Wang and Guo *Boundary Value Problems* (2019) 2019:48
https://doi.org/10.1186/s13661-019-1161-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A new approach to convergence analysis of linearized finite element method for nonlinear hyperbolic equation

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Abstract

We study a new way to convergence results for a nonlinear hyperbolic equation with bilinear element. Such equation is transformed into a parabolic system by setting the original solution u as $u_t = q$. A linearized backward Euler finite element method (FEM) is introduced, and the splitting skill is exploited to get rid of the restriction on the ratio between h and τ. The boundedness of the solutions about the time-discrete system in H^2-norm is proved skillfully through temporal error. The spatial error is derived without the mesh-ratio, where some new techniques are utilized to deal with the problems caused by the new parabolic system. The final unconditional optimal error results of u and q are deduced at the same time. Finally, a numerical example is provided to support the theoretical analysis. Here h is the subdivision parameter, and τ is the time step.

Keywords: Nonlinear hyperbolic equation; Parabolic system; Bilinear element; Linearized FEM; Optimal error results

1 Introduction

Consider the following nonlinear hyperbolic equation:

$$\begin{cases} u_{tt} - \nabla \cdot (a(u)\nabla u) = f(u), & (X,t) \in \Omega \times (0,T], \\ u = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), \qquad u_t(X,0) = u_1(X), & X \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a rectangle with boundary $\partial\Omega$ parallel to the coordinate axes, $0 < T < \infty$, $X = (x,y)$, and $a(u)$ and $f(u)$ are known smooth functions on $\mathbb{R}$, for which we assume that $0 < a_0 \le a(u) \le a_1$.

A nonlinear hyperbolic equation is a kind of important problems on nonlinear vibration, the permeation fluid mechanics, and so on. Indeed, such partial differential equations (PDEs) have attracted lots of attention to various methods, especially numerical methods. For example, the two-grid method was studied for solving a type of nonlinear hyperbolic equations, and the error estimate in H^1-norm was deduced in [1]. Newton's modified method was utilized to a nonlinear wave equation depending on different norms of the initial conditions in [2], and optimal error results were given in the L^2- and H^1-norms. The interpolation theory and integral identity skill were used to obtain a superclose result for the nonlinear hyperbolic equations with nonlinear boundary condition in [3]. Moreover, the global superconvergence was also obtained through the interpolated postprocessing technique. The Galerkin alternating-direction method was applied to a three-dimensional nonlinear hyperbolic equation in [4], and the error estimates in the H^1- and L^2-norms were deduced. A mixed FEM was discussed in [5] and [6], and optimal error estimates were derived.

The inverse inequality is usually employed to discuss the boundedness of numerical solution U_h^n in a nonlinear evolution equation, and such an issue usually results in some time-step restrictions, such as $\tau = O(h)$, $h^r = O(\tau)$ $(1 \leq r \leq k+1, k \geq 0)$, and $\tau = O(h^2)$ in [4] and [6], respectively. To get rid of such a restriction, [7, 8] took advantage of a special inequality for getting unconditional superclose results for nonlinear Sobolev equations. In [9] a corresponding time-discrete system to split the error into two parts, the temporal error and the spatial error, is introduced. Then the spatial error leads to the unconditional boundedness of a numerical solution in the L^∞-norm. Subsequently, this so-called splitting technique was also applied to the other nonlinear parabolic type equations in [10–18]. Later, in [19] and [20] a second-order scheme for the nonlinear hyperbolic equation and the unconditional superconvergence analysis by using the splitting skill were given. It can be seen that constructing a linearized form for a nonlinear hyperbolic equation is not an easy task in comparison with nonlinear parabolic equations. In fact, there are lots of literature referring to parabolic equations [21–24]. In [24] a special technique to change sine-Gordon equation into a parabolic system through $u_t = q$ was used, and optimal order error estimates of the Crank–Nicolson fully discrete scheme were obtained.

Inspired by [24], in this paper, we consider the unconditional convergent estimates for (1), which is a much more general nonlinear model than that in [24], with a bilinear element. First of all, we change a nonlinear hyperbolic equation into a nonlinear parabolic system. Such a practice can be used to avoid the difficulty in constructing a linearized scheme for a nonlinear hyperbolic equation and also give the error analysis for u and $q = u_t$ at the same time. Then we develop a linearized backward Euler FE scheme for the nonlinear parabolic system and apply the idea of splitting technique in [10–20] to split the error into the temporal and spatial errors. We obtain a temporal error, which implies the regularities of the solutions about the time-discrete equations. The spatial error result is exploited to get rid of the restriction on the ratio between h and τ. The unconditional optimal error results of u and q are simultaneously deduced. Note that, differently from [17, 18], we utilize some new tricks such as rewriting some error terms, the new mean-value technique, and some other skills to handle new difficulties brought by the special nonlinear parabolic system during the process. Further, the results in this paper also hold for linear conforming triangular elements but do not hold for some other particular elements; for example, the biquadratic finite element for $\Delta v_h|_K = 0$ cannot be true, where v_h belongs to the FE space. Some numerical results in the last section also show the validity of the theoretical analysis.

Throughout this paper, we denote the natural inner production in $L^2(\Omega)$ by $(\cdot,\cdot)$ and the norm by $\|\cdot\|_0$, and let $H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$. Further, we use the classical Sobolev spaces $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$, denoted by $W^{m,p}$, with norm $\|\cdot\|_{m,p}$. When $p = 2$, we simply write $\|\cdot\|_{m,p}$ as $\|\cdot\|_m$. Besides, we define the space $L^p(a,b;Y)$ with norm $\|f\|_{L^p(a,b;Y)} = (\int_a^b \|f(\cdot,t)\|_Y^p\,dt)^{\frac{1}{p}}$, and if $p = \infty$, the integral is replaced by the essential supremum.

2 Conforming FE approximation scheme

Let Ω be a rectangle in the (x, y) plane with edges parallel to the coordinate axes, and let Γ_h be a regular rectangular subdivision. Given $K \in \Gamma_h$, let the four vertices and edges be a_i, $i = 1 \sim 4$, and $l_i = \overline{a_i a_{i+1}}$, $i = 1 \sim 4 \pmod 4$, respectively. Let V_h be the usual bilinear FE space, and let $V_{h0} = \{v_h \in V_h, v_h|_{\partial\Omega} = 0\}$. Also, it can be found in [25] that if $u \in H^2(\Omega)$, then

$$\big(\nabla(u - I_h u), \nabla v_h\big) = 0, \quad v_h \in V_{h0}, \tag{2}$$

where I_h be the so-called Ritz projection operator on V_{h0}.

Set $\{t_n : t_n = n\tau; 0 \le n \le N\}$ be a uniform partition of $[0, T]$ with time step $\tau = T/N$. We denote $\sigma^n = \sigma(X, t_n)$. For a sequence of functions $\{\sigma^n\}_{n=0}^N$, we denote $\bar{\partial}_t \sigma^n = \frac{\sigma^n - \sigma^{n-1}}{\tau}$, $n = 1, 2, \dots, N$. With these notations, setting $u_t = q$, the weak form of (1) is seeking $u, q \in H_0^1(\Omega)$ such that, for all $v \in H_0^1(\Omega)$,

$$\begin{cases} (\bar{\partial}_t u^n, v) = (q^n, v) + (R_1^n, v), & v \in H_0^1(\Omega), \\ (\bar{\partial}_t q^n, v) + (a(u^{n-1})\tau \sum_{i=1}^n \nabla q^i, \nabla v) + (a(u^{n-1})\nabla u^0, \nabla v) & \\ \quad = (f(u^{n-1}), v) + (R_2^n + R_3^n + R_4^n, v), & v \in H_0^1(\Omega), \end{cases} \tag{3}$$

where

$$R_1^n = \bar{\partial}_t u^n - u_t^n, \qquad R_2^n = \bar{\partial}_t q^n - q_t^n, \qquad R_4^n = -\big(f\big(u^{n-1}\big) - f\big(u^n\big)\big),$$
$$R_3^n = -\nabla \cdot \left(a\big(u^{n-1}\big)\tau \sum_{i=1}^n \nabla q^i - a\big(u^n\big) \int_0^{t_n} \nabla q \, ds\right) - \nabla \cdot \big(\nabla u^0 \big(a\big(u^{n-1}\big) - a\big(u^n\big)\big)\big).$$

We develop the linearized Galerkin FEM to problem (3): seek $U_h^n, Q_h^n \in V_{h0}$ such that

$$\begin{cases} (\bar{\partial}_t U_h^n, v_h) = (Q_h^n, v_h), & v_h \in V_{h0}, \\ (\bar{\partial}_t Q_h^n, v_h) + (a(U_h^{n-1})\tau \sum_{i=1}^n \nabla Q_h^i, \nabla v_h) & \\ \quad + (a(U_h^{n-1})\nabla U_h^0, \nabla v_h) = (f(U_h^{n-1}), v_h), & v_h \in V_{h0}, \end{cases} \tag{4}$$

where $U_h^0 = I_h u_0$ and $Q_h^0 = I_h u_1$. A well-known consequence is that the linear system (4) may always be solved for U_h^n and Q_h^n; see [26].

3 Error estimates for the time-discrete system

To get rid of the ratio restriction between h and τ, we introduce a time-discrete system as follows:

$$\begin{cases} \bar{\partial}_t U^n = Q^n, & (X, t) \in \Omega, \\ \bar{\partial}_t Q^n - \nabla \cdot (a(U^{n-1})\tau \sum_{i=1}^n \nabla Q^i) - \nabla \cdot (a(U^{n-1})\nabla U^0) & \\ \quad = f(U^{n-1}), & (X, t) \in \Omega, \\ U^n = 0, \qquad Q^n = 0, & (X, t) \in \partial\Omega, \\ U^0 = u_0(X), \qquad Q^0 = u_1(X), & X \in \Omega. \end{cases} \tag{5}$$

The existence and uniqueness of solutions for this linear elliptic system (5) are obvious. To show the unconditional results, the regularities of U^n and Q^n are inevitable, and we therefore need some estimates for $u^n - U^n$ and $q^n - Q^n$. In what follows, we set $e^n \triangleq u^n - U^n$, $\delta^n \triangleq q^n - Q^n$ ($n = 1, 2, \ldots, N$), analyze the temporal errors and give the regularity results for U^n and Q^n. It is easy to see that $e^0 = \delta^0 = 0$.

Theorem 1 *Let u^m and U^m ($m = 0, 1, 2, \ldots, N$) be solutions of (1) and (5), respectively, $u, q \in L^2(0, T; H^3(\Omega))$, $u_t, q_t \in L^\infty(0, T; H^2(\Omega))$, and $u_{tt} \in L^2(0, T; L^2(\Omega))$. Then for $m = 1, \ldots, N$, there exists τ_0 such that when $\tau \le \tau_0$, we have*

$$\|e^m\|_2 + \tau\left(\sum_{i=2}^m \|\bar{\partial}_t e^i\|_2^2\right)^{\frac{1}{2}} + \|\delta^m\|_1 + \tau\left\|\sum_{i=1}^m \delta^i\right\|_2 + \tau\left(\sum_{i=2}^m \|\delta^i\|_2^2\right)^{\frac{1}{2}} \le C_0\tau, \tag{6}$$

$$\|\bar{\partial}_t U^m\|_2 + \|Q^m\|_2 \le C_0, \tag{7}$$

where C_0 is a positive constant independent of m, h, and τ.

Proof Setting $K_0 \triangleq 1 + \max_{1\le m\le N}(\|u^m\|_{0,\infty} + \sqrt{\tau}(\sum_{i=1}^m \|\bar{\partial}_t u^i\|_{0,\infty}^2)^{\frac{1}{2}})$, we begin to prove (6)–(7) by mathematical induction. When $m = 1$, by (1) and (5) we have the error equation

$$\begin{cases} \bar{\partial}_t e^1 = \delta^1 + R_1^1, \\ \bar{\partial}_t \delta^1 - \nabla \cdot (a(u^0)\tau\nabla\delta^1) = R_2^1 + R_3^1. \end{cases} \tag{8}$$

With $\delta^0 = 0$, multiplying the second equation of (8) by $\Delta\delta^1$ and integrating it over Ω, we get

$$\begin{aligned} \frac{1}{\tau}\|\nabla\delta^1\|_0^2 + \tau\|a^{\frac{1}{2}}(u^0)\Delta\delta^1\|_0^2 &= -\left(a_u(u^0)\nabla u^0\tau\nabla\delta^1, \Delta\delta^1\right) - \left(R_2^1 + R_3^1, \Delta\delta^1\right) \\ &\le C\tau\|\nabla\delta^1\|_0\|\Delta\delta^1\|_0 + C\tau\|\Delta\delta^1\|_0. \end{aligned} \tag{9}$$

Further, since $e^1 \in H^2(\Omega) \cap H_0^1(\Omega)$, using the first equation of (8), we get

$$\|e^1\|_2 = \tau\|\bar{\partial}_t e^1\|_2 \le C\tau\|\Delta\delta^1\|_0 + C\tau\|R_1^1\|_2. \tag{10}$$

Thus there exist positive constants τ_1, τ_2, C_1, C_2 such that when $\tau \le \tau_1$, we have

$$\|e^1\|_2 + \tau\|\bar{\partial}_t e^1\|_2 + \|\delta^1\|_1 + \tau\|\delta^1\|_2 \le C_1\tau, \tag{11}$$

which implies

$$\left\|\frac{U^1 - U^0}{\tau}\right\|_2 + \|Q^1\|_2 \le C_2, \tag{12}$$

$$\|U^1\|_{0,\infty} \le \|e^1\|_{0,\infty} + \|u^1\|_{0,\infty} \le CC_1\tau + \|u^1\|_{0,\infty} \le K_0, \tag{13}$$

where $\tau \le \tau_2 \le 1/CC_1$.

By mathematical induction we assume that (6) and (7) hold for $m \le n-1$. Then there exists τ_3 such that

$$
\begin{aligned}
&\|U^m\|_{0,\infty} + \sqrt{\tau}\left(\sum_{i=1}^{m}\|\bar{\partial}_t U^i\|_{0,\infty}^2\right)^{\frac{1}{2}} \\
&\quad \le C\|e^m\|_2 + C\sqrt{\tau}\left(\sum_{i=1}^{m}\|\bar{\partial}_t e^i\|_2^2\right)^{\frac{1}{2}} + \|u^m\|_{0,\infty} + \sqrt{\tau}\left(\sum_{i=1}^{m}\|\bar{\partial}_t u^i\|_{0,\infty}^2\right)^{\frac{1}{2}} \\
&\quad \le CC_0\tau + CC_0\sqrt{\tau} + \|u^m\|_{0,\infty} + \sqrt{\tau}\left(\sum_{i=1}^{m}\|\bar{\partial}_t u^i\|_{0,\infty}^2\right)^{\frac{1}{2}} \le K_0,
\end{aligned}
\tag{14}
$$

where $\tau \le \tau_3 = \min\{1/2CC_0, 1/4C^2C_0^2\}$.

Then we begin to prove (6) and (7) for $m = n$. Subtracting (5) from (1), we obtain

$$
\begin{cases}
\bar{\partial}_t e^n = \delta^n + R_1^n, \\
\bar{\partial}_t \delta^n - \nabla\cdot(a(U^{n-1})\tau\sum_{i=1}^n \nabla\delta^i) - \nabla\cdot(\tau\sum_{i=1}^n \nabla q^i(a(u^{n-1}) - a(U^{n-1}))) \\
\qquad - \nabla\cdot(\nabla u^0(a(u^{n-1}) - a(U^{n-1}))) \\
\quad = f(u^{n-1}) - f(U^{n-1}) + R_2^n + R_3^n + R_4^n.
\end{cases}
\tag{15}
$$

Multiplying the second equation of (15) by $\Delta\delta^n$ and integrating, we get

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\|\nabla\delta^n\|_0^2 - \|\nabla\delta^{n-1}\|_0^2\right) + \left(a\left(U_h^{n-1}\right)\tau\sum_{i=1}^{n}\Delta\delta^i, \Delta\delta^n\right) \\
&\quad = -\left(a_u\left(U^{n-1}\right)\nabla U^{n-1}\left(\tau\sum_{i=1}^{n}\nabla\delta^i\right), \Delta\delta^n\right) \\
&\qquad - \left(\nabla\cdot\left(\tau\sum_{i=1}^{n}\nabla q^i\left(a\left(u^{n-1}\right) - a\left(U^{n-1}\right)\right)\right), \Delta\delta^n\right) \\
&\qquad - \left(\nabla\cdot\left(\nabla u^0\left(a\left(u^{n-1}\right) - a\left(U^{n-1}\right)\right)\right), \Delta\delta^n\right) \\
&\qquad - \left(f\left(u^{n-1}\right) - f\left(U^{n-1}\right), \Delta\delta^n\right) - \left(R_2^n + R_3^n + R_4^n, \Delta\delta^n\right).
\end{aligned}
\tag{16}
$$

Observe that $(a(U^{n-1})\tau\sum_{i=1}^n \Delta\delta^i, \Delta\delta^n)$ cannot be bounded directly; we rewrite it as

$$
\begin{aligned}
&\left(a\left(U^{n-1}\right)\tau\sum_{i=1}^{n}\Delta\delta^i, \Delta\delta^n\right) \\
&\quad = \tau\int_\Omega a\left(U^{n-1}\right)\sum_{i=1}^{n-1}\Delta\delta^i\cdot\Delta\delta^n + \tau\left\|a^{\frac{1}{2}}\left(U^{n-1}\right)\Delta\delta^n\right\|_0^2 \\
&\quad = \frac{1}{2}\tau\int_\Omega a\left(U^{n-1}\right)\left(\sum_{i=1}^{n}\Delta\delta^i\right)^2 - \frac{1}{2}\tau\int_\Omega a\left(U^{n-1}\right)\left(\sum_{i=1}^{n-1}\Delta\delta^i\right)^2 \\
&\qquad + \frac{1}{2}\tau\left\|a^{\frac{1}{2}}\left(U^{n-1}\right)\Delta\delta^n\right\|_0^2.
\end{aligned}
$$

Then we have

$$\begin{aligned}
&\frac{1}{2\tau}\left(\|\nabla\delta^n\|_0^2-\|\nabla\delta^{n-1}\|_0^2\right)+\frac{1}{2}\tau\left\|a^{\frac{1}{2}}\left(U^{n-1}\right)\sum_{i=1}^{n}\Delta\delta^i\right\|_0^2-\frac{1}{2}\tau\left\|a^{\frac{1}{2}}\left(U^{n-2}\right)\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2\\
&\quad+\frac{1}{2}\tau\left\|a^{\frac{1}{2}}\left(U^{n-1}\right)\Delta\delta^n\right\|_0^2\\
&\quad\le C\tau^2\|\bar\partial_t U^{n-1}\|_2\left\|\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2-\left(a_u\left(U^{n-1}\right)\nabla U^{n-1}\left(\tau\sum_{i=1}^{n}\nabla\delta^i\right),\Delta\delta^n\right)\\
&\quad-\left(\tau\sum_{i=1}^{n}\Delta q^i\left(a\left(u^{n-1}\right)-a\left(U^{n-1}\right)\right),\Delta\delta^n\right)-\left(\tau\sum_{i=1}^{n}\nabla q^i a_u\left(U^{n-1}\right)\nabla e^{n-1},\Delta\delta^n\right)\\
&\quad-\left(\tau\sum_{i=1}^{n}\nabla q^i\nabla u^{n-1}\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right),\Delta\delta^n\right)\\
&\quad-\left(\Delta u^0\left(a\left(u^{n-1}\right)-a\left(U^{n-1}\right)\right),\Delta\delta^n\right)-\left(\nabla u^0\left(a_u\left(U^{n-1}\right)\nabla e^{n-1}\right),\Delta\delta^n\right)\\
&\quad-\left(\nabla u^0\nabla u^{n-1}\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right),\Delta\delta^n\right)-\left(f\left(u^{n-1}\right)-f\left(U^{n-1}\right),\Delta\delta^n\right)\\
&\quad-\left(R_2^n+R_3^n+R_4^n,\Delta\delta^n\right)\triangleq\sum_{i=1}^{10}A_i.
\end{aligned}$$

In what follows, we will bound A_i, $i=2\sim 10$, one by one. Note the particularity of $\Delta\delta^n$ on the left-hand side, so we have to use new ways to handle $\Delta\delta^n$ on the right-hand side instead of applying the Young inequality directly. In view of Green's formula, it follows that

$$\begin{aligned}
A_9&=-\left(f_u\left(\mu_1^{n-1}\right)e^{n-1},\Delta\delta^n\right)\\
&=\left(f_{uu}\left(\mu_1^{n-1}\right)\nabla\mu_1^{n-1}e^{n-1},\nabla\delta^n\right)+\left(f_u\left(\mu_1^{n-1}\right)\nabla e^{n-1},\nabla\delta^n\right)\\
&\le C\|\nabla e^{n-1}\|_0^2+C\|\nabla\delta^n\|_0^2,
\end{aligned}$$

where $\mu_1^{n-1}=U^{n-1}+\lambda_1^{n-1}e^{n-1}$ and $0<\lambda_1^{n-1}<1$.

For $A_2\sim A_8$, A_{10}, it is not so obvious to be dealt with. We choose to rewrite $\Delta\delta^n$ by $\tau\sum_{i=1}^n\Delta\bar\partial_t\delta^i$ and then try to transfer τ from one side in the inner product to the other; more precisely,

$$\begin{aligned}
A_4&=-\left(\tau\sum_{i=1}^{n}\nabla q^i a_u\left(U^{n-1}\right)\nabla e^{n-1},\tau\sum_{i=1}^{n}\bar\partial_t\Delta\delta^i\right)\\
&=\left(a_u\left(U^{n-1}\right)\nabla e^{n-1}\nabla q^n,\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)+\left(\tau\sum_{i=1}^{n-1}\nabla q^i a_u\left(U^{n-2}\right)\bar\partial_t\nabla e^{n-1},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&\quad+\left(\tau\sum_{i=1}^{n-1}\nabla q^i\nabla e^{n-1}\frac{a_u(U^{n-1})-a_u(U^{n-2})}{\tau},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&\quad-\bar\partial_t\left(\tau\sum_{i=1}^{n}\nabla q^i a_u\left(U^{n-1}\right)\nabla e^{n-1},\tau\sum_{i=1}^{n}\Delta\delta^i\right)\\
&\le C\|\bar\partial_t\nabla e^{n-1}\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0+C\|\nabla e^{n-1}\|_0\|\bar\partial_t U^{n-1}\|_{0,\infty}\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0
\end{aligned}$$

$$
\begin{aligned}
&+ C\|\nabla e^{n-1}\|_0 \left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 - \bar{\partial}_t\left(\tau\sum_{i=1}^{n}\nabla q^i a_u(U^{n-1})\nabla e^{n-1}, \tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&\le C\|\bar{\partial}_t\nabla e^{n-1}\|_0^2 + C\|\nabla e^{n-1}\|_0^2 + C\|\bar{\partial}_t U^{n-1}\|_{0,\infty}^2\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2\\
&+ C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2 - \bar{\partial}_t\left(\tau\sum_{i=1}^{n}\nabla q^i a_u(U^{n-1})\nabla e^{n-1}, \tau\sum_{i=1}^{n}\Delta\delta^i\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
A_{10} &= \left(\bar{\partial}_t R_2^n + \bar{\partial}_t R_3^n + \bar{\partial}_t R_4^n, \tau\sum_{i=1}^{n-1}\Delta\delta^i\right) - \bar{\partial}_t\left(R_2^n + R_3^n + R_4^n, \tau\sum_{i=1}^{n}\Delta\delta^i\right)\\
&\le C\tau^2 + C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2 - \bar{\partial}_t\left(R_2^n + R_3^n + R_4^n, \tau\sum_{i=1}^{n}\Delta\delta^i\right).
\end{aligned}
$$

For A_2, we rewrite it as follows:

$$
\begin{aligned}
A_2 &= -\left(a_u(U^{n-1})\nabla U^{n-1}\left(\tau\sum_{i=1}^{n}\nabla\delta^i\right), \tau\sum_{i=1}^{n}\Delta\bar{\partial}_t\delta^i\right)\\
&= \left(a_u(U^{n-2})\nabla U^{n-2}\nabla\delta^n, \tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&+ \left(\left(\tau\sum_{i=1}^{n}\nabla\delta^i\right)a_u(U^{n-2})\bar{\partial}_t\nabla U^{n-1}, \tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&+ \left(\left(\tau\sum_{i=1}^{n}\nabla\delta^i\right)\nabla U^{n-1}\frac{a_u(U^{n-1}) - a_u(U^{n-2})}{\tau}, \tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&- \bar{\partial}_t\left(a_u(U^{n-1})\nabla U^{n-1}\left(\tau\sum_{i=1}^{n}\nabla\delta^i\right), \tau\sum_{i=1}^{n}\Delta\delta^i\right) \triangleq A_{2i}.
\end{aligned}
$$

In view of the embedding theorem, this yields

$$
\begin{aligned}
A_{22} &\le C\left\|\tau\sum_{i=1}^{n}\Delta\delta^i\right\|_0 \|\bar{\partial}_t\Delta U^{n-1}\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0\\
&\le C\left\|\tau\sum_{i=1}^{n}\Delta\delta^i\right\|_0^2 + C\|\bar{\partial}_t\Delta U^{n-1}\|_0^2\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2.
\end{aligned}
$$

To get round the need of $U^i \in H^3(\Omega)$, $i = 1, 2, \dots, n-1$, we split U^i, $i = 1, 2, \dots, n-1$, into two parts; with inductive assumption (14), it reduces to

$$
\begin{aligned}
A_{21} &= -\left(a_u(U^{n-2})\nabla e^{n-2}\nabla\delta^n, \tau\sum_{i=1}^{n-1}\Delta\delta^i\right) + \left(a_u(U^{n-2})\nabla u^{n-2}\nabla\delta^n, \tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&\le C\|\Delta e^{n-2}\|_0\|\Delta\delta^n\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 + C\|\nabla\delta^n\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0
\end{aligned}
$$

$$\begin{aligned}
&\le C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2+\frac{a_0}{4}\left\|\Delta e^{n-2}\right\|_0^2\left\|\Delta\delta^n\right\|_0^2+C\left\|\nabla\delta^n\right\|_0^2\\
&\le C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2+\frac{a_0}{4}\tau\left\|\Delta\delta^n\right\|_0^2+C\left\|\nabla\delta^n\right\|_0^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
A_{23}&\le C\left\|\Delta U^{n-1}\right\|_0\left\|\bar{\partial}_t\Delta U^{n-1}\right\|_0\left\|\tau\sum_{i=1}^{n}\Delta\delta^i\right\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0\\
&\le C\left\|\tau\sum_{i=1}^{n}\Delta\delta^i\right\|_0^2+C\left\|\bar{\partial}_t\Delta U^{n-1}\right\|_0^2\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2.
\end{aligned}$$

We split A_3 as

$$\begin{aligned}
A_3&=-\left(\tau\sum_{i=1}^{n}\Delta q^i\big(a\big(u^{n-1}\big)-a\big(U^{n-1}\big)\big),\tau\sum_{i=1}^{n}\bar{\partial}_t\Delta\delta^i\right)\\
&=\left(\tau\sum_{i=1}^{n-1}\Delta q^i\frac{(a(u^{n-1})-a(U^{n-1}))-(a(u^{n-2})-a(U^{n-2}))}{\tau},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&\quad+\left(\big(a\big(u^{n-1}\big)-a\big(U^{n-1}\big)\big)\Delta q^n,\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&\quad-\bar{\partial}_t\left(\tau\sum_{i=1}^{n}\Delta q^i\big(a\big(u^{n-1}\big)-a\big(U^{n-1}\big)\big),\tau\sum_{i=1}^{n}\Delta\delta^i\right)\triangleq\sum_{i=1}^{3}A_{3i}.
\end{aligned}$$

We can see that

$$A_{32}=\left(\big(a\big(u^{n-1}\big)-a\big(U^{n-1}\big)\big)\Delta q^n,\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\le C\left\|\nabla e^{n-1}\right\|_0^2+C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2.$$

Since

$$\begin{aligned}
&\frac{(a(u^{n-1})-a(U^{n-1}))-(a(u^{n-2})-a(U^{n-2}))}{\tau}\\
&\quad=a'\big(\mu_2^{n-1}\big)\bar{\partial}_t e^{n-1}+\bar{\partial}_t u^{n-1}\big(a'\big(\mu_3^{n-1}\big)-a'\big(\mu_2^{n-1}\big)\big),
\end{aligned}\tag{17}$$

where

$$\begin{aligned}
&\mu_3^{n-1}=u^{n-2}+\tau\lambda_3^{n-1}\bar{\partial}_t u^{n-1},\qquad \mu_2^{n-1}=U^{n-2}+\tau\lambda_2^{n-1}\bar{\partial}_t U^{n-1},\\
&\quad 0<\lambda_2^{n-1}<1, 0<\lambda_3^{n-1}<1,
\end{aligned}$$

and

$$\mu_3^{n-1}-\mu_2^{n-1}=e^{n-2}+\tau\lambda_2^{n-1}\bar{\partial}_t e^{n-1}+\bar{\partial}_t u^{n-1}\tau\big(\lambda_3^{n-1}-\lambda_2^{n-1}\big),$$

we see that

$$\begin{aligned}
A_{31} &\le \left\|\tau\sum_{i=1}^{n-1}\Delta q^i\right\|_{0,4}\left\|\frac{(a(u^{n-1})-a(U^{n-1}))-(a(u^{n-2})-a(U^{n-2}))}{\tau}\right\|_{0,4}\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 \\
&\le C\|\bar\partial_t\nabla e^{n-1}\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 + C\|\nabla e^{n-2}\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 + C\tau\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 \\
&\le C\tau^2 + C\|\bar\partial_t\nabla e^{n-1}\|_0^2 + C\|\nabla e^{n-2}\|_0^2 + C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2,
\end{aligned}$$

whence

$$\begin{aligned}
A_3 &\le C\tau^2 + C\|\bar\partial_t\nabla e^{n-1}\|_0^2 + C\|\nabla e^{n-2}\|_0^2 + C\|\nabla e^{n-1}\|_0^2 + C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2 \\
&\quad - \bar\partial_t\left(\tau\sum_{i=1}^{n}\Delta q^i\big(a\big(u^{n-1}\big)-a\big(U^{n-1}\big)\big),\tau\sum_{i=1}^{n}\Delta\delta^i\right).
\end{aligned}$$

Rewriting A_5, A_6, A_8, with (17), we obtain

$$\begin{aligned}
A_6 &= \left(\Delta u^0\frac{(a(u^{n-1})-a(U^{n-1}))-(a(u^{n-2})-a(U^{n-2}))}{\tau},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right) \\
&\quad - \bar\partial_t\left(\Delta u^0\big(a\big(u^{n-1}\big)-a\big(U^{n-1}\big)\big),\tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\le \|\Delta u^0\|_{0,4}\left\|\frac{(a(u^{n-1})-a(U^{n-1}))-(a(u^{n-2})-a(U^{n-2}))}{\tau}\right\|_{0,4}\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 \\
&\quad - \bar\partial_t\left(\Delta u^0\big(a\big(u^{n-1}\big)-a\big(U^{n-1}\big)\big),\tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\le C\|\bar\partial_t\nabla e^{n-1}\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 + C\|\nabla e^{n-2}\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 \\
&\quad + C\tau\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0 - \bar\partial_t\left(\Delta u^0\big(a\big(u^{n-1}\big)-a\big(U^{n-1}\big)\big),\tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\le C\tau^2 + C\|\bar\partial_t\nabla e^{n-1}\|_0^2 + C\|\nabla e^{n-2}\|_0^2 + C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2 \\
&\quad - \bar\partial_t\left(\Delta u^0\big(a\big(u^{n-1}\big)-a\big(U^{n-1}\big)\big),\tau\sum_{i=1}^{n}\Delta\delta^i\right), \\
A_5 &= -\left(\tau\sum_{i=1}^{n}\nabla q^i\nabla u^{n-1}\big(a_u\big(u^{n-1}\big)-a_u\big(U^{n-1}\big)\big),\tau\sum_{i=1}^{n}\bar\partial_t\Delta\delta^i\right) \\
&= \left(\tau\sum_{i=1}^{n-1}\nabla q^i\nabla u^{n-2}\frac{(a_u(u^{n-1})-a_u(U^{n-1}))-(a_u(u^{n-2})-a_u(U^{n-2}))}{\tau},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right)\tau\sum_{i=1}^{n-1}\nabla q^i\bar{\partial}_t\nabla u^{n-1},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&+\left(\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right)\nabla u^{n-1}\nabla q^n,\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&-\bar{\partial}_t\left(\tau\sum_{i=1}^{n}\nabla q^i\nabla u^{n-1}\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right),\tau\sum_{i=1}^{n}\Delta\delta^i\right)\\
&\le C\left\|\frac{(a_u(u^{n-1})-a_u(U^{n-1}))-(a_u(u^{n-2})-a_u(U^{n-2}))}{\tau}\right\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0\\
&+C\left\|e^{n-1}\right\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0-\bar{\partial}_t\left(\tau\sum_{i=1}^{n}\nabla q^i\nabla u^{n-1}\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right),\tau\sum_{i=1}^{n}\Delta\delta^i\right)\\
&\le C\tau^2+C\left\|\bar{\partial}_t\nabla e^{n-1}\right\|_0^2+C\left\|\nabla e^{n-1}\right\|_0^2+C\left\|\nabla e^{n-2}\right\|_0^2+C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2\\
&-\bar{\partial}_t\left(\tau\sum_{i=1}^{n}\nabla q^i\nabla u^{n-1}\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right),\tau\sum_{i=1}^{n}\Delta\delta^i\right),
\end{aligned}
$$

and

$$
\begin{aligned}
A_8={}&\left(\nabla u^0\nabla u^{n-2}\frac{(a_u(u^{n-1})-a_u(U^{n-1}))-(a_u(u^{n-2})-a_u(U^{n-2}))}{\tau},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&+\left(\nabla u^0\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right)\bar{\partial}_t\nabla u^{n-1},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&-\bar{\partial}_t\left(\nabla u^0\nabla u^{n-1}\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right),\tau\sum_{i=1}^{n}\Delta\delta^i\right)\\
\le{}& C\tau^2+C\left\|\bar{\partial}_t\nabla e^{n-1}\right\|_0^2+C\left\|\nabla e^{n-1}\right\|_0^2+C\left\|\nabla e^{n-2}\right\|_0^2\\
&+C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2-\bar{\partial}_t\left(\nabla u^0\nabla u^{n-1}\left(a_u\left(u^{n-1}\right)-a_u\left(U^{n-1}\right)\right),\tau\sum_{i=1}^{n}\Delta\delta^i\right).
\end{aligned}
$$

Finally, A_7 can be bounded as

$$
\begin{aligned}
A_7={}&\left(\nabla u^0a_u\left(U^{n-2}\right)\bar{\partial}_t\nabla e^{n-1},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&+\left(\nabla u^0\nabla e^{n-1}\frac{a_u(U^{n-1})-a_u(U^{n-2})}{\tau},\tau\sum_{i=1}^{n-1}\Delta\delta^i\right)\\
&-\bar{\partial}_t\left(\nabla u^0a_u\left(U^{n-1}\right)\nabla e^{n-1},\tau\sum_{i=1}^{n}\Delta\delta^i\right)\\
\le{}& C\left\|\bar{\partial}_t\nabla e^{n-1}\right\|_0\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0+C\left\|\nabla e^{n-1}\right\|_0\left\|\bar{\partial}_tU^{n-1}\right\|_{0,\infty}\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0
\end{aligned}
$$

$$
\begin{aligned}
&- \bar{\partial}_t\left(\nabla u^0 a_u\big(U^{n-1}\big)\nabla e^{n-1}, \tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\leq C\big\|\bar{\partial}_t\nabla e^{n-1}\big\|_0^2 + C\big\|\nabla e^{n-1}\big\|_0^2 + C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2 \\
&\quad + C\big\|\bar{\partial}_t U^{n-1}\big\|_{0,\infty}^2\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2 - \bar{\partial}_t\left(\nabla u^0 a_u\big(U^{n-1}\big)\nabla e^{n-1}, \tau\sum_{i=1}^{n}\Delta\delta^i\right).
\end{aligned}
$$

Moreover, because

$$
\begin{aligned}
&\big\|\nabla\bar{\partial}_t e^n\big\|_0 = \big\|\nabla\big(\delta^n + R_1^n\big)\big\|_0 \leq C\big\|\nabla\delta^n\big\|_0 + C\big\|\nabla R_1^n\big\|_0 \leq C\big\|\nabla\delta^n\big\|_0 + C\tau, \\
&\big\|\Delta\bar{\partial}_t e^n\big\|_0 = \big\|\Delta\big(\delta^n + R_1^n\big)\big\|_0 \leq C\big\|\Delta\delta^n\big\|_0 + C\big\|\Delta R_1^n\big\|_0 \leq C\big\|\Delta\delta^n\big\|_0 + C\tau, \\
&\big\|\nabla e^n\big\|_0 \leq C\sqrt{\tau}\left(\sum_{i=1}^{n}\big\|\nabla\bar{\partial}_t e^i\big\|_0^2\right)^{\frac{1}{2}} \leq C\sqrt{\tau}\left(\sum_{i=1}^{n}\big\|\nabla\delta^i\big\|_0^2\right)^{\frac{1}{2}} + C\tau,
\end{aligned}
\tag{18}
$$

we have

$$
\begin{aligned}
&\frac{1}{\tau}\big(\big\|\nabla\delta^n\big\|_0^2 - \big\|\nabla\delta^{n-1}\big\|_0^2\big) + \tau\left\|a^{\frac{1}{2}}\big(U^{n-1}\big)\sum_{i=1}^{n}\Delta\delta^i\right\|_0^2 - \tau\left\|a^{\frac{1}{2}}\big(U^{n-2}\big)\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2 + \tau\big\|\Delta\delta^n\big\|_0^2 \\
&\leq C\tau^2 + C\big\|\nabla\delta^n\big\|_0^2 + C\big\|\nabla\delta^{n-1}\big\|_0^2 + C\tau\sum_{i=1}^{n}\big\|\nabla\delta^i\big\|_0^2 \\
&\quad + C\left\|\tau\sum_{i=1}^{n}\Delta\delta^i\right\|_0^2 + C\big\|\bar{\partial}_t U^{n-1}\big\|_2^2\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2 \\
&\quad + C\left\|\tau\sum_{i=1}^{n-1}\Delta\delta^i\right\|_0^2 - \bar{\partial}_t\left(a_u\big(U^{n-1}\big)\nabla U^{n-1}\left(\tau\sum_{i=1}^{n}\nabla\delta^i\right), \tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\quad - \bar{\partial}_t\left(\tau\sum_{i=1}^{n}\nabla q^i a_u\big(U^{n-1}\big)\nabla e^{n-1}, \tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\quad - \bar{\partial}_t\left(\tau\sum_{i=1}^{n}\Delta q^i\big(a\big(u^{n-1}\big) - a\big(U^{n-1}\big)\big), \tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\quad - \bar{\partial}_t\left(\Delta u^0\big(a\big(u^{n-1}\big) - a\big(U^{n-1}\big)\big), \tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\quad - \bar{\partial}_t\left(\nabla u^0 a_u\big(U^{n-1}\big)\nabla e^{n-1}, \tau\sum_{i=1}^{n}\Delta\delta^i\right) - \bar{\partial}_t\left(R_2^n + R_3^n + R_4^n, \tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\quad - \bar{\partial}_t\left(\tau\sum_{i=1}^{n}\nabla q^i\nabla u^{n-1}\big(a_u\big(u^{n-1}\big) - a_u\big(U^{n-1}\big)\big), \tau\sum_{i=1}^{n}\Delta\delta^i\right) \\
&\quad - \bar{\partial}_t\left(\nabla u^0\nabla u^{n-1}\big(a_u\big(u^{n-1}\big) - a_u\big(U^{n-1}\big)\big), \tau\sum_{i=1}^{n}\Delta\delta^i\right).
\end{aligned}
$$

Summing this inequality from 2 to n, we get

$$
\begin{aligned}
&\|\nabla\delta^n\|_0^2+\left\|\tau\sum_{i=1}^n\Delta\delta^i\right\|_0^2+\sum_{i=2}^n\|\tau\Delta\delta^i\|_0^2\\
&\le\|\nabla\delta^1\|_0^2+\tau^2\|\Delta\delta^1\|_0^2+C\tau^2+C\tau\sum_{i=1}^n\|\nabla\delta^i\|_0^2+C\tau\sum_{i=1}^n\left\|\tau\sum_{j=1}^{i-1}\Delta\delta^j\right\|_0^2\\
&\quad+C\tau^2\sum_{i=2}^n\sum_{j=1}^{i-1}\|\nabla\delta^j\|_0^2+C\tau\sum_{i=1}^n\|\bar\partial_t\Delta U^{i-1}\|_0^2\left\|\tau\sum_{j=1}^{i-1}\Delta\delta^j\right\|_0^2\\
&\quad-\left(a_u(U^{n-1})\nabla U^{n-1}\left(\tau\sum_{i=1}^n\nabla\delta^i\right),\tau\sum_{i=1}^n\Delta\delta^i\right)\\
&\quad+\left(a_u(U^0)\nabla U^0(\tau\nabla\delta^1),\tau\Delta\delta^1\right)-\left(\tau\sum_{i=1}^n\Delta q^i\left(a(u^{n-1})-a(U^{n-1})\right),\tau\sum_{i=1}^n\Delta\delta^i\right)\\
&\quad-\left(\tau\sum_{i=1}^n\nabla q^i a_u(U^{n-1})\nabla e^{n-1},\tau\sum_{i=1}^n\Delta\delta^i\right)\\
&\quad-\left(\Delta u^0\left(a(u^{n-1})-a(U^{n-1})\right),\tau\sum_{i=1}^n\Delta\delta^i\right)\\
&\quad-\left(\nabla u^0 a_u(U^{n-1})\nabla e^{n-1},\tau\sum_{i=1}^n\Delta\delta^i\right)-\left(R_2^n+R_3^n+R_4^n,\tau\sum_{i=1}^n\Delta\delta^i\right)\\
&\quad+\left(R_2^1+R_3^1+R_4^1,\tau\Delta\delta^1\right)-\left(\tau\sum_{i=1}^n\nabla q^i\nabla u^{n-1}\left(a_u(u^{n-1})-a_u(U^{n-1})\right),\tau\sum_{i=1}^n\Delta\delta^i\right)\\
&\quad-\left(\nabla u^0\nabla u^{n-1}\left(a_u(u^{n-1})-a_u(U^{n-1})\right),\tau\sum_{i=1}^n\Delta\delta^i\right).
\end{aligned}
\tag{19}
$$

Due to

$$
\begin{aligned}
&\left(a_u(U^{n-1})\nabla U^{n-1}\left(\tau\sum_{i=1}^n\nabla\delta^i\right),\tau\sum_{i=1}^n\Delta\delta^i\right)\\
&=\left(a_u(U^{n-1})\nabla u^{n-1}\left(\tau\sum_{i=1}^n\nabla\delta^i\right),\tau\sum_{i=1}^n\Delta\delta^i\right)\\
&\quad-\left(a_u(U^{n-1})\nabla e^{n-1}\left(\tau\sum_{i=1}^n\nabla\delta^i\right),\tau\sum_{i=1}^n\Delta\delta^i\right)\\
&\le C\tau^{\frac14}\left\|\tau\sum_{i=1}^n\Delta\delta^i\right\|_0^2+C\left\|\tau\sum_{i=1}^n\nabla\delta^i\right\|_0^2+\frac14\left\|\tau\sum_{i=1}^n\Delta\delta^i\right\|_0^2\\
&\le C\tau^{\frac12}\left\|\tau\sum_{i=1}^n\Delta\delta^i\right\|_0^2+\frac12\left\|\tau\sum_{i=1}^n\Delta\delta^i\right\|_0^2+C\tau\sum_{i=1}^n\|\nabla\delta^i\|_0^2,
\end{aligned}
\tag{20}
$$

after obvious estimates and a kickback of $\tau\sum_{i=1}^{n}\|\nabla\delta^i\|_0^2$, together with our earlier estimate for $n = 1$, we obtain

$$\|\nabla\delta^n\|_0^2+\tau^2\left\|\sum_{i=1}^{n}\Delta\delta^i\right\|_0^2+\tau^2\sum_{i=2}^{n}\|\Delta\delta^i\|_0^2\le C\tau^2. \tag{21}$$

Here by (18) we have

$$\tau\left\|\sum_{i=1}^{n}\bar{\partial}_t e^i\right\|_2+\tau\left(\sum_{i=2}^{n}\|\bar{\partial}_t e^i\|_2^2\right)^{\frac{1}{2}}\le C\tau \tag{22}$$

and, further,

$$\|e^n\|_2=\tau\left\|\sum_{i=1}^{n}\bar{\partial}_t e^i\right\|_2\le C\tau. \tag{23}$$

Then we conclude that there exist τ_4, τ_5, C_3, C_4 such that when $\tau\le\tau_4$, we have

$$\|e^n\|_2+\tau\left(\sum_{i=2}^{n}\|\bar{\partial}_t e^i\|_2^2\right)^{\frac{1}{2}}+\|\delta^n\|_1+\tau\left\|\sum_{i=1}^{n}\delta^i\right\|_2+\tau\left(\sum_{i=2}^{n}\|\delta^i\|_2^2\right)^{\frac{1}{2}}\le C_3\tau, \tag{24}$$

which leads to

$$\|e^n\|_2\le\tau^{\frac{1}{4}},\qquad\|\bar{\partial}_t U^n\|_2\le C_4, \tag{25}$$

$$\begin{aligned}
&\|U^n\|_{0,\infty}+\sqrt{\tau}\left(\sum_{i=1}^{n}\|\bar{\partial}_t U^i\|_{0,\infty}^2\right)^{\frac{1}{2}}\\
&\quad\le C\|e^n\|_2+C\sqrt{\tau}\left(\sum_{i=1}^{n}\|\bar{\partial}_t e^i\|_2^2\right)^{\frac{1}{2}}+\|u^n\|_{0,\infty}+\sqrt{\tau}\left(\sum_{i=1}^{n}\|\bar{\partial}_t u^i\|_{0,\infty}^2\right)^{\frac{1}{2}}\\
&\quad\le CC_3\tau+CC_3\sqrt{\tau}+\|u^n\|_{0,\infty}+\sqrt{\tau}\left(\sum_{i=1}^{n}\|\bar{\partial}_t u^i\|_{0,\infty}^2\right)^{\frac{1}{2}}\le K_0,
\end{aligned} \tag{26}$$

where $\tau\le\tau_5=\min\{1/2CC_3,1/4C^2C_3^2\}$. Clearly, C_3, C_4 have nothing to do with C_0, and thus (6) and (7) hold for $m = n$ if we take $C_0\ge\sum_{i=1}^{4}C_i$ and $\tau_0\le\min_{1\le\tau\le5}\tau_i$. Then the induction is closed. The proof is completed. □

Remark 1 The special method used to tackle the left-hand side of (16) is important to deduce the regularities of U^n and Q^n in the H^2-norm. Further, the terms including $\Delta\delta^n$ on the right-hand side needs innovative technologies to treat.

4 Error estimates for spatial-discrete system and optimal error results

In this section, we will establish τ-independent optimal error results for u^n and q^n through the spatial results. We decompose the errors as follows:

$$U^i-U_h^i=U^i-I_hU^i+I_hU^i-U_h^i\triangleq\eta^i+\xi^i,$$

$$Q^i - Q_h^i = Q^i - I_h Q^i + I_h Q^i - Q_h^i \triangleq r^i + \theta^i, \quad i = 1, 2, \dots, n,$$

and we are now ready for the unconditional spatial results.

Theorem 2 *Let u^m and U_h^m be solutions of (3) and (4), respectively, for $m = 1, 2, \dots, N$. Under the conditions of Theorem 1, there exist τ_0', h_0' such that, for $\tau \le \tau_0'$ and $h \le h_0'$, we have*

$$\|u^m - U_h^m\|_0 + \|q^m - Q_h^m\|_0 = O(h^2 + \tau) \tag{27}$$

and

$$\|\nabla(u^m - U_h^m)\|_0 + \|\nabla(q^m - Q_h^m)\|_0 = O(h + \tau). \tag{28}$$

Proof Before discussing (27) and (28), we shall pause to give the results

$$\|\xi^m\|_0 + \|\theta^m\|_0 + \tau\left(\sum_{i=1}^m \|\nabla\theta^i\|_0^2\right)^{\frac{1}{2}} \le C_0' h\big(h + \tau^{\frac{1}{2}}\big) \tag{29}$$

by mathematical induction, where C_0' is a positive constant independent of m, τ, and h. Since $\|I_h U^m\|_{0,\infty} + \sqrt{\tau}(\sum_{i=2}^m \|\bar{\partial}_t I_h U^i\|_{0,\infty}^2)^{\frac{1}{2}} \le C$, let $K_0' \triangleq 1 + \|I_h U^m\|_{0,\infty} + \sqrt{\tau} \times (\sum_{i=2}^m \|\bar{\partial}_t I_h U^i\|_{0,\infty}^2)^{\frac{1}{2}}$. We begin with $m = 1$:

$$\begin{aligned}
&(\bar{\partial}_t\theta^1, v_h) + (a(U^0)\tau\nabla\theta^1, \nabla v_h) \\
&\quad = -(\bar{\partial}_t r^1, v_h) - (a(U^0)\tau\nabla r^1, \nabla v_h) \\
&\qquad - ((a(U^0) - a(U_h^0))\tau\nabla Q_h^1, \nabla v_h) - (a(U_h^0)\nabla\eta^0, \nabla v) \\
&\qquad - (\nabla U^0(a(U^0) - a(U_h^0)), \nabla v) + (f(U^0) - f(U_h^0), v_h).
\end{aligned} \tag{30}$$

Taking $v_h = \theta^1$ in (30), we get

$$\begin{aligned}
&\frac{1}{\tau}\|\theta^1\|_0^2 + \tau\|a^{\frac{1}{2}}(U^0)\nabla\theta^1\|_0^2 \\
&\quad = -(\bar{\partial}_t r^1, \theta^1) - (a(U^0)\tau\nabla r^1, \nabla\theta^1) \\
&\qquad - ((a(U^0) - a(U_h^0))\tau\nabla Q_h^1, \nabla\theta^1) - (a(U_h^0)\nabla\eta^0, \nabla\theta^1) \\
&\qquad - (\nabla U^0(a(U^0) - a(U_h^0)), \nabla\theta^1) + (f(U^0) - f(U_h^0), \theta^1).
\end{aligned} \tag{31}$$

It is easy to see that

$$\begin{aligned}
&(\bar{\partial}_t r^1, \theta^1) \le Ch^2\|\bar{\partial}_t U^1\|_2\|\theta^1\|_0 \le Ch^4 + C\|\theta^1\|_0^2, \\
&(\nabla U^0(a(U^0) - a(U_h^0)), \nabla\theta^1) \le C\|r^0\|_0\|\nabla\theta^1\|_0 \le Ch^2\tau + \frac{1}{8\tau}\|\theta^1\|_0^2, \\
&(f(U^0) - f(U_h^0), \theta^1) \le C\|r^0\|_0\|\theta^1\|_0 \le Ch^4 + C\|\theta^1\|_0^2.
\end{aligned}$$

Denoting $\overline{\gamma(X)}|_K = \frac{1}{|K|}\int_K \gamma(X)\,dX$ and then using the mean-value technique, we obtain

$$\begin{aligned}
&\big(a(U^0)\tau\nabla r^1,\nabla\theta^1\big)\\
&\quad=\sum_K\big(\big(a(U^0)-\overline{a(U^0)}\big)\tau\nabla r^1,\nabla\theta^1\big)_K\\
&\qquad-\sum_K\overline{a(U^0)}|_K\big(\tau\big(\nabla e^1-\nabla I_he^1\big),\nabla\theta^1\big)_K+\sum_K\overline{a(U^0)}|_K\big(\tau\big(\nabla u^1-\nabla I_hu^1\big),\nabla\theta^1\big)_K\\
&\quad\le Ch^2\tau\|U^1\|_2\|\nabla\theta^1\|_0+Ch\tau\|e^1\|_2\|\nabla\theta^1\|_0\le Ch^4+Ch^2\tau^2+C\tau^2\|\nabla\theta^1\|_0^2,\\
&\big(a(U_h^0)\nabla\eta^0,\nabla\theta^1\big)\\
&\quad=\sum_K\big(\big(a(U_h^0)-\overline{a(U_h^0)}\big)\nabla\eta^0,\nabla\theta^1\big)_K+\sum_K\overline{a(U_h^0)}|_K\big(\nabla\eta^0,\nabla\theta^1\big)_K\\
&\quad\le Ch^2\|u^0\|_2\|\nabla\theta^1\|_0\le Ch\sqrt{\tau}\frac{1}{\sqrt{\tau}}\|\theta^1\|_0\le Ch^2\tau+\frac{1}{8\tau}\|\theta^1\|_0^2.
\end{aligned}$$

By Theorem 1 we have

$$\begin{aligned}
&\big(\big(a(U^0)-a(U_h^0)\big)\tau\nabla Q_h^1,\nabla\theta^1\big)\\
&\quad=-\big(\big(a(U^0)-a(U_h^0)\big)\tau\nabla\theta^1,\nabla\theta^1\big)\\
&\qquad-\big(\big(a(U^0)-a(U_h^0)\big)\tau\nabla r^1,\nabla\theta^1\big)-\big(\big(a(U^0)-a(U_h^0)\big)\tau\nabla\delta^1,\nabla\theta^1\big)\\
&\qquad+\big(\big(a(U^0)-a(U_h^0)\big)\tau\nabla q^1,\nabla\theta^1\big)\\
&\quad\le Ch^2\tau\|U^0\|_2\|\nabla\theta^1\|_{0,\infty}\|\nabla\theta^1\|_0\\
&\qquad+Ch^3\tau\|U^0\|_2\|U^1\|_2\|\nabla\theta^1\|_{0,\infty}+Ch^2\tau\|\nabla\delta^1\|_0\|\nabla\theta^1\|_{0,\infty}\\
&\qquad+Ch^2\tau\|\nabla q^1\|_{0,\infty}\|\nabla\theta^1\|_0\\
&\quad\le Ch^4+Ch^2\tau^2+Ch\tau\|\nabla\theta^1\|_0^2+C\tau^2\|\nabla\theta^1\|_0^2.
\end{aligned}$$

Allocating all the estimates obtained, we have

$$\begin{aligned}
&\frac{1}{\tau}\|\theta^1\|_0^2+\tau\|\nabla\theta^1\|_0^2\\
&\quad\le Ch^4+Ch^2\tau+Ch\tau\|\nabla\theta^1\|_0^2+C\|\theta^1\|_0^2+C\tau^2\|\nabla\theta^1\|_0^2.
\end{aligned}\tag{32}$$

Thus there exist $\tau_1',\tau_2',h_1',h_2',C_1'$ such that, for $\tau\le\tau_1'$ and $h\le h_1'$, we have

$$\|\theta^1\|_0+\tau\|\nabla\theta^1\|_0\le C_1'h(h+\sqrt{\tau}),\tag{33}$$

which implies

$$\begin{aligned}
\|U_h^1\|_{0,\infty}&\le Ch^{-1}\|\xi^1\|_0+\|I_hU^1\|_{0,\infty}\\
&\le CC_1'h+CC_1'\sqrt{\tau}+\|I_hU^1\|_{0,\infty}\le K_0',
\end{aligned}\tag{34}$$

where $h\le h_2'\le 1/2CC_1'$ and $\tau\le\tau_2'\le 1/2CC_1'$.

By mathematical induction we assume that (29) holds for $m \le n-1$. Then there exist τ_3' and h_3' such that

$$\begin{aligned}
&\|U_h^m\|_{0,\infty} + \sqrt{\tau}\left(\sum_{i=2}^{m}\|\bar{\partial}_t U_h^i\|_{0,\infty}^2\right)^{\frac{1}{2}} \\
&\le Ch^{-1}\left(\|\xi^m\|_0 + \sqrt{\tau}\left(\sum_{i=2}^{m}\|\bar{\partial}_t\xi^i\|_0^2\right)^{\frac{1}{2}}\right) \\
&\quad + \left(\|I_hU^m\|_{0,\infty} + \sqrt{\tau}\left(\sum_{i=2}^{m}\|\bar{\partial}_t I_hU^i\|_{0,\infty}^2\right)^{\frac{1}{2}}\right) \\
&\le 2CC_0'h + 2CC_0'\sqrt{\tau} + \left(\|I_hU^m\|_{0,\infty} + \sqrt{\tau}\left(\sum_{i=2}^{m}\|\bar{\partial}_t I_hU^i\|_{0,\infty}^2\right)^{\frac{1}{2}}\right) \le K_0',
\end{aligned} \tag{35}$$

where $h \le h_3' \le 1/4CC_0'$ and $\tau \le \tau_3' \le 1/6(CC_0')^2$.

Then we prove that (29) also holds for $m = n$. By (4) and (5) we derive the error equations

$$\begin{cases}
(\bar{\partial}_t\xi^n, v_h) = -(\bar{\partial}_t\eta^n, v_h) + (\theta^n, v_h) + (r^n, v_h), \\
(\bar{\partial}_t\theta^n, v_h) + (a(U_h^{n-1})\tau\sum_{i=1}^{n}\nabla\theta^i, \nabla v_h) \\
\quad = -(\bar{\partial}_t r^n, v_h) - (a(U^{n-1})\tau\sum_{i=1}^{n}\nabla r^i, \nabla v_h) \\
\qquad - ((a(U^{n-1}) - a(U_h^{n-1}))\tau\sum_{i=1}^{n}\nabla I_hQ^i, \nabla v_h) - (a(U_h^{n-1})\nabla\eta^0, \nabla v) \\
\qquad - (\nabla U^0(a(U^{n-1}) - a(U_h^{n-1})), \nabla v) + (f(U^{n-1}) - f(U_h^{n-1}), v_h).
\end{cases} \tag{36}$$

For $v_h = \theta^n$ in the second equation of (36), we have

$$\begin{aligned}
&\tau\left(a(U_h^{n-1})\sum_{i=1}^{n}\nabla\theta^i, \nabla\theta^n\right) \\
&= \tau\int_\Omega a(U_h^{n-1})\sum_{i=1}^{n-1}\nabla\theta^i\cdot\nabla\theta^n + \tau\|a^{\frac{1}{2}}(U_h^{n-1})\nabla\theta^n\|_0^2 \\
&= \frac{1}{2}\tau\int_\Omega a(U_h^{n-1})\left(\sum_{i=1}^{n}\nabla\theta^i\right)^2 - \frac{1}{2}\tau\int_\Omega a(U_h^{n-1})\left(\sum_{i=1}^{n-1}\nabla\theta^i\right)^2 \\
&\quad + \frac{1}{2}\tau\|a^{\frac{1}{2}}(U_h^{n-1})\nabla\theta^n\|_0^2,
\end{aligned}$$

and hence we find

$$\begin{aligned}
&\frac{1}{2\tau}(\|\theta^n\|_0^2 - \|\theta^{n-1}\|_0^2) + \frac{1}{2}\tau\|a^{\frac{1}{2}}(U_h^{n-1})\nabla\theta^n\|_0^2 \\
&\quad + \frac{1}{2}\tau\left\|a^{\frac{1}{2}}(U_h^{n-1})\sum_{i=1}^{n}\nabla\theta^i\right\|_0^2 - \frac{1}{2}\tau\left\|a^{\frac{1}{2}}(U_h^{n-2})\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2 \\
&\le C\tau^2\|\bar{\partial}_t U_h^{n-1}\|_{0,\infty}\left\|\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2 - (\bar{\partial}_t r^n, \theta^n) - \left(a(U^{n-1})\tau\sum_{i=1}^{n}\nabla r^i, \nabla\theta^n\right)
\end{aligned}$$

$$
\begin{aligned}
&-\left(\big(a\big(U^{n-1}\big)-a\big(U_h^{n-1}\big)\big)\tau\sum_{i=1}^{n}\nabla I_hQ^i,\nabla\theta^n\right)-\big(a\big(U_h^{n-1}\big)\nabla\eta^0,\nabla\theta^n\big)\\
&-\big(\nabla U^0\big(a\big(U^{n-1}\big)-a\big(U_h^{n-1}\big)\big),\nabla\theta^n\big)-\big(f\big(U^{n-1}\big)-f\big(U_h^{n-1}\big),\theta^n\big)\triangleq\sum_{i=1}^{7}B_i.
\end{aligned}
\tag{37}
$$

Obviously,

$$
\begin{aligned}
&B_1\le C\tau^2\left\|\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2,\\
&B_2\le Ch^2\big\|\bar\partial_tU_h^n\big\|_2\big\|\theta^n\big\|_0\le Ch^4+C\big\|\theta^n\big\|_0^2,\\
&B_7\le Ch^4+C\big\|\xi^{n-1}\big\|_0^2+C\big\|\theta^n\big\|_0^2.
\end{aligned}
\tag{38}
$$

Similarly to the proof of $A_2\sim A_8$ and A_{10}, we rewrite θ^n by $\tau\sum_{i=1}^{n}\bar\partial_t\theta^i$ and then try to transfer τ from one side to the other in the inner product. For simplicity and concreteness, with the help of (2), we show that

$$
\begin{aligned}
B_5={}&\left(a_u\big(\mu_4^{n-1}\big)\bar\partial_tU_h^{n-1}\nabla\eta^0,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&-\frac{1}{\tau}\int_\Omega\nabla\eta^0\left(a\big(U_h^{n-1}\big)\tau\sum_{i=1}^{n}\nabla\theta^i-a\big(U_h^{n-2}\big)\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
={}&\sum_K\left(\big(a_u\big(\mu_4^{n-1}\big)\bar\partial_t\xi^{n-1}-\overline{a_u\big(\mu_4^{n-1}\big)\bar\partial_t\xi^{n-1}}\big)\nabla\eta^0,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)_K\\
&+\sum_K\overline{a_u\big(\mu_4^{n-1}\big)\bar\partial_t\xi^{n-1}}|_K\left(\nabla\eta^0,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)-\left(a_u\big(\mu_4^{n-1}\big)\bar\partial_t\eta^{n-1}\nabla\eta^0,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&+\sum_K\left(\big(a_u\big(\mu_4^{n-1}\big)\bar\partial_tU^{n-1}-\overline{a_u\big(\mu_4^{n-1}\big)\bar\partial_tU^{n-1}}\big)\nabla\eta^0,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)_K\\
&+\sum_K\overline{a_u\big(\mu_4^{n-1}\big)\bar\partial_tU^{n-1}}|_K\left(\nabla\eta^0,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&-\frac{1}{\tau}\int_\Omega\nabla\eta^0\left(a\big(U_h^{n-1}\big)\tau\sum_{i=1}^{n}\nabla\theta^i-a\big(U_h^{n-2}\big)\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
\le{}&Ch^2\big\|\bar\partial_t\xi^{n-1}\big\|_1\big\|u^0\big\|_{2,4}\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_{0,4}+Ch^2\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0\\
&-\frac{1}{\tau}\int_\Omega\nabla\eta^0\left(a\big(U_h^{n-1}\big)\tau\sum_{i=1}^{n}\nabla\theta^i-a\big(U_h^{n-2}\big)\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
\le{}&Ch^4+C\big\|\bar\partial_t\xi^{n-1}\big\|_0^2+C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2\\
&-\frac{1}{\tau}\int_\Omega\nabla\eta^0\left(a\big(U_h^{n-1}\big)\tau\sum_{i=1}^{n}\nabla\theta^i-a\big(U_h^{n-2}\big)\tau\sum_{i=1}^{n-1}\nabla\theta^i\right),
\end{aligned}
$$

where $\mu_4^{n-1} = U_h^{n-2} + \tau\lambda_4^{n-1}\bar{\partial}_t U_h^{n-1}$ and $0 < \lambda_4^{n-1} < 1$. Again transferring τ from one part of the inner product to the other, we have

$$\begin{aligned}
B_3 &= -\left(a\big(U^{n-1}\big)\tau\sum_{i=1}^{n}\nabla r^i, \tau\sum_{i=1}^{n}\nabla\bar{\partial}_t\theta^i\right) \\
&= \left(a\big(U^{n-2}\big)\nabla r^n, \tau\sum_{i=1}^{n-1}\nabla\theta^i\right) + \left(\frac{a(U^{n-1}) - a(U^{n-2})}{\tau}\tau\sum_{i=1}^{n}\nabla r^i, \tau\sum_{i=1}^{n-1}\nabla\theta^i\right) \\
&\quad - \bar{\partial}_t\left(a\big(U^{n-1}\big)\tau\sum_{i=1}^{n}\nabla r^i, \tau\sum_{i=1}^{n}\nabla\theta^i\right) \\
&\triangleq \sum_{i=1}^{3} B_{3i}.
\end{aligned}$$

We split B_{31} and B_{32} and estimate them as follows:

$$\begin{aligned}
B_{31} &= \sum_K\left(\big(a\big(U^{n-2}\big) - \overline{a\big(U^{n-2}\big)}\big)\nabla r^n, \tau\sum_{i=1}^{n-1}\nabla\theta^i\right)_K \\
&\quad - \sum_K\overline{a\big(U^{n-2}\big)}|_K\left(\nabla e^i - \nabla I_h e^i, \tau\sum_{i=1}^{n-1}\nabla\theta^i\right)_K \\
&\quad + \sum_K\overline{a\big(U^{n-2}\big)}|_K\left(\nabla u^i - \nabla I_h u^i, \tau\sum_{i=1}^{n-1}\nabla\theta^i\right)_K \\
&\leq Ch^2\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0 + Ch\sqrt{\tau}\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0 \\
&\leq Ch^4 + Ch^2\tau + C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2, \\
B_{32} &= \left(\frac{a(U^{n-1}) - a(U^{n-2})}{\tau}\tau\sum_{i=1}^{n}\big(\nabla e^i - \nabla I_h e^i\big), \tau\sum_{i=1}^{n-1}\nabla\theta^i\right) \\
&\quad + \sum_K\left(\left(\frac{a(U^{n-1}) - a(U^{n-2})}{\tau} - \overline{\frac{a(U^{n-1}) - a(U^{n-2})}{\tau}}\right)\right. \\
&\quad \left.\times\tau\sum_{i=1}^{n}\big(\nabla u^i - \nabla I_h u^i\big), \tau\sum_{i=1}^{n-1}\nabla\theta^i\right)_K \\
&\quad + \sum_K\overline{\frac{a(U^{n-1}) - a(U^{n-2})}{\tau}}\bigg|_K\left(\tau\sum_{i=1}^{n}\big(\nabla u^i - \nabla I_h u^i\big), \tau\sum_{i=1}^{n-1}\nabla\theta^i\right)_K \\
&\leq Ch\sqrt{\tau}\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0 + Ch^2\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0 \\
&\leq Ch^4 + Ch^2\tau + C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2.
\end{aligned}$$

Then we have

$$B_3 \le Ch^4 + Ch^2\tau + C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2 - \bar{\partial}_t\left(a\left(U^{n-1}\right)\tau\sum_{i=1}^{n}\nabla r^i, \tau\sum_{i=1}^{n}\nabla\theta^i\right).$$

Note that

$$\begin{aligned}
B_6 &= \left(\nabla U^0 a_u\left(\mu_5^{n-2}\right)\left(\bar{\partial}_t\xi^{n-1} + \bar{\partial}_t\eta^{n-1}\right), \tau\sum_{i=1}^{n-1}\nabla\theta^i\right) \\
&\quad + \left(\nabla U^0\left(\xi^{n-1} + \eta^{n-1}\right)\frac{a_u(\mu_5^{n-1}) - a_u(\mu_5^{n-2})}{\tau}, \tau\sum_{i=1}^{n-1}\nabla\theta^i\right) \\
&\quad - \bar{\partial}_t\left(\nabla U^0 a_u\left(\mu_5^{n-1}\right)\left(\xi^{n-1} + \eta^{n-1}\right), \tau\sum_{i=1}^{n}\nabla\theta^i\right) \\
&\le Ch^4 + C\left\|\xi^{n-1}\right\|_0^2 + C\left\|\bar{\partial}_t\xi^{n-1}\right\|_0^2 + C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2 \\
&\quad - \bar{\partial}_t\left(\nabla U^0 a_u\left(\mu_5^{n-1}\right)\left(\xi^{n-1} + \eta^{n-1}\right), \tau\sum_{i=1}^{n}\nabla\theta^i\right),
\end{aligned}$$

where

$$\mu_5^{n-1} = U^{n-1} + \lambda_5^{n-1}\left(\xi^{n-1} + \eta^{n-1}\right), \quad 0 < \lambda_5^{n-1} < 1,$$

and

$$\left|\frac{a_u(\mu_5^{n-1}) - a_u(\mu_5^{n-2})}{\tau}\right| \le \left|\bar{\partial}_t U^{n-1}\right| + \lambda_5^{n-1}\left(\left|\bar{\partial}_t\xi^{n-1}\right| + \left|\bar{\partial}_t\eta^{n-1}\right|\right).$$

Rewriting B_4 and splitting it into several parts, we obtain

$$\begin{aligned}
B_4 &= \left(\left(a\left(U^{n-1}\right) - a\left(U_h^{n-1}\right)\right)\tau\sum_{i=1}^{n}\nabla r^i, \tau\sum_{i=1}^{n}\nabla\bar{\partial}_t\theta^i\right) \\
&\quad - \left(\left(a\left(U^{n-1}\right) - a\left(U_h^{n-1}\right)\right)\tau\sum_{i=1}^{n}\nabla Q^i, \nabla\theta^n\right) \\
&= -\left(\left(a\left(U^{n-2}\right) - a\left(U_h^{n-2}\right)\right)\nabla r^n, \tau\sum_{i=1}^{n-1}\nabla\theta^i\right) \\
&\quad - \left(\tau\sum_{i=1}^{n}\nabla r^i\frac{(a(U^{n-1}) - a(U_h^{n-1})) - (a(U^{n-2}) - a(U_h^{n-2}))}{\tau}, \tau\sum_{i=1}^{n-1}\nabla\theta^i\right) \\
&\quad + \bar{\partial}_t\left(\left(a\left(U^{n-1}\right) - a\left(U_h^{n-1}\right)\right)\tau\sum_{i=1}^{n}\nabla r^i, \tau\sum_{i=1}^{n}\nabla\theta^i\right) \\
&\quad - \left(\left(a\left(U^{n-1}\right) - a\left(U_h^{n-1}\right)\right)\tau\sum_{i=1}^{n}\nabla Q^i, \nabla\theta^n\right).
\end{aligned}$$

It is obvious that

$$\begin{aligned}
&-\left(\left(a\left(U^{n-2}\right)-a\left(U_h^{n-2}\right)\right)\nabla r^n,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&\quad\le Ch\|U^n\|_2\left(Ch^2\|U^{n-2}\|_2+\|\xi^{n-2}\|_0\right)\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_{0,\infty}\\
&\quad\le Ch^4+C\|\xi^{n-2}\|_0^2+C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2.
\end{aligned}$$

Because

$$\begin{aligned}
&\frac{(a(U^{n-1})-a(U_h^{n-1}))-(a(U^{n-2})-a(U_h^{n-2}))}{\tau}\\
&\quad=a_u\left(\mu_6^{n-1}\right)\bar\partial_t U^{n-1}-a_u\left(\mu_7^{n-1}\right)\bar\partial_t U_h^{n-1}\\
&\quad=a_u\left(\mu_7^{n-1}\right)\left(\bar\partial_t\xi^{n-1}+\bar\partial_t\eta^{n-1}\right)\\
&\qquad+\bar\partial_t U^{n-1}\left(a_u\left(\mu_6^{n-1}\right)-a_u\left(\mu_7^{n-1}\right)\right),
\end{aligned}\tag{39}$$

where

$$\begin{aligned}
&\mu_6^{n-1}=U^{n-2}+\tau\lambda_6^{n-1}\bar\partial_t U^{n-1},\qquad \mu_7^{n-1}=U_h^{n-2}+\tau\lambda_7^{n-1}\bar\partial_t U_h^{n-1},\\
&\quad 0<\mu_6^{n-1},\mu_7^{n-1}<1,
\end{aligned}$$

and

$$\begin{aligned}
\mu_6^{n-1}-\mu_7^{n-1}&=\xi^{n-2}+\eta^{n-2}+\tau\lambda_7^{n-1}\left(\bar\partial_t\xi^{n-1}+\bar\partial_t\eta^{n-1}\right)\\
&\quad+\tau\bar\partial_t U^{n-1}\left(\lambda_6^{n-1}-\lambda_7^{n-1}\right),
\end{aligned}$$

it follows that

$$\begin{aligned}
&\left(\tau\sum_{i=1}^{n}\nabla r^i\frac{(a(U^{n-1})-a(U_h^{n-1}))-(a(U^{n-2})-a(U_h^{n-2}))}{\tau},\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&\quad\le Ch^2\left\|\tau\sum_{i=1}^{n}Q^i\right\|_2\left\|\frac{(a(U^{n-1})-a(U_h^{n-1}))-(a(U^{n-2})-a(U_h^{n-2}))}{\tau}\right\|_0\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_{0,\infty}\\
&\quad\le\left(\|\bar\partial_t\xi^{n-1}\|_0+\|\bar\partial_t\eta^{n-1}\|_0+\|\xi^{n-2}\|_0+\|\eta^{n-2}\|_0+\tau\right)Ch\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0\\
&\quad\le Ch^4+Ch^2\tau+\|\bar\partial_t\xi^{n-1}\|_0^2+\|\xi^{n-2}\|_0^2+C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2.
\end{aligned}$$

The fourth part of B_4 can be found:

$$\begin{aligned}
&\left(\left(a\left(U^{n-1}\right)-a\left(U_h^{n-1}\right)\right)\tau\sum_{i=1}^{n}\nabla Q^i,\nabla\theta^n\right)\\
&\quad=\left(a_u\left(\mu_8^{n-1}\right)\eta^{n-1}\tau\sum_{i=1}^{n}\nabla Q^i,\nabla\theta^n\right)+\left(a_u\left(\mu_8^{n-1}\right)\xi^{n-1}\tau\sum_{i=1}^{n}\nabla Q^i,\nabla\theta^n\right),
\end{aligned}\tag{40}$$

where $\mu_8^{n-1}=U^{n-1}+\lambda_8^{n-1}(U_h^{n-1}-U^{n-1})$ and $0<\lambda_8^{n-1}<1$.

On one hand,

$$\begin{aligned}
&\left(a_u\left(\mu_8^{n-1}\right)\eta^{n-1}\tau\sum_{i=1}^{n}\nabla Q^i,\nabla\theta^n\right)\\
&\quad=\left(a_u\left(\mu_8^{n-1}\right)\eta^{n-1}\tau\sum_{i=1}^{n}\nabla Q^i,\tau\sum_{i=1}^{n}\bar{\partial}_t\nabla\theta^i\right)\\
&\quad=\left(a_u\left(\mu_8^{n-2}\right)\eta^{n-2}\nabla\delta^n,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)+\left(a_u\left(\mu_8^{n-2}\right)\tau\sum_{i=1}^{n}\nabla\delta^i\bar{\partial}_t\eta^{n-1},\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&\qquad+\left(\eta^{n-1}\tau\sum_{i=1}^{n}\nabla\delta^i\frac{a_u(\mu_8^{n-1})-a_u(\mu_6^{n-2})}{\tau},\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&\qquad-\left(a_u\left(\mu_8^{n-2}\right)\eta^{n-2}\nabla q^n,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)-\left(a_u\left(\mu_8^{n-2}\right)\tau\sum_{i=1}^{n}\nabla q^i\bar{\partial}_t\eta^{n-1},\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&\qquad-\left(\eta^{n-1}\tau\sum_{i=1}^{n}\nabla q^i\frac{a_u(\mu_8^{n-1})-a_u(\mu_6^{n-2})}{\tau},\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&\qquad+\bar{\partial}_t\left(a_u\left(\mu_8^{n}\right)\eta^{n}\tau\sum_{i=1}^{n}\nabla Q^i,\tau\sum_{i=1}^{n}\nabla\theta^i\right)\\
&\quad\le Ch^4+Ch^2\tau+C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2+\bar{\partial}_t\left(a_u\left(\mu_8^{n}\right)\eta^{n}\tau\sum_{i=1}^{n}\nabla Q^i,\tau\sum_{i=1}^{n}\nabla\theta^i\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&-\left(a_u\left(\mu_8^{n-1}\right)\xi^{n-1}\tau\sum_{i=1}^{n}\nabla Q^i,\tau\sum_{i=1}^{n-1}\bar{\partial}_t\nabla\theta^i\right)\\
&\quad=-\left(a_u\left(\mu_8^{n-2}\right)\xi^{n-2}\nabla Q^n,\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)-\left(a_u\left(\mu_8^{n-2}\right)\tau\sum_{i=1}^{n}\nabla Q^i\bar{\partial}_t\xi^{n-1},\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&\qquad-\left(\tau\sum_{i=1}^{n}\nabla Q^i\xi^{n-1}\frac{a_u(\mu_8^{n-1})-a_u(\mu_8^{n-2})}{\tau},\tau\sum_{i=1}^{n-1}\nabla\theta^i\right)\\
&\qquad+\bar{\partial}_t\left(a_u\left(\mu_8^{n-1}\right)\xi^{n-1}\tau\sum_{i=1}^{n}\nabla Q^i,\tau\sum_{i=1}^{n}\nabla\theta^i\right)\triangleq\sum_{i=1}^{4}D_i.
\end{aligned}\tag{41}$$

Now we make use of the mean-value technique:

$$
\begin{aligned}
D_1 &= \sum_K \left(\left(a_u\left(\mu_8^{n-2}\right) - \overline{a_u\left(\mu_8^{n-2}\right)} \right) \xi^{n-2} \nabla Q^n, \tau \sum_{i=1}^{n-1} \nabla \theta^i \right)_K \\
&\quad + \sum_K \overline{a_u\left(\mu_8^{n-2}\right)}|_K \left(\xi^{n-2} \nabla Q^n, \tau \sum_{i=1}^{n-1} \nabla \theta^i \right)_K .
\end{aligned}
$$

Since $Q^n \in H^2$, we cannot use the mean-value of ∇Q^n directly as before, thus with the first equation of (36), we try to tackle it as follows:

$$
\begin{aligned}
&\sum_K \overline{a_u\left(\mu_8^{n-2}\right)}|_K \left(\xi^{n-2} \nabla Q^n, \tau \sum_{i=1}^{n-1} \nabla \theta^i \right)_K \\
&\quad = \sum_K \overline{a_u\left(\mu_8^{n-2}\right)}|_K \left(\nabla Q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i - \overline{\nabla Q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i}, \xi^{n-2} \right)_K \\
&\quad\quad - \sum_K \left(\overline{a_u\left(\mu_8^{n-2}\right)} \overline{\nabla Q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i} \right) \Bigg|_K \tau \sum_{i=1}^{n-2} \left(1, \bar{\partial}_t \eta^i\right)_K \\
&\quad\quad - \sum_K \left(\overline{a_u\left(\mu_8^{n-2}\right)} \overline{\nabla Q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i} \right) \Bigg|_K \tau \sum_{i=1}^{n-2} \left(1, \theta^i\right)_K \\
&\quad\quad - \sum_K \left(\overline{a_u\left(\mu_8^{n-2}\right)} \overline{\nabla Q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i} \right) \Bigg|_K \tau \sum_{i=1}^{n-2} \left(1, r^i\right)_K .
\end{aligned}
$$

Because $\Delta \theta^i|_K = 0$, with the help of Theorem 1, we have

$$
\begin{aligned}
&\sum_K \overline{a_u\left(\mu_8^{n-2}\right)}|_K \left(\nabla Q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i - \overline{\nabla Q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i}, \xi^{n-2} \right)_K \\
&\quad \le Ch \left\| \nabla Q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i \right\|_1 \left\| \xi^{n-2} \right\|_0 \le Ch \left\| Q^n \right\|_2 \left\| \tau \sum_{i=1}^{n-1} \nabla \theta^i \right\|_{0,\infty} \left\| \xi^{n-2} \right\|_0 \\
&\quad \le C \left\| \tau \sum_{i=1}^{n-1} \nabla \theta^i \right\|_0^2 + C \left\| \xi^{n-2} \right\|_0^2 .
\end{aligned}
\tag{42}
$$

Further,

$$
\begin{aligned}
&\sum_K \left(\overline{a_u\left(\mu_8^{n-2}\right)} \overline{\nabla Q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i} \right) \Bigg|_K \tau \sum_{i=1}^{n-2} \left(1, \bar{\partial}_t \eta^i\right)_K \\
&\quad = \sum_K \overline{a_u\left(\mu_8^{n-2}\right)} \frac{1}{|K|} \int_K \nabla \delta^n \tau \sum_{i=1}^{n-1} \nabla \theta^i \, dx\, dy \tau \sum_{i=1}^{n-2} \int_K \bar{\partial}_t \eta^i \, dx\, dy \\
&\quad\quad + \sum_K \overline{a_u\left(\mu_8^{n-2}\right)} \frac{1}{|K|} \int_K \nabla q^n \tau \sum_{i=1}^{n-1} \nabla \theta^i \, dx\, dy \tau \sum_{i=1}^{n-2} \int_K \bar{\partial}_t \eta^i \, dx\, dy
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_K \frac{1}{|K|}\|\nabla\delta^n\|_{0,4}\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_{0,4}|K|^{\frac{1}{2}}\tau\sum_{i=1}^{n-2}\left(\int_K|\bar{\partial}_t\eta^i|^2\,dx\,dy\right)^{\frac{1}{2}}|K|^{\frac{1}{2}}\\
&\quad+C\sum_K\frac{1}{|K|}\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0|K|^{\frac{1}{2}}\tau\sum_{i=1}^{n-2}\left(\int_K|\bar{\partial}_t\eta^i|^2\,dx\,dy\right)^{\frac{1}{2}}|K|^{\frac{1}{2}}\\
&\le C\sum_K\|\delta^n\|_2\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_{0,4}\tau\sum_{i=1}^{n-2}\|\bar{\partial}_t\eta^i\|_0+C\sum_K\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0\tau\sum_{i=1}^{n-2}\|\bar{\partial}_t\eta^i\|_0\\
&\le Ch^{\frac{3}{2}}\|\delta^n\|_2\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0+Ch^2\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0\\
&\le Ch^4+Ch^3\|\delta^n\|_2^2+C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2.
\end{aligned}
$$

Similarly, we have

$$
\sum_K\left(\overline{\overline{a_u(\mu_8^{n-2})}\nabla Q^n\tau\sum_{i=1}^{n-1}\nabla\theta^i}\right)\bigg|_K\tau\sum_{i=1}^{n-2}(1,r^i)_K\le Ch^4+Ch^3\|\delta^n\|_2^2+C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2,
$$

$$
\begin{aligned}
&\sum_K\left(\overline{\overline{a_u(\mu_8^{n-2})}\nabla Q^n\tau\sum_{i=1}^{n-1}\nabla\theta^i}\right)\bigg|_K\tau\sum_{i=1}^{n-2}(1,\theta^i)_K\\
&\quad\le C\sum_K\frac{1}{|K|}\|\nabla Q^n\|_{0,4}\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0|K|^{\frac{1}{4}}\tau\sum_{i=1}^{n-2}\left(\int_K|\theta^i|^4\,dx\,dy\right)^{\frac{1}{4}}|K|^{\frac{3}{4}}\\
&\quad\le C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0\tau\sum_{i=1}^{n-2}\|\theta^i\|_{0,4}\le C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2+C\tau\sum_{i=1}^{n-2}\|\nabla\theta^i\|_0^2.
\end{aligned}
$$

Thus we have

$$
D_1\le C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2+C\|\xi^{n-2}\|_0^2+Ch^4+Ch^3\|\delta^n\|_2^2+C\tau\sum_{i=1}^{n-2}\|\nabla\theta^i\|_0^2.
$$

By a similar method we have

$$
\begin{aligned}
D_2+D_3\le{}& C\left\|\tau\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2+C\|\bar{\partial}_t\xi^{n-1}\|_0^2+C\|\xi^{n-1}\|_0^2\\
&+Ch^4+Ch^3\|\delta^n\|_2^2+C\tau\sum_{i=1}^{n-2}\|\nabla\theta^i\|_0^2.
\end{aligned}
$$

Altogether,

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\|\theta^n\|_0^2-\|\theta^{n-1}\|_0^2\right)+\frac{1}{2}\tau\|a^{\frac{1}{2}}(U^{n-1})\nabla\theta^n\|_0^2\\
&\quad+\frac{1}{2}\tau\left\|a^{\frac{1}{2}}(U^{n-1})\sum_{i=1}^{n}\nabla\theta^i\right\|_0^2-\frac{1}{2}\tau\left\|a^{\frac{1}{2}}(U^{n-2})\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&\le Ch^4 + Ch^2\tau + C\|\xi^{n-1}\|_0^2 + C\|\xi^{n-2}\|_0^2 + C\|\theta^n\|_0^2 + C\|\bar\partial_t\xi^{n-1}\|_0^2 \\
&\quad + C\tau^2\left\|\sum_{i=1}^{n-1}\nabla\theta^i\right\|_0^2 + Ch\tau\|\nabla\theta^n\|_0^2 + C\tau\bigl(h^{\frac14}+\tau^{\frac14}\bigr)\|\nabla\theta^n\|_0^2 \\
&\quad - \frac{1}{\tau}\int_\Omega \nabla\eta^0\left(a\bigl(U_h^{n-1}\bigr)\tau\sum_{i=1}^{n}\nabla\theta^i - a\bigl(U_h^{n-2}\bigr)\tau\sum_{i=1}^{n-1}\nabla\theta^i\right) \\
&\quad - \bar\partial_t\left(a\bigl(U^{n-1}\bigr)\tau\sum_{i=1}^{n}\nabla r^i, \tau\sum_{i=1}^{n}\nabla\theta^i\right) \\
&\quad + \bar\partial_t\left(\bigl(a\bigl(U^{n-1}\bigr)-a\bigl(U_h^{n-1}\bigr)\bigr)\tau\sum_{i=1}^{n}\nabla r^i, \tau\sum_{i=1}^{n}\nabla\theta^i\right) \\
&\quad + \frac12\int_\Omega\bigl(a\bigl(U^{n-1}\bigr)-a\bigl(U_h^{n-1}\bigr)\bigr)\tau\left(\sum_{i=1}^{n}\nabla\theta^i\right)^2 \\
&\quad - \frac12\int_\Omega\bigl(a\bigl(U^{n-2}\bigr)-a\bigl(U_h^{n-2}\bigr)\bigr)\tau\left(\sum_{i=1}^{n-1}\nabla\theta^i\right)^2 \\
&\quad - \bar\partial_t\left(a_u\bigl(\mu_8^{n-1}\bigr)\bigl(\xi^{n-1}+\eta^{n-1}\bigr)\tau\sum_{i=1}^{n}\nabla Q^i, \tau\sum_{i=1}^{n-1}\nabla\theta^i\right) \\
&\quad - \bar\partial_t\left(\nabla U^0 a_u\bigl(\mu_8^{n-1}\bigr)\bigl(\xi^{n-1}+\eta^{n-1}\bigr), \tau\sum_{i=1}^{n}\nabla\theta^i\right)
\end{aligned}
$$

with θ^1 estimated earlier. Using the Gronwall lemma, we have

$$
\|\theta^n\|_0^2 + \tau^2\left\|\sum_{i=1}^{n}\nabla\theta^i\right\|_0^2 + \sum_{i=2}^{n}\tau^2\|\nabla\theta^i\|_0^2 \le Ch^4 + Ch^2\tau. \tag{43}
$$

Again using the first equation of (36) with $v_h = \bar\partial_t\xi^n$, for ξ^n, we obtain

$$
\begin{aligned}
\|\bar\partial_t\xi^n\|_0^2 &= -\bigl(\bar\partial_t\eta^n, \bar\partial_t\xi^n\bigr) + \bigl(\theta^n, \bar\partial_t\xi^n\bigr) + \bigl(r^n, \bar\partial_t\xi^n\bigr) \\
&\le Ch^4\|U^n\|_2^2 + Ch^4\|\bar\partial_t U^n\|_2^2 + C\|\theta^n\|_0^2 + \frac12\|\bar\partial_t\xi^n\|_0^2,
\end{aligned} \tag{44}
$$

which implies

$$
\tau\sum_{i=2}^{n}\|\bar\partial_t\xi^i\|_0^2 \le Ch^4 + Ch^4\tau\sum_{i=2}^{n}\|\bar\partial_t U^i\|_2^2 + \tau\sum_{i=2}^{n}\|\theta^i\|_0^2 \le Ch^4 + Ch^2\tau, \tag{45}
$$

or

$$
\|\xi^n\|_0^2 = \tau^2\left\|\sum_{i=2}^{n}\bar\partial_t\xi^i\right\|_0^2 \le C\tau^2\left(\sum_{i=2}^{n}\|\bar\partial_t\xi^i\|_0\right)^2 \le Ch^4 + Ch^2\tau. \tag{46}
$$

Then there exist τ_4', τ_5', h_4', h_5', C_2' such that, for $\tau \le \tau_4'$ and $h \le h_4'$, we have

$$\sqrt{\tau}\left(\sum_{i=2}^{n}\|\bar{\partial}_t\xi^i\|_0^2\right)^{\frac{1}{2}} + \|\xi^n\|_0 + \|\theta^n\|_0 + \tau\left(\sum_{i=1}^{n}\|\nabla\theta^i\|_0^2\right)^{\frac{1}{2}} \le C_2'h\bigl(h+\tau^{\frac{1}{2}}\bigr), \tag{47}$$

from which we deduce

$$\begin{aligned}
&\|U_h^n\|_{0,\infty} + \sqrt{\tau}\left(\sum_{i=2}^{n}\|\bar{\partial}_t U_h^n\|_{0,\infty}^2\right)^{\frac{1}{2}} \\
&\quad\le Ch^{-1}\left(\|\xi^n\|_0 + \sqrt{\tau}\left(\sum_{i=2}^{n}\|\bar{\partial}_t\xi^i\|_{0,\infty}^2\right)^{\frac{1}{2}}\right) \\
&\qquad + \left(\|I_hU^n\|_{0,\infty} + \sqrt{\tau}\left(\sum_{i=2}^{n}\|\bar{\partial}_t I_hU^i\|_{0,\infty}^2\right)^{\frac{1}{2}}\right) \\
&\quad\le 2CC_2'h + 2CC_2'\sqrt{\tau} + \left(\|I_hU^n\|_{0,\infty} + \sqrt{\tau}\left(\sum_{i=2}^{n}\|\bar{\partial}_t I_hU^i\|_{0,\infty}^2\right)^{\frac{1}{2}}\right) \le K_0',
\end{aligned} \tag{48}$$

where $h \le h_5' \le 1/2CC_2'$ and $\tau \le \tau_5' \le 1/4(CC_2')^2$. Clearly, C_2' has nothing to do with C_0', and thus (29) holds for $m = n$ if we take $C_0' \ge \sum_{i=1}^{2} C_i'$, $\tau_0' \le \min_{1\le\tau\le5}\tau_i'$, and $h_0' \le \min_{1\le\tau\le5} h_i'$. Then the induction is closed.

The desired estimate for u^n and q^n in (27) and (28) are thus consequences of (29) combined with the triangle inequality. The proof is completed. □

Remark 2 It is precious to point out that to avoid the restriction involved by the regularities of Q^n, we try to use the new mean-value technique in the proof of $D_1 \sim D_3$.

Remark 3 It can be seen that (27) and (28) do not hold for the elements dissatisfying (42), such as the biquadratic finite element.

5 Numerical results

In this section, we consider the hyperbolic equation

$$\begin{cases} u_{tt} - \nabla\cdot(a(u)\nabla u) - f(u) = g(X,t), & (X,t)\in\Omega\times(0,T], \\ u = 0, & (X,t)\in\partial\Omega\times(0,T], \\ u(X,0) = u_0(X), \quad u_t(X,0) = u_1(X), & X\in\Omega, \end{cases} \tag{49}$$

with $\Omega = [0,1]\times[0,1]$, $a(u) = \sin u + 0.1$, $f(u) = u^2$, and $g(X,t)$ chosen corresponding to the exact solution $u = e^t xy(1-x)(1-y)$. Setting $q = u_t$, (49) is changed into a parabolic system. A uniform rectangular partition with $m+1$ nodes in each direction is used in our computation. We solve the system by the linearized Galerkin method with bilinear element.

To confirm our error analysis for (27) and (28), we choose $\tau = 5h^2$ for the backward Euler FEM with bilinear FE. Therefore, from our theoretical analysis, the L^2-norm errors for u and q are $O(h^2+\tau)\sim O(h^2)$, and the H^1-norm errors for u and q are $O(h+\tau)\sim O(h)$. We present the numerical results with respect to time $t = 0.25, 0.5, 0.75, 1.0$ in Tables 1–4,

Table 1 Results for U_h^n and Q_h^n when $t = 0.25$ ($\tau = 5h^2$)

$m \times m$	$\|u^n - U_h^n\|_0$	Order	$\|u^n - U_h^n\|_1$	Order	$\|q^n - Q_h^n\|_0$	Order	$\|q^n - Q_h^n\|_1$	Order
5×5	2.9760×10^{-3}	–	3.9028×10^{-2}	–	2.9757×10^{-3}	–	3.9028×10^{-2}	–
10×10	7.3473×10^{-4}	2.0181	1.9230×10^{-2}	1.0211	7.3473×10^{-4}	2.0179	1.9230×10^{-2}	1.0211
20×20	1.8429×10^{-4}	1.9952	9.5818×10^{-3}	1.0050	1.8429×10^{-4}	1.9953	9.5818×10^{-3}	1.0050
40×40	4.6110×10^{-5}	1.9988	4.7867×10^{-3}	1.0013	4.6110×10^{-5}	1.9988	4.7867×10^{-3}	1.0013

Table 2 Results for U_h^n and Q_h^n when $t = 0.5$ ($\tau = 5h^2$)

$m \times m$	$\|u^n - U_h^n\|_0$	Order	$\|u^n - U_h^n\|_1$	Order	$\|q^n - Q_h^n\|_0$	Order	$\|q^n - Q_h^n\|_1$	Order
5×5	3.8700×10^{-3}	–	5.0155×10^{-2}	–	3.8696×10^{-3}	–	5.0155×10^{-2}	–
10×10	9.3055×10^{-4}	2.0562	2.4686×10^{-2}	1.0227	9.3055×10^{-4}	2.0560	2.4686×10^{-2}	1.0227
20×20	2.3338×10^{-4}	1.9954	1.2303×10^{-2}	1.0048	2.3338×10^{-4}	1.9954	1.2303×10^{-2}	1.0048
40×40	5.8391×10^{-5}	1.9989	6.1461×10^{-3}	1.0012	5.8391×10^{-5}	1.9989	6.1461×10^{-3}	1.0012

Table 3 Results for U_h^n and Q_h^n when $t = 0.75$ ($\tau = 5h^2$)

$m \times m$	$\|u^n - U_h^n\|_0$	Order	$\|u^n - U_h^n\|_1$	Order	$\|q^n - Q_h^n\|_0$	Order	$\|q^n - Q_h^n\|_1$	Order
5×5	5.0318×10^{-3}	–	6.4456×10^{-2}	–	5.0313×10^{-3}	–	6.4456×10^{-2}	–
10×10	1.1784×10^{-3}	2.0943	3.1692×10^{-2}	1.0242	1.1784×10^{-3}	2.0941	3.1692×10^{-2}	1.0242
20×20	2.9550×10^{-4}	1.9956	1.5796×10^{-2}	1.0045	2.9549×10^{-4}	1.9956	1.5796×10^{-2}	1.0045
40×40	7.3929×10^{-5}	1.9989	7.8917×10^{-3}	1.0012	7.3929×10^{-5}	1.9989	7.8917×10^{-3}	1.0011

Table 4 Results for U_h^n and Q_h^n when $t = 1.0$ ($\tau = 5h^2$)

$m \times m$	$\|u^n - U_h^n\|_0$	Order	$\|u^n - U_h^n\|_1$	Order	$\|q^n - Q_h^n\|_0$	Order	$\|q^n - Q_h^n\|_1$	Order
5×5	5.8966×10^{-3}	–	8.2298×10^{-2}	–	5.8959×10^{-3}	–	8.2297×10^{-2}	–
10×10	1.4919×10^{-3}	1.9827	4.0685×10^{-2}	1.0164	1.4919×10^{-3}	1.9826	4.0685×10^{-2}	1.0164
20×20	3.7407×10^{-4}	1.9958	2.0281×10^{-2}	1.0043	3.7407×10^{-4}	1.9958	2.0281×10^{-2}	1.0043
40×40	9.3585×10^{-5}	1.9989	1.0133×10^{-2}	1.0011	9.3585×10^{-5}	1.9989	1.0133×10^{-2}	1.0011

Table 5 Results for $\|u^n - U_h^n\|_1$ ($h = \frac{1}{160}$, $\tau = kh$)

t	$k = 1$	$k = 5$	$k = 10$	$k = 20$	$k = 40$
0.25	2.396051×10^{-3}	2.509776×10^{-3}	2.868899×10^{-3}	4.142662×10^{-3}	7.852834×10^{-3}
0.50	3.093389×10^{-3}	3.652433×10^{-3}	5.078443×10^{-3}	8.767703×10^{-3}	1.632563×10^{-2}
0.75	4.009582×10^{-3}	5.496080×10^{-3}	8.725847×10^{-3}	1.598647×10^{-2}	2.828099×10^{-2}
1.00	5.216721×10^{-3}	8.300234×10^{-3}	1.426017×10^{-2}	2.670072×10^{-2}	4.519938×10^{-2}

Table 6 Results for $\|q^n - Q_h^n\|_1$ ($h = \frac{1}{160}$, $\tau = kh$)

t	$k = 1$	$k = 5$	$k = 10$	$k = 20$	$k = 40$
0.25	2.396041×10^{-3}	2.502978×10^{-3}	2.774917×10^{-3}	3.325243×10^{-3}	7.807021×10^{-3}
0.50	3.093358×10^{-3}	3.634416×10^{-3}	4.873726×10^{-3}	7.011551×10^{-3}	6.395837×10^{-3}
0.75	4.009512×10^{-3}	5.462226×10^{-3}	8.391812×10^{-3}	1.334560×10^{-2}	1.239224×10^{-2}
1.00	5.216582×10^{-3}	8.244615×10^{-3}	1.376092×10^{-2}	2.292862×10^{-2}	2.288225×10^{-2}

respectively. It can be seen that $\|u^n - U_h^n\|_0$ and $\|q^n - Q_h^n\|_0$ are convergent at rate $O(h^2)$ and $\|u^n - U_h^n\|_1$ and $\|q^n - Q_h^n\|_1$ are convergent at rate $O(h)$, which indicate the optimal convergence rates of the methods. Further, to show the unconditional convergence results, we test the FEM with $h = 1/160$ and the large time steps $\tau = h, 5h, 10h, 20h, 40h$, respectively. We present the numerical results in Tables 5–6, which suggest that the scheme is stable for large time steps. All these results are in good agreement with our theoretical analysis.

6 Conclusion

In this paper, we have established unconditional error estimates for a nonlinear hyperbolic equation. A striking feature of our analysis is that we transform the nonlinear hyperbolic equation into a parabolic system. Then a linearized backward Euler FEM is constructed for the nonlinear parabolic equation. It is shown in this paper that such an idea avoids the difficulty in constructing a linearized first-order scheme for a nonlinear hyperbolic equation, and we can also give the error analysis for u and $q = u_t$ at the same time. Splitting skill is exploited to derive the final unconditional convergent results. Some special methods are utilized to derive the boundedness of the solutions about the time-discrete system in H^2-norm, which may play a crucial role for getting rid of the restriction on the ratio between h and τ. Since the new parabolic system caused lots of problems for our the spatial errors analysis, several new techniques, such as rewriting the error equations, are introduced. It should be noted that the results in this paper also hold for linear conforming triangular elements but not hold for some other particular elements, such as the biquadratic finite element.

Acknowledgements
The authors would like to thank the referees for their valuable suggestions, which helped to improve this work.

Funding
This work was supported by the National Natural Science Foundation of China (No. 11671369), the Doctoral Starting Foundation of Pingdingshan University (No. PXY-BSQD-2019001), and the University Cultivation Foundation of Pingdingshan (No. PXY-PYJJ-2019006).

Abbreviations
FEM, finite element method; PDEs, partial differential equations.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JW carried out theoretical calculation, participated in the design of the study, and drafted the manuscript. LG participated in its design and helped to draft the manuscript. Both authors read and approved the final manuscript.

Publisher's Note

Received: 29 October 2018 Accepted: 26 February 2019 Published online: 04 March 2019

References

1. Chen, C., Liu, W.: A two-grid method for finite volume element approximations of second-order nonlinear hyperbolic equations. J. Comput. Appl. Math. **233**(11), 2975–2984 (2010)
2. Rincon, M., Quintino, N.: Numerical analysis and simulation for a nonlinear wave equation. J. Comput. Appl. Math. **296**, 247–264 (2016)
3. Shi, D., Li, Z.: Superconvergence analysis of the finite element method for nonlinear hyperbolic equations with nonlinear boundary condition. Appl. Math. J. Chin. Univ. Ser. A **23**(4), 455–462 (2008)
4. Lai, X., Yuan, Y.: Galerkin alternating-direction method for a kind of three-dimensional nonlinear hyperbolic problems. Comput. Math. Appl. **57**(3), 384–403 (2009)
5. Zhou, Z., Wang, W., Chen, H.: An H^1-Galerkin expanded mixed finite element approximation of second-order nonlinear hyperbolic equations. Abstr. Appl. Anal. **2013**, Article ID 657952 (2013)
6. Chen, Y., Huang, Y.: The full-discrete mixed finite element methods for nonlinear hyperbolic equations. Commun. Nonlinear Sci. Numer. Simul. **3**(3), 152–155 (1998)
7. Shi, D., Yan, F., Wang, J.: Unconditional superconvergence analysis of a new mixed finite element method for nonlinear Sobolev equation. Appl. Math. Comput. **274**(1), 182–194 (2016)
8. Shi, D., Wang, J., Yan, F.: Unconditional superconvergence analysis of an H^1-Galerkin mixed finite element method for nonlinear Sobolev equations. Numer. Methods Partial Differ. Equ. **34**(1), 145–166 (2018)
9. Li, B., Sun, W.: Unconditional convergence and optimal error estimates of a Galerkin-mixed FEM for incompressible miscible flow in porous media. SIAM J. Numer. Anal. **51**(4), 1959–1977 (2013)

10. Wang, J.: A new error analysis of Crank–Nicolson Galerkin FEMs for a generalized nonlinear Schrödinger equation. J. Sci. Comput. **60**(2), 390–407 (2014)
11. Wang, J., Si, Z., Sun, W.: A new error analysis of characteristics-mixed FEMs for miscible displacement in porous media. SIAM J. Numer. Anal. **52**(6), 3000–3020 (2013)
12. Gao, H.: Optimal error analysis of Galerkin FEMs for nonlinear Joule heating equations. J. Sci. Comput. **58**(3), 627–647 (2014)
13. Li, B., Gao, H., Sun, W.: Unconditionally optimal error estimates of a Crank–Nicolson Galerkin method for the nonlinear thermistor equations. SIAM J. Numer. Anal. **52**(2), 933–954 (2014)
14. Gao, H.: Unconditional optimal error estimates of BDF–Galerkin FEMs for nonlinear thermistor equations. J. Sci. Comput. **66**(2), 504–527 (2016)
15. Si, Z., Wang, J., Sun, W.: Unconditional stability and error estimates of modified characteristics FEMs for the Navier–Stokes equations. Numer. Math. **134**(1), 139–161 (2016)
16. Shi, D., Wang, J.: Unconditional superconvergence analysis of a Crank–Nicolson Galerkin FEM for nonlinear Schrödinger equation. J. Sci. Comput. **72**(3), 1093–1118 (2017)
17. Shi, D., Wang, J., Yan, F.: Unconditional superconvergence analysis for nonlinear parabolic equation with $EQ_1^{\mathbf{rot}}$ nonconforming finite element. J. Sci. Comput. **70**(1), 85–111 (2017)
18. Shi, D., Wang, J.: Unconditional superconvergence analysis of conforming finite element for nonlinear parabolic equation. Appl. Math. Comput. **294**, 216–226 (2017)
19. Shi, D., Wang, J.: Unconditional superconvergence analysis for nonlinear hyperbolic equation with nonconforming finite element. Appl. Math. Comput. **305**, 1–16 (2017)
20. Shi, D., Wang, J.: Unconditional superconvergence analysis of a linearized Galerkin FEM for nonlinear hyperbolic equations. Comput. Math. Appl. **74**(4), 634–651 (2017)
21. Jung, C.Y., Park, E., Temam, R.: Boundary layer analysis of nonlinear reaction–diffusion equations in a smooth domain. Adv. Nonlinear Anal. **6**(3), 277–300 (2017)
22. Grabowski, P.: Small-gain theorem for a class of abstract parabolic systems. Opusc. Math. **38**(5), 651–680 (2018)
23. Strani, M.: Semigroup estimates and fast–slow dynamics in parabolic–hyperbolic systems. Adv. Nonlinear Anal. **7**(1), 117–138 (2018)
24. Shi, D., Pei, L.: Nonconforming quadrilateral finite element method for a class of nonlinear sine-Gordon equations. Appl. Math. Comput. **219**, 9447–9460 (2013)
25. Lin, Q., Lin, J.: Finite Element Methods: Accuracy and Improvement. Science Press, Beijing (2006)
26. Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Springer Series in Computational Mathematics. Springer, Berlin (2000)