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Symmetries and conservation laws of the Yao–Zeng two-component short-pulse equation

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Abstract

This paper uses the Lie group method to analyze the symmetries of the Yao–Zeng two-component short-pulse equation which describes the propagation of polarized ultrashort light pulses in cubically nonlinear anisotropic optical fibers. Similarity reductions and exact solutions are obtained by constructing an optimal system of one-dimensional subalgebras. Moreover, the explicit solutions are constructed by the power series method and the convergence of power series solutions is proved. In addition, nonlinear self-adjointness and conservation laws of this system are discussed.

Keywords: The Yao–Zeng two-component short-pulse equation; Symmetries; The power series solutions; Conservation laws

1 Introduction

In the last decades, the short pulse equation has attracted much attention. It was introduced as a model equation to describe the propagation of ultrashort optical pules in silica optical fibers [1] which has the form

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \tag{1}$$

where u represents the magnitude of the electric field, subscripts denote partial differentiation. In order to obtain more information to study Eq. (1), a number of works have been presented. For instance, its integrability in [2–4], solitary wave solutions in [5], periodic and traveling wave solutions in [6], two-loop soliton solutions in [7], periodic and multiloop solitons in [8].

To describe the propagation of polarized ultra-short pulses, the short pulse equation has been generalized to the multi-component integrable systems. Among them, Matsuno presented the two-component system [9]

$$\begin{cases} u_{xt} = u + \frac{1}{2}(uvu_x)_x, \\ v_{xt} = v + \frac{1}{2}(uvv_x)_x, \end{cases} \tag{2}$$

which is addressed from its multi-component model.

Another integrable coupled short-pulse equation is given by Feng [10] as follows:

$$\begin{cases} u_{xt} = u + \frac{1}{6}(u^3)_{xx} + \frac{1}{2}v^2 u_{xx}, \\ v_{xt} = v + \frac{1}{6}(v^3)_{xx} + \frac{1}{2}u^2 v_{xx}. \end{cases} \tag{3}$$

Let $u = v$, Eq. (2) degenerates to Eq. (1); $v = 0$, Eq. (3) recasts to Eq. (1).

In this paper, we consider the Yao–Zeng two-component short-pulse equation [11]

$$\begin{cases} u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \\ v_{xt} = v + \frac{1}{2}(u^2 v_x)_x, \end{cases} \tag{4}$$

which describes the propagation of polarized ultrashort light pulses in cubically nonlinear anisotropic optical fibers. The Hamiltonian structure of Eq. (4) was established in [11], the zero-curvature representation of Eq. (4) was analyzed in [12]. For the sake of getting more information about Eq. (4), the goal of the present paper is to analyze symmetries, invariant solutions, and conservation laws of Eq. (4). The Lie group method [13–15] is considered to be one of the most effective methods to obtain exact solutions for lots of nonlinear partial differential equations(PDEs). Another important research topic is related to conservation laws of PDEs. For the PDEs which do not admit a Lagrangian, based on symmetries [16], Ibragimov presented the concept of an adjoint equation to investigate conservation laws by using the conservation law theorem in [17]. There has been a lot of success in this direction to construct conservation laws for PDEs [16, 18, 19]. By means of conservation laws, one can determine exact solutions of PDEs [20, 21].

The rest of the paper is arranged as follows. In Sect. 2, we present the Lie point symmetries of Eq. (4). In Sect. 3, the similarity reductions are made, exact solutions are considered by means of the Lie group method. In Sect. 4, the explicit solutions for the reduced equations are presented via the power series method, and the detailed proof for the convergence of the power series solutions is provided. In Sect. 5, we prove that Eq. (4) is nonlinearly self-adjoint and construct its conservation laws by applying Ibragimov's method. Finally, we have a summary of the paper.

2 Lie point symmetries

In this section, we apply the Lie point symmetry method to Eq. (4) and determine its infinitesimal generators and the commutation table of Lie algebras.

First of all, let us consider a one-parameter Lie group admitted by Eq. (4) with a generator of the Lie algebras of the form

$$X = \xi \partial_x + \tau \partial_t + \phi \partial_u + \varphi \partial_v, \tag{5}$$

where ξ, τ, ϕ, φ are functions of x, t, u, v and are described as infinitesimals of the symmetry groups.

The invariance criterion for Eq. (4) with respect to operator (5) is read as [13, 22]

$$\operatorname{pr} X^{(2)}\left[u_{xt} - u - \frac{1}{6}\left(u^3\right)_{xx}\right] = 0,$$

$$\operatorname{pr} X^{(2)}\left[v_{xt} - v - \frac{1}{2}\left(u^2 v_x\right)_x\right] = 0.$$

The symbol $\mathrm{pr}\,X^{(2)}$ is the usual 2th-order prolongation of the operator [13, 22]. In this case,

$$\mathrm{pr}\,X^{(2)} = X + \phi_x^{(1)}\frac{\partial}{\partial u_x} + \varphi_x^{(1)}\frac{\partial}{\partial v_x} + \phi_{xx}^{(2)}\frac{\partial}{\partial u_{xx}} + \varphi_{xx}^{(2)}\frac{\partial}{\partial v_{xx}} + \phi_{xt}^{(2)}\frac{\partial}{\partial u_{xt}} + \varphi_{xt}^{(2)}\frac{\partial}{\partial v_{xt}},$$

where

$$\begin{aligned}
\phi_x^{(1)} &= D_x\phi - u_xD_x\xi - u_tD_x\tau,\\
\varphi_x^{(1)} &= D_x\varphi - v_xD_x\xi - v_tD_x\tau,\\
\phi_{xx}^{(2)} &= D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt},\\
\varphi_{xx}^{(2)} &= D_x^2(\varphi - \xi v_x - \tau v_t) + \xi v_{xxx} + \tau v_{xxt},\\
\phi_{xt}^{(2)} &= D_xD_t(\phi - \xi u_x - \tau u_t) + \xi u_{xxt} + \tau u_{xtt},\\
\varphi_{xt}^{(2)} &= D_xD_t(\varphi - \xi v_x - \tau v_t) + \xi v_{xxt} + \tau v_{xtt},
\end{aligned}$$

and D_x, D_t are the total derivative operators, e.g.,

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + \cdots.$$

Substituting $\mathrm{pr}\,X^{(2)}$ into Eq. (4) yields the following over-determining equations for the unknown functions ξ, τ, ϕ, and φ:

$$\begin{aligned}
&\xi_t = \xi_u = \xi_v = 0, \qquad \xi_x = -\tau_t,\\
&\tau_x = \tau_u = \tau_v = 0, \qquad \tau_{tt} = 0,\\
&\phi = -u\tau_t,\\
&\varphi_x = \varphi_t = 0, \qquad \varphi_{vv} = 0, \qquad \varphi_u = \frac{(1-v)\varphi}{u}.
\end{aligned} \tag{6}$$

Solving Eq. (6), we get

$$\xi = -c_1x + c_3, \qquad \tau = c_1t + c_2, \qquad \phi = -c_1u, \qquad \varphi = c_5v + c_4u,$$

where c_1, c_2, c_3, c_4, and c_5 are arbitrary constants. The Lie algebra of infinitesimal symmetry of Eq. (4) is spanned by the following vector fields:

$$X_1 = -x\partial_x + t\partial_t - u\partial_u, \qquad X_2 = \partial_t, \qquad X_3 = \partial_x, \qquad X_4 = u\partial_v, \qquad X_5 = v\partial_v.$$

Furthermore, in order to classify all the group-invariant solutions, we determine an optimal system of one-dimensional subalgebras of Eq. (4) by using the method in [23, 24], which only relies on the commutator table.

First, the commutator relations of X_1, X_2, X_3, X_4, X_5 are represented in Table 1 by applying the commutator operator $[X_m, X_n] = X_mX_n - X_nX_m$.

An arbitrary operator $X \in L_5$ is written as

$$X = l_1X_1 + l_2X_2 + l_3X_3 + l_4X_4 + l_5X_5.$$

Table 1 Table of Lie brackets

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-X_2$	X_3	$-X_4$	0
X_2	X_2	0	0	0	0
X_3	$-X_3$	0	0	0	0
X_4	X_4	0	0	0	X_4
X_5	0	0	0	$-X_4$	0

To discuss the linear transformations of the vector $l = (l_1, l_2, l_3, l_4, l_5)$, we apply the following generator:

$$E_i = c_{ij}^k l_j \partial_{l_k}, \quad i = 1, 2, 3, 4, 5, \tag{7}$$

where c_{ij}^k is defined by the formula $[X_i, X_j] = c_{ij}^k X_k$. Based on Eq. (7) and Table 1, E_1, E_2, E_3, E_4, E_5 can be represented as

$$\begin{aligned}
&E_1 = -l_2\partial_{l_2} + l_3\partial_{l_3} - l_4\partial_{l_4},\\
&E_2 = l_1\partial_{l_2},\\
&E_3 = -l_1\partial_{l_3},\\
&E_4 = (l_1 + l_5)\partial_{l_4},\\
&E_5 = -l_4\partial_{l_4}.
\end{aligned}$$

For E_1, E_2, E_3, E_4, E_5, the Lie equations with parameters a_1, a_2, a_3, a_4, a_5 and the initial condition $\tilde{l}|_{a_i=0} = l$, $i = 1, 2, 3, 4, 5$ are shown as follows:

$$\begin{aligned}
&\frac{d\tilde{l}_1}{da_1} = 0, \quad \frac{d\tilde{l}_2}{da_1} = -\tilde{l}_2, \quad \frac{d\tilde{l}_3}{da_1} = \tilde{l}_3, \quad \frac{d\tilde{l}_4}{da_1} = -\tilde{l}_4, \quad \frac{d\tilde{l}_5}{da_1} = 0,\\
&\frac{d\tilde{l}_1}{da_2} = 0, \quad \frac{d\tilde{l}_2}{da_2} = \tilde{l}_1, \quad \frac{d\tilde{l}_3}{da_2} = 0, \quad \frac{d\tilde{l}_4}{da_2} = 0, \quad \frac{d\tilde{l}_5}{da_2} = 0,\\
&\frac{d\tilde{l}_1}{da_3} = 0, \quad \frac{d\tilde{l}_2}{da_3} = 0, \quad \frac{d\tilde{l}_3}{da_3} = -\tilde{l}_1, \quad \frac{d\tilde{l}_4}{da_3} = 0, \quad \frac{d\tilde{l}_5}{da_3} = 0,\\
&\frac{d\tilde{l}_1}{da_4} = 0, \quad \frac{d\tilde{l}_2}{da_4} = 0, \quad \frac{d\tilde{l}_3}{da_4} = 0, \quad \frac{d\tilde{l}_4}{da_4} = \tilde{l}_1 + \tilde{l}_5, \quad \frac{d\tilde{l}_5}{da_4} = 0,\\
&\frac{d\tilde{l}_1}{da_5} = 0, \quad \frac{d\tilde{l}_2}{da_5} = 0, \quad \frac{d\tilde{l}_3}{da_5} = 0, \quad \frac{d\tilde{l}_4}{da_5} = -\tilde{l}_4, \quad \frac{d\tilde{l}_5}{da_5} = 0.
\end{aligned}$$

The solutions of these equations give the transformations

$$\begin{aligned}
&T_1: \tilde{l}_1 = l_1, \quad \tilde{l}_2 = e^{-a_1} l_2, \quad \tilde{l}_3 = e^{a_1} l_3, \quad \tilde{l}_4 = e^{-a_1} l_4, \quad \tilde{l}_5 = l_5,\\
&T_2: \tilde{l}_1 = l_1, \quad \tilde{l}_2 = a_2 l_1 + l_2, \quad \tilde{l}_3 = l_3, \quad \tilde{l}_4 = l_4, \quad \tilde{l}_5 = l_5,\\
&T_3: \tilde{l}_1 = l_1, \quad \tilde{l}_2 = l_2, \quad \tilde{l}_3 = -a_3 l_1 + l_3, \quad \tilde{l}_4 = l_4, \quad \tilde{l}_5 = l_5,\\
&T_4: \tilde{l}_1 = l_1, \quad \tilde{l}_2 = l_2, \quad \tilde{l}_3 = l_3, \quad \tilde{l}_4 = a_4(l_1 + l_5) + l_4, \quad \tilde{l}_5 = l_5,\\
&T_5: \tilde{l}_1 = l_1, \quad \tilde{l}_2 = l_2, \quad \tilde{l}_3 = l_3, \quad \tilde{l}_4 = e^{-a_5} l_4, \quad \tilde{l}_5 = l_5.
\end{aligned}$$

The determination of the optimal system demands a simplification of the vector

$$l = (l_1, l_2, l_3, l_4, l_5) \tag{8}$$

by using the transformations $T_1 - T_5$. We will obtain the simplest representative of each class of similar vectors (8). Take two cases into account.

Case 2.1 $l_1 \neq 0$

Let $a_2 = -\frac{l_2}{l_1}$ and $a_3 = \frac{l_3}{l_1}$ in the transformations T_2 and T_3, we have $\tilde{l}_2 = \tilde{l}_3 = 0$. Vector (8) is simplified as the form

$$l = (l_1, 0, 0, l_4, l_5). \tag{9}$$

2.1.1 $l_1 + l_5 \neq 0$

By taking $a_4 = -\frac{l_4}{l_1+l_5}$ in the transformation T_4, we have $\tilde{l}_4 = 0$. Vector (9) is hence reduced to the form

$$l = (l_1, 0, 0, 0, l_5). \tag{10}$$

We get the following representatives:

$$X_1, X_1 + X_5. \tag{11}$$

2.1.2 $l_1 + l_5 = 0$

We get the following representatives:

$$X_1 - X_5, X_1 - X_5 \pm X_4. \tag{12}$$

Case 2.2 $l_1 = 0$

We obtain vector (8) of the form

$$l = (0, l_2, l_3, l_4, l_5). \tag{13}$$

2.2.1 $l_5 \neq 0$

Let $a_4 = -\frac{l_4}{l_5}$ in the transformations T_4, we have $\tilde{l}_4 = 0$. Vector (13) is simplified as the form

$$l = (0, l_2, l_3, 0, l_5). \tag{14}$$

As a result, we get the following representatives:

$$X_5, X_5 \pm X_2, , X_5 \pm X_3, , X_5 \pm X_2 \pm X_3. \tag{15}$$

2.2.2 $l_5 = 0$

We obtain vector (13) of the form

$$l = (0, l_2, l_3, l_4, 0). \tag{16}$$

Considering all the possible combinations, we get the following representatives:

$$X_2, X_3, X_4, X_2 \pm X_3, X_2 \pm X_4, X_3 \pm X_4, X_2 \pm X_3 \pm X_4. \tag{17}$$

Finally, by putting together all the operators (11), (12), (15), and (17), we achieve the following theorem.

Theorem 2.1 *The operators in* $\{X_1, X_2, X_3, X_4, X_5\}$ *create an optimal system*:

$$\begin{aligned} &X_1, X_1 \pm X_5, X_1 - X_5 \pm X_4, X_5, X_5 \pm X_2, X_5 \pm X_3, X_5 \pm X_2 \pm X_3, \\ &X_2, X_3, X_4, X_2 \pm X_3, X_2 \pm X_4, X_3 \pm X_4, X_2 \pm X_3 \pm X_4. \end{aligned}$$

3 Similarity reductions and exact solutions

Based on Theorem 2.1, we will discuss the reductions and exact solutions of Eq. (4) in this section.

Case 3.1 Reduction by X_1.

Solving the characteristic equation for X_1, we get similarity variables

$$z = tx, \qquad p = \frac{u}{x}, \qquad q = v,$$

and the group-invariant solution is $p = f(z)$, $q = g(z)$, i.e.,

$$u = xf(z), \qquad v = g(z). \tag{18}$$

Using Eq. (18) in Eq. (4), we have

$$\begin{cases} f + f^3 - 2f' + 3zf^2 f' - zf'' + z^2 ff'^2 + \frac{1}{2}z^2 f^2 f'' = 0, \\ g - g' + zf^2 g' - zg'' + z^2 ff' g' + \frac{1}{2}z^2 f^2 g'' = 0, \end{cases} \tag{19}$$

where $f' = \frac{df}{dz}$, $g' = \frac{dg}{dz}$.

Case 3.2 Reduction by $X_2 + X_5$.

Similarly, we have $z = x$, $u = f(z)$, $v = g(z)e^t$. The corresponding reduction equation is

$$\begin{cases} f + ff'^2 + \frac{1}{2}f^2 f'' = 0, \\ g - g' + ff' g' + \frac{1}{2}f^2 g'' = 0, \end{cases} \tag{20}$$

where $f' = \frac{df}{dz}$, $g' = \frac{dg}{dz}$. Therefore, Eq. (4) has a solution $u = 0$, $v = c_1 e^{x+t}$, where c_1 is an arbitrary constant.

Case 3.3 Reduction by $X_2 + X_3 + X_5$.

We have $u = f(z)$, $v = g(z)e^x$ in which $z = x - t$. Substituting group invariant solution into Eq. (4), we get

$$\begin{cases} f + ff'^2 + f'' + \frac{1}{2}f^2 f'' = 0, \\ g + g' + g'' + \frac{1}{2}f^2 g + ff' g + ff' g' + \frac{1}{2}f^2 g'' + f^2 g' = 0, \end{cases} \tag{21}$$

where $f' = \frac{df}{dz}$, $g' = \frac{dg}{dz}$.

Table 2 Similarity reductions of the Yao–Zeng two-component short-pulse equation

Generators	Similarity variables	Reduced equations
X_1+X_5	$z=tx$, $u=xf(z)$, $v=\frac{g(z)}{x}$.	$f+f^3-2f'-zf''+3zf^2f'+z^2ff'^2+\frac{1}{2}z^2f^2f''=0$, $g-zg''-zff'g+z^2ff'g'+\frac{1}{2}z^2f^2g''=0$.
$X_1-X_5+X_4$	$z=tx$, $u=xf(z)$, $v=xg(z)-x\ln xf(z)$.	$f+f^3-2f'-zf''-3zf^2f'-z^2ff'^2-\frac{1}{2}z^2f^2f''=0$, $g-\frac{3}{2}f^3+f'-2g'+f^2g-zg''-2zf^2f'+2zf^2g'+zff'g+\frac{1}{2}z^2f^2g''+z^2ff'g'=0$.
X_2	$z=x$, $u=f(z)$, $v=g(z)$.	$f+ff'^2+\frac{1}{2}f^2f''=0$, $g+ff'g'+\frac{1}{2}f^2g''=0$.
X_2+X_3	$z=x-t$, $u=f(z)$, $v=g(z)$.	$f+f''+ff'^2+\frac{1}{2}f^2f''=0$, $g+g''+ff'g'+\frac{1}{2}f^2g''=0$.
X_2+X_4	$z=x$, $u=f(z)$, $v=g(z)+tf(z)$.	$f+ff'^2+\frac{1}{2}f^2f''=0$, $g-f'+ff'g'+\frac{1}{2}f^2g''=0$.
$X_2+X_3+X_4$	$z=x-t$, $u=f(z)$, $v=g(z)+xf(z)$.	$f+f''+ff'^2+\frac{1}{2}f^2f''=0$, $g+f'+g''+2f^2f'+ff'g'+\frac{1}{2}f^2g''=0$.

Case 3.4 Reduction by X_3+X_4.

We have $u=f(z)$, $v=g(z)+xf(z)$ in which $z=t$. Substituting group invariant solution into Eq. (4), we get

$$\begin{cases} f=0, \\ g=0. \end{cases} \tag{22}$$

Therefore, Eq. (4) has a solution $u=0$, $v=0$. Obviously, the solution is not meaningful.

Some of the similarity reductions for the optimal system of one-dimensional subalgebra are represented in Table 2.

4 The explicit power series solutions

In Sect. 3, we obtained the reduction equations by using symmetry analysis. The power series can be used to treat differential equations, including many complicated differential equations with nonconstant coefficients [25]. In this section, we solve the nonlinear ODE (19) by the power series method. For other reduction equations, power series solutions can also be obtained similarly. For more details on power series solutions, see [15, 26].

Now, we seek a solution of Eq. (19) in a power series of the form

$$f(z)=\sum_{n=0}^{\infty}p_nz^n, \qquad g(z)=\sum_{n=0}^{\infty}q_nz^n, \tag{23}$$

where the coefficients p_n and q_n are all constants to be determined.

Substituting (23) into (19), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}p_nz^n+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}p_lp_{k-l}p_{n-k}z^n-2\sum_{n=0}^{\infty}(n+1)p_{n+1}z^n\\&\quad+3z\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}(n-k+1)p_lp_{k-l}p_{n-k+1}z^n\end{aligned}$$

$$
\begin{aligned}
& - z\sum_{n=0}^{\infty}(n+1)(n+2)p_{n+2}z^n \\
& + z^2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}(k-l+1)(n-k+1)p_l p_{k-l+1}p_{n-k+1}z^n \\
& + \frac{1}{2}z^2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}(n-k+1)(n-k+2)p_l p_{k-l}p_{n-k+2}z^n = 0, \qquad (24)\\
& \sum_{n=0}^{\infty}q_n z^n - \sum_{n=0}^{\infty}(n+1)q_{n+1}z^n + z\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}(n-k+1)p_l p_{k-l}q_{n-k+1}z^n \\
& - z\sum_{n=0}^{\infty}(n+1)(n+2)q_{n+2}z^n \\
& + z^2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}(k-l+1)(n-k+1)p_l p_{k-l+1}q_{n-k+1}z^n \\
& + \frac{1}{2}z^2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}(n-k+1)(n-k+2)p_l p_{k-l}q_{n-k+2}z^n = 0.
\end{aligned}
$$

From (24), comparing coefficients, we obtain

$$
\begin{aligned}
& p_1 = \frac{1}{2}p_0\left(1+p_0^2\right), \qquad q_1 = q_0; \\
& p_2 = \frac{1}{6}p_1\left(1+6p_0^2\right), \qquad q_2 = \frac{1}{4}q_1\left(1+p_0^2\right).
\end{aligned} \tag{25}
$$

Generally, for $n \geq 0$, we have

$$
\begin{aligned}
p_{n+3} = {} & \frac{1}{(n+3)(n+4)}\Bigg\{p_{n+2} + \sum_{k=0}^{n}\sum_{l=0}^{k}(n-k+1)p_l\bigg[(k-l+1)p_{k-l+1}p_{n-k+1} \\
& + \frac{1}{2}(n-k+2)p_{k-l}p_{n-k+2}\bigg] + \sum_{k=0}^{n+1}\sum_{l=0}^{k}(3n-3k+7)p_l p_{k-l}p_{n-k+2} \\
& + \sum_{l=0}^{n+2}p_l p_{n+2-l}p_0\Bigg\}, \qquad (26)\\
q_{n+3} = {} & \frac{1}{(n+3)^2}\Bigg\{q_{n+2} + \sum_{k=0}^{n}\sum_{l=0}^{k}\bigg[\frac{1}{2}(n-k+2)(n-k+3)p_l p_{k-l}q_{n-k+2} \\
& + (k-l+1)(n-k+1)p_l p_{k-l+1}q_{n-k+1}\bigg] + \sum_{l=0}^{n+1}p_l p_{n+1-l}q_1\Bigg\}.
\end{aligned}
$$

In view of Eq. (26), we can obtain all the coefficients p_i, q_i $(i \geq 3)$ of the power series (23), e.g.,

$$
\begin{aligned}
& p_3 = \frac{1}{12}\left(p_2 + 10p_0^2 p_2 + 10p_0 p_1^2\right), \\
& q_3 = \frac{1}{9}\left(q_2 + 3p_0^2 q_2 + 3p_0 p_1 q_1\right).
\end{aligned} \tag{27}
$$

Therefore, for arbitrary chosen constant numbers p_0 and q_0, the other terms of the sequence $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ can be determined by (25) and (26). This implies that for Eq. (19), there is a power series solution (23) with the coefficients constructed by (25) and (26).

Now we show the convergence of the power series solution (23) of Eq. (19). In fact, from (26), we have

$$\begin{aligned}|p_{n+3}| \le M\Bigg[&|p_{n+2}| + \sum_{k=0}^{n}\sum_{l=0}^{k}|p_l|\big(|p_{k-l+1}||p_{n-k+1}| + |p_{k-l}||p_{n-k+2}|\big)\\&+\sum_{k=0}^{n+1}\sum_{l=0}^{k}|p_l||p_{k-l}||p_{n-k+2}| + \sum_{l=0}^{n+2}|p_l||p_{n+2-l}|\Bigg], \quad n = 0, 1, \ldots,\end{aligned}$$

where $M = \max\{1, \frac{1}{2}, 3, p_0\}$. Similarly, from (26), we have

$$\begin{aligned}|q_{n+3}| \le N\Bigg[&|q_{n+2}| + \sum_{k=0}^{n}\sum_{l=0}^{k}\big(|p_l||p_{k-l}||q_{n-k+2}| + |p_l||p_{k-l+1}||q_{n-k+1}|\big)\\&+\sum_{l=0}^{n+1}|p_l||p_{n+1-l}|\Bigg], \quad n = 0, 1, \ldots,\end{aligned} \tag{28}$$

where $N = \max\{1, \frac{1}{2}, q_1\}$.

Now, we define two power series $R = R(z) = \sum_{n=0}^{\infty} r_n z^n$ and $S = S(z) = \sum_{n=0}^{\infty} s_n z^n$ by

$$r_i = |p_i|, \qquad s_j = |q_j|, \quad i, j = 0, 1, 2,$$

and

$$\begin{aligned}r_{n+3} &= M\Bigg[r_{n+2} + \sum_{k=0}^{n}\sum_{l=0}^{k}(r_l r_{k-l+1} r_{n-k+1} + r_l r_{k-l} r_{n-k+2})\\&\quad+\sum_{k=0}^{n+1}\sum_{l=0}^{k} r_l r_{k-l} r_{n-k+2} + \sum_{l=0}^{n+2} r_l r_{n+2-l}\Bigg],\\ s_{n+3} &= N\Bigg[s_{n+2} + \sum_{k=0}^{n}\sum_{l=0}^{k}(r_l r_{k-l} s_{n-k+2} + r_l r_{k-l+1} s_{n-k+1})\\&\quad+\sum_{l=0}^{n+1} r_l r_{n+1-l}\Bigg],\end{aligned} \tag{29}$$

where $n = 0, 1, \ldots$. Then, it is easily seen that

$$|p_n| \le r_n, \qquad |q_n| \le s_n, \quad n = 0, 1, 2, \ldots.$$

Thus, the two series $R = R(z) = \sum_{n=0}^{\infty} r_n z^n$ and $S = S(z) = \sum_{n=0}^{\infty} s_n z^n$ are majorant series of (23), respectively. Next, we show that the series $R = R(z)$ and $S = S(z)$ have positive radius of convergence.

$$\begin{aligned}
R(z) &= r_0 + r_1 z + r_2 z^2 + \sum_{n=0}^{\infty} r_{n+3} z^{n+3} = r_0 + r_1 z + r_2 z^2 \\
&\quad + M\Bigg[\sum_{n=0}^{\infty} r_{n+2} z^{n+3} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k} (r_l r_{k-l+1} r_{n-k+1} + r_l r_{k-l} r_{n-k+2}) z^{n+3} \\
&\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\sum_{l=0}^{k} r_l r_{k-l} r_{n-k+2} z^{n+3} + \sum_{n=0}^{\infty}\sum_{l=0}^{n+2} r_l r_{n+2-l} z^{n+3}\Bigg] \\
&= r_0 + r_1 z + r_2 z^2 + M\big[z(R - r_0 - r_1 z) + zR(R - r_0)^2 + zR^2(R - r_0 - r_1 z) \\
&\quad + z\big(R^2(R - r_0) - r_0^2 r_1 z\big) + z\big((R + r_0)(R - r_0 - r_1 z) + r_1 z(R - r_0)\big)\big],
\end{aligned}$$

and

$$\begin{aligned}
S(z) &= s_0 + s_1 z + s_2 z^2 + \sum_{n=0}^{\infty} s_{n+3} z^{n+3} \\
&= s_0 + s_1 z + s_2 z^2 + N\Bigg[\sum_{n=0}^{\infty} s_{n+2} z^{n+3} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k} (r_l r_{k-l} s_{n-k+2} + r_l r_{k-l+1} s_{n-k+1}) z^{n+3} \\
&\quad + \sum_{n=0}^{\infty}\sum_{l=0}^{n+1} r_l r_{n+1-l} z^{n+3}\Bigg] \\
&= s_0 + s_1 z + s_2 z^2 + N\big[z(S - s_0 - s_1 z) \\
&\quad + zR^2(S - s_0 - s_1 z) + zR(RS - r_0 S - R s_0 + r_0 s_0) + z^2\big(R^2 - r_0^2\big)\big].
\end{aligned}$$

Consider now the implicit functional system with respect to the independent variable z:

$$\begin{aligned}
F(z, R, S) &= R - r_0 - r_1 z - r_2 z^2 - M\big[z(R - r_0 - r_1 z) + zR(R - r_0)^2 + zR^2(R - r_0 - r_1 z) \\
&\qquad + z\big(R^2(R - r_0) - r_0^2 r_1 z\big) + z\big((R + r_0)(R - r_0 - r_1 z) + r_1 z(R - r_0)\big)\big], \\
G(z, R, S) &= S - s_0 - s_1 z - s_2 z^2 - N\big[z(S - s_0 - s_1 z) + zR^2(S - s_0 - s_1 z) \\
&\qquad + zR(RS - r_0 S - R s_0 + r_0 s_0) + z^2\big(R^2 - r_0^2\big)\big].
\end{aligned}$$

Since F, G are analytic in the neighborhood of $(0, r_0, s_0)$ and $F(0, r_0, s_0) = 0$, $G(0, r_0, s_0) = 0$. Furthermore, the Jacobian determinant

$$\left.\frac{\partial(F, G)}{\partial(R, S)}\right|_{(0, r_0, s_0)} = 1 \neq 0,$$

if we choose the parameters $r_0 = |p_0|$ and $s_0 = |q_0|$ properly. By the implicit function theorem [27], we see that $R = R(z)$ and $S = S(z)$ are analytic in a neighborhood of the point $(0, r_0, s_0)$ and with the positive radius. This implies that the two power series (23) converge in a neighborhood of the point $(0, r_0, s_0)$. This completes the proof.

Hence, the power series solution (23) for Eq. (19) is an analytic solution. The power series solution of Eq. (19) can be written as the following:

$$
\begin{aligned}
f(z) &= p_0 + p_1 z + p_2 z^2 + \sum_{n=0}^{\infty} p_{n+3} z^{n+3} \\
&= p_0 + \frac{1}{2} p_0 (1 + p_0^2) z + \frac{1}{6} p_1 (1 + 6p_0^2) z^2 \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{(n+3)(n+4)} \Biggl\{ p_{n+2} + \sum_{k=0}^{n} \sum_{l=0}^{k} (n-k+1) p_l \biggl[(k-l+1) p_{k-l+1} p_{n-k+1} \\
&\quad + \frac{1}{2}(n-k+2) p_{k-l} p_{n-k+2} \biggr] + \sum_{k=0}^{n+1} \sum_{l=0}^{k} (3n-3k+7) p_l p_{k-l} p_{n-k+2} \\
&\quad + \sum_{l=0}^{n+2} p_l p_{n+2-l} p_0 \Biggr\} z^{n+3}, \\
g(z) &= q_0 + q_1 z + q_2 z^2 + \sum_{n=0}^{\infty} q_{n+3} z^{n+3} \\
&= q_0 + q_0 z + \frac{1}{4} q_1 (1 + p_0^2) z^2 \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{(n+3)^2} \Biggl\{ q_{n+2} + \sum_{k=0}^{n} \sum_{l=0}^{k} \biggl[\frac{1}{2}(n-k+2)(n-k+3) p_l p_{k-l} q_{n-k+2} \\
&\quad + (k-l+1)(n-k+1) p_l p_{k-l+1} q_{n-k+1} \biggr] + \sum_{l=0}^{n+1} p_l p_{n+1-l} q_1 \Biggr\} z^{n+3}.
\end{aligned}
$$

Thus, the explicit power series solution of Eq. (4) is

$$
\begin{aligned}
u(x,t) &= p_0 x + \frac{1}{2} p_0 (1 + p_0^2) t x^2 + \frac{1}{6} p_1 (1 + 6p_0^2) t^2 x^3 \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{(n+3)(n+4)} \Biggl\{ p_{n+2} + \sum_{k=0}^{n} \sum_{l=0}^{k} (n-k+1) p_l \biggl[(k-l+1) p_{k-l+1} p_{n-k+1} \\
&\quad + \frac{1}{2}(n-k+2) p_{k-l} p_{n-k+2} \biggr] + \sum_{k=0}^{n+1} \sum_{l=0}^{k} (3n-3k+7) p_l p_{k-l} p_{n-k+2} \\
&\quad + \sum_{l=0}^{n+2} p_l p_{n+2-l} p_0 \Biggr\} t^{n+3} x^{n+4}, \\
v(x,t) &= q_0 + q_0 t x + \frac{1}{4} q_1 (1 + p_0^2) t^2 x^2 \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{(n+3)^2} \Biggl\{ q_{n+2} + \sum_{k=0}^{n} \sum_{l=0}^{k} \biggl[\frac{1}{2}(n-k+2)(n-k+3) p_l p_{k-l} q_{n-k+2} \\
&\quad + (k-l+1)(n-k+1) p_l p_{k-l+1} q_{n-k+1} \biggr] + \sum_{l=0}^{n+1} p_l p_{n+1-l} q_1 \Biggr\} t^{n+3} x^{n+3},
\end{aligned}
$$

where p_0 and q_0 are arbitrary constants, the other coefficients p_n, q_n ($n \geq 1$) depend on (25) and (26) completely.

Remark 4.1 The power series solutions can greatly enrich the solutions of Eq. (4) and converge quickly, so it is convenient for computations in both theory and applications.

5 Nonlinear self-adjointness and conservation law

In this section, we prove the nonlinear self-adjointness of Eq. (4) and determine its conservation laws.

5.1 Preliminaries

Consider the rth-order system of m PDEs:

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \quad \alpha = 1, 2, \dots, m, \tag{30}$$

where $x = (x^1, \dots, x^n)$, $u = (u^1, \dots, u^m)$, $u_i^\alpha = \partial u^\alpha / \partial x^i$, $u_{ij}^\alpha = \partial^2 u^\alpha / \partial x^i \partial x^j$, and $u_{(i)}$ denotes the collection of all ith-order partial derivatives of u with respect to x.

The adjoint equations of Eq. (30) are defined by [28]

$$E_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(r)}, v_{(r)}) = \frac{\delta \mathcal{L}}{\delta u^\sigma} = 0, \tag{31}$$

where $v = (v^1, \dots, v^m)$, $\mathcal{L} = v^\beta E_\beta(x, u, u_{(1)}, \dots, u_{(r)})$ is the formal Lagrangian, and $\delta/\delta u^\sigma$ is the Euler–Lagrange operator defined by [28]

$$\frac{\delta}{\delta u^\sigma} = \frac{\partial}{\partial u^\sigma} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 \dots i_s}^\sigma},$$

where D_i denotes the total derivative operators with respect to x_i.

Definition 5.1 ([16]) System (30) is said to be nonlinearly self-adjoint if the adjoint system (31) is satisfied for all solutions u of Eq. (30) upon a substitution

$$v^\alpha = \varphi^\alpha(x, u), \quad \alpha = 1, \dots, m, \tag{32}$$

such that $\varphi(x, u) = (\varphi^1, \dots, \varphi^m) \neq 0$.

Definition 5.1 is identical to the following identities:

$$E_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(r)}, v_{(r)}) = \lambda_\alpha^\beta E_\beta(x, u, u_{(1)}, \dots, u_{(r)}), \alpha, \quad \beta = 1, \dots, m, \tag{33}$$

where λ_α^β is a certain function.

The following theorem will be used to obtain conservation laws [17].

Theorem 5.2 *Any infinitesimal symmetry* (*local and nonlocal*)

$$X = \xi^i(x, u, u_{(1)}, \dots) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \dots) \frac{\partial}{\partial u^\alpha}$$

admitted by Eq. (30) *gives rise to a conservation law* $D_i(C^i) = 0$, *where* C^i *is constructed by the formula*

$$C^i = W^\alpha\left[\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \cdots\right] + D_j\left(W^\alpha\right)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_jD_k\left(W^\alpha\right)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots\right] + \cdots, \tag{34}$$

with $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$. *The Lagrangian* $\mathcal{L}$ *should be written in the symmetric form with respect to all mixed derivatives* $u_{ij}^\alpha, u_{ijk}^\alpha, \ldots$.

5.2 Nonlinear self-adjointness

Following Definition 5.1 and Theorem 5.2, we can prove the nonlinear self-adjointness of Eq. (4) and thus obtain its conservation laws.

Theorem 5.3 *Equation* (4) *is nonlinearly self-adjoint under the substitution*

$$\Lambda_1 = c_1 v, \qquad \Lambda_2 = -c_1 u,$$

where c_1 *is an arbitrary constant.*

Proof Let the formal Lagrangian of Eq. (4) be of the form

$$\mathcal{L} = \Lambda_1\left(u_{xt} - u - \frac{1}{6}\left(u^3\right)_{xx}\right) + \Lambda_2\left(v_{xt} - v - \frac{1}{2}\left(u^2 v_x\right)_x\right),$$

where Λ_1, Λ_2 are two new dependent variables.

Using the equivalent formula (33) of the definition of nonlinear self-adjointness, the identities

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta u} &= \lambda_1^1\left(u_{xt} - u - \frac{1}{6}\left(u^3\right)_{xx}\right) + \lambda_1^2\left(v_{xt} - v - \frac{1}{2}\left(u^2 v_x\right)_x\right),\\ \frac{\delta\mathcal{L}}{\delta v} &= \lambda_2^1\left(u_{xt} - u - \frac{1}{6}\left(u^3\right)_{xx}\right) + \lambda_2^2\left(v_{xt} - v - \frac{1}{2}\left(u^2 v_x\right)_x\right),\end{aligned} \tag{35}$$

are established under the substitution $\Lambda_1 = \Lambda_1(x,t,u,v)$, $\Lambda_2 = \Lambda_2(x,t,u,v)$.

By splitting Eq. (35) with respect to the coefficients of different order derivatives of u and v, we obtain a system in the unknown variables Λ_1, Λ_2 whose solutions are

$$\Lambda_1 = c_1 v, \qquad \Lambda_2 = -c_1 u,$$

where c_1 is an arbitrary constant. This completes the proof. □

5.3 Construction of conservation laws

For the infinitesimal operator $X = \xi\partial_x + \tau\partial_t + \phi\partial_u + \varphi\partial_v$, by Theorem 5.2, the conservation law of Eq. (4) is represented in the form $D_xC^x + D_tC^t = 0$. Moreover, we have

$$\begin{aligned} C^x &= W^u\left[\frac{\partial\mathcal{L}}{\partial u_x} - D_t\left(\frac{\partial\mathcal{L}}{\partial u_{xt}}\right) - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right)\right] + D_t\left(W^u\right)\frac{\partial\mathcal{L}}{\partial u_{xt}} + D_x\left(W^u\right)\frac{\partial\mathcal{L}}{\partial u_{xx}} \\ &\quad + W^v\left[\frac{\partial\mathcal{L}}{\partial v_x} - D_t\left(\frac{\partial\mathcal{L}}{\partial v_{xt}}\right) - D_x\left(\frac{\partial\mathcal{L}}{\partial v_{xx}}\right)\right] + D_t\left(W^v\right)\frac{\partial\mathcal{L}}{\partial v_{xt}} + D_x\left(W^v\right)\frac{\partial\mathcal{L}}{\partial v_{xx}}, \\ C^t &= W^u\left[-D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xt}}\right)\right] + D_x\left(W^u\right)\frac{\partial\mathcal{L}}{\partial u_{xt}} \\ &\quad + W^v\left[-D_x\left(\frac{\partial\mathcal{L}}{\partial v_{xt}}\right)\right] + D_x\left(W^v\right)\frac{\partial\mathcal{L}}{\partial v_{xt}}, \end{aligned}$$

where $W^u = \phi - \xi u_x - \tau u_t$, $W^v = \varphi - \xi v_x - \tau v_t$, the Lagrangian $\mathcal{L} = \Lambda_1(u_{xt} - u - \frac{1}{6}(u^3)_{xx}) + \Lambda_2(v_{xt} - v - \frac{1}{2}(u^2v_x)_x) = c_1v(u_{xt} - u - \frac{1}{6}(u^3)_{xx}) - c_1u(v_{xt} - v - \frac{1}{2}(u^2v_x)_x)$.

By simplifying C^x, C^t, we have

$$\begin{aligned} C^x &= \left(-c_1uvu_x + \frac{3}{2}c_1u^2v_x - c_1v_t\right)W^u + c_1vD_t\left(W^u\right) - \frac{1}{2}c_1u^2vD_x\left(W^u\right) \\ &\quad + \left(-\frac{1}{2}c_1u^2u_x + c_1u_t\right)W^v - c_1uD_t\left(W^v\right) + \frac{1}{2}c_1u^3D_x\left(W^v\right), \end{aligned} \tag{36}$$

$$C^t = -c_1v_xW^u + c_1vD_x\left(W^u\right) + c_1u_xW^v - c_1uD_x\left(W^v\right). \tag{37}$$

Case 5.1 For the generator $X_1 = -x\partial_x + t\partial_t - u\partial_u$, we obtain $W^u = -u + xu_x - tu_t$, $W^v = xv_x - tv_t$. According to Eqs. (36) and (37), we have

$$\begin{aligned} C^x &= \left(-c_1uvu_x + \frac{3}{2}c_1u^2v_x - c_1v_t\right)(-u + xu_x - tu_t) + c_1v(-2u_t + xu_{tx} - tu_{tt}) \\ &\quad - \frac{1}{2}c_1u^2v(xu_{xx} - tu_{tx}) + \left(-\frac{1}{2}c_1u^2u_x + c_1u_t\right)(xv_x - tv_t) - c_1u(-v_t - tv_{tt} + xv_{xt}) \\ &\quad + \frac{1}{2}c_1u^3(v_x - tv_{tx} + xv_{xx}), \\ C^t &= -c_1v_x(-u + xu_x - tu_t) + c_1v(xu_{xx} - tu_{tx}) + c_1u_x(xv_x - tv_t) - c_1u(v_x - tv_{tx} + xv_{xx}). \end{aligned}$$

Case 5.2 For the generator $X_2 = \partial_t$, we obtain $W^u = -u_t$, $W^v = -v_t$. According to Eqs. (36) and (37), we have

$$\begin{aligned} C^x &= -\left(-c_1uvu_x + \frac{3}{2}c_1u^2v_x - c_1v_t\right)u_t - c_1vu_{tt} + \frac{1}{2}c_1u^2vu_{tx} \\ &\quad - \left(-\frac{1}{2}c_1u^2u_x + c_1u_t\right)v_t + c_1uv_{tt} - \frac{1}{2}c_1u^3v_{tx}, \\ C^t &= c_1v_xu_t - c_1vu_{tx} - c_1u_xv_t + c_1uv_{tx}. \end{aligned}$$

Case 5.3 For the generator $X_3 = \partial_x$, we obtain $W^u = -u_x$, $W^v = -v_x$. According to Eqs. (36) and (37), we have

$$\begin{aligned}
C^x &= -\left(-c_1 uvu_x + \frac{3}{2}c_1 u^2 v_x - c_1 v_t\right)u_x - c_1 vu_{xt} + \frac{1}{2}c_1 u^2 vu_{xx} \\
&\quad - \left(-\frac{1}{2}c_1 u^2 u_x + c_1 u_t\right)v_x + c_1 uv_{xt} - \frac{1}{2}c_1 u^3 v_{xx}, \\
C^t &= c_1 v_x u_x - c_1 vu_{xx} - c_1 u_x v_x + c_1 uv_{xx}.
\end{aligned}$$

Case 5.4 For the generator $X_4 = u\partial_v$, we obtain $W^u = 0$, $W^v = u$. According to Eqs. (36) and (37), we have

$$\begin{aligned}
C^x &= \left(-\frac{1}{2}c_1 u^2 u_x + c_1 u_t\right)u - c_1 uu_t + \frac{1}{2}c_1 u^3 u_x, \\
C^t &= c_1 u_x u - c_1 uu_x.
\end{aligned}$$

Case 5.5 For the generator $X_5 = v\partial_v$, we obtain $W^u = 0$, $W^v = u$. According to Eqs. (36) and (37), we have

$$\begin{aligned}
C^x &= \left(-\frac{1}{2}c_1 u^2 u_x + c_1 u_t\right)v - c_1 uv_t + \frac{1}{2}c_1 u^3 v_x, \\
C^t &= c_1 u_x v - c_1 uv_x.
\end{aligned}$$

6 Conclusions

In this paper, the Lie symmetry analysis is applied for the Yao–Zeng two-component short-pulse equation. New invariant solutions are constructed based on the optimal system. Moreover, we use the properties of nonlinear self-adjointness of Eq. (4) to obtain general formulae of conservation law. Our results can be applied to describe the propagation of polarized ultrashort light pulses in cubically nonlinear anisotropic optical fibers.

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Authors' contributions

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