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Multiplicity results for p -Laplacian boundary value problem with jumping nonlinearities

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Abstract

We investigate the multiplicity of solutions for one-dimensional p -Laplacian Dirichlet boundary value problem with jumping nonlinearities. We obtain three theorems: The first states that there exists exactly one solution when nonlinearities cross no eigenvalue. The second guarantees that there exist exactly two solutions, exactly one solution and no solution, depending on the source term, when nonlinearities cross just the first eigenvalue. The third claims that there exist at least three solutions, exactly one solution and no solution, depending on the source term, when nonlinearities cross the first and second eigenvalues. We obtain the first and second theorem by considering the eigenvalues and the corresponding normalized eigenfunctions of the p -Laplacian eigenvalue problem, and the contraction mapping principle in the p -Lebesgue space (when $p \geq 2$). We obtain the third result by Leray–Schauder degree theory.

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1 Introduction

The p -Laplacian boundary value problems with p -growth conditions arise in applications of nonlinear elasticity theory, electro-rheological fluids, and in non-Newtonian fluid theory of a porous medium (cf. [6, 7, 13]). A typical model of elliptic equation with p -growth conditions is

$$-\operatorname{div}((\alpha + |\nabla u|)^{p-2} \nabla u) = f(x, u).$$

In particular, when $\alpha = 0$, the operator

$$-\Delta_p = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

is called the p -Laplacian. In general, when $p = p(x)$, the $p(x)$ -Laplacian problems are inhomogeneous, so they may have singular phenomena like $\inf \Lambda = 0$, where Λ is the set of the eigenvalues of the $p(x)$ -Laplacian eigenvalue problem

$$-\operatorname{div}((\alpha + |\nabla u|)^{p(x)-2} \nabla u) = \lambda |u|^{p(x)-2} u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in R^N , $N \geq 1$, with a smooth boundary $\partial\Omega$. But the eigenvalue problem when $\alpha = 0$, $p(x) = p$ constant and $1 < p < \infty$ has no singular phenomena, i.e., $\inf \Lambda > 0$. It was proved in [9] that the eigenvalue problem when $\alpha = 0$, $p(x) = p$ constant and $1 < p < \infty$ has a nondecreasing sequence of nonnegative eigenvalues λ_j , obtained by the Ljusternik–Schnirelman principle, tending to ∞ as $j \rightarrow \infty$, and the corresponding orthonormalized eigenfunctions ϕ_j , $j = 1, 2, \dots$, where the first eigenvalue λ_1 is positive and simple and only eigenfunctions associated with λ_1 do not change sign, the set of eigenvalues is closed, the first eigenvalue λ_1 is isolated. Thus there are two sequences of eigenfunctions $(\phi_j)_j$ and $(\mu_j)_j$ corresponding to the eigenvalues λ_j such that the first eigenfunction $\phi_1 > 0$ in the sequence $(\phi_j)_j$ and the first eigenfunction $\mu_1 < 0$ in the sequence $(\mu_j)_j$.

Now in this paper, let ϕ_1 be the first positive orthonormalized eigenfunction corresponding to λ_1 .

In this paper we consider the multiplicity of solutions $u \in W^{1,p}(\Omega)$ for the following p -Laplacian Dirichlet boundary value problem with jumping nonlinearities when $N = 1$ and $p(x) = p$;

$$\begin{aligned} -(|u'|^{p-2}u')' &= b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega = (c, d) \subset R$, $c < d$, is an open interval, $2 \leq p < \infty$ and p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$, $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$, $s \in R$, $L^p(\Omega)$ is the p -Lebesgue space, with its dual space $L^{p'}(\Omega)$, and $W^{1,p}(\Omega)$ is the Lebesgue–Sobolev space.

Our problems are characterized as a jumping problem, which was first suggested in the suspension bridge equation as a model of the nonlinear oscillations in a differential equation

$$\begin{aligned} u_{tt} + K_1u_{xxxx} + K_2u^+ &= W(x) + \epsilon f(x, t), \\ u(0, t) = u(L, t) = 0, \quad u_{xx}(0, t) = u_{xx}(L, t) &= 0. \end{aligned} \tag{1.2}$$

This equation represents a bending beam supported by cables under a load f . The constant b represents the restoring force if the cables stretch. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression. Choi and Jung (cf. [2–4]) and McKenna and Walter (cf. [12]) investigated the existence and multiplicity of solutions for the single nonlinear suspension bridge equation with a Dirichlet boundary condition. In [1], the authors investigated the multiplicity of solutions of a semilinear equation

$$\begin{aligned} Au + bu^+ - au^- &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Omega, \end{aligned}$$

where Ω is a bounded domain in R^n , $n \geq 1$, with a smooth boundary $\partial\Omega$, and A is a second-order linear partial differential operator when the forcing term is a multiple $s\phi_1$, $s \in R$, of the positive eigenfunction and the nonlinearity crosses eigenvalues.

Our main theorems are as follows:

Theorem 1.1

- (i) If $2 \leq p < \infty$, $a < b$, $-\infty < a, b < \lambda_1$, then (1.1) has exactly one nontrivial solution for all s in a bounded interval. In particular, we have that if $2 \leq p < \infty$, $a < b$, $-\infty < a, b < \lambda_1$ and $s > 0$, then $u = (\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1 > 0$ is a solution and if $-\infty < a, b < \lambda_1$ and $s < 0$, then $u = (\frac{s}{\lambda_1 - a})^{\frac{1}{p-1}} \phi_1 < 0$ is a solution.
- (ii) If $2 \leq p < \infty$, $a < b$, $\lambda_j < a, b < \lambda_{j+1}$, $s \in R$ is bounded, $j = 1, 2, \dots$, (1.1) has exactly one nontrivial solution for all s in a bounded interval.

Theorem 1.2

- (i) If $1 < p < \infty$, $a < b$, $-\infty < a < \lambda_1 < b < \lambda_2$, $s \in R$ and $s > 0$, then (1.1) has no solution,
- (ii) If $1 < p < \infty$, $a < b$, $-\infty < a < \lambda_1 < b < \lambda_2$, $s \in R$ and $s = 0$, then (1.1) has exactly one solution $u = 0$.
- (iii) If $2 \leq p < \infty$, $a < b$, $-\infty < a < \lambda_1 < b < \lambda_2$, $s \in R$, then there exists $s_1 < 0$ such that for any s with $s_1 \leq s < 0$, (1.1) has exactly two solutions.

Theorem 1.3

- (i) If $1 < p < \infty$, $a < b$, $-\infty < a < \lambda_1, \lambda_2 < b < \lambda_3$, $s \in R$ and $s > 0$, then (1.1) has no solution,
- (ii) If $1 < p < \infty$, $a < b$, $-\infty < a < \lambda_1, \lambda_2 < b < \lambda_3$, $s \in R$ and $s = 0$, then (1.1) has exactly one solution $u = 0$.
- (iii) If $2 \leq p < \infty$, $a < b$, $-\infty < a < \lambda_1, \lambda_2 < b < \lambda_3$, $s \in R$, then there exists $s_1 < 0$ such that for any s with $s_1 \leq s < 0$, (1.1) has at least three solutions.

For the proofs of (i) and (ii) of Theorem 1.1, (iii) of Theorem 1.2 and (iii) of Theorem 1.3, we use the contraction mapping principle under the condition $p \geq 2$ and direct computations. The outline of the proofs of Theorems 1.1, 1.2 and 1.3 is as follows: In Sect. 2, we introduce some preliminaries and prove Theorem 1.1 by direct computations using the first eigenfunction and the contraction mapping principle. In Sect. 3, we prove Theorem 1.2 by a similar method to that of Theorem 1.1. In Sect. 4, we prove Theorem 1.3 by direct computations using the first eigenfunction, contraction mapping principle and Leray–Schauder degree theory under the condition $p \geq 2$.

2 Preliminaries and proof of Theorem 1.1

Let $L^p(\Omega)$ be the Lebesgue space defined by

$$L^p(\Omega) = \left\{ u \mid u : \Omega \rightarrow R \text{ is measurable, } \int_c^d |u|^p dx < \infty \right\}$$

and $W^{1,p}(\Omega)$ be the Lebesgue–Sobolev space defined by

$$W^{1,p}(\Omega) = \{ u \in L^p(\Omega) \mid u'(x) \in L^p(\Omega) \}.$$

We introduce norms on $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively, by

$$\|u\|_{L^p(\Omega)} = \inf \left\{ \lambda > 0 \mid \int_c^d \left| \frac{u(x)}{\lambda} \right|^p \leq 1 \right\},$$

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_c^d |u'(x)|^p dx \right]^{\frac{1}{p}} + \left[\int_c^d |u(x)|^p dx \right]^{\frac{1}{p}}.$$

Let us define the operator $-L_p$ by

$$-L_p u = -(|u'|^{p-2} u')'.$$

We first consider the problem:

$$\begin{aligned} -(|u'|^{p-2} u')' &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \partial\Omega, \end{aligned} \tag{2.1}$$

where $1 < p < \infty$ and $f \in L^r(\Omega)$, $r > 1$. Then (2.1) has a unique solution $u \in C^1(\bar{\Omega})$ which is of the form

$$u(x) = \int_c^x g_p^{-1} \left(c_f - \int_a^y f(\tau) d\tau \right) dy,$$

where $g_p(t) = |t|^{p-2}t$ for $t \neq 0$, $g_p(0) = 0$ and its inverse g_p^{-1} is $g_p^{-1}(t) = t^{\frac{1}{p-1}}$ if $t > 0$ and $g_p^{-1}(t) = -|t|^{\frac{1}{p-1}}$ if $t < 0$, and c_f is the unique constant such that $u(d) = 0$. By [10, Lemma 2.1] or [11, Lemma 4.2], the solution operator S is such that $S : L^p(\Omega) \rightarrow C^1(\bar{\Omega})$ is continuous and, by [5, Corollary 2.3], the embedding $S : L^p(\Omega) \rightarrow C(\bar{\Omega})$ is continuous and compact. By [8], we also have a Poincaré-type inequality.

Lemma 2.1 *Let $1 < p < \infty$. Then the embedding*

$$H^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

is continuous and compact, and for every $u \in C_0^\infty(\Omega)$ we have

$$\|u\|_{L^p(\bar{\Omega})} \leq C \|u\|_{W^{1,p}(\bar{\Omega})},$$

for a positive constant C independent of u .

By Lemma 2.1, we obtain the following:

Lemma 2.2 *Assume that $1 < p < \infty$, $f(x, u) \in L^p(\Omega)$. Then the solutions of the problem*

$$\begin{aligned} -(|u'|^{p-2} u')' &= f(x, u) \quad \text{in } L^p(\Omega), \\ u &= 0 \quad \partial\Omega \end{aligned}$$

belong to $W^{1,p}(\Omega)$.

For given $v(u) \in L^p(\Omega)$ and $h(x) \in L^p(\Omega)$, the equation

$$-L_p u = v(u) + h(x) \quad \text{in } L^p(\Omega)$$

is equivalent to the equation

$$u = (-L_p)^{-1}(v(u) + h(x)).$$

We observe that

$$\begin{aligned} \|(-L_p)^{-1}(v(u) + h(x))\|_{W^{1,p}(\Omega)} &\leq \frac{1}{\lambda_1} \|g_p^{-1}(v(u) + h(x))\|_{W^{1,p}(\Omega)} \\ &\leq \frac{1}{\lambda_1} \|(v(u) + h(x))\|_{L^p(\Omega)}^{p-1}. \end{aligned} \tag{2.2}$$

Proof of Theorem 1.1 (i) We assume that $-\infty < a, b < \lambda_1$. Let us choose $\mu > 0$ and $\epsilon > 0$ so that $-\mu + \epsilon < a, b < \lambda_1 - \epsilon$, and choose $\tau = \frac{a+b}{2}$. Then problem (1.1) can be rewritten as

$$-L_p - \tau |u|^{p-2}u = (b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- + s\phi_1^{p-1},$$

or equivalently,

$$u = (-L_p - \tau g_p)^{-1}((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- + s\phi_1^{p-1}). \tag{2.3}$$

We have that

$$\begin{aligned} &\|(-L_p - \tau g_p)^{-1}((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- + s\phi_1^{p-1})\|_{L^p(\Omega)} \\ &\leq \left\| \frac{1}{\lambda_1 - \tau} g_p^{-1}((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- + s\phi_1^{p-1}) \right\|_{L^p(\Omega)}. \end{aligned}$$

Let us set the right-hand side of (2.3) as

$$T(u) = (-L_p - \tau g_p)^{-1}((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- + s\phi_1^{p-1}).$$

Then $T(u)$ satisfies

$$\begin{aligned} &\|T(u) - T(v)\|_{L^p(\Omega)} \\ &= \|(-L_p - \tau g_p)^{-1}((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- + s\phi_1^{p-1}) \\ &\quad - (-L_p - \tau g_p)^{-1}((b - \tau)|v|^{p-2}v^+ - (a - \tau)|v|^{p-2}v^- + s\phi_1^{p-1})\|_{L^p(\Omega)} \\ &\leq \frac{1}{\lambda_1 - \tau} \|g_p^{-1}((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- + s\phi_1^{p-1}) \\ &\quad - g_p^{-1}((b - \tau)|v|^{p-2}v^+ - (a - \tau)|v|^{p-2}v^- + s\phi_1^{p-1})\|_{L^p(\Omega)} \\ &= \frac{1}{\lambda_1 - \tau} \left\| g_p^{-1}\left(\frac{b-a}{2}|u|^{p-1} + s\phi_1^{p-1}\right) - g_p^{-1}\left(\frac{b-a}{2}|v|^{p-1} + s\phi_1^{p-1}\right) \right\|_{L^p(\Omega)}. \end{aligned}$$

Since $|\cdot|^{p-1}$ is continuous, for given $u \in W^{1,p}(\Omega)$, there exists $v \in W^{1,p}(\Omega)$ in a small neighborhood of u such that $\frac{b-a}{2}|u|^{p-1} + s\phi_1^{p-1}$ and $\frac{b-a}{2}|v|^{p-1} + s\phi_1^{p-1}$ have the same sign. We can also check that for any $2 \leq p < \infty, s < 0$ and for any u and v , where v is in a small neighborhood of u such that $\frac{b-a}{2}|u|^{p-1} + s\phi_1^{p-1}$ and $\frac{b-a}{2}|v|^{p-1} + s\phi_1^{p-1}$ have the same sign, we have

$$\begin{aligned} &\left\| g_p^{-1}\left(\frac{b-a}{2}|u|^{p-1} + s\phi_1^{p-1}\right) - g_p^{-1}\left(\frac{b-a}{2}|v|^{p-1} + s\phi_1^{p-1}\right) \right\|_{L^p(\Omega)} \\ &\leq \left\| g_p^{-1}\left(\frac{b-a}{2}|u|^{p-1}\right) - g_p^{-1}\left(\frac{b-a}{2}|v|^{p-1}\right) \right\|_{L^p(\Omega)}. \end{aligned}$$

Thus for any u and v , where v is in a small neighborhood of u such that $\frac{b-a}{2}|u|^{p-1} + s\phi_1^{p-1}$ and $\frac{b-a}{2}|v|^{p-1} + s\phi_1^{p-1}$ have the same sign, we have

$$\begin{aligned} & \|T(u) - T(v)\|_{L^p(\Omega)} \\ & \leq \frac{1}{\lambda_1 - \tau} \left\| g_p^{-1} \left(\frac{b-a}{2}|u|^{p-1} + s\phi_1^{p-1} \right) - g_p^{-1} \left(\frac{b-a}{2}|v|^{p-1} + s\phi_1^{p-1} \right) \right\|_{L^p(\Omega)} \\ & \leq \frac{1}{\lambda_1 - \tau} \left(\frac{b-a}{2} \right)^{\frac{1}{p-1}} \|g_p^{-1}(u^{p-1}) - g_p^{-1}(|v|^{p-1})\|_{L^p(\Omega)} \\ & = \frac{\left(\frac{b-a}{2}\right)^{\frac{1}{p-1}}}{\lambda_1 - \tau} \| |u| - |v| \|_{L^p(\Omega)} \leq \frac{\left(\frac{b-a}{2}\right)^{p-1}}{\lambda_1 - \tau} \|u - v\|_{L^p(\Omega)}. \end{aligned}$$

Since $\lambda_1 - \tau > \frac{b-a}{2}$ and $2 \leq p < \infty$, we have $\frac{\left(\frac{b-a}{2}\right)^{\frac{1}{p-1}}}{\lambda_1 - \tau} < 1$. Thus T is a contraction mapping on $L^p(\Omega)$. Thus T has a unique solution in $L^p(\Omega)$. Thus (1.1) has a unique solution.

(ii) We assume that $\lambda_j < a, b < \lambda_{j+1}, j = 1, 2, \dots$. Let us choose $\epsilon > 0$ so that $\lambda_j + \epsilon < a, b < \lambda_{j+1} - \epsilon$. Let us set $\tau = \frac{a+b}{2}$. Then (1.1) can be rewritten as

$$u = (-L_p - \tau g_p)^{-1} \left((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- + s\phi_1^{p-1} \right).$$

We observe that

$$\begin{aligned} & \|(-L_p - \tau g_p)^{-1} \left((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- \right)\|_{L^p(\Omega)} \\ & \leq \left\| \left(\frac{2}{\lambda_{j+1} - \lambda_j} \right)^{-1} g_p^{-1} \left((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- \right) \right\|_{L^p(\Omega)}. \end{aligned}$$

By the same process as that of the proof of Theorem 1.1(i), the mapping on $L^p(\Omega)$ given by

$$N(u) = (-L_p - \tau g_p)^{-1} \left((b - \tau)|u|^{p-2}u^+ - (a - \tau)|u|^{p-2}u^- + s\phi_1^{p-1} \right)$$

satisfies

$$\begin{aligned} & \|N(u) - N(v)\|_{L^p(\Omega)} \\ & = \|(-L_p - \tau g_p)^{-1} \left((b - \tau)(|u|^{p-2}u^+ - |v|^{p-2}v^+) - (a - \tau)(|u|^{p-2}u^- - |v|^{p-2}v^-) + s\phi_1^{p-1} \right)\|_{L^p(\Omega)} \\ & = \frac{2}{\lambda_{j+1} - \lambda_j} \left\| g_p^{-1} \left(\frac{b-a}{2}(|u|^{p-1} + s\phi_1^{p-1}) \right) - g_p^{-1} \left(\frac{b-a}{2}(|v|^{p-1} + s\phi_1^{p-1}) \right) \right\|_{L^p(\Omega)} \\ & \leq \frac{2}{\lambda_{j+1} - \lambda_j} \left\| g_p^{-1} \left(\frac{b-a}{2}(|u|^{p-1}) \right) - g_p^{-1} \left(\frac{b-a}{2}(|v|^{p-1}) \right) \right\|_{L^p(\Omega)} \\ & \leq \frac{2}{\lambda_{j+1} - \lambda_j} \left(\frac{b-a}{2} \right)^{\frac{1}{p-1}} \| |u| - |v| \|_{L^p(\Omega)} \leq \frac{2}{\lambda_{j+1} - \lambda_j} \left(\frac{b-a}{2} \right)^{\frac{1}{p-1}} \|u - v\|_{L^p(\Omega)} \end{aligned}$$

for any u and v , where v is in a small neighborhood of u such that $\frac{b-a}{2}|u|^{p-1} + s\phi_1^{p-1}$ and $\frac{b-a}{2}|v|^{p-1} + s\phi_1^{p-1}$ have the same sign, and $2 \leq p < \infty$. Since $\frac{\lambda_{j+1} - \lambda_j}{2} > \frac{b-a}{2}$ and $2 \leq p < \infty$, we

have $\frac{2}{\lambda_{j+1}-\lambda_j}(\frac{b-a}{2})^{\frac{1}{p-1}} < 1$. It follows that N is a contraction mapping on $L^p(\Omega)$. Therefore N has a unique solution in $L^p(\Omega)$. Thus (1.1) has a unique solution. \square

3 Proof of Theorem 1.2

(i) (For the case $s > 0$) We assume that $1 < p < \infty$, $a < b$, $-\infty < a < \lambda_1 < b < \lambda_2$ and $s > 0$. Then (3.1) can be rewritten as

$$-(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u = (b - \lambda_1)|u|^{p-2}u^+ - (a - \lambda_1)|u|^{p-2}u^- + s\phi_1^{p-1}. \tag{3.1}$$

Taking the inner product with ϕ_1 , we have

$$\langle -(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u, \phi_1 \rangle = \langle (b - \lambda_1)|u|^{p-2}u^+ - (a - \lambda_1)|u|^{p-2}u^- + s\phi_1^{p-1}, \phi_1 \rangle. \tag{3.2}$$

The left-hand side of (3.1) is equal to 0. On the other hand, the right-hand side of (3.1) is positive because $b - \lambda_1 > 0$, $-(a - \lambda_1) > 0$ and $s\phi_1^{p-1} > 0$ for $s > 0$. Thus if $s > 0$, then there is no solution for (1.1).

(ii) (For the case $s = 0$) If $s = 0$, then (3.2) is reduced to the equation

$$\langle -(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u, \phi_1 \rangle = \langle (b - \lambda_1)|u|^{p-2}u^+ - (a - \lambda_1)|u|^{p-2}u^-, \phi_1 \rangle,$$

i.e.,

$$\begin{aligned} & \int_c^d [-(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u]\phi_1 dx \\ & = 0 = \int_c^d [(b - \lambda_1)|u|^{p-2}u^+ - (a - \lambda_1)|u|^{p-2}u^-]\phi_1 dx. \end{aligned} \tag{3.3}$$

Since $b - \lambda_1 > 0$ and $-(a - \lambda_1) > 0$, the only possibility for (3.3) to hold is that $u = 0$.

(iii) (For the case $s < 0$) We assume that $2 \leq p < \infty$, $a < b$, $-\infty < a < \lambda_1 < b < \lambda_2$ and $s < 0$. Let V be a subspace of $L^p(\Omega)$ spanned by ϕ_1 and W be the orthogonal complement of V in $L^p(\Omega)$. Then

$$L^p(\Omega) = V \oplus W.$$

Let P be a orthogonal projection in $L^p(\Omega)$ onto V and $I - P$ be the orthogonal projection onto W . Then

$$Pu = \left(\int_c^d u\phi_1 \right) \phi_1 \quad \text{for all } u \in L^p(\Omega).$$

Let $u \in L^p(\Omega)$. Then u can be written as

$$u = v + w, \quad v = Pu, \quad w = (I - P)u.$$

We note that P commutes with $D = \frac{d}{dx}$. Thus (1.1) is equivalent to a pair of equations

$$\begin{aligned} & (I - P)(-(|v + w'|^{p-2}(v + w)')') \\ & = (I - P)(b|v + w|^{p-2}(v + w)^+ - a|v + w|^{p-2}(v + w)^-), \end{aligned} \tag{3.4}$$

$$\begin{aligned}
 &P(-(|(v+w)'|^{p-2}(v+w)')') \\
 &= P(b|v+w|^{p-2}(v+w)^+ - a|v+w|^{p-2}(v+w)^- + s\phi_1^{p-1}). \tag{3.5}
 \end{aligned}$$

We claim that for fixed $v \in V$, (3.4) has a unique solution $w(v)$ when $2 \leq p \leq \infty$. In fact, we suppose that (3.4) has two solutions w_1, w_2 for fixed $v \in V$. Let us set $\alpha_v(w) = b|v+w|^{p-2}(v+w)^+ - a|v+w|^{p-2}(v+w)^-$. Then we have

$$\begin{aligned}
 &(I - P)[(-(|(v+w_1)'|^{p-2}(v+w_1)')') - (-(|(v+w_2)'|^{p-2}(v+w_2)')')]) \\
 &= (I - P)[(b|v+w_1|^{p-2}(v+w_1)^+ - a|v+w_1|^{p-2}(v+w_1)^- \\
 &\quad - (b|v+w_2|^{p-2}(v+w_2)^+ - a|v+w_2|^{p-2}(v+w_2)^-)]. \tag{3.6}
 \end{aligned}$$

Taking the inner product of (3.6) with $w_1 - w_2$, we have

$$\begin{aligned}
 &\langle (I - P)[(-(|(v+w_1)'|^{p-2}(v+w_1)')') - (-(|(v+w_2)'|^{p-2}(v+w_2)')')], w_1 - w_2 \rangle \\
 &= \langle (I - P)[(b|v+w_1|^{p-2}(v+w_1)^+ - a|v+w_1|^{p-2}(v+w_1)^- \\
 &\quad - (b|v+w_2|^{p-2}(v+w_2)^+ - a|v+w_2|^{p-2}(v+w_2)^-)], w_1 - w_2 \rangle. \tag{3.7}
 \end{aligned}$$

The right-hand side of (3.7) is equal to

$$\begin{aligned}
 &\langle (I - P)[(b|v+w_1|^{p-2}(v+w_1)^+ - a|v+w_1|^{p-2}(v+w_1)^- \\
 &\quad - (b|v+w_2|^{p-2}(v+w_2)^+ - a|v+w_2|^{p-2}(v+w_2)^-)], w_1 - w_2 \rangle \\
 &\leq (p-1)b \int_c^d [I - P] |v+w_1 + \theta(w_2-w_1)|^{p-2} (w_1-w_2)^2 dx \tag{3.8}
 \end{aligned}$$

for $0 < \theta < 1$. On the other hand, the left-hand side of (3.6) is equal to

$$\begin{aligned}
 &\langle (I - P)[(-(|(v+w_1)'|^{p-2}(v+w_1)')') - (-(|(v+w_2)'|^{p-2}(v+w_2)')')], w_1 - w_2 \rangle \\
 &= (p-1) \int_c^d [(I - P)(|(v+w_1)' + \theta(w_2-w_1)'|^{p-2} ((w_1-w_2)')^2)] dx \\
 &\geq (p-1)\lambda_2 \int_c^d [(I - P)(|(v+w_1) + \theta(w_2-w_1)|^{p-2} (w_1-w_2)^2)] dx, \tag{3.9}
 \end{aligned}$$

by mean value theorem. On the other hand, by (3.8) and (3.9), we have

$$\begin{aligned}
 &(p-1)\lambda_2 \int_c^d [(I - P)(|(v+w_1) + \theta(w_2-w_1)|^{p-2} (w_1-w_2)^2)] dx \\
 &\leq (p-1)b \int_c^d [I - P] |v+w_1 + \theta(w_2-w_1)|^{p-2} (w_1-w_2)^2 dx,
 \end{aligned}$$

which is a contradiction because $b < \lambda_2$. Thus $w_1 = w_2$. Thus for fixed $v \in V$, every solution of (3.4) is unique. We note that $w = 0$ is a solution of (3.4) for every $v \in V = PH, v > 0$ or $v < 0$ everywhere in Ω . If $v > 0$, then

$$(I - P)(-(|(v+w)'|^{p-2}(v+w)')') = (I - P)(b|v|^{p-2}v^+ - a|v|^{p-2}v^- + s\phi_1^{p-1}). \tag{3.10}$$

If $v > 0$, then $(I - P)(b|v|^{p-2}v^+ - a|v|^{p-2}v^- + s\phi_1^{p-1}) = (I - P)b|v|^{p-2}v = 0$. If $v < 0$, then $(I - P)(b|v|^{p-2}v^+ - a|v|^{p-2}v^- + s\phi_1^{p-1}) = (I - P)(a|v|^{p-2}v + s\phi_1^{p-1}) = 0$. Thus (1.1) is reduced to

$$P(-(|v'|^{p-2}v')') = P(b|v|^{p-2}v^+ - a|v|^{p-2}v^- + s\phi_1^{p-1}),$$

where $v = c\phi_1, c \in R$.

If $c > 0$, then

$$\lambda c^{p-1} = bc^{p-1} + s, \quad c^{p-1} = \frac{s}{\lambda_1 - b}.$$

If $c < 0$, then

$$\lambda |c|^{p-2}c = a|c|^{p-2}c + s, \quad |c|^{p-2}c = \frac{s}{\lambda_1 - a}.$$

Thus (1.1) has exactly two solutions.

4 Proof of Theorem 1.3

Lemma 4.1 (A priori bound) *Assume that $1 < p < \infty, -\infty < a < \lambda_1, \lambda_2 < b < \lambda_3, s \in R$. Then there exist $s_1 < 0, s_2 > 0$ and a constant $C > 0$ depending only on a and b such that for any s with $s_1 \leq s \leq s_2$, any solution u of (1.1) satisfies $\|u\|_{W^{1,p}(\Omega)} < C$.*

Proof Let u be any solution of (1.1). Suppose that any solution of (1.1) is not bounded. Then there exists a sequence $(u_n)_n$ such that $\|u_n\|_{W^{1,p}(\Omega)} \rightarrow \infty$ so that

$$-(|u'_n|^{p-2}u'_n)' = b|u_n|^{p-2}u_n^+ - a|u_n|^{p-2}u_n^- + s\phi_1^{p-1} \quad \text{in } \Omega, \tag{4.1}$$

or equivalently,

$$u_n = (-L_p)^{-1}(b|u_n|^{p-2}u_n^+ - a|u_n|^{p-2}u_n^- + s\phi_1^{p-1}) \quad \text{in } \Omega.$$

Let us set $w_n = \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}}$. Then $(w_n)_n$ is bounded, so by passing to a subsequence if necessary, which we still denote by $(w_n)_n$, we get that $(w_n)_n \rightarrow w$ weakly for some w in $W^{1,p}(\Omega)$. Dividing (4.1) by $\|u_n\|_{W^{1,p}(\Omega)}^{p-1}$, we have

$$\frac{-(|u'_n|^{p-2}u'_n)' }{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} = b \frac{|u_n|^{p-2}u_n^+}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} - a \frac{|u_n|^{p-2}u_n^-}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} + \frac{s\phi_1^{p-1}}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} \quad \text{in } \Omega, \tag{4.2}$$

i.e.,

$$w_n = (-L_p)^{-1} \left(b|w_n|^{p-2}w_n^+ - a|w_n|^{p-2}w_n^- + \frac{s\phi_1^{p-1}}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} \right) \quad \text{in } \Omega.$$

Since, by Lemma 2.1, the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, and $(-L_p)^{-1}$ is compact operator, $(w_n)_n \rightarrow w$ strongly in $W^{1,p}(\Omega)$. Taking the limit of (4.2) as $n \rightarrow \infty$, we have

$$-(|w'|^{p-2}w')' = b|w|^{p-2}w^+ - a|w|^{p-2}w^-. \tag{4.3}$$

By Theorem 1.1(i), (4.3) has only the trivial solution, which is absurd, because $\|w\|_{W^{1,p}(\Omega)} = 1$. Thus the lemma is proved. \square

We shall consider the Leray–Schauder degree on a large ball.

Lemma 4.2 *Assume that $1 < p < \infty$, $-\infty < a < \lambda_1$, $\lambda_2 < b < \lambda_3$. Then there exist a constant $R > 0$ depending on $a, b, s, s_1 < 0$ and $s_2 > 0$ such that for any s with $s_1 \leq s \leq s_2$, the Leray–Schauder degree*

$$d_{LS}(u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_R(0), 0) = 0,$$

where $-L_p u = -(|u|^{p-2}u)'$.

Proof Let us consider the homotopy

$$F(x, u) = u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}).$$

By Theorem 1.3(ii), for any $s > 0$, (1.1) has no solution. Thus there exist $s_2 > 0$ and a large $R > 0$ such that (4.3) has no zero in $B_R(0)$ for any $s \geq s_2$, and by the a priori bound in Lemma 4.1, there exists $s_1 < 0$ such that for any s with $s_1 \leq s \leq s_2$, all solutions of

$$u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}) = 0$$

satisfy $\|u\|_{W^{1,p}(\Omega)} \leq R$ and (4.3) has no zero on ∂B_R for any s with $s_1 \leq s \leq s_2$. Since

$$d_{LS}(u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s_2\phi_1^{p-1}), B_R(0), 0) = 0,$$

by homotopy arguments, for any any s with $s_1 \leq s \leq s_2$, we have

$$\begin{aligned} & d_{LS}(u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_R(0), 0) \\ &= d_{LS}(u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1} + \lambda(s_2 - s)\phi_1^{p-1}), B_R(0), 0) \\ &= d_{LS}(u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s_2\phi_1^{p-1}), B_R(0), 0) = 0, \end{aligned}$$

for any $0 \leq \lambda \leq 1$. Thus the lemma is proved. \square

Lemma 4.3 *Let K be a compact set in $L^p(\Omega)$. Let $\xi > 0$ a.e. Then there exists a modulus of continuity $\alpha : R \rightarrow R$ depending only on K and ξ such that*

$$\left\| \left(|\tau| - \frac{\xi}{\eta} \right)^+ \right\|_{L^p(\Omega)} \leq \alpha(\eta) \quad \text{for all } \tau \in K.$$

It follows that

$$\|(\eta\tau + \xi)^-\|_{L^p(\Omega)} \leq \eta\alpha(\eta)$$

and

$$\|(\eta\tau - \xi)^+\|_{L^p(\Omega)} \leq \eta\alpha(\eta) \quad \text{for all } \tau \in K.$$

Proof For any $\tau \in K$, let $\tau_n = (|\tau| - \frac{\xi}{n})^+$. Since $0 \leq \tau_n \leq |\tau|$ and $\tau_n(x) \rightarrow 0$ as $n \rightarrow \infty$ a.e., it follows that $\|\tau_n\|_{L^p(\Omega)} \rightarrow 0$ for all $\tau \in K$. We claim that, for a given $\epsilon > 0$, there exists $\delta > 0$ such that if $\tau \in K$, then $\|\tau_n\|_{L^p(\Omega)} \leq 2\epsilon$ for all $n \in [0, \delta]$. Choose $\{\tau_i, i = 1, \dots, N\}$ as an ϵ net for K . Choose δ so that $\|(\tau_i)_\delta\|_{L^p(\Omega)} < \epsilon$ for $i = 1, \dots, N$. Then for any $\tau \in K$, there exists τ_k , α , $\|\alpha\|_{L^p(\Omega)} < \epsilon$ that $\tau = \tau_k + \alpha$. Since $(u + v)^+ \leq u^+ + v^+$, we have $\|\tau_\delta\|_{L^p(\Omega)} \leq (\tau_k)_\delta + \|\alpha\|_{L^p(\Omega)}$ and therefore $\|\tau_n\|_{L^p(\Omega)} \leq \|\tau_\delta\|_{L^p(\Omega)} + \|\alpha\|_{L^p(\Omega)} \leq 2\epsilon$. \square

Lemma 4.4 *Assume that $1 < p < \infty$, $-\infty < a < \lambda_1, \lambda_2 < b < \lambda_3$. Then there exist a constant $R > 0$ depending on a, b, s and $s_1 < 0$ such that for $s_1 \leq s < 0$, the Leray–Schauder degree*

$$d_{LS}(u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_{R_0}(u_0), 0) = 1,$$

where $u_0 = (\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1 > 0$ is a positive solution of (1.1).

Proof Let us set $M = (-L_p - bg_p)^{-1}$. Then (1.1) can be rewritten as

$$(-L_p - bg_p)(u) = b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u + s\phi_1^{p-1},$$

or equivalently,

$$u = M\left(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u + s\frac{\phi_1}{\lambda_1 - b}\right) = Tu, \tag{4.4}$$

where $M = (-L_p - bg_p)^{-1}$. The operator M is compact on $L^p(\Omega)$, and the set $K = M(\bar{B})$, where \bar{B} is the closed unit ball in $L^p(\Omega)$. Then K is a compact set. Let us set $\gamma = \min\{b - \lambda_2, \lambda_3 - b\}$. We can observe that $\|M(u)\|_{L^p(\Omega)} \leq \frac{1}{\gamma} \|g_p^{-1}(u)\|_{L^p(\Omega)}$. Let α be the modulus continuity given in Lemma 4.3 corresponding to K and $\xi = M(s\phi_1^{p-1}) = (\frac{s\phi_1}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1$, and choose $\epsilon > 0$ so that

$$\alpha\left(\left(\epsilon(b - a)\right)^{\frac{1}{p-1}} + \gamma\right) \leq \frac{\gamma}{4(b - a)^{\frac{1}{p-1}}\left(\left(b - a\right)^{\frac{1}{p-1}} + \gamma\right)}. \tag{4.5}$$

We have

$$\|b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u\|_{L^p(\Omega)} \leq (b - a)\| |u|^{p-2}u^-\|_{L^p(\Omega)}. \tag{4.6}$$

For $u \in (\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1 + (|s|\epsilon v)^{\frac{1}{p-1}}$ with $v \in \bar{B}$,

$$\|u^-\|_{L^p(\Omega)} = \left\| \left(\left(\frac{s}{\lambda_1 - b} \right)^{\frac{1}{p-1}} \phi_1 + (|s|\epsilon v)^{\frac{1}{p-1}} \right)^-\right\|_{L^p(\Omega)} \leq \|((|s|\epsilon v)^{\frac{1}{p-1}})^-\|_{L^p(\Omega)} \leq |s|\epsilon$$

since $(\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1 > 0$. Then $T(u) = M(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u + s\frac{\phi_1}{\lambda_1 - b})$ can be rewritten as

$$T(u) = \left(\frac{s}{\lambda_1 - b}\right)^{\frac{1}{p-1}} \phi_1 + (|s|\epsilon)^{\frac{1}{p-1}} \left((b - a)^{\frac{1}{p-1}} + \gamma\right)v, \quad v \in K.$$

If u is a solution of (4.4), then $u = Tu$ and, by Lemma 4.3,

$$\begin{aligned} \|u^-\|_{L^p(\Omega)} &= \left\| \left(\frac{s}{\lambda_1 - b} \right)^{\frac{1}{p-1}} \phi_1 + (|s|\epsilon)^{\frac{1}{p-1}} ((b-a)^{\frac{1}{p-1}} + \gamma)v^- \right\|_{L^p(\Omega)} \\ &\leq (|s|\epsilon)^{\frac{1}{p-1}} ((b-a)^{\frac{1}{p-1}} + \gamma)\alpha(\epsilon^{\frac{1}{p-1}} ((b-a)^{\frac{1}{p-1}} + \gamma)) \\ &< \frac{\gamma(|s|\epsilon)^{\frac{1}{p-1}}}{4(b-a)^{\frac{1}{p-1}}}. \end{aligned}$$

It follows that

$$\begin{aligned} \|M(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u)\|_{L^p(\Omega)} &\leq (b-a)^{\frac{1}{p-1}} \|u^-\|_{L^p(\Omega)} \\ &\leq \frac{1}{4} (|s|\epsilon)^{\frac{1}{p-1}} \leq \frac{1}{4} |s|\epsilon. \end{aligned}$$

Thus we have shown that any solution $u \in (\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1 + |s|\epsilon \bar{B}$ of (4.4) belongs to $(\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1 + \frac{1}{4}|s|\epsilon \bar{B}$. This estimate holds if we replace $b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u$ by $\lambda(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u)$ with $0 \leq \lambda \leq 1$. Thus the equation

$$u = (-L_p)^{-1}(s\phi_1^{p-1} + b|u|^{p-2}u + \lambda(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u))$$

has no solution on the boundary of the ball $B_{\epsilon|s|}((\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1)$ for $0 \leq \lambda \leq 1$. By the homotopy invariance degree,

$$\begin{aligned} d_{LS}(u - (-L_p)^{-1}(s\phi_1^{p-1} + b|u|^{p-2}u + \lambda(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u)), \\ B_{\epsilon|s|}\left(\left(\frac{s}{\lambda_1 - b}\right)^{\frac{1}{p-1}} \phi_1, 0\right) \end{aligned}$$

is defined for $0 \leq \lambda \leq 1$ and is independent of λ . For $\lambda = 0$,

$$d_{LS}(u - (-L)^{-1}(s\phi_1^{p-1} + b|u|^{p-2}u, B_{\epsilon|s|}\left(\left(\frac{s}{\lambda_1 - b}\right)^{\frac{1}{p-1}} \phi_1, 0\right)) = (-1) \times (-1) = +1,$$

since $u = (\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1$ is the unique solution of the equation, and, since there are 2 eigenvalues λ_1, λ_2 of $-L_p$ to the left of b , the operator $I - b(-L_p)^{-1}$ has two negative eigenvalues, while all the rest are positive. When $\lambda = 1$, we have

$$\begin{aligned} d_{LS}((u - (-L_p)^{-1}(s\phi_1^{p-1} + b|u|^{p-2}u^+ + 1(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u)), \\ B_{\epsilon|s|}\left(\left(\frac{s}{\lambda_1 - b}\right)^{\frac{1}{p-1}} \phi_1, 0\right)) \\ = d_{LS}(s\phi_1^{p-1} + b|u|^{p-2}u^+ - a|u|^{p-2}u^-, B_{\epsilon|s|}\left(\left(\frac{s}{\lambda_1 - b}\right)^{\frac{1}{p-1}} \phi_1, 0\right)). \end{aligned}$$

Thus by the homotopy invariance of degree, we have

$$\begin{aligned} & d_{LS} \left(s\phi_1^{p-1} + b|u|^{p-2}u^+ - a|u|^{p-2}u^-, B_{\epsilon|s|} \left(\left(\frac{s}{\lambda_1 - b} \right)^{\frac{1}{p-1}} \phi_1, 0 \right) \right) \\ &= d_{LS} \left(u - (-L)^{-1} \left(s\phi_1^{p-1} + b|u|^{p-2}u, B_{\epsilon|s|} \left(\frac{s}{\lambda_1 - b} \right)^{\frac{1}{p-1}} \phi_1 \right), 0 \right) = 1. \end{aligned}$$

Thus the lemma is proved. □

Lemma 4.5 *Assume that $1 < p < \infty$, $-\infty < a < \lambda_1$, $\lambda_2 < b < \lambda_3$ and $s_1 < 0$. Then there exist a constant $\epsilon > 0$ depending on a, b, s such that for $s_1 \leq s < 0$, the Leray–Schauder degree*

$$d_{LS}(u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_{|s|\epsilon}(u_1), 0) = 1,$$

where $u_1 = -(\frac{s}{a-\lambda_1})^{\frac{1}{p-1}} \phi_1 < 0$ is a negative solution of (1.1).

Proof We can prove this lemma by an almost identical argument as that of Lemma 4.4. □

Proof of Theorem 1.3 The proofs of Theorem 1.3(i)–(ii) are the same as those of Theorem 1.2(i)–(ii).

(iii) By Lemmas 4.4 and 4.5, there is a solution $(\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1 > 0$ in $B_{|s|\epsilon}((\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1)$ and a solution $-(\frac{s}{a-\lambda_1})^{\frac{1}{p-1}} \phi_1 < 0$ in $B_{|s|\epsilon}(-(\frac{s}{a-\lambda_1})^{\frac{1}{p-1}} \phi_1)$. We may assume that $\epsilon < (\frac{1}{b-\lambda_1})^{\frac{1}{p-1}}$ and $\epsilon < (\frac{1}{\lambda_1 - a})^{\frac{1}{p-1}}$. Then these two balls are disjoint. This gives two solutions for any n . There is a large ball B_R centered at the origin and containing $B_{|s|\epsilon}((\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1)$ and $B_{|s|\epsilon}(-(\frac{s}{a-\lambda_1})^{\frac{1}{p-1}} \phi_1)$. Since

$$\begin{aligned} & d_{LS}(u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_R(0), 0) = 0, \\ & d_{LS} \left(u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_R \left(\left(\frac{s}{\lambda_1 - b} \right)^{\frac{1}{p-1}} \phi_1, 0 \right) \right) \\ &= d_{LS} \left(u - (-L_p)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_R \left(- \left(\frac{s}{a - \lambda_1} \right)^{\frac{1}{p-1}} \phi_1, 0 \right) \right) = 1, \end{aligned}$$

we have

$$\begin{aligned} & d_{LS}(u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), \\ & B_R(0) \setminus \left(B_{|s|\epsilon} \left(\left(\frac{s}{\lambda_1 - b} \right)^{\frac{1}{p-1}} \phi_1 \right) \cup B_{|s|\epsilon} \left(- \left(\frac{s}{\lambda_1 - a} \right)^{\frac{1}{p-1}} \phi_1 \right), 0 \right) \\ &= -2. \end{aligned}$$

Thus there exists a third solution in $B_R(0) \setminus (B_{|s|\epsilon}((\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}} \phi_1) \cup B_{|s|\epsilon}(-(\frac{s}{a-\lambda_1})^{\frac{1}{p-1}} \phi_1))$. Thus we have proved Theorem 1.3(iii). □

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Authors' contributions

TJ introduced the main ideas of multiplicity study for this problem. Q-HC participated in applying the method for solving this problem and drafted the manuscript. All authors contributed equally to reading and approving the final manuscript.

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