i + d_i)\|u\|_{C^3} \quad (i = 1,2), \\
\left|(L_i u)''(t)\right| &\le 1.0480 \int_0^1 \left(a_i\left|u(s)\right| + b_i\left|u'(s)\right| + c_i\left|u''(s)\right| + d_i\left|u'''(s)\right|\right) ds \\
&\le 1.0480(a_i + b_i + c_i + d_i)\|u\|_{C^3} \quad (i = 1,2),
\end{aligned}$$

$$\begin{aligned}\left|(L_iu)'''(t)\right| &\leq \int_0^1 \left(a_i\left|u(s)\right| + b_i\left|u'(s)\right| + c_i\left|u''(s)\right| + d_i\left|u'''(s)\right|\right) ds \\ &\leq (a_i + b_i + c_i + d_i)\|u\|_{C^3} \quad (i = 1,2).\end{aligned}$$

Therefore the radius $r(L_i) \leq \|L_i\| \leq 1.0480(a_i + b_i + c_i + d_i) < 1$ if

$$a_i + b_i + c_i + d_i < 1.0480^{-1} \quad (i = 1,2). \tag{2.29}$$

On the other hand, we have from Lemma 2.1 and Lemma 2.2 that, for $u \in K \setminus \{0\}$ and $t \in [0,1]$,

$$\begin{aligned}(L_2u)(t) &\geq \int_0^1 k_S(t,s)a_2u(s)\,ds \geq a_2c_0(t)\int_0^1 \Phi_0(s)u(s)\,ds \\ &\geq a_2c_0(t)\int_0^1 \Phi_0(s)c_0(s)\|u\|_C\,ds = a_2c_0(t)\|u\|_C\int_0^1 c_0(s)\Phi_0(s)\,ds\end{aligned}$$

and

$$\left\|(L_2u)\right\|_C = (L_2u)(0) \geq \frac{1}{2}a_2\|u\|_C\int_0^1 c_0(s)\Phi_0(s)\,ds,$$

hence

$$\begin{aligned}\left(L_2^2u\right)(t) &\geq a_2\int_0^1 k_S(t,s)(L_2u)(s)\,ds \geq a_2c_0(t)\int_0^1 \Phi_0(s)(L_2u)(s)\,ds \\ &\geq a_2c_0(t)\int_0^1 \Phi_0(s)c_0(s)\left\|(L_2u)\right\|_C\,ds \geq \frac{1}{2}a_2^2c_0(t)\|u\|_C\left(\int_0^1 c_0(s)\Phi_0(s)\,ds\right)^2\end{aligned}$$

and

$$\left\|\left(L_2^2u\right)\right\|_C = \left(L_2^2u\right)(0) \geq \frac{1}{4}a_2^2\|u\|_C\left(\int_0^1 c_0(s)\Phi_0(s)\,ds\right)^2.$$

By induction,

$$\left\|\left(L_2^nu\right)\right\|_C = \left(L_2^nu\right)(0) \geq \left(\frac{a_2}{2}\right)^n\|u\|_C\left(\int_0^1 c_0(s)\Phi_0(s)\,ds\right)^n.$$

As a result, it follows that, for $u \in K \setminus \{0\}$,

$$\left\|L_2^n\right\|\|u\|_{C^3} \geq \left\|L_2^nu\right\|_{C^3} \geq \left\|L_2^nu\right\|_C \geq \left(\frac{a_2}{2}\right)^n\|u\|_C\left(\int_0^1 c_0(s)\Phi_0(s)\,ds\right)^n,$$

and according to Gelfand's formula, the spectral radius

$$\begin{aligned}r(L_2) &= \lim_{n\to\infty}\left\|L_2^n\right\|^{1/n} \\ &\geq \frac{a_2}{2}\left(\int_0^1 c_0(s)\Phi_0(s)\,ds\right)\lim_{n\to\infty}\left(\frac{\|u\|_C}{\|u\|_{C^3}}\right)^{1/n} = \frac{a_2}{2}\left(\int_0^1 c_0(s)\Phi_0(s)\,ds\right),\end{aligned}$$

which implies that $r(L_2) \geq 1$ when

$$a_2 \geq \frac{720}{13} = \frac{2}{\int_0^1 \frac{1}{2}(1-s^2)[\frac{1}{2}s(1-s)+\frac{1}{6}s^3]\,ds} \geq \frac{2}{\int_0^1 c_0(s)\Phi_0(s)\,ds}. \tag{2.30}$$

Example 2.1 If $f(t,x_1,x_2,x_3,x_4) = \sqrt[3]{x_1} - \sqrt[3]{x_4}$, then BVP (2.28) has a positive solution.

Proof Take $a_1 = \frac{1}{4}$, $b_1 = c_1 = 0$, $d_1 = \frac{1}{6}$, $C_0 = 2$ and $a_2 = 56$, $b_2 = c_2 = 0$, $d_2 = 1$, $r = 1/2500$. Obviously, (2.29) and (2.30) are satisfied, meanwhile conditions (2.11) and (2.12) are fulfilled. Then BVP (2.28) has a positive solution by Theorem 2.1. □

Example 2.2 If

$$f(t,x_1,x_2,x_3,x_4) = \frac{\frac{1}{2}{x_1}^4 + \frac{1}{10}x_2^4 + \frac{1}{8}x_3^4 + \frac{1}{9}x_4^4}{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2},$$

then BVP (2.28) has a positive solution.

Proof Take $a_2 = \frac{1}{2}$, $b_2 = \frac{1}{10}$, $c_2 = \frac{1}{8}$, $d_2 = \frac{1}{9}$, $r < 1$, it is easy to check that (2.14) and (2.29) are satisfied. Now take $a_1 = 56$, $b_1 = 10$, $c_1 = 3$, it is clear that

$$\begin{aligned}
&\frac{a_1}{4}\int_0^1 (1-s^2)\Phi_0(s)\,ds > \frac{a_1}{4}\int_0^1 (1-s^2)\left[\frac{1}{2}s(1-s)+\frac{1}{6}s^3\right]ds \\
&\qquad > \frac{1}{4}\times\frac{720}{13}\int_0^1 (1-s^2)\left[\frac{1}{2}s(1-s)+\frac{1}{6}s^3\right]ds = 1, \\
&b_1\int_0^1 s^2\Phi_1(s)\,ds > b_1\int_0^1 \frac{1}{2}s^4\,ds = 1, \\
&c_1\int_0^1 s\Phi_2(s)\,ds > c_1\int_0^1 s^2\,ds = 1,
\end{aligned}$$

so (2.15) is valid. It can be seen that (2.16) is satisfied for C_0 large enough. Let $H_M(\rho) = M^2 + \rho^2$ for (F_5). Then BVP (2.28) has a positive solution by Theorem 2.2. □

3 Positive solutions of BVP (1.2)

For BVP (1.2) we make the following assumption:

(C_1') $g : [0,1]\times\mathbb{R}_+^4 \to \mathbb{R}_+$ is continuous.

As in [15], there exists a solution to BVP (1.2) if and only if the integral equation

$$u(t) = \sum_{i=1}^3 \alpha_i[u]\delta_i(t) + \int_0^1 \widetilde{k}_0(t,s)g\bigl(s,u(s),u'(s),u''(s),u'''(s)\bigr)\,ds =: (\widetilde{T}u)(t),$$

where $\delta_1(t) = 1$, $\delta_2(t) = t$, $\delta_3(t) = \frac{1}{2}t^2$,

$$\widetilde{k}_0(t,s) = \begin{cases} \frac{1}{6}t^3, & 0 \leq t \leq s \leq 1, \\ \frac{1}{6}s(3t^2 - 3ts + s^2), & 0 \leq s \leq t \leq 1, \end{cases}$$

and $\alpha_i[u] = \int_0^1 u(t)\,dA_i(t)$ $(i = 1,2,3)$, has a solution in $C^3[0,1]$. We set

$$(Au)(t) =: \sum_{i=1}^{3} \alpha_i[u]\delta_i(t), \qquad (\widetilde{F}u)(t) =: \int_0^1 \widetilde{k}_0(t,s) g\bigl(s,u(s),u'(s),u''(s),u'''(s)\bigr)\,ds,$$

so that $(\widetilde{T}u)(t) = (Au)(t) + (\widetilde{F}u)(t)$.

We impose other hypotheses:

(C_2') A_i is of bounded variation, moreover

$$\widetilde{\mathcal{K}}_i(s) := \int_0^1 \widetilde{k}_0(t,s)\,dA_i(t) \geq 0, \quad \forall s \in [0,1]\ (i = 1,2,3);$$

(C_3') The 3×3 matrix $[A]$ is positive whose (i,j)th entry is $\alpha_i[\delta_j]$ and whose spectrum radius $r([A]) < 1$.

Writing $\langle \alpha, \delta \rangle = \sum_{i=1}^{3} \alpha_i \delta_i$ for the inner product in $\mathbb{R}^3$, we define the operator $\widetilde{S}$ as

$$(\widetilde{S}u)(t) = \bigl\langle \bigl(I - [A]\bigr)^{-1} \alpha[\widetilde{F}u], \delta(t) \bigr\rangle + (\widetilde{F}u)(t), \tag{3.1}$$

and thus $\widetilde{S}$ can be written as follows:

$$\begin{aligned}
&(\widetilde{S}u)(t) \\
&\quad = \int_0^1 \bigl(\bigl\langle \bigl(I - [A]\bigr)^{-1} \widetilde{\mathcal{K}}(s), \delta(t) \bigr\rangle + \widetilde{k}_0(t,s) \bigr) g\bigl(s,u(s),u'(s),u''(s),u'''(s)\bigr)\,ds \\
&\quad =: \int_0^1 \widetilde{k}_S(t,s) g\bigl(s,u(s),u'(s),u''(s),u'''(s)\bigr)\,ds,
\end{aligned}$$

that is,

$$\widetilde{k}_S(t,s) = \bigl\langle \bigl(I - [A]\bigr)^{-1} \widetilde{\mathcal{K}}(s), \delta(t) \bigr\rangle + \widetilde{k}_0(t,s) = \sum_{i=1}^{3} \widetilde{\kappa}_i(s)\delta_i(t) + \widetilde{k}_0(t,s),$$

where $\widetilde{\kappa}_i(s)$ is the ith component of $(I - [A])^{-1}\widetilde{\mathcal{K}}(s)$.

Lemma 3.1 *If* (C_2') *and* (C_3') *hold, then* $\widetilde{\kappa}_i(s) \geq 0$ $(i = 1,2,3)$ *and, for* $t,s \in [0,1]$,

$$\widetilde{c}_0(t)\widetilde{\Phi}_0(s) \leq \widetilde{k}_S(t,s) \leq \widetilde{\Phi}_0(s), \tag{3.2}$$

where

$$\widetilde{\Phi}_0(s) = \sum_{i=1}^{3} \widetilde{\kappa}_i(s) + \frac{1}{6}s^3 + \frac{1}{2}s(1-s), \qquad \widetilde{c}_0(t) = \frac{1}{2}t^3,$$

and

$$\widetilde{c}_1(t)\widetilde{\Phi}_1(s) \leq \frac{\partial \widetilde{k}_S(t,s)}{\partial t} \leq \widetilde{\Phi}_1(s), \qquad \widetilde{c}_2(t)\widetilde{\Phi}_2(s) \leq \frac{\partial^2 \widetilde{k}_S(t,s)}{\partial t^2} \leq \widetilde{\Phi}_2(s), \tag{3.3}$$

where

$$\widetilde{\Phi}_1(s)=\sum_{i=2}^{3}\widetilde{\kappa}_i(s)+\frac{1}{2}s(2-s),\qquad \widetilde{c}_1(t)=t^2,\qquad \widetilde{\Phi}_2(s)=\widetilde{\kappa}_3(s)+s,\qquad \widetilde{c}_2(t)=t.$$

Proof $\widetilde{\kappa}_i(s)\geq 0$ ($i=1,2,3$) are due to [15]. (3.2) and (3.3) come directly from the inequalities

$$\begin{aligned}
&\frac{1}{2}t^3\sum_{i=1}^{3}\widetilde{\kappa}_i(s)\leq\frac{1}{2}t^2\sum_{i=1}^{3}\widetilde{\kappa}_i(s)\leq\sum_{i=1}^{3}\widetilde{\kappa}_i(s)\delta_i(t)\leq\sum_{i=1}^{3}\widetilde{\kappa}_i(s),\\
&\frac{1}{2}t^3\left(\frac{1}{6}s^3+\frac{1}{2}s(1-s)\right)\leq t^3\left(\frac{1}{6}s^3+\frac{1}{2}s(1-s)\right)\leq\widetilde{k}_0(t,s)\leq\frac{1}{6}s^3+\frac{1}{2}s(1-s)
\end{aligned}$$

and

$$\begin{aligned}
&t^2\sum_{i=2}^{3}\widetilde{\kappa}_i(s)\leq t\sum_{i=2}^{3}\widetilde{\kappa}_i(s)\leq\sum_{i=1}^{3}\widetilde{\kappa}_i(s)\delta_i'(t)\leq\sum_{i=2}^{3}\widetilde{\kappa}_i(s),\\
&\frac{1}{2}t^2s(2-s)\leq\frac{\partial\widetilde{k}_0(t,s)}{\partial t}\leq\frac{1}{2}s(2-s),\\
&t\widetilde{\kappa}_3(s)\leq\widetilde{\kappa}_3(s)=\sum_{i=1}^{3}\widetilde{\kappa}_i(s)\delta_i''(t),ts\leq\frac{\partial^2\widetilde{k}_0(t,s)}{\partial t^2}\leq s
\end{aligned}$$

for $t,s\in[0,1]$. □

In $C^3[0,1]$ we denote the subsets

$$\widetilde{P}=\left\{u\in C^3[0,1]:u(t)\geq 0,u'(t)\geq 0,u''(t)\geq 0,u'''(t)\geq 0,\forall t\in[0,1]\right\},\tag{3.4}$$

$$\begin{aligned}
\widetilde{K}=\Big\{&u\in\widetilde{P}:u(t)\geq c_0(t)\|u\|_C,u'(t)\geq c_1(t)\left\|u'\right\|_C,\\
&u''(t)\geq c_2(t)\left\|u''\right\|_C,\forall t\in[0,1];\\
&\alpha_1[u]\geq 0,\alpha_2[u]\geq 0,\alpha_3[u]\geq 0,u'''(1)=0\Big\}.
\end{aligned}\tag{3.5}$$

It is easy to verify that $\widetilde{P}$ and $\widetilde{K}$ are cones in $C^3[0,1]$ with $\widetilde{K}\subset\widetilde{P}$. Now define the following linear operators:

$$\begin{aligned}
&(\widetilde{L}_iu)(t)=\int_0^1\widetilde{k}_S(t,s)\big(\widetilde{a}_iu(s)+\widetilde{b}_iu'(s)+\widetilde{c}_iu''(s)+\widetilde{d}_iu'''(s)\big)\,ds\quad(i=1,2),\\
&(\widetilde{L}_3u)(t)=\widetilde{a}_1\int_0^1\widetilde{k}_S(t,s)u(s)\,ds,
\end{aligned}\tag{3.6}$$

where $\widetilde{a}_i,\widetilde{b}_i,\widetilde{c}_i,\widetilde{d}_i$ ($i=1,2$) are nonnegative constants.

By the routine method we can prove the following Lemma 3.2 via Lemma 3.1.

Lemma 3.2 *If (C_1')–(C_3') hold, then $\widetilde{S}:\widetilde{P}\to\widetilde{K}$ and $\widetilde{L}_i:C^3[0,1]\to C^3[0,1]$ are all completely continuous, and $\widetilde{L}_i(\widetilde{P})\subset\widetilde{K}$ ($i=1,2,3$).*

Lemma 3.3 ([15]) *If (C_1')–(C_3') hold, then $\widetilde{S}$ has the same fixed points in $\widetilde{K}$ as $\widetilde{T}$. Furthermore, the positive solutions to BVP (1.2) are equivalent to fixed points of $\widetilde{S}$ in $\widetilde{K}$.*

Theorem 3.1 *Suppose that (C_1')–(C_3') hold and that*

$(\widetilde{F}_1)$ *there are constants $\widetilde{a}_1, \widetilde{b}_1, \widetilde{c}_1, \widetilde{d}_1, \widetilde{C}_0 \geq 0$ such that*

$$g(t,x_1,x_2,x_3,x_4) \leq \widetilde{a}_1 x_1 + \widetilde{b}_1 x_2 + \widetilde{c}_1 x_3 + \widetilde{d}_1 x_4 + \widetilde{C}_0 \tag{3.7}$$

for $(t,x_1,x_2,x_3,x_4) \in [0,1] \times \mathbb{R}_+^4$, the spectral radius $r(\widetilde{L}_1) < 1$;

$(\widetilde{F}_2)$ *there are constants $\widetilde{a}_2, \widetilde{b}_2, \widetilde{c}_2, \widetilde{d}_2 \geq 0$, and $\widetilde{r} > 0$ such that*

$$g(t,x_1,x_2,x_3,x_4) \geq \widetilde{a}_2 x_1 + \widetilde{b}_2 x_2 + \widetilde{c}_2 x_3 + \widetilde{d}_2 x_4 \tag{3.8}$$

for $(t,x_1,x_2,x_3,x_4) \in [0,1] \times [0,\widetilde{r}]^4$, the spectral radius $r(\widetilde{L}_2) \geq 1$; here $\widetilde{L}_i : C^3[0,1] \to C^3[0,1]$ $(i = 1,2)$ are defined by (3.6).

Then BVP (1.2) has a positive solution in K.

Proof It is easy to verify that $\widetilde{P}$ defined by (3.4) is a solid cone, and we define the partial ordering induced by $\widetilde{P}$ such as $u \preceq v$ if and only if $v - u \in \widetilde{P}$. The rest is similar to the proof of Theorem 2.1. □

Theorem 3.2 *Suppose that (C_1')–(C_3') hold and that*

$(\widetilde{F}_3)$ *there are constants $\widetilde{a}_2, \widetilde{b}_2, \widetilde{c}_2, \widetilde{d}_2 \geq 0$, and $\widetilde{r} > 0$ such that*

$$g(t,x_1,x_2,x_3,x_4) \leq \widetilde{a}_2 x_1 + \widetilde{b}_2 x_2 + \widetilde{c}_2 x_3 + \widetilde{d}_2 x_4 \tag{3.9}$$

for $(t,x_1,x_2,x_3,x_4) \in [0,1] \times [0.\widetilde{r}]^4$, the spectral radius $r(\widetilde{L}_2) < 1$, where $\widetilde{L}_2$ is defined by (3.6),

$(\widetilde{F}_4)$ *there are positive constants $\widetilde{a}_1$, $\widetilde{b}_1$, $\widetilde{c}_1$, $\widetilde{C}_0$ satisfying*

$$\min\left\{\frac{\widetilde{a}_1}{4}\int_0^1 s^3 \widetilde{\Phi}_0(s)\,ds, \widetilde{b}_1 \int_0^1 s^2 \widetilde{\Phi}_1(s)\,ds, \widetilde{c}_1 \int_0^1 s\widetilde{\Phi}_2(s)\,ds\right\} > 1 \tag{3.10}$$

such that

$$g(t,x_1,x_2,x_3,x_4) \geq \widetilde{a}_1 x_1 + \widetilde{b}_1 x_2 + \widetilde{c}_1 x_3 - \widetilde{C}_0 \tag{3.11}$$

for $(t,x_1,x_2,x_3,x_4) \in [0,1] \times \mathbb{R}_+^4$.

If the condition of Nagumo type is fulfilled, i.e.,

$(\widetilde{F}_5)$ *for any $M > 0$, there exists a positive continuous function $\widetilde{H}_M(\rho)$ on $\mathbb{R}_+$ which satisfies*

$$\int_0^{+\infty} \frac{\rho\,d\rho}{\widetilde{H}_M(\rho) + 1} = +\infty$$

such that

$$g(t,x_1,x_2,x_3,x_4) \leq \widetilde{H}_M(x_4), \quad \forall (t,x_1,x_2,x_3,x_4) \in [0,1] \times [0,M]^3 \times \mathbb{R}_+, \tag{3.12}$$

then BVP (1.2) has a positive solution in K.

Proof Let

$$M=\max\left\{\frac{4\widetilde{C}_0\int_0^1\widetilde{\Phi}_0(s)\,ds}{\widetilde{a}_1\int_0^1 s^3\widetilde{\Phi}_0(s)\,ds-4},\frac{\widetilde{C}_0\int_0^1\widetilde{\Phi}_1(s)\,ds}{\widetilde{b}_1\int_0^1 s^2\widetilde{\Phi}_1(s)\,ds-1},\frac{\widetilde{C}_0\int_0^1\widetilde{\Phi}_2(s)\,ds}{\widetilde{c}_1\int_0^1 s\widetilde{\Phi}_2(s)\,ds-1}\right\}$$

and the rest is similar to the proof of Theorem 2.2 in which $h(t)=\frac{1}{2}t^3$ for step (iv). □

In the following, we consider the fourth-order boundary problem under mixed multi-point and integral boundary conditions with sign-changing coefficients

$$\begin{cases}-u^{(4)}(t)=g(t,u(t),u'(t),u''(t),u'''(t)), & t\in[0,1],\\ u(0)=\frac{1}{2}u(\frac{1}{4})-\frac{1}{160}u(\frac{3}{4}), & u'(0)=\int_0^1(t-\frac{1}{8})u(t)\,dt,\\ u''(0)=\frac{1}{2}u(\frac{1}{2})-\frac{1}{14}u(\frac{3}{4}), & u'''(1)=0,\end{cases}\tag{3.13}$$

thus $\alpha_1[u]=\frac{1}{2}u(\frac{1}{4})-\frac{1}{160}u(\frac{3}{4})$, $\alpha_2[u]=\int_0^1(t-\frac{1}{8})u(t)\,dt$, $\alpha_3[u]=\frac{1}{2}u(\frac{1}{2})-\frac{1}{14}u(\frac{3}{4})$. We also estimate some coefficients as in the previous section.

$$\begin{aligned}\widetilde{\mathcal{K}}_1(s)&=\int_0^1\widetilde{k}_0(t,s)\,dA_1(t)=\frac{1}{2}\widetilde{k}_0\left(\frac{1}{4},s\right)-\frac{1}{160}\widetilde{k}_0\left(\frac{3}{4},s\right)\\ &=\begin{cases}\frac{79}{960}s^3-\frac{77}{1280}s^2+\frac{71}{5120}s, & 0\le s\le\frac{1}{4},\\ \frac{1}{768}-\frac{9}{5120}s+\frac{3}{1280}s^2-\frac{1}{960}s^3, & \frac{1}{4}<s\le\frac{3}{4},\\ \frac{53}{61{,}440}, & \frac{3}{4}<s\le 1,\end{cases}\end{aligned}$$

and hence $0\le\widetilde{\mathcal{K}}_1(s)<0.0011$;

$$\widetilde{\mathcal{K}}_2(s)=\frac{1}{6}\int_0^s\left(t-\frac{1}{8}\right)t^3\,dt+\frac{1}{6}\int_s^1\left(t-\frac{1}{8}\right)s\left(3t^2-3ts+s^2\right)dt<0.0282;$$

$$\begin{aligned}\widetilde{\mathcal{K}}_3(s)&=\int_0^1\widetilde{k}_0(t,s)\,dB_3(t)=\frac{1}{2}\widetilde{k}_0\left(\frac{1}{2},s\right)-\frac{1}{14}\widetilde{k}_0\left(\frac{3}{4},s\right)\\ &=\begin{cases}\frac{1}{14}s^3-\frac{11}{112}s^2+\frac{19}{448}s, & 0\le s\le\frac{1}{2},\\ \frac{1}{96}-\frac{9}{448}s+\frac{3}{112}s^2-\frac{1}{84}s^3, & \frac{1}{2}<s\le\frac{3}{4},\\ \frac{29}{5376}, & \frac{3}{4}<s\le 1,\end{cases}\end{aligned}$$

and hence $0\le\widetilde{\mathcal{K}}_3(s)<0.0060$.

The 3×3 matrix

$$[A]=\begin{pmatrix}\alpha_1[\delta_1] & \alpha_1[\delta_2] & \alpha_1[\delta_3]\\ \alpha_2[\delta_1] & \alpha_2[\delta_2] & \alpha_2[\delta_3]\\ \alpha_3[\delta_1] & \alpha_3[\delta_2] & \alpha_3[\delta_3]\end{pmatrix}=\begin{pmatrix}\frac{79}{160} & \frac{77}{640} & \frac{71}{5120}\\ \frac{3}{8} & \frac{13}{48} & \frac{5}{48}\\ \frac{3}{7} & \frac{11}{56} & \frac{19}{448}\end{pmatrix}$$

and its spectrum radius $r([A])=0.6600<1$. Those mean that (C_2') and (C_3') are satisfied.

Now we take into account the constants in Theorem 3.1 and Theorem 3.2.

$$\left(I-[A]\right)^{-1}<\begin{pmatrix}2.3438 & 0.4079 & 0.0784\\ 1.3962 & 1.6558 & 0.2004\\ 1.3354 & 0.5223 & 1.1205\end{pmatrix}$$

and

$$\big(I-[A]\big)^{-1}\widetilde{\mathcal{K}}(s)<\begin{pmatrix}0.0145\\0.0494\\0.0229\end{pmatrix},$$

thus $\widetilde{k}_S(t,s)<0.0145+0.0494t+0.0229\times\frac{1}{2}t^2+\widetilde{k}_0(t,s)<0.2421$. So, for $u\in C^3[0,1]$ and $t\in[0,1]$,

$$\begin{aligned}\big|(\widetilde{L}_iu)(t)\big|&\le 0.2421\int_0^1\big(\widetilde{a}_i\big|u(s)\big|+\widetilde{b}_i\big|u'(s)\big|+\widetilde{c}_i\big|u''(s)\big|+\widetilde{d}_i\big|u'''(s)\big|\big)\,ds\\&\le 0.2421(\widetilde{a}_i+\widetilde{b}_i+\widetilde{c}_i+\widetilde{d}_i)\|u\|_{C^3}\quad(i=1,2),\end{aligned}$$

here $\widetilde{L}_i$ is defined in (3.6) ($i=1,2$). Since all the terms are nonnegative in the first, second, and third derivatives of $\widetilde{k}_S(t,s)$ with respect to t, we also have that, for $u\in C^3[0,1]$ and $t\in[0,1]$,

$$\begin{aligned}\big|(\widetilde{L}_iu)'(t)\big|&\le 0.5732\int_0^1\big(\widetilde{a}_i\big|u(s)\big|+\widetilde{b}_i\big|u'(s)\big|+\widetilde{c}_i\big|u''(s)\big|+\widetilde{d}_i\big|u'''(s)\big|\big)\,ds\\&\le 0.5732(\widetilde{a}_i+\widetilde{b}_i+\widetilde{c}_i+\widetilde{d}_i)\|u\|_{C^3}\quad(i=1,2),\\\big|(\widetilde{L}_iu)''(t)\big|&\le 1.0229\int_0^1\big(\widetilde{a}_i\big|u(s)\big|+\widetilde{b}_i\big|u'(s)\big|+\widetilde{c}_i\big|u''(s)\big|+\widetilde{d}_i\big|u'''(s)\big|\big)\,ds\\&\le 1.0229(\widetilde{a}_i+\widetilde{b}_i+\widetilde{c}_i+\widetilde{d}_i)\|u\|_{C^3}\quad(i=1,2),\\\big|(\widetilde{L}_iu)'''(t)\big|&\le\int_0^1\big(\widetilde{a}_i\big|u(s)\big|+\widetilde{b}_i\big|u'(s)\big|+\widetilde{c}_i\big|u''(s)\big|+\widetilde{d}_i\big|u'''(s)\big|\big)\,ds\\&\le(\widetilde{a}_i+\widetilde{b}_i+\widetilde{c}_i+\widetilde{d}_i)\|u\|_{C^3}\quad(i=1,2).\end{aligned}$$

Therefore the radius $r(\widetilde{L}_i)\le\|\widetilde{L}_i\|\le 1.0229(\widetilde{a}_i+\widetilde{b}_i+\widetilde{c}_i+\widetilde{d}_i)<1$ if

$$\widetilde{a}_i+\widetilde{b}_i+\widetilde{c}_i+\widetilde{d}_i<1.0229^{-1}\quad(i=1,2).\tag{3.14}$$

By the same reasoning as in the last section, we have from Lemma 3.1 and Lemma 3.2 that, for $u\in\widetilde{K}\setminus\{0\}$ and $t\in[0,1]$,

$$\big\|\big(\widetilde{L}_2^nu\big)\big\|_C=\big(\widetilde{L}_2^nu\big)(1)\ge\left(\frac{\widetilde{a}_2}{2}\right)^n\|u\|_C\left(\int_0^1\widetilde{c}_0(s)\widetilde{\Phi}_0(s)\,ds\right)^n$$

and the spectral radius

$$r(\widetilde{L}_2)\ge\frac{\widetilde{a}_2}{2}\left(\int_0^1\widetilde{c}_0(s)\widetilde{\Phi}_0(s)\,ds\right),$$

which implies that $r(\widetilde{L}_2)\ge 1$ when

$$\widetilde{a}_2\ge\frac{1680}{17}=\frac{2}{\int_0^1\frac{1}{2}s^3[\frac{1}{2}s(1-s)+\frac{1}{6}s^3]\,ds}\ge\frac{2}{\int_0^1\widetilde{c}_0(s)\widetilde{\Phi}_0(s)\,ds}.\tag{3.15}$$

Example 3.1 If $g(t,x_1,x_2,x_3,x_4) = \sqrt{x_1} + \sqrt{x_4}$, then BVP (3.13) has a positive solution.

Proof Take $\widetilde{a}_1 = \frac{1}{4}, \widetilde{b}_1 = \widetilde{c}_1 = 0, \widetilde{d}_1 = \frac{1}{5}, \widetilde{C}_0 = \frac{9}{4}$ and $\widetilde{a}_2 = 100, \widetilde{b}_2 = \widetilde{c}_2 = 0, \widetilde{d}_2 = 1, \widetilde{r} = 1/40{,}000$. Obviously, (3.14) and (3.15) are satisfied, meanwhile conditions (3.7) and (3.8) are fulfilled. Then BVP (3.13) has a positive solution by Theorem 3.1. □

Example 3.2 If

$$g(t,x_1,x_2,x_3,x_4) = \frac{\frac{1}{4}{x_1}^4 + \frac{1}{20}x_2^4 + \frac{1}{20}x_3^4 + \frac{1}{5}x_4^4}{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2},$$

then BVP (3.13) has a positive solution.

Proof Take $\widetilde{a}_2 = \frac{1}{4}, \widetilde{b}_2 = \frac{1}{20}, \widetilde{c}_2 = \frac{1}{20}, \widetilde{d}_2 = \frac{1}{5}, \widetilde{r} < 1$, it is easy to check that (3.9) and (3.14) are satisfied. Now take $\widetilde{a}_1 = 99, \widetilde{b}_1 = 7, \widetilde{c}_1 = 3$, it is clear that

$$\begin{aligned}
\frac{\widetilde{a}_1}{4}\int_0^1 s^3\widetilde{\Phi}_0(s)\,ds &> \frac{\widetilde{a}_1}{4}\int_0^1 s^3\left[\frac{1}{2}s(1-s) + \frac{1}{6}s^3\right]ds \\
&> \frac{1}{4}\times\frac{1680}{17}\int_0^1 s^3\left[\frac{1}{2}s(1-s) + \frac{1}{6}s^3\right]ds = 1, \\
\widetilde{b}_1\int_0^1 s^2\widetilde{\Phi}_1(s)\,ds &> \widetilde{b}_1\int_0^1 \frac{1}{2}s^3(2-s)\,ds > \frac{20}{3}\int_0^1 \frac{1}{2}s^3(2-s)\,ds = 1, \\
\widetilde{c}_1\int_0^1 s\widetilde{\Phi}_2(s)\,ds &> \widetilde{c}_1\int_0^1 s^2\,ds = 1,
\end{aligned}$$

so (3.10) is valid. It can be seen that (3.11) is satisfied for $\widetilde{C}_0$ large enough. Let $\widetilde{H}_M(\rho) = M^2 + \rho^2$ for $(\widetilde{F}_5)$. Then BVP (3.13) has a positive solution by Theorem 3.2. □

4 Conclusion

By the theory of fixed point index on cones in $C^3[0,1]$, we in this paper give the sufficient conditions for the existence of positive solutions to two classes of fourth-order problems with dependence on all derivatives in nonlinearities subject to Stieltjes integral boundary conditions. These sufficient conditions include some inequality ones on nonlinearities and the spectral radius ones of linear operators so that the nonlinearities have superlinear or sublinear growth. The derivatives, from the first to third order, of the positive solutions are nonpositive for one class and are nonnegative for the other class respectively. Some examples are also presented to illustrate the theorems under mixed multi-point and integral boundary conditions with sign-changing coefficients.

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Abbreviations

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