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Infinitely many solutions for fractional Schrödinger equation with potential vanishing at infinity

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Abstract
The paper investigates the following fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = K(x)f(u), \quad x \in \mathbb{R}^N,$$

where $0 < s < 1$, $2s < N$, $(-\Delta)^s$ is the fractional Laplacian operator of order s. $V(x)$, $K(x)$ are nonnegative continuous functions and $f(x)$ is a continuous function satisfying some conditions. The existence of infinitely many solutions for the above equation is presented by using a variant fountain theorem, which improves the related conclusions on this topic. The interesting result of this paper is the potential $V(x)$ vanishing at infinity, i.e., $\lim_{|x|\to+\infty} V(x) = 0$.

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1 Introduction

This paper deals with the following fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = K(x)f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $0 < s < 1$, $2s < N$, $V(x)$, $K(x)$ are nonnegative continuous functions, which satisfy some conditions, and the function f is continuous on $\mathbb{R}^N$. $(-\Delta)^s$ is the fractional Laplacian operator of order s, i.e., $(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s}\mathscr{F}u)$, where $\mathscr{F}$ is the usual Fourier transform in $\mathbb{R}^N$. Especially, the potential $V(x)$ vanishes at infinity, i.e., $\lim_{|x|\to+\infty} V(x) = 0$ (shortly $V(\infty) = 0$).

Since Laskin in [1, 2], for the first time, studied the existence of standing wave solutions for fractional Schrödinger equation of the form

$$ih\frac{\partial\psi}{\partial t} = (-\Delta)^s\psi + V(x)\psi - f(x,\psi), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.2}$$

the fractional Schrödinger equation has become an important model in fractional quantum mechanics. Recently, the fractional Schrödinger equation has attracted a large number of studies for its property, a lot of papers deal with the existence of solutions of it by using critical point theory, see [3–8] and the references therein.

When $s = 1$, the fractional Schrödinger equation (1.1) reduces to the following classical nonlinear Schrödinger equation:

$$-\Delta u + V(x)u = K(x)f(u), \quad x \in \mathbb{R}^N. \tag{1.3}$$

There have been many works focusing on the existence of solutions of equation (1.3) in the last decades, such as sign-changing solutions, ground state solutions, radially symmetric solutions, multiplicity of standing wave solutions, and so on, see [9, 10]. Alves and Souto in [9] studied equation (1.3) with the potential function $V(x)$ vanishing at infinity. They proposed the following assumptions for $V(x)$, $K(x)$:

(I) For any $x \in \mathbb{R}^N$, $V(x), K(x) > 0$ holds and $K(x) \in L^\infty(\mathbb{R}^N)$.

(II) For some $R > 0$ and $|A_n| \leq R$, then $\lim_{r\to+\infty} \int_{A_n \cap B_r^c(0)} K(x)\,dx = 0$, where $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets.

(III) $\frac{K(x)}{V(x)} \in L^\infty(\mathbb{R}^N)$.

(IV) For $p \in (2, 2^*)$, then

$$\frac{K(x)}{V(x)^{\frac{2^*-p}{2^*-2}}} \to 0 \quad \text{as } |x| \to +\infty.$$

Then they obtained a positive ground state solution of equation (1.3) by using the variational method.

In [9], Alves and Souto studied the classical nonlinear Schrödinger equation (1.3), but for fractional Schrödinger equation of the form (1.1), since it is difficult to get compact embedding in fractional Sobolev spaces, especially when the potential function $V(x)$ vanishes at infinity. There are few papers dealing with this aspect. In [11], Li et al. obtained a positive solution of the fractional Schrödinger equation with vanishing at infinity by using the variational method. Based on the variational method in [12] of Yang and Zhao, three solutions have been got for a class of fractional Schrödinger equations with vanishing at infinity. Especially, as far as we know, there are no results that study the existence of infinitely many solutions for problem (1.1). We focus on this topic and obtain some new conclusions. We make the following assumptions for the function f. Note that we do not need all assumptions to hold simultaneously.

(f1) For all $t \in \mathbb{R}, f(-t) = -f(t)$ holds.

(f2) $f(t)t \geq 0$ holds for all $t \in \mathbb{R}$, there exists $\nu \in (1,2)$ such that

$$\left|f(t)\right| \leq c_1\left(1 + |t|^{\nu-1}\right) \quad \text{for all } t \in \mathbb{R} \text{ and some } c_1 > 0.$$

(f3) $\lim_{|t|\to 0} \frac{f(t)}{t} = 0$.

(f4) There exists a constant $d > 0$ such that $\lim_{|t|\to\infty} \inf \frac{F(t)}{|t|} \geq d$, where $F(t) = \int_0^t f(s)\,ds$.

(f5) $f(t)t \geq 0$ holds for all $t \in \mathbb{R}$, and there exist $c_2 > 0$ and $p \in (2, 2_s^*)$ such that

$$\left|f(t)\right| \leq c_2\left(1 + |t|^{p-1}\right) \quad \text{for all } t \in \mathbb{R} \text{ and some } c_2 > 0.$$

(f6) $\lim_{|t|\to\infty} \frac{F(t)}{|t|^2} = \infty$.

(f7) There exist two constants $\mu > 2$ and $c_3 > 0$ such that

$$f(t)t - \mu F(t) \geq c_3\left(1 + t^2\right) \quad \text{for all } t \in \mathbb{R}.$$

In the following, we present the main results of this paper.

Theorem 1.1 *If the functions V, K satisfy* (I)–(III) *and f satisfies* (f1)–(f4), *then* (1.1) *has infinitely many small energy solutions $u_k \in H$ for every $k \in \mathbb{N}$.*

Theorem 1.2 *Assume that V, K satisfy* (I)–(III), *f satisfies* (f1) *and* (f5)–(f7), *then* (1.1) *admits infinitely many high energy solutions $u^k \in H$ for every $k \geq k_0$ ($k_0 \in \mathbb{N}$).*

Throughout this paper, $s \in (0,1)$ is a fixed constant, $\|\cdot\|_p$ is the usual norm of $L^p(\mathbb{R}^N)$, $B_r(x)$ is an open ball centered at x with radius r, c_i ($i = 1, 2, \dots$), and C represents different positive constants.

We organize this paper as follows. The preliminaries of a fractional Sobolev space and the variant fountain theorems are presented in Sect. 2. In Sect. 3, we give the proofs of Theorems 1.1 and 1.2.

2 Preliminaries

In this section, we firstly provide a short review of fractional Sobolev spaces. For more information, we refer to [13, 14].

For $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s\left(\mathbb{R}^N\right) := \left\{u \in L^2\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} \left(1 + |\xi|^2\right)^s \left|\mathscr{F}u(\xi)\right|^2 d\xi < +\infty\right\},$$

and the norm is

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^N} u^2\,dx + \int_{\mathbb{R}^N} |\xi|^{2s}\left|\mathscr{F}u(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}.$$

For problem (1.1), the working space is defined as follows:

$$H := \left\{u \in H^s\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} |\xi|^{2s}\left|\mathscr{F}u(\xi)\right|^2 d\xi + \int_{\mathbb{R}^N} V(x)u^2\,dx < +\infty\right\}.$$

Apparently, H is a Hilbert space with the inner product

$$(u,v) := \int_{\mathbb{R}^N} |\xi|^{2s}\mathscr{F}u(\xi)\mathscr{F}v(\xi)\,d\xi + \int_{\mathbb{R}^N} V(x)uv\,dx$$

and the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} |\xi|^{2s}\left|\mathscr{F}u(\xi)\right|^2 d\xi + \int_{\mathbb{R}^N} V(x)u^2\,dx\right)^{\frac{1}{2}}.$$

Denote by $L_k^r(\mathbb{R}^N)$ the weighted Lebesgue space

$$L_k^r\left(\mathbb{R}^N\right) := \left\{u \in \mathbb{R}^N : \int_{\mathbb{R}^N} K(x)|u|^r\,dx < +\infty\right\}$$

equipped with the norm

$$\|u\|_{L_k^r(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} K(x)|u|^r\,dx\right)^{\frac{1}{r}}.$$

In the following, we present the definition of a weak solution of (1.1).

Definition 2.1 We say that $u \in H$ is a weak solution of (1.1) if

$$\int_{\mathbb{R}^N} |\xi|^{2s}\mathscr{F}u(\xi)\mathscr{F}v(\xi)d\xi + \int_{\mathbb{R}^N} V(x)uv\,dx = \int_{\mathbb{R}^N} K(x)f(u)v\,dx, \quad \forall v \in H.$$

Define the energy functional for problem (1.1) by

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F(u)\,dx.$$

Under our assumptions, $I \in C^1(H,\mathbb{R})$ is well-defined and Gâteaux-differentiable on H, that is,

$$\begin{aligned}\langle I'(u,v)\rangle = {}& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)uv\,dx \\ & - \int_{\mathbb{R}^N} K(x)f(u)v\,dx \quad \text{for all } u,v \in H.\end{aligned}$$

Moreover, the critical points of energy functional I are solutions to problem (1.1).

Lemma 2.2 ([13]) *Let $0 < s < 1$, $N \geq 1$ such that $2s < N$, then*

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)} \leq C\|u\|_{H^s(\mathbb{R}^N)}, \quad \forall u \in H^s(\mathbb{R}^N),$$

where $C = C(N,s) > 0$ and $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent. Furthermore, the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for every $2 \leq q \leq 2_s^*$ and is locally compact for all $2 \leq q < 2_s^*$.

Lemma 2.3 ([11]) *Assume that* (I)–(III) *are satisfied, then the embedding $H \hookrightarrow L_K^q(\mathbb{R}^N)$ is compact whenever $q \in [2, 2_s^*)$.*

Remark 2.4 From Lemma 2.3, we obtain the following inequality:

$$\|u\|^q_{L_k^q(\mathbb{R}^N)} \leq d_1\|u\|^q \quad \textit{for } q \in [2, 2_s^*), \tag{2.1}$$

where d_1 is the best constant for $H \hookrightarrow L_K^q(\mathbb{R}^N)$.

In the rest of this section, we state two variant fountain theorems which will be used later.

Let X be a Banach space and the norm is $\|\cdot\|$. Denote by $\{X_j\}$ a sequence of subspaces of X and $\dim X_j < \infty$ for each $j \in \mathbb{N}$; moreover, $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$. We define

$$M_k := \bigoplus_{j=0}^{k} X_j, \qquad N_k := \overline{\bigoplus_{j=k}^{\infty} X_j},$$

and

$$Y_k := \bigl\{u \in M_k : \|u\| \leq \phi_k\bigr\}, \qquad Z_k := \bigl\{u \in N_k : \|u\| = \psi_k\bigr\},$$

where $\phi_k > \psi_k > 0$. For $\lambda \in [1,2]$, $I_\lambda : H \to R$ is a family of C^1-functionals defined by

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

where $\Phi(u) := \frac{1}{2}\|u\|^2$ and $\Psi(u) := \int_{\mathbb{R}^N} K(x)F(u)\,dx$.

Lemma 2.5 ([15]) *Let X be a Banach space. If the functional $I_\lambda(u)$ satisfies:*

(A1) *For $\lambda \in [1,2]$, $I_\lambda(u)$ maps bounded sets into bounded sets uniformly and $I_\lambda(-u) = I_\lambda(u)$ holds for all $u \in X$.*

(A2) *$\Psi(u) \geq 0$ holds for all $u \in X$, and $\Psi(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of X.*

(A3) *For $\lambda \in [1,2]$, there exist $\phi_k > \psi_k > 0$ such that*

$$\alpha_k(\lambda) = \inf_{u\in N_k, \|u\|=\phi_k} I_\lambda(u) \geq 0 > \beta_k(\lambda) = \inf_{u\in M_k, \|u\|=\psi_k} I_\lambda(u)$$

and

$$\gamma_k(\lambda) = \inf_{u\in N_k, \phi_k \geq \|u\|} I_\lambda(u) \quad \text{as } k \to \infty.$$

Then there exist $\lambda_n \to 1$, $u_n(\lambda_n) \in M_n$ such that

$$I'_{\lambda_n}|_{M_n}\bigl(u(\lambda_n)\bigr) = 0 \quad \text{and} \quad I_{\lambda_n}\bigl(u(\lambda_n)\bigr) \to \omega_k \in \bigl[\gamma_k(2), \beta_k(1)\bigr] \quad \text{as } n \to \infty.$$

In particular, if $\{u(\lambda_n)\}$ possesses a convergent subsequence, then I_1 has infinitely many nontrivial critical points $\{u_k\} \in X \setminus \{0\}$ satisfying $I_1(u_k) \to 0^-$ as $k \to \infty$.

Lemma 2.6 ([16]) *Let X be a Banach space. If the functional $I_\lambda(u)$ satisfies:*

(B1) *For $\lambda \in [1,2]$, $I_\lambda(u)$ maps bounded sets into bounded sets uniformly and $I_\lambda(-u) = I_\lambda(u)$ holds for all $u \in X$.*

(B2) *$\Psi(u) \geq 0$ holds for all $u \in X$, $\Phi(u) \to \infty$ or $\Psi(u) \to \infty$ when $\|u\| \to \infty$.*

(B3) *There exists $\phi_k > \psi_k > 0$ such that*

$$e_k(\lambda) = \inf_{u\in N_k, \|u\|=\psi_k} I_\lambda(u) > g_k(\lambda) = \max_{u\in M_k, \|u\|=\phi_k} I_\lambda(u), \quad \forall\lambda \in [1,2].$$

Then

$$e_k(\lambda) \le h_k(\lambda) = \inf_{\eta\in\Gamma_k}\max_{u\in Y_k} I_\lambda\big(\gamma(u)\big), \quad \forall\lambda\in[1,2],$$

where $\Gamma_k = \{\eta\in C(Y_k,X):\eta$ *is odd*, $\eta|_{\partial Y_k} = \mathrm{id}, k\ge 2\}$. *Furthermore, for a.e.* $\lambda\in[1,2]$, *there exists* $\{u_n^k(\lambda)\}_{n=1}^\infty$ *such that*

$$\sup_n\big\|u_n^k(\lambda)\big\| < \infty, \qquad I'_\lambda\big(u_n^k(\lambda)\big)\to\infty \quad \textit{and} \quad I_\lambda\big(u_n^k(\lambda)\big)\to g_k(\lambda) \quad \textit{as } n\to\infty.$$

3 Proofs of the main results

In this section, by using the variant fountain theorem, we prove Theorem 1.1 and Theorem 1.2, respectively.

3.1 Proof of Theorem 1.1

To begin with, we give some lemmas.

Lemma 3.1 *If* $p\in(1,2_s^*)$, *then we have that*

$$\zeta_k = \sup_{u\in N_k,\|u\|=1}\left(\int_{\mathbb{R}^N}K(x)|u|^p\,dx\right)^{\frac{1}{p}} \to 0, \quad k\to\infty.$$

The proof is similar to Lemma 3.8 in [17], here we omit it.

Lemma 3.2 *If* (f2) *and* (f4) *hold, then for all* $u\in H$, $\Psi(u)\ge 0$ *holds and if* $\|u\|\to\infty$, *then* $\Psi(u)\to\infty$ *on any finite dimensional subspace of* H.

Proof From (f2), it is clear that $\Psi(u)\ge 0$ holds for all $u\in H$. Let H_0 be any finite dimensional subspace of H. In the following we will prove that $\Psi(u)\to\infty$ when $\|u\|\to\infty$ on H_0.

Firstly, we show that, for all $u\in H_0\subset H$, there is

$$\operatorname{meas}\big\{x\in\mathbb{R}^N : K(x)\big|u(x)\big|\ge\theta\|u\|\big\}\ge\theta, \tag{3.1}$$

where θ is a constant with $\theta>0$. To prove it, arguing by contradiction, we assume that there exists $u_n\in H_0\setminus\{0\}$ such that

$$\operatorname{meas}\left\{x\in\mathbb{R}^N : K(x)\big|u_n(x)\big|\ge\frac{1}{n}\|u_n\|\right\}<\frac{1}{n}, \quad \forall n\in\mathbb{N}.$$

Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and

$$\operatorname{meas}\left\{x\in\mathbb{R}^N : K(x)\big|v_n(x)\big|\ge\frac{1}{n}\right\}<\frac{1}{n}, \quad \forall n\in\mathbb{N}. \tag{3.2}$$

Since $\|v_n\| = 1$, there exists $v\in H_0$ with $\|v\| = 1$, $v_n\to v$ holds in H. Using Lemma 2.3, we have

$$\int_{\mathbb{R}^N}K(x)\big|v_n(x)-v(x)\big|^2\,dx\to 0, \quad \text{as } n\to\infty. \tag{3.3}$$

From $v \neq 0$ and $K(x) > 0$, there exist two constants $\delta_0 > 0$ and $\delta_1 > 0$ such that

$$\operatorname{meas}\{x \in \mathbb{R}^N : |v(x)| \geq \delta_0\} \geq \delta_0, \tag{3.4}$$

and

$$\operatorname{meas}\{x \in \mathbb{R}^N : K(x)|v(x)| \geq \delta_1\} \geq \delta_1. \tag{3.5}$$

For any $n \in \mathbb{R}^N$, we denote

$$D_n := \left\{x \in \mathbb{R}^N : K(x)|v_n(x)| < \frac{1}{n}\right\},$$
$$D_n^C := \left\{x \in \mathbb{R}^N : K(x)|v_n(x)| \geq \frac{1}{n}\right\},$$

and

$$D_0 := \{x \in \mathbb{R}^N : |v(x)| \geq \delta_0\}.$$

For n sufficiently large, by (3.2), (3.4), and (3.5), we obtain

$$\operatorname{meas}(D_n \cap D_0) \geq \operatorname{meas}(D_0) - \operatorname{meas}(D_n^C) \geq \frac{2\delta_0}{3}.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^N} K(x)|v_n(x) - v(x)|^2 \, dx \\
&\quad \geq \int_{D_n \cap D_0} K(x)|v_n(x) - v(x)|^2 \, dx \\
&\quad \geq \int_{D_n \cap D_0} K(x)\left[|v(x)|^2 - 2|v_n(x)||v(x)|\right] dx \\
&\quad \geq \delta_0\left(\delta_1 - \frac{2}{n}\right) \operatorname{meas}\{D_n \cap D_0\} \\
&\quad \geq \frac{2\delta_0}{3}\left(\delta_1 - \frac{2}{n}\right) \\
&\quad \geq \frac{2\delta_0\delta_1}{9} > 0,
\end{aligned}$$

which contradicts (3.3), thus, (3.1) holds.

By condition (f4), for all $x \in \mathbb{R}^N$ and $\|u\| \geq \delta_2 > 0$, one has

$$F(u) \geq d|u|. \tag{3.6}$$

Define $D_u := \{x \in \mathbb{R}^N : K(x)|u(x)| \geq \theta\|u\|\}$ for $u \in H_0 \setminus \{0\}$. By (3.1), for any $u \in H_0$ with $\|u\| \geq \frac{\delta_2}{\theta}$, $K(x)|u(x)| \geq \delta_2$ holds as $x \in D_u$. Consequently, from (3.6) we obtain

$$\begin{aligned}\Psi(u) &\geq \int_{\mathbb{R}^N} K(x)F(u)\,dx \\ &\geq \int_{D_u} K(x)F(u)\,dx \\ &\geq d\int_{D_u} K(x)\big|u(x)\big|\,dx \\ &\geq d\theta\|u\|\operatorname{meas}\{D_u\} \\ &\geq d\theta^2\|u\|,\end{aligned}$$

which implies that $\Psi(u) \to \infty$ when $\|u\| \to \infty$ on $H_0 \subset H$. □

Lemma 3.3 *If* (f2)–(f4) *hold, then for* $\lambda \in [1,2]$, *there exists* $\phi_k > \psi_k > 0$ *such that*

$$\alpha_k(\lambda) := \inf_{u\in N_k,\|u\|=\phi_k} I_\lambda(u) \geq 0 > \beta_k(\lambda) := \inf_{u\in M_k,\|u\|=\psi_k} I_\lambda(u)$$

and

$$\gamma_k(\lambda) := \inf_{u\in N_k,\phi_k\geq\|u\|} I_\lambda(u) \quad \text{as } k \to \infty.$$

Proof From Lemma 3.1, for $p \in (1, 2_s^*)$, we obtain

$$\int_{\mathbb{R}^N} K(x)|u|^p\,dx \leq \zeta_k^p\|u\|^p. \tag{3.7}$$

By (f2) and (f3), for any $\varepsilon > 0$, one has

$$F(u) \leq \varepsilon|u|^2 + c_\varepsilon|u|^\nu, \tag{3.8}$$

where $c_\varepsilon > 0$ depending on ε. From (2.1) and (3.8), for $u \in N_k$, we have

$$\begin{aligned}I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \lambda\int_{\mathbb{R}^N} K(x)\big[\varepsilon|u|^2 + c_\varepsilon|u|^\nu\big]\,dx \\ &\geq \frac{1}{2}\|u\|^2 - 2\varepsilon\int_{\mathbb{R}^N} K(x)|u|^2\,dx - 2c_\varepsilon\int_{\mathbb{R}^N} K(x)|u|^\nu\,dx \\ &\geq \frac{1}{2}\|u\|^2 - 2d_1\varepsilon\|u\|^2 - 2c_\varepsilon\zeta_k^\nu\|u\|^\nu.\end{aligned}$$

Let $d_1\varepsilon < \frac{1}{12}$, then for any $u \in N_k$ with $\|u\| < 1$, we know that

$$I_\lambda(u) \geq \frac{1}{3}\|u\|^2 - 2c_\varepsilon\zeta_k^\nu\|u\|^\nu. \tag{3.9}$$

Choose $\phi_k = (12c_\varepsilon\zeta_k^\nu)^{\frac{1}{2-\nu}}$, then $\phi_k > 0$ and $\phi_k \to 0$ as $k \to \infty$. For any $\lambda \in [1,2]$, we obtain

$$\alpha_k(\lambda) = \inf_{u\in N_k,\|u\|=\phi_k} I_\lambda(u) \geq \frac{1}{6}\big(12c_\varepsilon\zeta_k^\nu\big)^{\frac{2}{2-\nu}} = \frac{1}{6}\phi_k^2 > 0.$$

From (3.9), for any $u \in N_k$ with $\|u\| \le \phi_k$ and $\lambda \in [1,2]$, one has

$$I_\lambda(u) \ge -2c_\varepsilon \zeta_k^\nu \|u\|^\nu \ge -2c_\varepsilon \zeta_k^\nu \phi_k^\nu,$$

which means that

$$\gamma_k(\lambda) = \inf_{u\in N_k, \|u\|\le\phi_k} I_\lambda(u) \ge -2c_\varepsilon \zeta_k^\nu \phi_k^\nu \to 0^+ \quad \text{as } k \to \infty.$$

Hence, for $\phi_k \to 0$ as $k \to \infty$, we get

$$\gamma_k(\lambda) = \inf_{u\in N_k, \|u\|\le\phi_k} I_\lambda(u) \to 0 \quad \text{as } k \to \infty.$$

By (f2)–(f4), there is

$$F(u) \ge d|u| - \varepsilon|u|^2 - c_\varepsilon|u|^\nu. \tag{3.10}$$

Hence, if $u \in M_k$, by the equivalence of norms on a finite dimensional space, one has

$$\begin{aligned} I_\lambda(u) &\le \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F(u)\,dx \\ &\le \frac{1}{2}\|u\|^2 - d\int_{\mathbb{R}^N} K(x)|u|\,dx + \varepsilon\int_{\mathbb{R}^N} K(x)|u|^2\,dx + c_\varepsilon\int_{\mathbb{R}^N} K(x)|u|^\nu\,dx \\ &\le \|u\|^2 - c_4\|u\| + c_5\|u\|^\nu. \end{aligned}$$

Therefore, we choose $\psi_k > 0$ small enough satisfying $\phi_k > \psi_k > 0$ such that

$$\beta_k(\lambda) = \inf_{u\in M_k, \|u\|=\psi_k} I_\lambda(u) < 0, \quad \forall\lambda \in [1,2]. \qquad \square$$

Lemma 3.4 *If* (f1)–(f3) *hold and for* $\lambda \in [1,2]$, $k \in \mathbb{N}$, *there exist* $\lambda_n \to 1$ *and* $u(\lambda_n) \in M_n$ *such that*

$$I'_{\lambda_n}|_{M_n}\big(u(\lambda_n)\big) = 0 \quad \textit{and} \quad I_{\lambda_n}\big(u(\lambda_n)\big) \to \omega_k \in \big[\gamma_k(2), \beta_k(1)\big] \quad \textit{as } n \to \infty,$$

then $\{u(\lambda_n)\}$ *possesses a convergent subsequence in* H.

Proof Firstly, we show that $\{u(\lambda_n)\}$ is bounded in H. Using (3.8) and the Hölder inequality, for $\lambda_n \in [1,2]$ with $\lambda_n \to 1$ as $n \to \infty$, one has

$$\begin{aligned} \big\|u(\lambda_n)\big\|^2 &= 2I_{\lambda_n}\big(u(\lambda_n)\big) + 2\lambda_n\int_{\mathbb{R}^N} K(x)F\big(u(\lambda_n)\big)\,dx \\ &\le 2(\omega_k+1) + 2\lambda_n\int_{\mathbb{R}^N} K(x)\big[\varepsilon\big|u(\lambda_n)\big|^2 + c_\varepsilon\big|u(\lambda_n)\big|^\nu\big]\,dx \\ &\le 2(\omega_k+1) + c_6\big\|u(\lambda_n)\big\|^2 + c_7\big\|u(\lambda_n)\big\|^\nu. \end{aligned}$$

This implies that $\{u(\lambda_n)\}$ is bounded in H. Passing to a subsequence, without loss of generality, still denoted by $\{u(\lambda_n)\}$, we may assume that $u(\lambda_n) \rightharpoonup u$ weakly in H. By Lemma 2.3, we have $u(\lambda_n) \to u$ strongly in $L_k^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*)$. Let $P_n : H \to Y_n$ denote the orthogonal projection operator, then

$$0 = I'_{\lambda_n}|_{Y_n}\big(u(\lambda_n)\big) = P_n I'_{\lambda_n}\big(u(\lambda_n)\big).$$

Thus $\langle P_n I'_{\lambda_n}(u(\lambda_n)), u(\lambda_n) - u\rangle = 0$. From $u(\lambda_n) \rightharpoonup u$ in H, we know that $\langle I'_1(u), u(\lambda_n) - u\rangle = 0$ as $n \to \infty$. According to the property of the orthogonal projection operator, one finds

$$\begin{aligned}\|u(\lambda_n) - u\|^2 &= \|P_n\big(u(\lambda_n)\big) - u\|^2 \\ &= \big\langle P_n I_{\lambda_n}\big(u(\lambda_n)\big), u(\lambda_n) - u\big\rangle - \big\langle I'_1(u), u(\lambda_n) - u\big\rangle \\ &\quad + \lambda_n \int_{\mathbb{R}^N} K(x) f\big(u(\lambda_n)\big) P_n\big(u(\lambda_n) - u\big)\, dx \\ &\quad - \int_{\mathbb{R}^N} K(x) f(u)\big(u(\lambda_n) - u\big)\, dx.\end{aligned}$$

From conditions (f1) and (f2), for $\varepsilon > 0$ small, we get

$$\big|f(u)\big| \le \varepsilon |u| + c_\varepsilon |u|^{\nu-1}, \tag{3.11}$$

where $c_\varepsilon > 0$ depending on ε. Using the Hölder inequality, we obtain

$$\begin{aligned}&\left|\lambda_n \int_{\mathbb{R}^N} K(x) f\big(u(\lambda_n)\big) P_n\big(u(\lambda_n) - u\big)\, dx\right| \\ &\quad \le \lambda_n \int_{\mathbb{R}^N} K(x)\big[\varepsilon\big|u(\lambda_n)\big| + c_\varepsilon\big|u(\lambda_n)\big|^{\nu-1}\big] P_n\big(u(\lambda_n) - u\big)\, dx \\ &\quad \le 2\varepsilon \int_{\mathbb{R}^N} K(x)\big|u(\lambda_n)\big| P_n\big(u(\lambda_n) - u\big)\, dx + 2c_\varepsilon \int_{\mathbb{R}^N} K(x)\big|u(\lambda_n)\big|^{\nu-1} P_n\big(u(\lambda_n) - u\big)\, dx \\ &\quad \le 2\varepsilon \|K\|_\infty \int_{\mathbb{R}^N} \big|u(\lambda_n)\big| P_n\big(u(\lambda_n) - u\big)\, dx + 2c_\varepsilon \|K\|_\infty \int_{\mathbb{R}^N} \big|u(\lambda_n)\big|^{\nu-1} P_n\big(u(\lambda_n) - u\big)\, dx \\ &\quad \le 2\varepsilon \|K\|_\infty \|u_n\|_2 \big\|P_n\big(u(\lambda_n) - u\big)\big\|_2 + 2c_\varepsilon \|K\|_\infty \|u_n\|_2^{\nu-1} \big\|P_n\big(u(\lambda_n) - u\big)\big\|_2 \to 0\end{aligned}$$

as $n \to \infty$. From (3.11) and the Hölder inequality, one has

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} K(x) f(u)\big(u(\lambda_n) - u\big)\, dx\right| \\ &\quad \le \int_{\mathbb{R}^N} K(x)\big[\varepsilon |u| + c_\varepsilon |u|^{\nu-1}\big]\big(u(\lambda_n) - u\big)\, dx \\ &\quad \le \varepsilon \|K\|_\infty \int_{\mathbb{R}^N} |u|\big(u(\lambda_n) - u\big)\, dx + c_\varepsilon \|K\|_\infty \int_{\mathbb{R}^N} |u|^{\nu-1}\big(u(\lambda_n) - u\big)\, dx \\ &\quad \le \varepsilon \|K\|_\infty \|u_n\|_2 \big\|u(\lambda_n) - u\big\|_2 + c_\varepsilon \|K\|_\infty \|u_n\|_2^{\nu-1} \big\|u(\lambda_n) - u\big\|_2 \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Consequently, $u_n \to u$ in H as $n \to \infty$. □

Proof of Theorem 1.1 By (f2)–(f3) and $\lambda \in [1, 2]$, we know clearly that $I_\lambda(u)$ maps bounded sets into bounded sets. Condition (f1) implies that $I_\lambda(-u) = I_\lambda(u)$, then (A1) of Lemma 2.5 holds.

From Lemmas 3.2 and 3.3, we have (A2) of Lemma 2.5 holds. Thanks to Lemma 2.5, for each $k \in \mathbb{N}$, there exist $\lambda_n \to 1$, $u_n(\lambda_n) \in M_n$ such that

$$I'_{\lambda_n}|_{Y_n}\big(u(\lambda_n)\big) = 0 \quad \text{and} \quad I_{\lambda_n}\big(u(\lambda_n)\big) \to \omega_k \in \big[\gamma_k(2), \beta_k(1)\big] \quad \text{as } n \to \infty.$$

It follows from Lemma 3.4 that $\{u(\lambda_n)\}$ has a convergent subsequence for any $k \in \mathbb{N}$. Using Lemma 2.5, I_1 has infinitely many nontrivial critical points $\{u_k\}$ satisfying

$$I_1(u_k) = \frac{1}{2}\|u_k\|^2 - \int_{\mathbb{R}^N} K(x)F(u_k)\,dx \to 0^- \quad \text{as } k \to \infty$$

for any $k \in \mathbb{N}$. Consequently, problem (1.1) owns infinitely many small energy solutions. The proof of Theorem 1.1 is completed. □

3.2 Proof of Theorem 1.2

Lemma 3.5 *Assume that* (f3), (f5), *and* (f6) *hold. Then there exists* $\phi_k > \psi_k > 0$ *such that*

$$e_k(\lambda) = \inf_{u \in N_k, \|u\| = \psi_k} I_\lambda(u) > g_k(\lambda) = \max_{u \in M_k, \|u\| = \phi_k} I_\lambda(u), \quad \forall \lambda \in [1,2].$$

Proof By (f3) and (f5), for any $\varepsilon > 0$, we get

$$\big|f(u)\big| \le \varepsilon|u| + c_\varepsilon |u|^{p-1}, \tag{3.12}$$

where the constant $c_\varepsilon > 0$ depending on ε.

By (3.7), for $u \in N_k$ and $\varepsilon > 0$ with $\lambda\varepsilon < \frac{1}{3}$, one has

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{2}\|u\|^2 - \frac{\lambda\varepsilon}{2}\int_{\mathbb{R}^N} K(x)|u|^2\,dx - \frac{\lambda C_\varepsilon}{p}\int_{\mathbb{R}^N} K(x)|u|^p\,dx \\ &\ge \frac{1}{3}\|u\|^2 - c_8\zeta_k^p\|u\|^p. \end{aligned}$$

Choose $\gamma_k = (6C_8\zeta_k^p\|u\|^p)^{\frac{1}{2-p}}$, then for any $u \in N_k$ with $\|u\| = \psi_k$, we can deduce

$$I_\lambda(u) \ge \frac{1}{6}\big(6C_8\zeta_k^p\big)^{\frac{2}{2-p}} = \frac{1}{6}\psi_k^2 > 0. \tag{3.13}$$

Therefore,

$$e_k(\lambda) = \inf_{u \in N_k, \|u\| = \psi_k} I_\lambda(u) > \frac{1}{6}\psi_k^2 > 0, \quad \forall \lambda \in [1,2]. \tag{3.14}$$

Notice that (3.1) still holds here. Hence, there exists $\varepsilon_k > 0$ such that

$$\operatorname{meas}(G_u) \ge \varepsilon_k, \quad \forall u \in N_k \setminus \{0\}, \tag{3.15}$$

where $G_u := \{x \in \mathbb{R}^N : K(x)|u(x)| \ge \varepsilon_k\|u\|\}$. By (f5), there exists a constant $R_k > 0$ with $|u| \ge R_k$ such that

$$F(u) \ge \frac{1}{\varepsilon_k^3}|u|^2. \tag{3.16}$$

Consequently, using (3.15), for any $u \in N_k$ with $\|u\| \geq \frac{R_k}{\varepsilon_k}$, $|u(x)| \geq R_k$ holds for all $x \in G_u$. Therefore, by (3.15) and (3.16), one has

$$\begin{aligned} I_\lambda(u) &\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F(u)\,dx \\ &\leq \frac{1}{2}\|u\|^2 - \int_{G_u} K(x)F(u)\,dx \\ &\leq \frac{1}{2}\|u\|^2 - \frac{1}{\varepsilon_k^3}\int_{G_u} K(x)|u|^2\,dx \\ &\leq \frac{1}{2}\|u\|^2 - \frac{R_k\|u\|}{\varepsilon_k^2}\operatorname{meas}\{G_u\} \\ &\leq \frac{1}{2}\|u\|^2 - \frac{R_k\|u\|}{\varepsilon_k} \\ &\leq \frac{1}{2}\|u\|^2 - \|u\|^2 \\ &= -\frac{1}{2}\|u\|^2. \end{aligned}$$

Finally, for any $\|u\| = \phi_k > \max\{\psi_k, \frac{R_k}{\varepsilon_k}\}$, we obtain

$$g_k(\lambda) = \max_{u \in W_k, \|u\|=\rho_k} I_\lambda(u) \leq -\frac{r_k^2}{2} < 0, \quad \forall \lambda \in [1,2].$$ □

Proof of Theorem 1.2 From (f6) and (f7), for $\lambda \in [1,2]$, we know that $I_\lambda(u)$ maps bounded sets into bounded sets. By (f1), for all $(\lambda, u) \in [1,2] \times H$, we have $I_\lambda(-u) = I_\lambda(u)$. Therefore, (B1) of Lemma 2.6 is satisfied.

From condition (f6)

$$\lim_{|t|\to\infty} \frac{F(t)}{|t|^2} = \infty,$$

we know that $\Psi(u) \geq 0$ holds for $u \in H$. By the definition of $\Phi(u)$, $\Phi(u) \to \infty$ holds when $\|u\| \to \infty$, which means that (B2) of Lemma 2.6 holds. Using Lemma 3.5, we have (B3) of Lemma 2.6 holds. Therefore, by Lemma 2.6, for a.e. $\lambda \in [1,2]$, there exists $\{u_n^k(\lambda)\}_{n=1}^\infty$ for $k \geq k_0$ with $k \in \mathbb{N}$, one has

$$\sup_n \left\|u_n^k(\lambda)\right\| < \infty, \qquad I'_\lambda\left(u_n^k(\lambda)\right) \to \infty \quad \text{and} \quad I_\lambda\left(u_n^k(\lambda)\right) \to g_k(\lambda) \quad \text{as } n \to \infty. \tag{3.17}$$

Using Lemma 2.6, for all $\lambda \in [1,2]$, we also have

$$e_k(\lambda) \leq h_k(\lambda) = \inf_{\delta \in \Gamma_k} \max_{u \in Y_k} I_\lambda\left(\gamma(u)\right).$$

Let $\xi_k = (6C_7\zeta_k^p\|u\|^p)^{\frac{1}{2-p}}$, then $\xi_k \to \infty$ as $k \to \infty$. For $k \geq k_0$ with $k \in \mathbb{N}$, by (3.13), we have $h_k(\lambda) \geq e_k(\lambda) \geq \xi_k$, then

$$h_k(\lambda) \in [\xi_k, \rho_k], \tag{3.18}$$

where $\rho_k := \max_{u\in Y_k} I_\lambda(\gamma(u))$, $\Gamma_k := \{\delta \in C(B_k,H) : \delta \text{ is odd}, \delta|_{\partial Y_k} = \mathrm{id}, k \ge 2\}$ with $Y_k := \{u \in Y_k : \|u\| \le \xi_k\}$.

Choose $\lambda_m \to 1$ as $m \to \infty$ for $\lambda_m \in [1,2]$. Thanks to (3.17), we know that $\{u_n^k(\lambda_m)\}$ is bounded, which implies $\{u_n^k(\lambda_m)\}$ possesses a weakly convergent subsequence. Similar to the proof of Lemma 3.4, we know that $\{u_n^k(\lambda_m)\}$ has a strong convergent subsequence as $n \to \infty$. Assume that $\lim_{n\to\infty} u_n^k(\lambda_m) = u^k(\lambda_m)$ for $m \in \mathbb{N}$ and $k \ge k_0$. By (3.17) and (3.18), one has

$$I'_{\lambda_m}\big(u^k(\lambda_m)\big) = 0 \quad \text{and} \quad I_{\lambda_m}\big(u^k(\lambda_m)\big) \in [\xi_k, \rho_k] \quad \text{for } k \ge k_0. \tag{3.19}$$

We claim that $\{u_n^k(\lambda_m)\}$ is bounded in H. If the claim is not true, we have $\|u^k(\lambda_m)\| \to \infty$ as $m \to \infty$. Let $v_m := \frac{u^k(\lambda_m)}{\|u^k(\lambda_m)\|}$, then $\|v_m\| = 1$. Without loss of generality, we assume $v_m \to v$ in H. Thanks to Lemma 2.3, we obtain $v_m \to v$ in $L_k^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*)$. In the following, we divide two steps to show our assumption is not true.

Step 1. Assume $v(x) \ne 0$ in H. Since

$$\begin{aligned}\frac{1}{2} - \frac{I_{\lambda_m}(u^k(\lambda_m))}{\|u^k(\lambda_m)\|^2} &= \lambda_m \int_{\mathbb{R}^N} \frac{K(x)F(u^k(\lambda_m))}{\|u^k(\lambda_m)\|^2}\,dx \\ &= \lambda_m \int_{\mathbb{R}^N} \frac{|v_m(x)|K(x)F(u^k(\lambda_m))}{|u^k(\lambda_m)|^2}\,dx,\end{aligned}$$

where $v_m(x) \ne 0$, $m = 1,2,3,\dots$. By (3.19), (f6), and Fatou's lemma, one has

$$\frac{1}{2} = \liminf_{m\to\infty} \lambda_m \int_{\mathbb{R}^N} \frac{|v_m(x)|K(x)F(u^k(\lambda_m))}{|u^k(\lambda_m)|^2}\,dx \to 0,$$

which is in contradiction with our claim.

Step 2. Assume $v(x) = 0$ in H. Using (3.19), we have

$$\begin{aligned}\frac{\mu}{2} - 1 &= \frac{\mu I_{\lambda_m}(u^k(\lambda_m)) - \langle I'_{\lambda_m}(u^k(\lambda_m)), u^k(\lambda_m)\rangle}{\|u^k(\lambda_m)\|^2} \\ &\quad + \lambda_m \int_{\mathbb{R}^N} \frac{|v_m(x)|^2 K(x)[\mu F(u^k(\lambda_m)) - f(u^k(\lambda_m))u^k(\lambda_m)]}{|u^k(\lambda_m)|^2}\,dx.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N} \frac{|v_m(x)|^2 K(x)[\mu F(u^k(\lambda_m)) - f(u^k(\lambda_m))u^k(\lambda_m)]}{|u^k(\lambda_m)|^2}\,dx \to \frac{\mu}{2} - 1 \tag{3.20}$$

as $m \to \infty$. However, according to (f7), one has

$$\limsup_{m\to\infty} \frac{|v_m(x)|^2 K(x)[\mu F(u^k(\lambda_m)) - f(u^k(\lambda_m))u^k(\lambda_m)]}{|u^k(\lambda_m)|^2} \le 0. \tag{3.21}$$

Combining with (3.20) and (3.21), we have $\frac{\mu}{2} - 1 \le 0$, i.e., $\mu \le 2$, we deduce a contradiction to our assumption. Therefore, $\{u_n^k(\lambda_m)\}$ is bounded in H. Similar to the proof of Lemma 3.4, we know that $\{u_n^k(\lambda_m)\}$ owns a strongly convergent subsequence with $u^k \in H$ for all $k \ge k_0$ and $k \in \mathbb{N}$. Assume

$$\lim_{m\to\infty} u^k(\lambda_m) = u^k(1) = u^k \in H.$$

Using (3.19) and $\beta_k \to \infty$ as $k \to \infty$, for all $k \geq k_0$ and $k \in \mathbb{N}$, we have

$$I_1(u^k) = 0, \qquad I_1(u^k) \in [\beta_k, \delta_k] \to \infty \quad \text{as } k \to \infty,$$

which implies that u^k is a nontrivial critical point of I_1. Hence, for $k \geq k_0$, $k \in \mathbb{N}$, infinitely many nontrivial critical points u^k of I_1 are obtained. From the above discussion, we know that problem (1.1) has infinitely many nontrivial solutions with high energy, that is,

$$I_1(u^k) = \frac{1}{2}\|u^k\|^2 - \int_{\mathbb{R}^N} K(x)F(u^k)\,dx \to \infty \quad \text{as } k \to \infty.$$

The proof of Theorem 1.2 is completed. □

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