Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

A further study on the coupled Allen–Cahn/Cahn–Hilliard equations

Jiaqi Yang[1*] and Changchun Liu[1]

*Correspondence:
yangjq18@mails.jlu.edu.cn
[1]Department of Mathematics, Jilin University, Changchun, China

Abstract

In this paper, we will show that solutions of the initial boundary value problem for the coupled system of Allen–Cahn/Cahn–Hilliard equations continuously depend on parameters of the system, and under some restrictions on the parameters all solutions of the initial boundary value problem for Allen–Cahn/Cahn–Hilliard equations tend to zero with an exponential rate as $t \to \infty$.

MSC: Primary 35M30; secondary 35B35; 35B40

Keywords: Decay estimate; Continuous dependence; Allen–Cahn/Cahn–Hilliard equations

1 Introduction

In this paper, we consider the following Allen–Cahn/Cahn–Hilliard system:

$$\partial_t A - \alpha A - \partial_x^2 A + k|A|^2 A = Ah, \quad x \in (0,l), t > 0, \tag{1.1}$$

$$\partial_t h + |A|\partial_x h = m\partial_x^2 \mu, \quad x \in (0,l), t > 0, \tag{1.2}$$

$$A(x,0) = A_0(x), \qquad h(x,0) = h_0(x), \quad x \in (0,l), \tag{1.3}$$

$$A(0,t) = A(l,t) = \vec{0}, \qquad \partial_x h(x,t)|_{x=0,l} = \partial_x^3 h(x,t)|_{x=0,l} = 0, \quad t > 0, \tag{1.4}$$

where $\mu = f'(h) - \gamma \partial_x^2 h$, $f'(h) = h^3 - h$, and $\vec{0}$ is a zero vector of R^N, $mk > \frac{5}{4}$, $m, k, \alpha > 0$ are given numbers, $A(x,t) = (A_1(x,t), \dots, A_N(x,t))$ is the unknown vector function, $h(x,t)$ is the unknown scalar function, $A_0(x)$ and $h_0(x)$ are given initial data.

System (1.1)–(1.4) was introduced to model simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point [1]. Here, h denotes the concentration of one of the components, while A is an order parameter. The Allen–Cahn equation and the Cahn–Hilliard equation have been intensively studied [2–5]. Miranville, Saoud, and Talhouk [5] studied the long time behavior, in terms of finite-dimensional attractors, of a coupled Allen–Cahn/Cahn–Hilliard system. In particular, they proved the existence of an exponential attractor and, as a consequence, the existence of a global attractor with finite fractal dimension. Çelebi and Kalantarov [6] proved the decay of solutions and structural stability for the coupled Kuramoto–Sivashinsky–Ginzburg–Landau equations.

The large time behavior and the structural stability of solutions are important for the study of a higher-order parabolic system. Many papers have already been published to study the decay and the structural stability of solutions [7–9]. In this paper, we consider the asymptotic behavior of solutions and the continuous dependence of solutions for system (1.1)–(1.4). We are going to show the continuous dependence when the coefficient changes, which helps us to know whether a coefficient in the system can cause a large change in the solution.

The following is the main result of the paper.

Theorem 1.1 *If*

$$\alpha < \lambda_1 \quad \textit{and} \quad \lambda_1\left(\gamma m - \frac{1}{2\lambda_1^2}\right) - \frac{3m}{2} > 0, \tag{1.5}$$

then all solutions of problem (1.1)–(1.4) *tend to zero with an exponential rate as* $t \to \infty$.

Theorem 1.1 implies that the concentration of one of the components and the order parameter will tend to zero as $t \to \infty$. Hence one of the components will disappear and the system will become disorder in a background point of view.

To prove Theorem 1.1, the basic a priori estimates are the L^2 norm estimates on h and $\partial_x h$. The main difficulties are caused by the nonlinearity of both the diffusive and the convective factors in equation (1.2). To overcome such difficulty, we establish two new functionals $E_1(t)$ and $E_2(t)$. Our method is based on the global energy estimates and require some delicate local integral estimates.

This paper is arranged as follows. We first study a priori estimates in Sect. 2, and then establish the exponential decay of solution in Sect. 3. Subsequently, we discuss the continuous dependence results in Sect. 4.

2 A priori estimates

Similar to [10], we know that problem (1.1)–(1.4) has a unique global solution. The first step is to obtain a priori estimates of solutions of system (1.1), (1.2). Applying the operator P^2 to both sides of equation (1.2), here P^2 is the inverse operator of the operator $L = -\frac{d^2}{dx^2}$ with the domain of definition $D(L) = H^2(0,l) \cap H_0^1(0,l)$, we get the following problem:

$$\partial_t A - \alpha A - \partial_x^2 A + k|A|^2 A = Ah, \quad x \in (0,l), t > 0, \tag{2.1}$$

$$P^2 \partial_t h + P^2\big(|A|\partial_x h\big) = -m\mu, \quad x \in (0,l), t > 0, \tag{2.2}$$

$$A(x,0) = A_0(x), \qquad h(x,0) = h_0(x), \quad x \in (0,l), \tag{2.3}$$

$$A(0,t) = A(l,t) = \vec{0}, \qquad \partial_x h(0,t) = \partial_x h(l,t) = 0, \quad t > 0. \tag{2.4}$$

Multiplying equation (2.1) by A and (2.2) by h shows

$$\frac{1}{2}\frac{d}{dt}\|A(t)\|^2 - \alpha\|A(t)\|^2 + \|\partial_x A(t)\|^2 + k\int_0^l |A(x,t)|^4\,dx = \big(|A(t)|^2, h(t)\big) \tag{2.5}$$

and

$$\frac{1}{2}\frac{d}{dt}\|Ph(t)\|^2 + \int_0^l \big(|A(x,t)|\partial_x h(x,t)\big)P^2 h(x,t)\,dx$$

$$= -m\int_0^l \left|h(x,t)\right|^4 dx + m\left\|h(t)\right\|^2 - \gamma m\left\|\partial_x h(t)\right\|^2. \tag{2.6}$$

Adding the two resulting equations together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|A(t)\right\|^2 + \left\|Ph(t)\right\|^2\right) - \alpha\left\|A(t)\right\|^2 + \left\|\partial_x A(t)\right\|^2 + m\int_0^l \left|h(x,t)\right|^4 dx \\
&\quad + k\int_0^l \left|A(x,t)\right|^4 dx + \gamma m\left\|\partial_x h(t)\right\|^2 \\
&= m\left\|h(t)\right\|^2 + \int_0^l \left|A(x,t)\right|^2 h(x,t)\,dx - \int_0^l \left(\left|A(x,t)\right|\partial_x h(x,t)\right)P^2 h(x,t)\,dx \\
&\leq m\left\|h(t)\right\|^2 + \frac{1}{2m}\int_0^l \left|A(x,t)\right|^4 dx + \frac{m}{2}\left\|h(t)\right\|^2 + \frac{1}{8m}\int_0^l \left|A(x,t)\right|^4 dx \\
&\quad + \frac{m\lambda_1^4}{2}\int_0^l \left|P^2 h(x,t)\right|^4 dx + \frac{1}{2\lambda_1^2}\int_0^l \left|\partial_x h(x,t)\right|^2 dx \\
&\leq \frac{3m}{2}\left\|h(t)\right\|^2 + \frac{5}{8m}\int_0^l \left|A(x)\right|^4 dx + \frac{1}{2\lambda_1^2}\left\|\partial_x h(t)\right\|^2 + \frac{m}{2}\int_0^l \left|h(x,t)\right|^4 dx,
\end{aligned} \tag{2.7}$$

that is,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|A(t)\right\|^2 + \left\|Ph(t)\right\|^2\right) + \left\|\partial_x A(t)\right\|^2 + \frac{m}{2}\int_0^l \left|h(x,t)\right|^4 dx \\
&\quad + \frac{k}{2}\int_0^l \left|A(x,t)\right|^4 dx + \left(\gamma m - \frac{1}{2\lambda_1^2}\right)\left\|\partial_x h(t)\right\|^2 \\
&\leq \alpha\left\|A(t)\right\|^2 + \frac{3}{2}m\left\|h(t)\right\|^2.
\end{aligned} \tag{2.8}$$

Due to the Cauchy–Schwarz inequality, we have

$$\|h\|^2 = \left(P^{-1}h, Ph\right) \leq \left\|P^{-1}h\right\|\|Ph\| \leq \varepsilon\|\partial_x h\|^2 + C_1(\varepsilon)\|Ph\|^2.$$

Choosing $\varepsilon = \frac{\gamma}{3} - \frac{1}{3m\lambda_1^2}$ and $C_1(\varepsilon) = \frac{1}{4\varepsilon}$ for the last inequality and using it in (2.8), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|A(t)\right\|^2 + \left\|Ph(t)\right\|^2\right) + 2\left\|\partial_x A(t)\right\|^2 + m\int_0^l \left|h(x,t)\right|^4 dx \\
&\quad + k\int_0^l \left|A(x,t)\right|^4 dx + \gamma m\left\|\partial_x h(t)\right\|^2 \\
&\leq 2\alpha\left\|A(t)\right\|^2 + C_2(\gamma, m, \lambda_1)\left\|Ph(t)\right\|^2.
\end{aligned} \tag{2.9}$$

Thanks to (2.9), it is true that

$$\begin{aligned}
&\left\|A(t)\right\|^2 + \left\|Ph(t)\right\|^2 \leq D_1(t), \quad \forall t \in R^+, \\
&\int_0^t \left\|\partial_x A(t)\right\|^2 ds, k\int_0^t \int_0^l \left|A(x,t)\right|^4 dx\,ds \leq D_2(t),
\end{aligned} \tag{2.10}$$

and

$$\gamma m\int_0^t \left\|\partial_x h(t)\right\|^2 ds, m\int_0^t\int_0^l \left|h(x,t)\right|^4 dx\,ds \le D_2(t), \quad \forall t\in R^+, \tag{2.11}$$

where $D_1(t) = [\|A_0\|^2 + \|Ph_0\|^2]e^{(2\alpha+C_2)t}$, $D_2(t) = D_1(t) + \|A_0\|^2 + \|Ph_0\|^2$.

Based on (2.9) and (2.10), we use the standard Faedo–Galerkin method to prove the existence of a global weak solution $[A,h]$ of problem (2.1)–(2.4) with the following properties:

$$A \in C\bigl(0,T;L^2(0,l)\bigr)\cap L^2\bigl(0,T;H_0^1(0,l)\bigr), \tag{2.12}$$

$$h \in C\bigl(0,T;H^{-1}(0,l)\bigr)\cap L^2\bigl(0,T;H_0^1(0,l)\bigr). \tag{2.13}$$

Multiplying (1.2) by h in $L^2(0,l)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|h(t)\right\|^2 &= -3m\int_0^l h^2(x,t)\partial_x h^2(x,t)\,dx + m\int_0^l \left|\partial_x h(x,t)\right|^2 dx \\ &\quad + \gamma m\int_0^l \partial_x h(x,t)\partial_x^3 h(x,t)\,dx - \int_0^l \left|A(x,t)\right|^2 \partial_x h(x,t)\cdot h\,dx.\end{aligned}$$

By Young's inequality, we see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|h(t)\right\|^2 + \gamma m\left\|\partial_x^2 h(t)\right\|^2 \\ &\quad\le m\left\|\partial_x h(t)\right\|^2 + \frac{1}{4m}\int_0^l \left|A(x,t)\right|^4 dx + \frac{1}{4m}\int_0^l \left|h(x,t)\right|^4 dx + \frac{m}{2}\left\|\partial_x h(t)\right\|^2 \\ &\quad\le \frac{3m}{2}\left\|\partial_x h(t)\right\|^2 + \frac{1}{4m}\int_0^l \left|A(x,t)\right|^4 dx + \frac{1}{4m}\int_0^l \left|h(x,t)\right|^4 dx.\end{aligned} \tag{2.14}$$

Applying (2.11), integrating inequality (2.14) in $(0,t)$ gives the following estimate:

$$\begin{aligned}&\frac{1}{2}\left\|h(t)\right\|^2 + \gamma m\int_0^t \left\|\partial_x^2 h(s)\right\|^2 ds \\ &\quad\le \left(\frac{1}{4mk} + \frac{1}{4m^2}\right)D_2(t) + \frac{3m}{2}\int_0^t \left\|\partial_x h(s)\right\|^2 ds + \frac{1}{2}\|h_0\|^2 \\ &\quad\le D_3(t), \quad \forall t\in[0,T],\end{aligned} \tag{2.15}$$

where $D_3(t) := (\frac{1}{4mk} + \frac{1}{4m^2} + \frac{3}{2\gamma})D_2(t) + \frac{1}{2}\|h_0\|^2$. Therefore, we deduce

$$\left\|h(t)\right\|^2, 2\gamma m\int_0^t \left\|\partial_x^2 h(s)\right\|^2 ds \le 2D_3(T), \quad \forall t\in[0,T]. \tag{2.16}$$

Multiplying equation (1.1) by $\partial_t A$ in $L^2(0,l)$, we see

$$\begin{aligned}&\left\|\partial_t A(t)\right\|^2 + \frac{d}{dt}\left[\frac{1}{2}\left\|\partial_x A(t)\right\|^2 - \frac{\mu}{2}\left\|A(t)\right\|^2 + \frac{k}{2}\int_0^l \left|A(x,t)\right|^4 dx\right] \\ &\quad= \bigl(\bigl\langle A(t),\partial_t A(t)\bigr\rangle, h(t)\bigr).\end{aligned} \tag{2.17}$$

Depending on Sobolev's imbedding theorem and estimates (2.11), (2.16), the term on the right-hand side of (2.17) can be estimated as follows:

$$\begin{aligned}\bigl|\bigl(\langle A(t),\partial_t A(t)\rangle,h(t)\bigr)\bigr| &\le \|A(t)\|_{L^\infty(0,l)}\|h(t)\|\|\partial_t A(t)\|\\ &\le \frac{1}{2}\|\partial_t A(t)\|^2+\frac{1}{2}\|A(t)\|^2_{L^\infty(0,l)}\|h(t)\|^2\\ &\le \frac{1}{2}\|\partial_t A(t)\|^2+\frac{l}{2}\|\partial_x A(t)\|^2\|h(t)\|^2\\ &\le \frac{1}{2}\|\partial_t A(t)\|^2+lD_3(t)\|\partial_x A(t)\|^2, \quad \forall t\in[0,T].\end{aligned}$$

Thus, according to (2.17),

$$\begin{aligned}&\|\partial_t A(t)\|^2+\frac{d}{dt}\biggl[\|\partial_x A(t)\|^2-\mu\|A(t)\|^2+k\int_0^l|A(x,t)|^4\,dx\biggr]\\ &\quad\le 2lD_3(t)\|\partial_x A(t)\|^2, \quad \forall t\in[0,T].\end{aligned}$$

It is easy to obtain the estimate

$$\int_0^t\|\partial_s A(s)\|^2\,ds,\ \|\partial_x A(t)\|^2\le D(T), \quad \forall t\in[0,T], \tag{2.18}$$

where

$$D(T):=2lD_3(T)D_2(t)+\alpha D_1(T)+\|\partial_x A_0\|^2-\alpha\|A_0\|^2+k\int_0^l|A_0(x)|^4\,dx.$$

Remark 2.1 If $\lambda_1(\gamma m-\frac{1}{2\lambda_1^2})-\frac{3m}{2}=r_0>0$, the following uniform estimate holds true:

$$\|A(t)\|^2+\|Ph(t)\|^2\le\bigl[\|A_0\|^2+\|Ph_0\|^2\bigr]e^{-\gamma_1 t}+\frac{\alpha^2 l}{2k\gamma_1}, \tag{2.19}$$

where $\gamma_1:=2\lambda_1\min\{1,r_0\}$.

Proof We deduce from (2.6) the inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Ph(t)\|^2\\ &\quad=-m\int_0^l|h(x,t)|^4\,dx+m\|h(t)\|^2-\gamma m\|\partial_x h(t)\|^2\\ &\qquad-\int_0^l\bigl(|A(x,t)|\partial_x h(x,t)\bigr)P^2h(x,t)\,dx\\ &\quad\le -m\int_0^l|h(x,t)|^4\,dx+m\|h(t)\|^2-\gamma m\|\partial_x h(t)\|^2+\frac{1}{8m}\int_0^l|A(x,t)|^4\,dx\\ &\qquad+\frac{m\lambda_1^4}{2}\int_0^l|P^2h(x,t)|^4\,dx+\frac{1}{2\lambda_1^2}\int_0^l|\partial_x h(x,t)|^2\,dx\\ &\quad\le -\frac{m}{2}\int_0^l|h(x,t)|^4\,dx+m\|h(t)\|^2-\biggl(\gamma m-\frac{1}{2\lambda_1^2}\biggr)\|\partial_x h(t)\|^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{8m}\int_0^l |A(x,t)|^4\,dx \\
&\le m\|h(t)\|^2-\left(\gamma m-\frac{1}{2\lambda_1^2}\right)\|\partial_x h(t)\|^2+\frac{1}{8m}\int_0^l |A(x,t)|^4\,dx.
\end{aligned}
$$

From (2.5), we know that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A(t)\|^2+\|\partial_x A(t)\|^2+k\int_0^l |A(x,t)|^4\,dx \\
&\quad\le \frac{1}{2m}\int_0^l |A(x,t)|^4\,dx+\frac{m}{2}\|h(t)\|^2+\frac{k}{2}\int_0^l |A(x,t)|^4\,dx+\frac{\alpha^2 l}{2k}.
\end{aligned}
$$

Adding the above two inequalities, we derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|Ph(t)\|^2+\|A(t)\|^2\right)+\|\partial_x A(t)\|^2+k\int_0^l |A(x,t)|^4\,dx \\
&\quad\le \frac{3m}{2}\|h(t)\|^2-\left(\gamma m-\frac{1}{2\lambda_1^2}\right)\|\partial_x h(t)\|^2+k\int_0^l |A(x,t)|^4\,dx+\frac{\alpha^2 l}{2k},
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|Ph(t)\|^2+\|A(t)\|^2\right)+\left(\gamma m-\frac{1}{2\lambda_1^2}\right)\|\partial_x h(t)\|^2 \\
&\quad-\frac{3m}{2}\|h(t)\|^2+\|\partial_x A(t)\|^2\le \frac{\alpha^2 l}{2k}.
\end{aligned}
$$

Hence

$$
\frac{1}{2}\frac{d}{dt}\left(\|Ph(t)\|^2+\|A(t)\|^2\right)+r_0\|h(t)\|^2+\|\partial_x A(t)\|^2\le \frac{\alpha^2 l}{2k}.
$$

Taking $\gamma_1=2\lambda_1\min\{1,r_0\}$, we have

$$
\frac{d}{dt}\left(\|Ph(t)\|^2+\|A(t)\|^2\right)+\gamma_1\left(\|Ph(t)\|^2+\|A(t)\|^2\right)\le \frac{\alpha^2 l}{2k}.
$$

We get (2.19) by integrating the last inequality. □

3 Exponential decay of solution

In this section, we are going to prove the exponential decay of solution.

Proof of Theorem 1.1. Multiplying in $L^2(0,l)$ (2.1) by A, (2.2) by h, and adding the obtained relations, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|Ph(t)\|^2+\|A(t)\|^2\right)+\|\partial_x A(t)\|^2+k\int_0^l |A(x,t)|^4\,dx-\alpha\|A(t)\|^2 \\
&\quad+\left(\gamma m-\frac{1}{2\lambda_1^2}\right)\|\partial_x h(t)\|^2-\frac{3m}{2}\|h(t)\|^2-\frac{5}{8m}\int_0^l |A(x,t)|^4\,dx\le 0, \qquad (3.1)
\end{aligned}
$$

that is,

$$\frac{1}{2}\frac{d}{dt}\left(\left\|Ph(t)\right\|^2+\left\|A(t)\right\|^2\right)+d_0\left\|A(t)\right\|^2+\frac{k}{2}\int_0^l\left|A(x,t)\right|^4\,dx+r_0\lambda_1\left\|Ph(t)\right\|^2\le 0,$$

which implies

$$\frac{1}{2}\frac{d}{dt}\left(\left\|Ph(t)\right\|^2+\left\|A(t)\right\|^2\right)+\gamma_0\left(\left\|Ph(t)\right\|^2+\left\|A(t)\right\|^2\right)+\frac{k}{2}\int_0^l\left|A(x,t)\right|^4\,dx\le 0, \tag{3.2}$$

where $d_0:=\lambda_1-\alpha$, $r_0:=\lambda_1(\gamma m-\frac{1}{2\lambda_1^2})-\frac{3m}{2}$, and $\gamma_0=\min\{d_0,r_0\lambda_1\}$. Hence, we have

$$\left\|Ph(t)\right\|^2+\left\|A(t)\right\|^2\le\left(\|Ph_0\|^2+\|A_0\|^2\right)e^{-2\gamma_0 t}. \tag{3.3}$$

We conclude from (3.1) that

$$\begin{aligned}&\frac{d}{dt}\left(\left\|Ph(t)\right\|^2+\left\|A(t)\right\|^2\right)+2d_0\lambda_1^{-1}\left\|\partial_x A(t)\right\|^2+2r_0\lambda_1^{-1}\left\|\partial_x h(t)\right\|^2\\&\quad+k\int_0^l\left|A(x,t)\right|^4\,dx\le 0.\end{aligned} \tag{3.4}$$

Integrating this inequality over the interval $(0,t)$ and employing estimate (3.3), we obtain

$$\frac{d_0}{\lambda_1}\int_0^t\left\|\partial_x A(\tau)\right\|^2\,d\tau+\frac{r_0}{\lambda_1}\int_0^t\left\|\partial_x h(\tau)\right\|^2\,d\tau\le\frac{1}{2}\left(\|Ph_0\|^2+\|A_0\|^2\right),\quad\forall t>0. \tag{3.5}$$

We can know that if $A_0,h_0\in H_0^1(0,l)$ then problem (1.1)–(1.4) has a unique weak solution such that

$$A,h\in C\left(0,T;H_0^1(0,l)\right)\cap L^2\left(0,T;H^2(0,l)\right),\quad\forall T>0.$$

Taking the inner product of (2.2) with $-\partial_x^2 h$, we have

$$\frac{d}{dt}\left\|h(t)\right\|^2+2\frac{1}{\lambda_1}(\lambda_1\gamma m-m)\left\|\partial_x^2 h(t)\right\|^2\le\frac{1}{2m}\int_0^l\left|A(x,t)\right|^2\,dx. \tag{3.6}$$

Besides, we know

$$\frac{d}{dt}\left(\left\|Ph(t)\right\|^2+\left\|A(t)\right\|^2\right)+2d_0\left\|A(t)\right\|^2+k\int_0^l\left|A(x,t)\right|^4\,dx+2r_0\left\|h(t)\right\|^2\le 0.$$

Let us multiply (3.6) by a positive parameter ε_1 and add it with the above inequality

$$\begin{aligned}&\frac{d}{dt}\left(\varepsilon_1\left\|h(t)\right\|^2+\left\|Ph(t)\right\|^2+\left\|A(t)\right\|^2\right)+\frac{2(\lambda_1\gamma m-m)\varepsilon_1}{\lambda_1}\left\|\partial_x^2 h(t)\right\|^2\\&\quad+\left(2d_0-\frac{\varepsilon_1}{2m}\right)\left\|A(t)\right\|^2+2r_0\left\|h(t)\right\|^2+k\int_0^l\left|A(x,t)\right|^4\,dx\le 0.\end{aligned}$$

Choosing $\varepsilon_1 = 2md_0$, we obtain the inequality

$$\begin{aligned}&\frac{d}{dt}E_1(t) + \delta_1 E_1(t) + \frac{2(\lambda_1\gamma m - m)\varepsilon_1}{\lambda_1}\|\partial_x^2 h(t)\|^2 + k\int_0^l |A(x,t)|^4\,dx \\ &\quad\le 0, \quad \forall t \ge 0,\end{aligned} \tag{3.7}$$

where

$$E_1(t) := \varepsilon_1\|h(t)\|^2 + \|Ph(t)\|^2 + \|A(t)\|^2,$$

and $\delta_1 := \min\{d_0, \lambda_1 r_0, \frac{r_0}{\varepsilon_1}\}$. Then we obtain

$$E_1(t) \le E_1(0)e^{-\delta_1 t}, \quad \forall t \ge 0. \tag{3.8}$$

Taking the inner product in $L^2(0,l)$ of (2.1) with $\partial_t A$, we get

$$\begin{aligned}&\|\partial_t A(t)\|^2 + \frac{d}{dt}\left(-\frac{\alpha}{2}\|A(t)\|^2 + \frac{1}{2}\|\partial_x A(t)\|^2 + \frac{k}{4}\int_0^l |A(x,t)|^4\,dx\right) \\ &\quad\le \|\partial_t A(t)\|^2 + \frac{1}{4}\int_0^l |A(x,t)|^2|h(x,t)|^2\,dx.\end{aligned}$$

According to [6],

$$\begin{aligned}&\frac{d}{dt}\left(-\frac{\alpha}{2}\|A(t)\|^2 + \frac{1}{2}\|\partial_x A(t)\|^2 + \frac{k}{4}\int_0^l |A(x,t)|^4\,dx\right) \\ &\quad\le \varepsilon_2\int_0^l |A(x,t)|^4\,dx + \varepsilon_3\|\partial_x A(t)\|^2 + \frac{1}{256\varepsilon_2^2\varepsilon_3}\|h(t)\|^6.\end{aligned}$$

Multiplying (3.4) by $\frac{1}{2}$ and adding with the inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}E_2(t) + \frac{d_0}{\lambda_1}\|\partial_x A(t)\|^2 + \left(\frac{r_0}{\lambda_1} - \varepsilon_3\right)\|\partial_x h(t)\|^2 + \left(\frac{k}{2} - \varepsilon_2\right)\int_0^l |A(x,t)|^4\,dx \\ &\quad\le \frac{1}{256\varepsilon_2^2\varepsilon_3}\|h(t)\|^6,\end{aligned}$$

where

$$E_2(t) := \frac{(1-\alpha)}{2}\|A(t)\|^2 + \frac{1}{2}\|Ph(t)\|^2 + \frac{1}{2}\|\partial_x A(t)\|^2 + \frac{k}{4}\int_0^l |A(x,t)|^4\,dx.$$

We choose in the last inequality $\varepsilon_2 = \frac{k}{4}$, $\varepsilon_3 = \frac{r_0}{2\lambda_1}$ and obtain the equality

$$\frac{d}{dt}E_2(t) + \delta_2 E_2(t) \le A_0\|A(t)\|^2 + A_1\|h(t)\|^6, \quad \forall t \ge 0,$$

where

$$\delta_2 := \min\left\{\frac{2d_0}{\lambda_1}, 1, 2r_0\lambda_1\right\}, \qquad A_0 = \frac{\delta_2|1-\alpha|}{2}, \qquad A_1 = \frac{1}{256\varepsilon_2^2\varepsilon_3}.$$

Using estimate (3.8), we have

$$\frac{d}{dt}E_2(t) + \delta_2 E_2(t) \le A_2 e^{-\delta_1 t}, \tag{3.9}$$

where $A_2 = A_1 E_1^3(0) + A_0 E_1(0)$. Integrating (3.9) and by Gronwall's inequality, we get

$$E_2(t) \le \frac{A_2}{\delta_1} e^{-\delta_2 t}. \tag{3.10}$$

Combining with (3.3), we have

$$\|\partial_x A(t)\|^2 \le \frac{2A_2}{\delta_1} e^{-\delta_2 t} + R_0 e^{-2\gamma_0 t}. \tag{3.11}$$

Multiplying (2.2) by $\partial_t h$, we obtain

$$\begin{aligned} &\|P\partial_t h(t)\|^2 + \frac{d}{dt}\left(-\frac{m}{2}\|h(t)\|^2 + \frac{\gamma m}{2}\|\partial_x h(t)\|^2\right) \\ &\quad \le -\frac{m}{4}\frac{d}{dt}\int_0^l |h(x.t)|^4\,dx \\ &\qquad + \frac{1}{4}\int_0^l \left[P\left(|A(x,t)|\partial_x h(x,t)\right)\right]^2 dx + \|P\partial_t h(t)\|^2. \end{aligned} \tag{3.12}$$

Adding $\frac{m}{2}$ (3.6) to (3.12), we get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\gamma m}{2}\|\partial_x h(t)\|^2 + \frac{m}{4}\int_0^l |h(x.t)|^4\,dx\right) + 2(\lambda_1\gamma m - m)\|\partial_x h(t)\|^2 \\ &\quad \le \frac{1}{2m}\int_0^l |A(x,t)|^2\,dx + \frac{l}{4\lambda_1}\|\partial_x A(t)\|^2\|\partial_x h(t)\|^2, \end{aligned}$$

thus

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\gamma m}{2}\|\partial_x h(t)\|^2 + \frac{m}{4}\int_0^l |h(x.t)|^4\,dx\right) + \left[2(\lambda_1\gamma m - m) - \frac{l}{4\lambda_1}\right]\|\partial_x h(t)\|^2 \\ &\quad \le \frac{1}{2m}\int_0^l |A(x,t)|^2\,dx. \end{aligned}$$

Note that

$$\int_0^l |h(x.t)|^4\,dx \le \|h(t)\|_\infty^2\|h(t)\|^2 \le l\|\partial_x h(t)\|^2\|h(t)\|^2 \le c\|\partial_x h(t)\|^2.$$

Hence, we know that

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\gamma m}{2}\|\partial_x h(t)\|^2 + \frac{m}{4}\int_0^l |h(x.t)|^4\,dx\right) \\ &\quad + c\left(\frac{\gamma m}{2}\|\partial_x h(t)\|^2 + \frac{m}{4}\int_0^l |h(x.t)|^4\,dx\right) \le \frac{E_2(0)}{2m}e^{-\delta_1 t}. \end{aligned} \tag{3.13}$$

Finally, we integrate (3.13) and get

$$\frac{\gamma m}{2}\|\partial_x h(t)\|^2+\frac{m}{4}\int_0^l|h(x.t)|^4\,dx\le c_1e^{-ct}, \tag{3.14}$$

where $c_1=-\frac{E_2(0)}{2m\delta_1}e^{-\delta_1 t}+\frac{m}{2}\|\partial_x h_0\|^2+\frac{m}{4}\int_0^l|h_0|^4\,dx-\frac{E_2(0)}{2m\delta_1}$. The theorem is true. □

4 Continuous dependence results

Assume that $[\tilde{A},\tilde{h}]$ is the weak solution of the problem

$$\partial_t\tilde{A}-\alpha\tilde{A}-\partial_x^2\tilde{A}+\tilde{k}|\tilde{A}|^2\tilde{A}=\tilde{A}\tilde{h},\quad x\in(0,l),t>0, \tag{4.1}$$

$$\partial_t\tilde{h}+|\tilde{A}|\partial_x\tilde{h}=m\partial_x^2\tilde{\mu},\quad x\in(0,l),t>0, \tag{4.2}$$

$$\tilde{A}(x,0)=A_0(x),\qquad \tilde{h}(x,0)=h_0(x),\quad x\in(0,l), \tag{4.3}$$

$$\tilde{A}(0,t)=\tilde{A}(l,t)=\vec{0},\qquad \partial_x\tilde{h}(0,t)=\partial_x^3\tilde{h}(0,t)=\partial_x\tilde{h}(l,t)=\partial_x^3\tilde{h}(l,t)=0,\quad t>0, \tag{4.4}$$

where $\tilde{\mu}=f'(\tilde{h})-\gamma\partial_x^2\tilde{h}, f'(\tilde{h})=\tilde{h}^3-\tilde{h}$.

Theorem 4.1 *Assume that $[A,h]$ is a solution of problem (1.1)–(1.4) and $[\tilde{A},\tilde{h}]$ is a solution of problem (4.1)–(4.4). Let $[a,H]=[A-\tilde{A},h-\tilde{h}]$, we have*

$$\|PH(t)\|^2+\|a(t)\|^2\le q_0e^{-\int_0^t R_1(s)\,ds}\int_0^t\|\partial_x A(s)\|^4\,ds,$$

where $q_0=\frac{2k_1^{\frac{4}{3}}C_0^4}{(k_2b_0)^{\frac{1}{3}}}+\frac{C_0^4}{4m^3\lambda_1^4\gamma^2}$,

$$R_1(t)=2C(\gamma,m)+2C_1(\gamma,m)+4l\|h(t)\|^2+2\alpha+\frac{l}{\lambda_1^2 m}\|\partial_x^2\tilde{h}(t)\|^2+l\|\partial_x^2\tilde{A}(t)\|^2.$$

Proof Note that $[a,H]=[A-\tilde{A},h-\tilde{h}]$ is a solution of the following problem:

$$\partial_t a-\alpha a-\partial_x^2 a+k|A|^2A-\tilde{k}|\tilde{A}|^2\tilde{A}=Ah-\tilde{A}\tilde{h},\quad x\in(0,l),t>0, \tag{4.5}$$

$$\begin{aligned}&P^2\partial_t H-mH-\gamma m\partial_x^2 H\\&\quad=-m\big(h^3-\tilde{h}^3\big)+P^2\big(|\tilde{A}|\partial_x\tilde{h}\big)-P^2\big(|A|\partial_x h\big),\quad x\in(0,l),t>0,\end{aligned} \tag{4.6}$$

$$a(x,0)=\vec{0},\qquad H(x,0)=0,\quad x\in(0,l),$$

$$a(0,t)=a(l,t)=\vec{0},\qquad \partial_x H(0,t)=\partial_x H(0,t),\quad x\in(0,l),t>0.$$

By

$$Ah-\tilde{A}\tilde{h}=Ah-\tilde{A}h+\tilde{A}h-\tilde{A}\tilde{h}=ah+\tilde{A}H$$

and

$$P^2\big(|\tilde{A}|\partial_x\tilde{h}\big)-P^2\big(|A|\partial_x h\big)=P^2\big(|\tilde{A}|\partial_x\tilde{h}-|A|\partial_x h\big)$$

$$= P^2\big(|\tilde{A}|\partial_x\tilde{h} - |A|\partial_x\tilde{h} + |A|\partial_x\tilde{h} - |A|\partial_x h\big)$$
$$= P^2\big(-|A|\partial_x H - \partial_x\tilde{h}\big(|A| - |\tilde{A}|\big)\big),$$

we see that $[a, H]$ satisfies the following system:

$$\partial_t a - \alpha a - \partial_x^2 a + k_2\big(|A|^2 A - |\tilde{A}|^2\tilde{A}\big) = k_1|A|^2 A + ah + \tilde{A}H, \tag{4.7}$$

$$\begin{aligned} &P^2\partial_t H - mH - \gamma m\partial_x^2 H \\ &\quad = -m\big(h^3 - \tilde{h}^3\big) - P^2\big(|A|\partial_x H + \partial_x\tilde{h}\big(|A| - |\tilde{A}|\big)\big), \end{aligned} \tag{4.8}$$

where $k_2 = \tilde{k}$, $k_1 = \tilde{k} - k$.

On the other hand, we know that

$$\big(\big|A(t)\big|^2 A(t) - \big|\tilde{A}(t)\big|^2\tilde{A}(t), A(t) - \tilde{A}(t)\big) \geq b_0\int_0^l \big|a(x,t)\big|^4\,dx. \tag{4.9}$$

Multiplying (4.7) by a and using inequality (4.9), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big\|a(t)\big\|^2 - \alpha\big\|a(t)\big\|^2 + \big\|\partial_x a(t)\big\|^2 + k_2 b_0\int_0^l\big|a(x,t)\big|^4\,dx \\ &\quad\leq k_1\int_0^l\big|A(x,t)\big|^2\big\langle A(x,t), a(x,t)\big\rangle\,dx + \int_0^l\big|a(x,t)\big|^2 h(x,t)\,dx \\ &\qquad+ \int_0^l\big\langle\tilde{A}(x,t), a(x,t)\big\rangle H(x,t)\,dx. \end{aligned} \tag{4.10}$$

We are going to estimate the first integral on the right-hand side of (4.10) by the Nirenberg inequality as follows:

$$\begin{aligned} \left|\int_0^l\big|a(x,t)\big|^2 h(x,t)\,dx\right| &\leq \big\|a(t)\big\|_\infty^2\int_0^l\big|h(x,t)\big|\,dx \\ &\leq \sqrt{l}\big\|a(t)\big\|_\infty^2\big\|h(t)\big\| \\ &\leq 2\sqrt{l}\big\|a(t)\big\|\big\|\partial_x a(t)\big\|\big\|h(t)\big\| \\ &\leq \frac{1}{2}\big\|\partial_x a(t)\big\|^2 + 2l\big\|a(t)\big\|^2\big\|h(t)\big\|^2. \end{aligned} \tag{4.11}$$

We can infer from the Nirenberg inequality and the Friedrichs inequality that the following estimate of the second term on the right-hand side of (4.10) is true:

$$\begin{aligned} &\left|\int_0^l\big\langle\tilde{A}(x,t), a(x,t)\big\rangle H(x,t)\,dx\right| \\ &\quad\leq \int_0^l\big|\tilde{A}(x,t)\big|\big|a(x,t)\big|\big|H(x,t)\big|\,dx \\ &\quad\leq \big\|\tilde{A}(t)\big\|_\infty\big\|a(t)\big\|\big\|H(t)\big\| \\ &\quad\leq \frac{1}{2}\big\|\tilde{A}(t)\big\|_\infty^2\big\|a(t)\big\|^2 + \frac{1}{2}\big\|H(t)\big\|^2 \\ &\quad\leq \frac{l}{2}\big\|\partial_x\tilde{A}(t)\big\|^2\big\|a(t)\big\|^2 + \varepsilon_2\big\|\partial_x H(t)\big\|^2 + C_1(\varepsilon_2)\big\|PH(t)\big\|^2. \end{aligned} \tag{4.12}$$

Employing Young's inequality and Sobolev's inequality, we have

$$\begin{aligned}
&\left| k_1 \int_0^l \left|A(x,t)\right|^2 \left\langle A(x,t), a(x,t)\right\rangle dx \right| \\
&\quad \le k_1 \int_0^l \left|A(x,t)\right|^3 \left|a(x,t)\right| dx \\
&\quad \le \frac{k_1^{\frac{4}{3}}}{(k_2 b_0)^{\frac{1}{3}}} \int_0^l \left|A(x,t)\right|^4 + k_2 b_0 \int_0^l \left|a(x,t)\right|^4 dx \\
&\quad \le \frac{k_1^{\frac{4}{3}} C_0^4}{(k_2 b_0)^{\frac{1}{3}}} \left\|\partial_x A(t)\right\|^4 dx + k_2 b_0 \int_0^l \left|a(x,t)\right|^4 dx.
\end{aligned}$$

Then employing the last inequality and (4.11), (4.12), we deduce the following inequality from (4.10):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|a(t)\right\|^2 + \left\|\partial_x a(t)\right\|^2 + k_2 b_0 \int_0^l \left|a(x,t)\right|^4 dx \\
&\quad \le \alpha \left\|a(t)\right\|^2 + \frac{k_1^{\frac{4}{3}} C_0^4}{(k_2 b_0)^{\frac{1}{3}}} \left\|\partial_x A(t)\right\|^4 + 2l \left\|h(t)\right\|^2 \left\|a(t)\right\|^2 + \frac{l}{2} \left\|\partial_x \tilde{A}(t)\right\|^2 \left\|a(t)\right\|^2 \\
&\qquad + \varepsilon \left\|\partial_x H(t)\right\|^2 + C(\varepsilon) \left\|PH(t)\right\|^2. \qquad (4.13)
\end{aligned}$$

Multiplying (4.8) in $L^2(0,l)$ by H, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|PH(t)\right\|^2 - m\left\|H(t)\right\|^2 + \gamma m \left\|\partial_x H(t)\right\|^2 \\
&\quad = -m\left(h^3 - \tilde{h}^3, H\right) + \left(|A|\partial_x H + \partial_x \tilde{h}\left(|A| - |\tilde{A}|\right), -P^2 H\right) \qquad (4.14)
\end{aligned}$$

and

$$\begin{aligned}
&\left(|A|\partial_x h + \partial_x \tilde{h}\left(|A| - |\tilde{A}|\right), -P^2 H\right) \\
&\quad = -\int_0^l |A|\partial_x H P^2 H\, dx - \int_0^l \partial_x \tilde{h}\left(|A| - |\tilde{A}|\right) P^2 H\, dx \\
&\quad \le \frac{1}{8m^3\lambda_1^4\gamma^2} \int_0^l \left|A(x,t)\right|^4 dx + m\lambda_1^4 \int_0^l \left|P^2 H(t)\right|^4 dx + \frac{\gamma m}{2} \int_0^l \left|\partial_x H(t)\right|^2 dx \\
&\qquad + \frac{l}{2\lambda_1^2 m} \left\|\partial_x^2 \tilde{h}(t)\right\|^2 \left\|a(t)\right\|^2 + \frac{\lambda_1^2 m}{2} \int_0^l \left|P^2 H(x,t)\right|^2 dx \\
&\quad \le m \int_0^l \left|H(x,t)\right|^4 dx + \frac{l}{2\lambda_1^2 m} \left\|\partial_x^2 \tilde{h}(t)\right\|^2 \left\|a(t)\right\|^2 + \frac{m}{2} \int_0^l \left|H(x,t)\right|^2 dx \\
&\qquad + \frac{\gamma m}{2} \int_0^l \left|\partial_x H(x,t)\right|^2 dx + \frac{1}{8m^3\lambda_1^4\gamma^2} \int_0^l \left|A(x,t)\right|^4 dx \\
&\quad \le m \int_0^l \left|H(x,t)\right|^4 dx + \frac{l}{2\lambda_1^2 m} \left\|\partial_x^2 \tilde{h}(t)\right\|^2 \left\|a(t)\right\|^2 + \frac{5\gamma m}{8} \left\|\partial_x H(t)\right\|^2 \\
&\qquad + C_1(\gamma, m) \left\|PH(t)\right\|^2 + \frac{1}{8m^3\lambda_1^4\gamma^2} \int_0^l \left|A(x,t)\right|^4 dx.
\end{aligned}$$

Employing Sobolev's imbedding theorem and the Nirenberg inequality, we get

$$\left(h^3 - \tilde{h}^3, H\right) = \int_0^l H^2\left(h^2 + h\tilde{h} + \tilde{h}^2\right) dx \geq \int_0^l H^4\, dx, \tag{4.15}$$

$$|m|\left\|H(t)\right\|^2 \leq \frac{\gamma m}{4}\left\|\partial_x H(t)\right\|^2 + C_1(m,\gamma)\left\|PH(t)\right\|^2. \tag{4.16}$$

Using (4.15), (4.16), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|PH(t)\right\|^2 + \frac{\gamma m}{8}\left\|\partial_x H(t)\right\|^2 \\
&\quad\leq C_1(\gamma, m)\left\|PH(t)\right\|^2 \\
&\qquad + \frac{l}{2\lambda_1^2 m}\left\|\partial_x^2 \tilde{h}(t)\right\|^2 \left\|a(t)\right\|^2 + \frac{1}{8m^3\lambda_1^4\gamma^2}\int_0^l \left|A(x,t)\right|^4 dx.
\end{aligned} \tag{4.17}$$

Taking $\varepsilon = \frac{\gamma m}{8}$ in (4.13) and adding it to (4.17), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|PH(t)\right\|^2 + \left\|a(t)\right\|^2\right) \\
&\quad\leq \left(\frac{2k_1^{\frac{4}{3}}C_0^4}{(k_2 b_0)^{\frac{1}{3}}} + \frac{C_0^4}{4m^3\lambda_1^4\gamma^2}\right)\left\|\partial_x A(t)\right\|^4 + R_1(t)\left(\left\|PH(t)\right\|^2 + \left\|a(t)\right\|^2\right),
\end{aligned} \tag{4.18}$$

where

$$R_1(t) = 2C(\gamma, m) + 2C_1(\gamma, m) + 4l\left\|h(t)\right\|^2 + 2\alpha + \frac{l}{\lambda_1^2 m}\left\|\partial_x^2 \tilde{h}(t)\right\|^2 + l\left\|\partial_x^2 \tilde{A}(t)\right\|^2.$$

From (4.18) we derive the estimate

$$\left\|PH(t)\right\|^2 + \left\|a(t)\right\|^2 \leq q_0 e^{-\int_0^t R_1(s)\,ds}\int_0^t \left\|\partial_x A(s)\right\|^4 ds,$$

where $q_0 = \frac{2k_1^{\frac{4}{3}}C_0^4}{(k_2 b_0)^{\frac{1}{3}}} + \frac{C_0^4}{4m^3\lambda_1^4\gamma^2}$. The proof is complete. □

Acknowledgements
Not applicable.

Funding
This work is supported by the Jilin Scientific and Technological Development Program [number 20170101143JC].

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 31 December 2018 Accepted: 8 March 2019 Published online: 15 March 2019

References

1. Cahn, J.W., Novick-Cohen, A.: Evolution equations for phase separation and ordering in binary alloys. J. Stat. Phys. **76**, 877–909 (1994)
2. Boldrini, J.L., da Silva, P.N.: A generalized solution to a Cahn–Hilliard/Allen–Cahn system. Electron. J. Differ. Equ. **2004**, 126 (2004)
3. Liu, A., Liu, C.: The Cauchy problem for the degenerate convective Cahn–Hilliard equation. Rocky Mt. J. Math. **48**, 2595–2623 (2018)
4. Liu, C., Tang, H.: Existence of periodic solution for a Cahn–Hilliard/Allen–Cahn equation in two space dimensions. Evol. Equ. Control Theory **6**, 219–237 (2017)
5. Miranville, A., Saoud, W., Talhouk, R.: Asymptotic behavior of a model for order-disorder and phase separation. Asymptot. Anal. **103**, 57–76 (2017)
6. Çelebi, O.A., Kalantarov, V.K.: Decay of solutions and structural stability for the coupled Kuramoto–Sivashinsky–Ginzburg–Landau equations. Appl. Anal. **94**, 2342–2354 (2015)
7. Çelebi, O.A., Gür, Ş., Kalantarov, V.K.: Structural stability and decay estimate for marine riser equations. Math. Comput. Model. **54**, 3182–3188 (2011)
8. Liu, A., Liu, C.: Cauchy problem for a sixth order Cahn–Hilliard type equation with inertial term. Evol. Equ. Control Theory **4**, 315–324 (2015)
9. Liu, C., Wang, J.: Some properties of solutions for a sixth order Cahn–Hilliard type equation with inertial term. Appl. Anal. **97**, 2332–2348 (2018)
10. Kalantarov, V.K.: Global solution of coupled Kuramoto–Sivashinsky and Ginzburg–Landau equations. In: Rozhkovskaya, T. (ed.) Nonlinear Problems in Mathematical Physics and Related Topics, II. Int. Math. Ser. (N.Y.), pp. 213–227. Kluwer/Plenum, New York (2002)