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Existence and asymptotic properties of positive solutions for a general quasilinear Schrödinger equation

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Abstract

By a change of variables with cut-off functions, we study the existence and the asymptotic behavior of positive solutions for a general quasilinear Schrödinger equation which arises from plasma physics. We extend the results of (Adv. Nonlinear Stud. 18(1):131-150, 2017) from $\alpha = 1$ to $\alpha > \frac{1}{2}$. Especially, we can consider the exponent p in $(2, 2^*)$ for all $N \geq 3$.

Keywords: Quasilinear Schrödinger equation; Existence; Asymptotic properties

1 Introduction

In this paper, we study the existence and asymptotic behavior of positive solutions for the following general quasilinear elliptic equation:

$$-\Delta u + V(x)u - \alpha\gamma\left(\Delta\left(|u|^{2\alpha}\right)\right)|u|^{2\alpha-2}u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1}$$

where $\alpha > \frac{1}{2}$ is a positive constant, $\gamma > 0$ is a parameter, $p > 2$ and $N \geq 3$.

Equation (1) is derived from a superfluid film equation in plasma physics [11]; see [7–9, 15] and the references therein for more physical backgrounds. When $\alpha = 1$, the existence of solutions for Eq. (1) was extensively considered in recent years [2, 3, 9, 14–16, 19–21] since the change in [9, 14] was introduced. Furthermore, using the change of variables, for general $\alpha > \frac{1}{2}$, the existence of solutions of (1) have been studied; see [1, 4, 12] and the references therein. Comparing with the semilinear elliptic equations, it is much more challenging and interesting because of the existence of the term $(\Delta(|u|^{2\alpha}))|u|^{2\alpha-2}u$. It is worth mentioning that the authors in [20] considered problem (1) with $\alpha = 1$. Using the change of variables introduced in [19] and the cut-off function technique in [5], the authors reduced Eq. (1) to a semilinear elliptic equation. Then the existence and boundedness of solution was obtained by the critical point theory when $p \in (2, 2^*)$ for $N \geq 4$ or $p \in (2, 4)$ for $N = 3$. Moreover, they got the asymptotic properties of the solution of (1) by using the arguments in [1, 3]. But in [20], what will happen when $p \in [4, 6)$ for $N = 3$?

In this paper, we want to address the existence of Eq. (1) with $\alpha > \frac{1}{2}$ by using the technique of [5, 19, 20]. Furthermore, we can discuss the exponent p from 2 to 2^* for any $N \geq 3$ by introducing different cut-off functions when $p < 4\alpha$ and $p \geq 4\alpha$. We also can get the asymptotic properties of the solution of (1) with the use of techniques in [1, 3, 20].

We assume that the potential function V satisfies (V_1) $0 < V_0 \leq V(x) \leq \lim_{|x|\to+\infty} V(x) = V_\infty < +\infty$.

Define the space $X = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{2(2\alpha-1)}|\nabla u|^2\,dx < \infty\}$. Then, for $u \in X$, the energy functional $I_\gamma(u)$ associated with (1) is

$$I_\gamma(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)|u|^2\right) dx + \alpha^2\gamma \int_{\mathbb{R}^N} |u|^{2(2\alpha-1)}|\nabla u|^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx. \tag{2}$$

Theorem 1.1 *Assume $V(x) = \mu > 0$, then Eq. (1) has a positive solution u_γ satisfying: (i) u_γ is spherically symmetric and u_γ decreases with respect to $|x|$; (ii) $u_\gamma \in C^2(\mathbb{R}^N)$; (iii) u_γ together with its derivatives up to order 2 have exponential decay at infinity $|D^\alpha u_\gamma| \leq Ce^{-\delta|x|}$, $x \in \mathbb{R}^N$, for some $C, \delta > 0$ and $|\alpha| \leq 2$. Passing to a subsequence if necessary, it follows that*

$$u_\gamma \to u_0 \quad \text{in } H^2(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \text{ as } \gamma \to 0^+,$$

where u_0 is the ground state of equation $-\Delta u + \mu u = |u|^{p-2}u$, $x \in \mathbb{R}^N$.

Theorem 1.2 *Assume that (V_1) holds and $p \in (2, 2^*)$. Then there exists a γ_0 such that, for $\gamma \in (0, \gamma_0)$, Eq. (1) has a positive solution u_γ satisfying $\max_{x\in\mathbb{R}^N} |\gamma^\mu u_\gamma(x)| \to 0$ as $\gamma \to 0^+$ for any $\mu > \frac{1}{2(2\alpha-1)}$.*

Remark 1.1 If $\alpha = 1$, the above theorem is essentially Theorem 1.1 of [20]. When $N = 3$, $p < 4$ is necessary in [20]. But in here, we extend this result to $p < 2^*$. Moreover, for general $\alpha > \frac{1}{2}$, [2, 15] obtain the existence of solutions of (1) for $p \geq 4\alpha$. But we can obtain the existence of solutions for the case $p < 4\alpha$.

In this paper, we use the following notations: C denotes constant, $\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)\,dx$ for $u \in H^1(\mathbb{R}^N)$, $\|u\|_p$ denotes the norm of the space $L^p(\mathbb{R}^N)$.

2 The cut-off technique and some lemmas

We introduce the cut-off function $\zeta(t) : \mathbb{R} \to \mathbb{R}$ such that $\zeta(t) = 0$ if $t \leq 0$, $\zeta(t) = \frac{e^{-\frac{1}{t}}}{e^{-\frac{1}{t}} + e^{-\frac{1}{1-t}}}$ if $0 < t < 1$ and $\zeta(t) = 1$ if $t \geq 1$. The basic property of the function was already used in [17, 18, 20]. It is easy to see that $\zeta(t) \in C^\infty(\mathbb{R}, [0,1])$, $0 \leq \zeta(t) \leq 1$ for all $t \in \mathbb{R}$. Moreover, $\zeta'(t) = \frac{(2t^2-2t+1)e^{\frac{1-2t}{t(1-t)}}}{t^2(1-t)^2[1+e^{\frac{1-2t}{t(1-t)}}]^2}$ if $0 < t < 1$ and $\zeta'(t) = 0$ if $t < 0$ or $t > 1$. Let $\zeta'(0) = \zeta'(1) = 0$, then $\zeta'(t) \geq 0$ is uniformly bounded in $[0,1]$. This means there exists some $C_0 > 0$ such that $|\zeta'(t)| \leq C_0$ for any $t \in \mathbb{R}$.

Case I: $4\alpha > p$. In this case, we assume that

$$\rho(t) = \zeta^2\left[\frac{2^{\frac{1}{2\alpha-1}}}{2^{\frac{1}{2\alpha-1}} - 1}\left(1 - \left(\frac{8\alpha^2\gamma(4\alpha - p)}{p-2}\right)^{\frac{1}{2(2\alpha-1)}} t\right)\right].$$

Then $\rho(t) \in C^\infty(\mathbb{R}^+, [0,1])$ and

$$\rho(t)\begin{cases} = 1 & \text{if } 0 \leq t < \left(\frac{p-2}{32\alpha^2\gamma(4\alpha-p)}\right)^{\frac{1}{2(2\alpha-1)}}, \\ \in (0,1) & \text{if } \left(\frac{p-2}{32\alpha^2\gamma(4\alpha-p)}\right)^{\frac{1}{2(2\alpha-1)}} \leq t < \left(\frac{p-2}{8\alpha^2\gamma(4\alpha-p)}\right)^{\frac{1}{2(2\alpha-1)}}, \\ = 0 & \text{if } t \geq \left(\frac{p-2}{8\alpha^2\gamma(4\alpha-p)}\right)^{\frac{1}{2(2\alpha-1)}}. \end{cases}$$

Moreover, for any $t \in \mathbb{R}^+$, we have $0 \geq \rho'(t) \geq -\frac{2^{\frac{2\alpha}{2\alpha-1}}}{2^{\frac{1}{2\alpha-1}}-1}(\frac{8\alpha^2\gamma(4\alpha-p)}{p-2})^{\frac{1}{2(2\alpha-1)}} C_0\sqrt{\rho(t)}$. Nextly, we assume that $\eta(t) = \rho(-t)$ if $t \leq 0$ and $\eta(t) = \rho(t)$ if $t \geq 0$. It means that

$$\eta(t)\begin{cases} = \eta(-t) & \text{if } t \leq 0, \\ = 1 & \text{if } 0 \leq t < (\frac{p-2}{32\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}}, \\ \in (0,1) & \text{if } (\frac{p-2}{32\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}} \leq t < (\frac{p-2}{8\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}}, \\ = 0 & \text{if } t \geq (\frac{p-2}{8\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}}, \end{cases} \tag{3}$$

$\eta(t) \in C_0^\infty(\mathbb{R}, [0,1])$ and $\eta'(t)t \leq 0$ for $t \in \mathbb{R}^+$. Furthermore, for $t \in \mathbb{R}^+$, we have

$$t\eta'(t) \geq \begin{cases} -\frac{1}{2^{\frac{1}{2\alpha-1}}-1}C_0 & \text{if } 0 \leq t < (\frac{p-2}{32\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}}, \\ -\frac{2^{\frac{1}{2\alpha-1}}}{2^{\frac{1}{2\alpha-1}}-1}C_0\sqrt{\eta(t)} & \text{if } (\frac{p-2}{32\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}} \leq t < (\frac{p-2}{8\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}}, \\ 0 & \text{if } t \geq (\frac{p-2}{8\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}}. \end{cases}$$

Case II: $p \geq 4\alpha$. In this case, we let

$$\rho(t) = \zeta^2\left[\frac{2^{\frac{1}{2\alpha-1}}}{2^{\frac{1}{2\alpha-1}}-1}\left(1-\left(\frac{8\alpha^2\gamma(6-p)}{p-2}\right)^{\frac{1}{2(2\alpha-1)}}t\right)\right].$$

Similar to the case I, we assume that $\eta(t) = \rho(-t)$ if $t \leq 0$ and $\eta(t) = \rho(t)$ if $t \geq 0$. Then $0 \geq \rho'(t) \geq -2\frac{2^{\frac{1}{2\alpha-1}}}{2^{\frac{1}{2\alpha-1}}-1}(\frac{8\alpha^2\gamma(6-p)}{p-2})^{\frac{1}{2(2\alpha-1)}} C_0\sqrt{\rho(t)}$ and

$$\eta(t)\begin{cases} = \eta(-t) & \text{if } t \leq 0, \\ = 1 & \text{if } 0 \leq t < (\frac{p-2}{32\alpha^2\gamma(6-p)})^{\frac{1}{2(2\alpha-1)}}, \\ \in (0,1) & \text{if } (\frac{p-2}{32\alpha^2\gamma(6-p)})^{\frac{1}{2(2\alpha-1)}} \leq t < (\frac{p-2}{8\alpha^2\gamma(6-p)})^{\frac{1}{2(2\alpha-1)}}, \\ = 0 & \text{if } t \geq (\frac{p-2}{8\alpha^2\gamma(6-p)})^{\frac{1}{2(2\alpha-1)}}. \end{cases} \tag{4}$$

For $p \in (2, 2^*)$, we construct an auxiliary function $g_\gamma(t)\colon \mathbb{R} \to \mathbb{R}^+$ just like:

$$g_\gamma(t) = \sqrt{\left(\frac{1}{2}+2\alpha^2\gamma|t|^{2(2\alpha-1)}\right)\eta(t)+\frac{1}{2}},$$

where $\eta(t)$ take the form (3) if $p < 4\alpha$ and the form (4) if $p \geq 4\alpha$. Then we know that $g_\gamma(0) = 1$, $\frac{\sqrt{2}}{2} \leq g_\gamma(t) \leq \sqrt{\frac{14-3p}{4(4-p)}}$ if $p \leq 4\alpha$, $\frac{\sqrt{2}}{2} \leq g_\gamma(t) \leq \sqrt{\frac{22-3p}{4(6-p)}}$ if $p \geq 4\alpha$,

$$g'_\gamma(t)t = \frac{(\frac{1}{2}+2\alpha^2\gamma|t|^{2(2\alpha-1)})\eta'(t)t + 4(2\alpha-1)\gamma|t|^{2(2\alpha-1)\eta(t)}}{2[(\frac{1}{2}+2\alpha^2\gamma|t|^{2(2\alpha-1)})\eta(t)+\frac{1}{2}]^{\frac{1}{2}}} \tag{5}$$

and $g'_\gamma(t)t = -g'_\gamma(-t)t$. Define $G_\gamma(t) = \int_0^t g_\gamma(s)\,ds$. Then the inverse function $G_\gamma^{-1}(t)$ exists and is an odd function. Furthermore, $G_\gamma, G_\gamma^{-1} \in C^\infty(\mathbb{R},\mathbb{R})$.

Lemma 2.1 *The following properties hold*:

$$\lim_{t\to 0}\frac{G_\gamma^{-1}(t)}{t} = 1; \qquad \lim_{t\to\infty}\frac{G_\gamma^{-1}(t)}{t} = \sqrt{2}; \tag{6}$$

$$\sqrt{\frac{4(4\alpha-p)}{16\alpha-2-3p}}|t| \le \left|G_\gamma^{-1}(t)\right| \le \sqrt{2}|t|, \quad \textit{for all } t\in\mathbb{R} \textit{ and } p\le 4\alpha; \tag{7}$$

$$\sqrt{\frac{4(6-p)}{22-3p}}|t| \le \left|G_\gamma^{-1}(t)\right| \le \sqrt{2}|t|, \quad \textit{for all } t\in\mathbb{R} \textit{ and } p\ge 4\alpha; \tag{8}$$

$$-C \le \frac{g_\gamma'(t)t}{g_\gamma(t)} \le \frac{(8\alpha-2-p)(p-2)}{16\alpha-2-3p}, \quad \textit{for all } t\in\mathbb{R} \textit{ and } p\le 4\alpha; \tag{9}$$

$$-C \le \frac{g_\gamma'(t)t}{g_\gamma(t)} \le \frac{(6-p)(p-2)}{14-3p}, \quad \textit{for all } t\in\mathbb{R} \textit{ and } p\ge 4\alpha. \tag{10}$$

Proof The proofs of (6)–(8) are similar to those of Lemma 2.1 in [20], so we omit them. For the case (9), By the definition of g_γ and (3), we obtain

$$\frac{g_\gamma'(t)t}{g_\gamma(t)} \ge \frac{-C(\frac{1}{2}+2\alpha^2\gamma t^{2(2\alpha-1)})\sqrt{\eta(t)}}{(1+4\alpha^2\gamma t^{2(2\alpha-1)})\eta(t)+1} \ge \begin{cases} -C & \text{if } 0\le t<(\frac{p-2}{8\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}}, \\ 0 & \text{if } t\ge(\frac{p-2}{8\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}}. \end{cases}$$

Moreover, for $0\le t<(\frac{p-2}{8\alpha^2\gamma(4\alpha-p)})^{\frac{1}{2(2\alpha-1)}}$, we know that $(p-2)+(4p-16\alpha)\alpha^2\gamma t^{2(2\alpha-1)} \ge \frac{p-2}{2} > 0$. Hence

$$\begin{aligned} &\frac{p-2}{2}-\frac{g_\gamma'(t)t}{g_\gamma(t)} \\ &\quad= \frac{[(p-2)+(4p-16\alpha)\alpha^2\gamma t^{2(2\alpha-1)}]\eta(t)-\eta'(t)t(1+4\alpha^2\gamma t^{2(2\alpha-1)})+p-2}{4g_\gamma^2(t)} \\ &\quad\ge \frac{p-2}{2[(1+4\alpha^2\gamma t^{2(2\alpha-1)})\eta(t)+1]} \ge \frac{(p-2)(4\alpha-p)}{16\alpha-2-3p}, \end{aligned}$$

which yields the result.

For the case (10), since $p\ge 4\alpha$, it is easy to see that $(p-2)+(4p-16\alpha)\alpha^2\gamma t^{2(2\alpha-1)} > 0$. Then

$$\frac{p-2}{2}-\frac{g_\gamma'(t)t}{g_\gamma(t)} \ge \frac{p-2}{2[(1+4\alpha^2\gamma t^{2(2\alpha-1)})\eta(t)+1]} \ge \frac{(p-2)(6-p)}{22-3p}. \qquad \square$$

According to the properties of g_γ, we will focus on the existence of positive solutions for the following general quasilinear Schrödinger equation:

$$-\operatorname{div}\bigl(g_\gamma^2(u)\nabla u\bigr)+g_\gamma(u)g_\gamma'(u)|\nabla u|^2+V(x)u=|u|^{p-2}u, \quad x\in\mathbb{R}^N. \tag{11}$$

The energy functional of (11) is

$$E_\gamma(u)=\frac{1}{2}\int_{\mathbb{R}^N} g_\gamma^2(u)|\nabla u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx-\frac{1}{p}\int_{\mathbb{R}^N}|u|^p\,dx.$$

Furthermore, we introduce $G_\gamma(t)=\int_0^t g_\gamma(s)\,ds$ and the change of variables $u=G_\gamma^{-1}(v)$. Then that functional E_γ can be rewritten as

$$J_\gamma(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N} V(x)\left|G_\gamma^{-1}(v)\right|^2 dx-\frac{1}{p}\int_{\mathbb{R}^N}\left|G_\gamma^{-1}(v)\right|^p dx.$$

This means that the function v is the solution of the following equation:

$$-\Delta v + V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} - \frac{|G_\gamma^{-1}(v)|^{p-2}G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} = 0, \quad x \in \mathbb{R}^N. \tag{12}$$

From Lemma 2.1, J_γ is well defined in $H^1(\mathbb{R}^N)$ and of class C^1.

Lemma 2.2 *Assume that* $V(x) = \mu > 0$ *and* $h(v) = \frac{|G_\gamma^{-1}(v)|^{p-2}G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} - \mu\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}$. *Then*

$$\lim_{v\to 0}\frac{h(v)}{v} = -\mu, \qquad \lim_{v\to\infty}\frac{h(v)}{v^{\frac{N+2}{N-2}}} = 0$$

and there is a $\xi > 0$ *such that* $H(\xi) = \int_0^\xi h(s)\,ds > 0$.

Proof From Lemma 2.1, we have $G_\gamma^{-1}(v) \to 0$ and $g_\gamma(G_\gamma^{-1}(v)) \to 1$ as $v \to 0$. $G_\gamma^{-1}(v) \to \infty$ and $g_\gamma(G_\gamma^{-1}(v)) \to \frac{1}{\sqrt{2}}$ as $v \to \infty$. Hence

$$\begin{aligned} \lim_{v\to 0}\frac{h(v)}{v} &= \lim_{v\to 0}\frac{|G_\gamma^{-1}(v)|^{p-2}G_\gamma^{-1}(v)}{vg_\gamma(G_\gamma^{-1}(v))} - \mu\lim_{v\to 0}\frac{G_\gamma^{-1}(v)}{vg_\gamma(G_\gamma^{-1}(v))} = -\mu, \\ \lim_{v\to\infty}\frac{h(v)}{v^{\frac{N+2}{N-2}}} &= \lim_{v\to\infty}\frac{|G_\gamma^{-1}(v)|^{p-2}G_\gamma^{-1}(v)}{G_\gamma^{-1}(v)^{\frac{N+2}{N-2}}}\frac{G_\gamma^{-1}(v)^{\frac{N+2}{N-2}}}{v^{\frac{N+2}{N-2}}g_\gamma(G_\gamma^{-1}(v))} - 0 = 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_0^{G_\gamma(\xi)} h(s)\,ds &= \int_0^{G_\gamma(\xi)} \left|G_\gamma^{-1}(s)\right|^{p-2}G_\gamma^{-1}(s)\,dG_\gamma^{-1}(s) - \mu\int_0^{G_\gamma(\xi)} G_\gamma^{-1}(s)\,dG_\gamma^{-1}(s) \\ &= \frac{\xi^p}{p} - \frac{\mu\xi}{2}. \end{aligned}$$

Hence, there is a $\xi > 0$ such that $H(\xi) = \int_0^\xi h(s)\,ds > 0$. □

Lemma 2.3 *Assume that* (V_1) *holds. Then any* (PS) *sequence* $\{v_n\}$ *of* J_γ *is bounded.*

Proof Let $\{v_n\}$ be a (PS) sequence, we have

$$\begin{aligned} J_\gamma(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)\left|G_\gamma^{-1}(v_n)\right|^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N}\left|G_\gamma^{-1}(v_n)\right|^p\,dx \\ &= c_\gamma + o_n(1), \end{aligned} \tag{13}$$

$$\begin{aligned} \langle J_\gamma'(v_n), \psi\rangle &= \int_{\mathbb{R}^N}\nabla v_n\nabla\psi\,dx + \int_{\mathbb{R}^N}V(x)\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}\psi\,dx \\ &\quad - \int_{\mathbb{R}^N}\frac{|G_\gamma^{-1}(v_n)|^{p-2}G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}\psi\,dx = o(\|\psi\|) \\ &\text{for all } \psi \in H^1(\mathbb{R}^N). \end{aligned} \tag{14}$$

Taking $\psi_n = G_\gamma^{-1}(v_n)g_\gamma(G_\gamma^{-1}(v_n))$. From Lemma 2.1, we can get

$$|\nabla\psi_n| = \left|\left[1 + \frac{G_\gamma^{-1}(v_n)g'_\gamma(G_\gamma^{-1}(v_n))}{g_\gamma(G_\gamma^{-1}(v_n))}\right]\nabla v_n\right| \le C_0|\nabla v_n|$$

and $|\psi_n| \le \sqrt{\frac{16\alpha-2-3p}{2(4\alpha-p)}}|v_n|$ if $p \le 4\alpha$, $|\psi_n| \le \sqrt{\frac{22-3p}{2(6-p)}}|v_n|$ if $p \ge 4\alpha$.

If $p \le 4\alpha$, combining (13), (14) and (9) of Lemma 2.1, we get

$$\begin{aligned} pc_\gamma + o(1) + o(1)\|v_n\| &\ge \frac{(p-2)(4\alpha-p)}{16\alpha-2-3p}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx + \frac{p-2}{2}\int_{\mathbb{R}^N}V(x)\left|G_\gamma^{-1}(v_n)\right|^2\,dx \\ &\ge \frac{(p-2)(4\alpha-p)}{16\alpha-2-3p}\|v\|^2. \end{aligned}$$

If $p \ge 4\alpha$, combining (13), (14) and (10) of Lemma 2.1, we get

$$\begin{aligned} pc_\gamma + o(1) + o(1)\|v_n\| &\ge \frac{(p-2)(6-p)}{22-3p}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx + \frac{p-2}{2}\int_{\mathbb{R}^N}V(x)\left|G_\gamma^{-1}(v_n)\right|^2\,dx \\ &\ge \frac{(p-2)(6-p)}{22-3p}\|v\|^2. \end{aligned}$$

This shows the boundedness of $\{v_n\}$ in $H^1(\mathbb{R}^N)$. □

3 The proof of theorems

Proof of Theorem 1.1 If $V(x) = \mu > 0$, from Lemma 2.2, a standard method similar to the proof of [6] indicates that there is a solution v_γ of Eq. (12) satisfies: (i) $v_\gamma > 0$ is spherically symmetric and v_γ decrease with respect to $|x|$; (ii) $v_\gamma \in C^2(\mathbb{R}^N)$; (iii) v_γ together with its derivatives up to order 2 have exponential decay at infinity: $|D^\alpha v_\gamma| \le Ce^{-\delta|x|}$, $x \in \mathbb{R}^N$, for some $C, \delta > 0$ and $|\alpha| \le 2$. Then, according the techniques of [2, 10, 20], we can deduce that $u_\gamma = G^{-1}(v_\gamma)$ is a solution of problem (1) and $\|\nabla u_\gamma\|_\infty \le C$. Moreover, there is a u_0, such that $u_\gamma = G^{-1}(v_\gamma) \to u_0$, where u_0 is a nonnegative solution of problem $-\triangle u + \mu u = |u|^{p-2}u$ in $\mathbb{R}^N$. Furthermore, similar to the proof of Lemma 4.5 in [20], we can deduce that $u_\gamma \to u_0$ in $H^2(\mathbb{R}^N)$.

Similar to the proof of Lemma 5.5 in [3] or Lemma 4.6 in [20], we know that $|v_\gamma| \le \frac{C}{|x|}\|v_\gamma\| \le \frac{C}{|x|}$, $|x| \ge 1$. Then, for any $\varepsilon > 0$ and $q > 2$, there exists $R > 0$ independent of γ, such that

$$\begin{aligned} &\left\|-\mu\frac{G_\gamma^{-1}(v_\gamma)}{g_\gamma(G_\gamma^{-1}(v_\gamma))} + \frac{|G_\gamma^{-1}(v_\gamma)|^{p-2}G_\gamma^{-1}(v_\gamma)}{g_\gamma(G_\gamma^{-1}(v_\gamma))}\right\|_{L^q(\mathbb{R}^N\setminus B_R(0))} < \varepsilon, \\ &\|\mu u_0\|_{L^q(\mathbb{R}^N\setminus B_R(0))} + \left\||u_0|^{p-2}u_0\right\|_{L^q(\mathbb{R}^N\setminus B_R(0))} < \varepsilon. \end{aligned}$$

From $\|u_\gamma\|_\infty = \|G_\gamma^{-1}(v_\gamma)\|_\infty \le C$, we get $G_\gamma^{-1}(v_\gamma) \to u_0$, a.e. in $\mathbb{R}^N$ and

$$-\mu\frac{G_\gamma^{-1}(v_\gamma)}{\sqrt{1 + 2\alpha^2\gamma|G_\gamma^{-1}(v_\gamma)|^{2(2\alpha-1)}}} \to -\mu u_0, \quad \text{a.e. in } \mathbb{R}^N.$$

Using the Lebesgue dominated convergence theorem, we have

$$\left\|-\mu\frac{G_\gamma^{-1}(v_\gamma)}{\sqrt{1+2\alpha^2\gamma|G_\gamma^{-1}(v_\gamma)|^{2(2\alpha-1)}}}-\mu u_0\right\|_{L^q(B_R(0))}+\left\||u_\gamma|^{p-2}u_\gamma-|u_0|^{p-2}u_0\right\|_{L^q(B_R(0))}\to 0.$$

Hence $\limsup_{\gamma\to 0^+}\|\Delta(v_\gamma-u_0)\|_{L^q}\le 2\varepsilon$. From the arbitrariness of ε, we have $v_\gamma\to u_0$ in $W^{2,q}(\mathbb{R}^N)$ for any $q>2$ as $\gamma\to 0^+$. From the Sobolev embedding, we get $v_\gamma\to u_0$ in $C^{1,\alpha}(\mathbb{R}^N)$. Moreover, the bootstrap arguments indicate that $v_\gamma\to u_0$ in $C^2(\mathbb{R}^N)$.

From the definition of v_γ, we have

$$|v_\gamma-u_\gamma|=\left|\int_0^{u_\gamma}\left(\sqrt{1+2\alpha^2\gamma|t|^{2(2\alpha-1)}}-1\right)dt\right|\le\frac{\alpha^2\gamma u_\gamma^{4\alpha-1}}{4\alpha-1}.$$

Hence $\sup_{x\in\mathbb{R}^N}|v_\gamma(x)-u_\gamma(x)|\le C\gamma\|u_\gamma\|_\infty^3\to 0$ as $\gamma\to 0$.

Furthermore, from the definition of v_γ, we know that $\nabla v_\gamma=g_\gamma(u_\gamma)\nabla u_\gamma$ and

$$\begin{aligned}\sup_{x\in\mathbb{R}^N}\left|\nabla v_\gamma(x)-\nabla u_\gamma(x)\right|&=\sup_{x\in\mathbb{R}^N}\left|\left(g_\gamma(u_\gamma)-1\right)\nabla u_\gamma\right|=\sup_{x\in\mathbb{R}^N}\left|\frac{2\alpha^2\gamma u_\gamma^{2(2\alpha-1)}\nabla u_\gamma}{\sqrt{1+2\alpha^2\gamma u_\gamma^{2(2\alpha-1)}}+1}\right|\\&\le\sup_{x\in\mathbb{R}^N}\left|\alpha^2\gamma u_\gamma^{2(2\alpha-1)}\nabla u_\gamma\right|\le\alpha^2\gamma\left||u_\gamma|\right|_\infty^{2(2\alpha-1)}\left||\nabla u_\gamma|\right|_\infty\to 0,\\ \sup_{x\in\mathbb{R}^N}\left|-\mu\frac{G_\gamma^{-1}(v_\gamma)}{g_\gamma(G_\gamma^{-1}(v_\gamma))}+\frac{|G_\gamma^{-1}(v_\gamma)|^{p-2}G_\gamma^{-1}(v_\gamma)}{g_\gamma(G_\gamma^{-1}(v_\gamma))}-\mu u_\gamma-|u_\gamma|^{p-2}u_\gamma\right|&\to 0\end{aligned}$$

as $\gamma\to 0$. On the other hand,

$$|\Delta u_\gamma|=\left|\frac{1}{1+2\alpha^2\gamma|u_\gamma|^{2(2\alpha-1)}}\left[2(2\alpha-1)\alpha^2\gamma|u_\gamma|^{4\alpha-4}u_\gamma|\nabla u_\gamma|^2-\mu u_\gamma+|u_\gamma|^{p-2}u_\gamma\right]\right|\le C.$$

It indicates that

$$\begin{aligned}&\sup_{x\in\mathbb{R}^N}\left|\Delta(v_\gamma-u_\gamma)\right|\\&\quad\le\sup_{x\in\mathbb{R}^N}\left|2\alpha^2\gamma u_\gamma^{2(2\alpha-1)}\Delta u_\gamma\right|+\sup_{x\in\mathbb{R}^N}\left|2(2\alpha-1)\alpha^2\gamma u_\gamma^{4\alpha-3}|\nabla u_\gamma|^2\right|\\&\qquad+\sup_{x\in\mathbb{R}^N}\left|-\mu\frac{G_\gamma^{-1}(v_\gamma)}{g_\gamma(G_\gamma^{-1}(v_\gamma))}+\frac{|G_\gamma^{-1}(v_\gamma)|^{p-2}G_\gamma^{-1}(v_\gamma)}{g_\gamma(G_\gamma^{-1}(v_\gamma))}-\mu u_\gamma-|u_\gamma|^{p-2}u_\gamma\right|\to 0.\end{aligned}\tag{15}$$

As in [3] Lemma 5.5, or [20] Lemma 4.6, (15) together with the Sobolev interpolation inequality yields

$$\sup_{x\in\mathbb{R}^N}\left|D^j(v_\gamma-u_\gamma)\right|\to 0,\quad |j|\le 2.$$

Multiplying u_γ by (1), we have

$$\int_{x\in\mathbb{R}^N}\left(1+4\alpha^3\gamma u_\gamma^2\right)|\nabla u_\gamma|^2+\mu u_\gamma^2-u_\gamma^p\,dx=0.$$

This implies that

$$\int_{x\in\mathbb{R}^N} \mu u_\gamma^2 - u_\gamma^p < 0.$$

If $u_\gamma(0) = \|u_\gamma\|_{L^\infty} \le \mu^{\frac{1}{p-2}}$, one has $\mu u_\gamma^2 - u_\gamma^p \ge 0$, from which we arrive at a contradiction. Then we get $u_\gamma(0) > \mu^{\frac{1}{p-2}}$. Since $u_\gamma \to u_0$ in C^2, we can obtain $u_0(0) \ge \mu^{\frac{1}{p-2}}$. By the maximum principle, we finally get $u_0 > 0$. □

The proof of Theorem 1.2 From Lemma 2.1, a standard discussion shows that J_γ satisfies the mountain pass geometric hypothesis. Hence, there exists a (PS) sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$, such that $J_\gamma(v_n) \to c_\gamma$ and $J'_\gamma(v_n) \to 0$, where $c_\gamma = \inf_{\xi\in\Gamma_\gamma} \sup_{t\in[0,1]} J_\gamma(\xi(t))$, $\Gamma_\gamma = \{\xi(t) \in C([0,1], H^1(\mathbb{R}^N)) : \xi(0) = 0, \xi(1) \neq 0, J_\gamma(\xi(1)) < 0\}$. Then, from Lemma 2.3, we see that the sequence $\{v_n\}$ is bounded. This indicates that there is a subsequence of $\{v_n\}$, denoted still by $\{v_n\}$, there is $v_\gamma \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v_\gamma$ in $H^1(\mathbb{R}^N)$, $v_n \to v_\gamma$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N), q \in [2, 2^*)$. Hence, using Lebesgue dominated convergence theorem, it is easy to see that $J'_\gamma(v_\gamma) = 0$. Furthermore, we can replace v_n by $|v_n|$. Hence, we can assume that $v_n \ge 0$ in $\mathbb{R}^N$ and $v_\gamma \ge 0$. If $v_\gamma \neq 0$, then v_γ is a positive solution of Eq. (12). By contradiction, we assume that $v_\gamma = 0$. In this time, consider the functional $J_\gamma^\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_\gamma^\infty = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V_\infty \left|G_\gamma^{-1}(v_n)\right|^2\right) dx - \frac{1}{p}\int_{\mathbb{R}^N} \left|G_\gamma^{-1}(v_n)\right|^p dx.$$

Then we get a contradiction as in a similar proof to [9, 19, 20] by using the compactness lemma [13]. Hence, v_γ is a nontrivial solution of Eq. (12). By using the fact that $G_\gamma^{-1}(t) \in C^2$ together with Lemma 2.1, a direct computation shows that $u = G_\gamma^{-1}(v) \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. If v_γ is a critical point for J_γ, we know that

$$\int_{\mathbb{R}^N} \left[\nabla v \nabla \psi + V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}\psi - \frac{|G_\gamma^{-1}(v)|^{p-2}G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}\psi\right] dx = 0 \quad \text{for all } \psi \in H^1\left(\mathbb{R}^N\right). \tag{16}$$

Taking $\psi = g_\gamma(u)\varphi \in C_0^2(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$ in (16), we have

$$\int_{\mathbb{R}^N} \left[g_\gamma^2(u)\nabla u \nabla \varphi + g_\gamma(u)g'_\gamma(u)|\nabla u|^2\varphi + V(x)u\varphi + |u|^{p-2}u\varphi\right] dx = 0.$$

It means that u is a classical solution of (11). In the next part of this section, we will prove that $u = G^{-1}(v_\gamma)$ is the solution of Eq. (1).

If $p \le 4\alpha$, we define the functional $P : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$P(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + 2V_\infty \int_{\mathbb{R}^N} |v|^2 dx - \frac{1}{p}\left[\frac{4(4\alpha - p)}{16 - 2\alpha - 3p}\right]^{\frac{p}{2}} \int_{\mathbb{R}^N} |v|^p dx.$$

Then the function v satisfies the equation

$$-\Delta v + 4V_\infty v = \left[\frac{4(4\alpha - p)}{16 - 2\alpha - 3p}\right]^{\frac{p}{2}} |v|^{p-2}v, \quad x \in \mathbb{R}^N. \tag{17}$$

From Jeanjean and Tanaka [10], if we consider the set $\Gamma = \{\xi \in C([0,1], H^1(\mathbb{R}^N)) : \xi(0) = 0, \xi(1) \neq 0, P(\xi(1)) < 0\}$. Then $m = \inf_{\xi \in \Gamma} \sup_{t\in[0,1]} P(\xi(t))$ is the least energy level of the functional $P(v)$.

Since v_γ is a critical point of J_γ, one has

$$pc_\gamma = pJ_\gamma(v_\gamma) - \langle J'_\gamma(v_\gamma), G_\gamma^{-1}(v_\gamma) g_\gamma\big(G_\gamma^{-1}(v_\gamma)\big)\rangle \geq \frac{(p-2)(4\alpha - p)}{16 - 2\alpha - 3p} \int_{\mathbb{R}^N} |\nabla v_\gamma|^2 \, dx.$$

This indicates that

$$\|\nabla v_\gamma\|_2^2 \leq \frac{p(16 - 2\alpha - 3p)}{(p-2)(4\alpha - p)} c_\gamma.$$

Furthermore, from the property (7) of Lemma 2.1, we can deduce that $J_\gamma(v) \leq P(v)$ and $\Gamma \subset \Gamma_\gamma$. Hence

$$c_\gamma = \inf_{\xi\in\Gamma_\gamma} \sup_{t\in[0,1]} J_\gamma\big(\xi(t)\big) \leq \inf_{\xi\in\Gamma} \sup_{t\in[0,1]} J_\gamma\big(\xi(t)\big) \leq \inf_{\xi\in\Gamma} \sup_{t\in[0,1]} P\big(\xi(t)\big) := m$$

and

$$\|\nabla v_\gamma\|_2^2 \leq \frac{p(16 - 2\alpha - 3p)}{(p-2)(4\alpha - p)} m. \tag{18}$$

Using the Sobolev inequality, we can get

$$\|v_\gamma\|_{2^*} \leq \sqrt{\frac{pm(16 - 2\alpha - 3p)}{S(p-2)(4\alpha - p)}}, \tag{19}$$

where S is the best Sobolev constant.

If $p \geq 4\alpha$, we define the function $P : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$P(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + 2V_\infty \int_{\mathbb{R}^N} |v|^2 \, dx - \frac{1}{p} \left[\frac{4(6-p)}{22 - 3p}\right]^{\frac{p}{2}} \int_{\mathbb{R}^N} |v|^p \, dx,$$

the set Γ and m are defined like $p \leq 4\alpha$. In this time, if v_γ is a critical point of J_γ,

$$pc_\gamma = pJ_\gamma(v_\gamma) - \langle J'_\gamma(v_\gamma), G_\gamma^{-1}(v_\gamma) g_\gamma\big(G_\gamma^{-1}(v_\gamma)\big)\rangle \geq \frac{(p-2)(6-p)}{22 - 3p} \int_{\mathbb{R}^N} |\nabla v_\gamma|^2 \, dx.$$

Hence, we can deduce that

$$\|\nabla v_\gamma\|_2^2 \leq \frac{p(22 - 3p)}{(p-2)(6-p)} m \tag{20}$$

and

$$\|v_\gamma\|_{2^*} \leq S^{-\frac{1}{2}} \|\nabla v_\gamma\|_2 \leq \sqrt{\frac{pm(22 - 3p)}{S(p-2)(6-p)}}. \tag{21}$$

Then, by the same proof as Proposition 3.6 of [20], we can deduce that there exists a constant $K > 0$ independent of γ such that $\|v_\gamma\|_\infty \le K$. If $p \le 4\alpha$, let $\gamma_0 := \frac{p-2}{32\alpha^2(4\alpha-p)(2K)^{2(2\alpha-1)}}$, we have

$$\|u_\gamma\|_\infty = \left\|G_\gamma^{-1}(v_\gamma)\right\| \le 2\|v_\gamma\|_\infty \le 2K \le \left(\frac{p-2}{32\alpha^2\gamma(4\alpha-p)}\right)^{\frac{1}{2(2\alpha-1)}} \quad \text{for all } \gamma \in (0,\gamma_0).$$

If $p \ge 4\alpha$, let $\gamma_0 := \frac{p-2}{32\alpha^2(6-p)(2K)^{2(2\alpha-1)}}$, we get

$$\|u_\gamma\|_\infty = \left\|G_\gamma^{-1}(v_\gamma)\right\| \le 2\|v_\gamma\|_\infty \le 2K \le \left(\frac{p-2}{32\alpha^2\gamma(6-p)}\right)^{\frac{1}{2(2\alpha-1)}} \quad \text{for all } \gamma \in (0,\gamma_0).$$

Hence, we can deduce that $g_\gamma(u_\gamma) = \sqrt{1+2\alpha^2\gamma|u_\gamma|^{2(2\alpha-1)}}$ if $\gamma \in (0,\gamma_0)$ and so $u_\gamma = G_\gamma^{-1}(v_\gamma)$ is a positive solution of (1). □

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