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Positive solutions of semilinear problems in an exterior domain of $\mathbb{R}^2$

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Abstract

The aim of this paper is to establish the existence and global asymptotic behavior of a positive continuous solution for the following semilinear problem:

$$\begin{cases} -\Delta u(x) = a(x)u^{\sigma}(x), & x \in D, \\ u > 0, & \text{in } D, \\ u(x) = 0, & x \in \partial D, \\ \lim_{|x|\to\infty} \frac{u(x)}{\ln |x|} = 0, \end{cases}$$

where $\sigma < 1$, D is an unbounded domain in $\mathbb{R}^2$ with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. As main tools, we use Kato class, Karamata regular variation theory and the Schauder fixed point theorem.

MSC: 34B16; 34B18; 34B27

Keywords: Singular positive solution; Blow-up; Green's function; Karamata class; Kato class

1 Introduction

There are many papers which have been dedicated to resolving questions of existence, uniqueness and asymptotic behavior for solutions of semilinear/quasilinear elliptic equations of the form

$$\Delta u(x) + \varphi(x,u) = 0, \quad x \in \Omega, \tag{1.1}$$

where Ω is either a bounded or unbounded domain of $\mathbb{R}^n$ ($n \geq 2$) (see, for instance, [1, 3, 5, 16, 20–22, 24, 25, 27–29, 32, 33, 37, 48, 49, 51, 52] and the references therein).

In [32, Theorem 3.2], Kusano and Swanson studied equation (1.1) for $\Omega = \Omega_T := \{x \in \mathbb{R}^2 : |x| > T > 1\}$. Using the sub-super solution method, they have proved that equation (1.1) has a positive solution $u(x)$ such that $\frac{u(x)}{\ln|x|}$ is bounded and bounded away from zero in Ω_T if

$$\phi\big(|x|,u\big) \leq \varphi(x,u) \leq \Phi\big(|x|,u\big), \quad \text{for all } (x,u) \in \Omega_T \times (0,\infty),$$

where $\phi(t,u)$ and $\Phi(t,u)$ are nonincreasing functions of u for each fixed $t > 0$ with $\int^{\infty} t\Phi(t, c\ln t)\,dt < \infty$, for some positive constant c.

Ufuktepe and Zhao [48, Theorem 1.1] considered (1.1) in the case $\Omega = D$ is an unbounded domain in $\mathbb{R}^2$ with a compact nonempty boundary ∂D consisting of finitely many Jordan curves, $\varphi(x,t)$ is a Borel measurable function in $\mathbb{R}^2 \times [0,\infty)$ continuous in the second variable for each fixed $x \in \mathbb{R}^2$ and satisfying

$$\left|\varphi(x,t)\right| \leq F(x,t), \quad (x,t) \in \mathbb{R}^2 \times [0,\infty),$$

where $F(x,t)$ is a positive, convex and continuously differentiable function in $\mathbb{R}^2 \times [0,\infty)$ with $F(x,0) = \frac{\partial}{\partial t}F(x,0) = 0$, and $\frac{\partial}{\partial t}F(x, \ln|x| + 1)$ belonging to the Kato class $K_2^\infty(D)$ (see [48]).

Then by using Brownian path integration and potential theory, they have proved that for a small $\lambda > 0$, Eq. (1.1) has a positive solution $u \in C(\overline{D})$ satisfying $u_{|\partial D} = 0$ and $\lim_{|x|\to\infty} \frac{u(x)}{\ln|x|} = \lambda$.

In [37, Theorem 1.1], Mâagli and Mâatoug improved the result obtained in [48] by introducing a Kato class $K(D)$ (see Definition 1.1), which properly contains $K_2^\infty(D)$, and adopting weaker hypotheses on φ.

The class $K(D)$ has been proved to be very useful in the study of various existence and multiplicity results for large classes of elliptic boundary value problems (see, e.g., [39, 41, 51]).

In [51, Theorem 3.5], Zeddini proved that for $\Omega = \{x \in \mathbb{R}^2 : |x| > 1\}$ and for each $\lambda > 0$, Eq. (1.1) has a positive solution $u \in C(\overline{\Omega})$ satisfying $u_{|\partial\Omega} = 0$ and $\lim_{|x|\to\infty} \frac{u(x)}{\ln|x|} = \lambda$, provided that φ is continuous and nonincreasing with respect to the second variable with $\varphi(\cdot, c) \in K(\Omega)$ for every $c > 0$. Several estimates of a such solution have been also obtained.

In [26, Theorems 1.1 and 1.2], by combining variational methods with the geometrical feature, Filippucci et al. established existence and non-existence results for quasilinear elliptic problems with nonlinear boundary conditions and lack of compactness.

In [11, Theorem 1.2], by using sub-supersolution method, Chhetri et al. studied the existence of positive solutions of $-\Delta_p u = K(x)\frac{f(u)}{u^\delta}$ in an exterior domain Ω of $\mathbb{R}^n$, $u = 0$ on $\partial\Omega$ $\lim_{|x|\to\infty} u(x) = 0$, where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < n$ and $0 \leq \delta < 1$. The weight function $K : \Omega \to (0,\infty)$ and the nonlinearity $f : [0,\infty) \to (0,\infty)$ satisfy the following hypotheses:

- There exists $K^* > 0$ such that $0 < K(x) < \frac{K^*}{|x|^\alpha}$ for $x \in \overline{\Omega}$, where $\alpha > n + \delta\frac{n-p}{p-1}$.
- f is continuous and $\lim_{s\to\infty} \frac{f(s)}{s^{p-1+\delta}} = 0$.

Sharp estimates have also been obtained. The uniqueness has also been proved under an additional assumption on f.

Recently, Carl et al. [7] considered Eq. (1.1) for $\Omega = \{x \in \mathbb{R}^2 : |x| > 1\}$ with $\varphi(x,u) = a(x)(\lambda u - g(u))$, where $\lambda > 0$, $a : \Omega \to \mathbb{R}$ is measurable with $|\operatorname{supp}(a)| > 0$ satisfying

$$0 \leq a(x) \leq \frac{c}{|x|^{2+\alpha}}, \quad \text{with } \alpha > 0 \text{ for some constant } c > 0,$$

and the growth for the continuous nonlinearity $g : \mathbb{R} \to \mathbb{R}$ at zero and at infinity is superlinear, which includes even exponential growth. By proving a Hopf-type lemma and employing the sub-supersolution method in the space $D_0^{1,2}(\Omega)$, which is the completion of $C_c^\infty(\Omega)$ with respect to the $\|\nabla.\|_{2,\Omega}$-norm, they have proved the existence of extremal constant-sign solutions u in $D_0^{1,2}(\Omega)$ for Eq. (1.1) subject to the boundary condition $u = 0$ on $\partial\Omega$. The obtained solutions are not decaying to zero at infinity, and instead are bounded away from zero.

In this paper, we will address the question of existence and global behavior of positive continuous solutions to the following nonlinear singular sublinear problem:

$$\begin{cases} -\Delta u(x) = a(x)u^\sigma(x), & x \in D \text{ (in the distributional sense)}, \\ u > 0, & \text{in } D, \\ u(x) = 0, & x \in \partial D, \\ \lim_{|x|\to\infty} \frac{u(x)}{\ln|x|} = 0, \end{cases} \tag{1.2}$$

where $\sigma < 1$, D is an unbounded domain in $\mathbb{R}^2$ with a compact nonempty boundary ∂D consisting of finitely many Jordan curves and the weight function $a(x)$ (which could be singular) is required to satisfy some adequate conditions related to the Karamata class (see Definition 1.3). In particular, we improve the sharp estimates obtained in [51]. We emphasize that the use of Karamata regular variation theory has been suggested by Cîrstea and Rădulescu, [13–17] in the study of various qualitative and asymptotic properties of solutions of nonlinear differential equations. Since then, this setting became a powerful tool in describing the asymptotic behavior of solutions of large classes of nonlinear equations (see [3, 8–10, 18, 23, 27–29, 35, 36, 38, 43–45, 50, 53]).

Before stating our main result, we need to fix some notation.

Notation

(i) D is an unbounded domain in $\mathbb{R}^2$ with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. That is, $\overline{D}^c = \bigcup_{j=1}^k D_j$, where (D_j) is a family of bounded domains satisfying:
- For $i \neq j$, $\overline{D_i} \cap \overline{D_j} = \emptyset$.
- For any $j \in \{1,\dots,k\}$, there exists a continuous function $f_j : [0,1] \to \mathbb{R}^2$ with $f_j(0) = f_j(1)$ and $f_j(t) \neq f_j(s)$ for any $0 < t < s < 1$ such that $\partial D_j = \{f_j(t), t \in [0,1)\}$.

(ii) For $x \in D$, $\delta_D(x)$ will denote the Euclidean distance from x to ∂D, $\rho_D(x) = \frac{\delta_D(x)}{\delta_D(x)+1}$ and $\lambda_D(x) = \delta_D(x)(\delta_D(x) + 1)$.

(iii) For $x, y \in D$, $G_D(x,y)$ will be the Green's function of the Laplace operator $u \to -\Delta u$ in D with zero boundary Dirichlet condition.

(iv) Let $a \in \mathbb{R}^2 \backslash \overline{D}$ and $r > 0$ such that $\overline{B(a,r)} \subset \mathbb{R}^2 \backslash \overline{D}$. Then we have for $x, y \in D$,

$$G_D(x,y) = G_{\frac{D-a}{r}}\left(\frac{x-a}{r}, \frac{y-a}{r}\right) \quad \text{and} \quad \delta_D(x) = r\delta_{\frac{D-a}{r}}\left(\frac{x-a}{r}\right).$$

So without loss of generality, we may assume throughout this paper that $\overline{B(0,1)} \subset \mathbb{R}^2 \backslash \overline{D}$.

(v) For any two nonnegative functions f and g on a set S,

$$f(x) \approx g(x), \quad x \in S \quad \Longleftrightarrow \quad \exists c > 0, \forall x \in S, \qquad \frac{1}{c} f(x) \leq g(x) \leq c f(x).$$

(vi) For $x \in D$, let $x^* = \frac{x}{|x|^2}$ be the Kelvin inversion from $D \cup \{\infty\}$ onto $D^* = \{x^* \in B(0,1) : x \in D \cup \{\infty\}\}$.
- From [12], D^* is a regular bounded domain containing 0 and

$$G_D(x,y) = G_{D^*}(x^*, y^*) \approx \ln\left(1 + \frac{\delta_{D^*}(x^*)\delta_{D^*}(y^*)}{|x^* - y^*|^2}\right), \quad \text{for } x, y \in D. \tag{1.3}$$

- From [37, Lemma 2.1 and Proposition 2.3], we have

$$\begin{cases} \delta_D(x) + 1 \approx |x|, & x \in D, \\ \rho_D(x) \approx \delta_{D^*}(x^*), & x \in D, \\ G_D(x,y) \approx \ln(1 + \frac{\lambda_D(x)\lambda_D(y)}{|x-y|^2}), & x, y \in D. \end{cases} \tag{1.4}$$

(vii) ω is a sufficiently large positive real number.

(viii) $\mathcal{B}(D)$ be the set of Borel measurable functions in D and $\mathcal{B}^+(D)$ be the set of nonnegative ones.

(ix) $C(\overline{D})$ is the set of all continuous functions in $\overline{D}$.

(x) $C_0(\overline{D}) := \{v \in C(\overline{D}), \lim_{x \to \xi \in \partial D} v(x) = 0 \text{ and } \lim_{|x| \to \infty} v(x) = 0\}$. Note that $C_0(\overline{D})$ is a Banach space with the uniform norm $\|v\|_\infty := \sup_{x \in D} |v(x)|$.

(xi) We define the potential kernel V on $\mathcal{B}^+(D)$ by

$$Vf(x) = \int_D G_D(x,y) f(y)\, dy.$$

We recall that for any function $f \in \mathcal{B}^+(D)$ such that $f \in L^1_{\mathrm{loc}}(D)$ and $Vf \in L^1_{\mathrm{loc}}(D)$, we have

$$-\Delta(Vf) = f \quad \text{in } D \text{ (in the distributional sense).} \tag{1.5}$$

From [12, Lemma 2.9] or [42, Theorem 6.6], we know that for any function $f \in \mathcal{B}^+(D)$ such that $Vf(x_0) < \infty$ for some $x_0 \in D$, we have $Vf \in L^1_{\mathrm{loc}}(D)$.

The letter C will be a generic positive constant which may vary from line to line.

Definition 1.1 A Borel measurable function q belongs to the class $K(D)$ if q satisfies the following conditions:

$$\lim_{r \to 0} \sup_{x \in D} \int_{B(x,r) \cap D} \frac{\rho_D(y)}{\rho_D(x)} G_D(x,y) \left|q(y)\right| dy = 0$$

and

$$\lim_{M \to +\infty} \sup_{x \in D} \int_{(|y| \geq M) \cap D} \frac{\rho_D(y)}{\rho_D(x)} G_D(x,y) \left|q(y)\right| dy = 0.$$

Remark 1.2 ([37, Proposition 3.6]) Let $\lambda, \mu \in \mathbb{R}$. Then

$$x \to |x|^{\lambda - \mu} \left(\delta(x)\right)^{-\lambda} \in K(D) \quad \text{if and only if} \quad \lambda < 2 < \mu.$$

Definition 1.3 ([30])

(i) A function $\mathcal{M}$ defined on $(0,\eta)$, for some $\eta > 0$, belongs to the Karamata class $\mathcal{K}_0$ if

$$\mathcal{M}(t) := c\exp\left(\int_t^\eta \frac{\nu(s)}{s}\,ds\right),$$

where $c > 0$ and $\nu \in C([0,\eta])$ with $\nu(0) = 0$.

(ii) A function $\mathcal{M}$ defined on $[1,\infty)$ belongs to the Karamata class $\mathcal{K}_\infty$ if

$$\mathcal{M}(t) := c\exp\left(\int_t^\eta \frac{\nu(s)}{s}\,ds\right),$$

where $c > 0$ and $\nu \in C([1,\infty))$ with $\lim_{t\to\infty}\nu(t) = 0$.

Remark 1.4

(i) The classes $\mathcal{K}_\infty$ and $\mathcal{K}_0$ are characterized respectively by

$$\mathcal{K}_\infty = \left\{\mathcal{M} : [1,\infty) \longrightarrow (0,\infty), \mathcal{M} \in C^1\big([1,\infty)\big) \text{ and } \lim_{t\to\infty}\frac{t\mathcal{M}'(t)}{\mathcal{M}(t)} = 0\right\},$$

and, for some $\eta > 0$,

$$\mathcal{K}_0 = \left\{\mathcal{M} : (0,\eta) \longrightarrow (0,\infty), \mathcal{M} \in C^1\big((0,\eta)\big) \text{ and } \lim_{t\to 0^+}\frac{t\mathcal{M}'(t)}{\mathcal{M}(t)} = 0\right\}.$$

(ii) Observe that the map $t \to \mathcal{M}(t)$ belongs to $\mathcal{K}_\infty$ if and only if the map $t \to \mathcal{M}(\frac{1}{t})$, defined on $(0,1]$, belongs to $\mathcal{K}_0$.

Nontrivial examples of functions belonging to the class $\mathcal{K}_0$ (see [4, 40, 46, 47]) include

- $\prod_{j=1}^m (\ln_j(\frac{\omega}{t}))^{\xi_j}$, for any integer $m \geq 1$, $\ln_j t = \ln\circ\ln\circ\cdots\circ\ln t$ (j times), $\xi_j \in \mathbb{R}$. Such functions are frequently used as weight functions (see, for example, [31] and [34]);
- $\exp((-\ln t)^\alpha)$ with $\alpha \in (0,1)$, $t \in (0,1)$.

The class $\mathcal{K}_\infty$ contains, for example, functions of the form $\prod_{j=1}^m (\ln_j(\omega t))^{\xi_j}$, where $\xi_j \in \mathbb{R}$.
Throughout this paper, we assume that the following conditions hold:

(H) a is a positive continuous function in D such that

$$a(x) \approx \big(\rho_D(x)\big)^{-\lambda}\mathcal{L}_0\big(\rho_D(x)\big)|x|^{-\mu}\mathcal{L}_\infty\big(|x|\big), \quad \text{for } x \in D, \tag{1.6}$$

where $\sigma < 1$ and $\lambda \leq 2 \leq \mu$.

Here,

- $\mathcal{L}_0 \in \mathcal{K}_0$ defined on $(0,\eta)$ $(\eta > 1)$ is such that

$$\int_0^\eta s^{1-\lambda}\mathcal{L}_0(s)\,ds < \infty.$$

- $\mathcal{L}_\infty(t) := \prod_{k=1}^m (\ln_k(\omega t))^{-\mu_k}$, with $\mu_k \in \mathbb{R}$ satisfying

$$\begin{cases} \text{either } \mu_1 > 1 + \sigma, \\ \text{or } \mu_1 = 1 + \sigma \text{ and there exists } p \geq 2 \text{ such that} \\ \mu_2 = \cdots = \mu_{p-1} = 1 \text{ and } \mu_p > 1. \end{cases}$$

We introduce the function θ defined on D by

$$\theta(x) := \big(\rho_D(x)\big)^{\min(1,\frac{2-\lambda}{1-\sigma})}\big(\widetilde{\mathcal{L}}_0\big(\rho_D(x)\big)\big)^{\frac{1}{1-\sigma}}\big(\widetilde{\mathcal{L}}_\infty\big(|x|\big)\big)^{\frac{1}{1-\sigma}}, \tag{1.7}$$

where $\widetilde{\mathcal{L}}_0$ is defined on $(0,\eta)$ by

$$\widetilde{\mathcal{L}}_0(t) := \begin{cases} 1, & \text{if } \lambda < 1, \\ \int_t^\eta \frac{\mathcal{L}_0(s)}{s}\,ds, & \text{if } \lambda = 1, \\ \mathcal{L}_0(t), & \text{if } 1 < \lambda < 2, \\ \int_0^t \frac{\mathcal{L}_0(s)}{s}\,ds, & \text{if } \lambda = 2, \end{cases}$$

and $\widetilde{\mathcal{L}}_\infty(t) = 1$, if $\mu > 2$ and for $\mu = 2$, $\widetilde{\mathcal{L}}_\infty$ is defined on $[1,\infty)$ as follows:

(i) If $\mu_1 = 1+\sigma$, $\mu_2 = \cdots = \mu_{p-1} = 1$ and $\mu_p > 1$,

$$\widetilde{\mathcal{L}}_\infty(t) := \big(\ln(\omega t)\big)^{1-\sigma}\big(\ln_p(\omega t)\big)^{1-\mu_p}\prod_{k=p+1}^{m}\big(\ln_k(\omega t)\big)^{-\mu_k}.$$

(ii) If $1+\sigma < \mu_1 < 2$,

$$\widetilde{\mathcal{L}}_\infty(t) := \big(\ln(\omega t)\big)^{2-\mu_1}\prod_{k=2}^{m}\big(\ln_k(\omega t)\big)^{-\mu_k}.$$

(iii) If $\mu_1 = 2$ and $\mu_i = 1$, for $2 \le i \le m$,

$$\widetilde{\mathcal{L}}_\infty(t) := \ln_{m+1}(\omega t).$$

(iv) If $\mu_1 = 2$, $\mu_2 = \cdots = \mu_{l-1} = 1$ and $\mu_l < 1$,

$$\widetilde{\mathcal{L}}_\infty(t) := \big(\ln_l(\omega t)\big)^{1-\mu_l}\prod_{k=l+1}^{m}\big(\ln_k(\omega t)\big)^{-\mu_k}.$$

(v) If ($\mu_1 = 2$, $\mu_2 = \cdots = \mu_{l-1} = 1$ and $\mu_l > 1$) or $\mu_1 > 2$,

$$\widetilde{\mathcal{L}}_\infty(t) := 1.$$

Using Karamata's theory and the Schauder fixed point theorem, we prove our main result.

Theorem 1.5 *Let $\sigma < 1$ and assume that function a satisfies assumption* (H). *Then problem* (1.2) *has at least one positive continuous solution u on D such that*

$$\frac{1}{c}\theta(x) \le u(x) \le c\theta(x), \tag{1.8}$$

for $x \in D$ and where c is a positive constant.

Remark 1.6 Since $u \approx \theta$, it is important to note that in the above cases (i)–(iv), $\lim_{|x|\to\infty} u(x) = \infty$.

2 Preliminaries and key tools

2.1 Kato class $K(D)$

In this subsection, we recall and prove some properties related to the Kato class $K(D)$.

Proposition 2.1 *Let $q \in K(D)$, $x_0 \in \overline{D}$, and let h be a positive superharmonic function in D. Then we have*

(i)

$$\lim_{r\to 0}\left(\sup_{\xi\in D}\frac{1}{h(\xi)}\int_{B(x_0,r)\cap D} G_D(\xi,y)h(y)\left|q(y)\right|dy\right) = 0, \tag{2.1}$$

$$\lim_{M\to +\infty}\left(\sup_{x\in D}\frac{1}{h(\xi)}\int_{(|y|\geq M)\cap D} G_D(\xi,y)h(y)\left|q(y)\right|dy\right) = 0. \tag{2.2}$$

(ii) *The potential Vq is bounded and the function $x \mapsto \rho_D(x)q(x)$ is in $L^1(D)$.*

Proof See [37, Proposition 3.4 and Corollary 3.5]. □

Lemma 2.2 *Let $M > 0$ and $\alpha > 0$. Then there exists a constant $C > 0$ such that for all $x, y \in D$ with $|x - y| \geq \alpha$, and $|y| \leq M$,*

$$\frac{\ln(\omega|y|)}{\ln(\omega|x|)}G_D(x,y) \leq C\delta(y).$$

Proof Let $x, y \in D$ with $|x - y| \geq \alpha$, and $|y| \leq M$. Since $|x| \geq 1$, by using (1.4) and the fact that $\ln(1 + t) \leq t$, for $t \geq 0$, we obtain

$$\begin{aligned}
\frac{\ln(\omega|y|)}{\ln(\omega|x|)}G_D(x,y) &\leq C\frac{\lambda_D(x)\lambda_D(y)}{|x-y|^2} \\
&\leq C\frac{(\delta_D(x)+1)^2}{|x-y|^2}\big(\delta_D(y)\big) \\
&\leq C\left[\sup_{|x|\leq M+1}\frac{(\delta_D(x)+1)^2}{\alpha^2} + \sup_{|x|\geq M+1}\frac{(\delta_D(x)+1)^2}{(|x|-M)^2}\right]\delta_D(y) \\
&\leq C\delta_D(y).
\end{aligned}$$

□

Proposition 2.3 *Let $q \in K(D)$. Then the function*

$$\vartheta(x) := \frac{1}{\ln(\omega|x|)}\int_D G_D(x,y)\ln\big(\omega|y|\big)q(y)\,dy$$

belongs to $C_0(\overline{D})$.

Proof Let $\varepsilon > 0$, $x_0 \in \overline{D}$ and $q \in K(D)$. Using Proposition 2.1 with $h(x) = \ln(\omega|x|)$, there exists $r > 0$ such that

$$\sup_{\xi\in D}\frac{1}{\ln(\omega|\xi|)}\int_{B(x_0,r)\cap D} G(\xi,y)\ln\big(\omega|y|\big)\left|q(y)\right|dy \leq \frac{\varepsilon}{8}$$

and

$$\sup_{\xi\in D}\frac{1}{\ln(\omega|\xi|)}\int_{(|y|\geq M)\cap D} G_D(\xi,y)\ln\big(\omega|y|\big)\left|q(y)\right|dy \leq \frac{\varepsilon}{8}.$$

If $x_0 \in D$ and $x \in B(x_0, \frac{r}{2}) \cap D$, then we have

$$\begin{aligned}
\bigl|\vartheta(x) - \vartheta(x_0)\bigr| &\le 2 \sup_{\xi \in D} \frac{1}{\ln(\omega|\xi|)} \int_{B(x_0,r)\cap D} G_D(\xi, y) \ln\bigl(\omega|y|\bigr)\bigl|q(y)\bigr|\,dy \\
&\quad + 2 \sup_{\xi \in D} \frac{1}{\ln(\omega|\xi|)} \int_{(|y|\ge M)\cap D} G_D(\xi, y) \ln\bigl(\omega|y|\bigr)\bigl|q(y)\bigr|\,dy \\
&\quad + \int_{D_0} \left|\frac{1}{\ln(\omega|x|)} G_D(x, y) - \frac{1}{\ln(\omega|x_0|)} G_D(x_0, y)\right| \ln\bigl(\omega|y|\bigr)\bigl|q(y)\bigr|\,dy \\
&\le \frac{\varepsilon}{2} + \int_{D_0} \left|\frac{1}{\ln(\omega|x|)} G_D(x, y) - \frac{1}{\ln(\omega|x_0|)} G_D(x_0, y)\right| \ln\bigl(\omega|y|\bigr)\bigl|q(y)\bigr|\,dy,
\end{aligned}$$

where $D_0 = D \cap B(0, M) \cap B^c(x_0, r)$.

By Lemma 2.2, there exists $C > 0$ such that for all $x \in B(x_0, \frac{r}{2}) \cap D$ and $y \in D_0$,

$$\frac{\ln(\omega|y|)}{\ln(\omega|x|)} G_D(x, y)\bigl|q(y)\bigr| \le C\delta(y)\bigl|q(y)\bigr|.$$

Moreover, $(x, y) \mapsto \frac{1}{\ln(\omega|x|)} G_D(x, y)$ is continuous on $(B(x_0, \frac{r}{2}) \cap D) \times D_0$. Then by Proposition 2.1(ii) and Lebesgue's dominated convergence theorem, we have

$$\lim_{x \to x_0} \int_{D_0} \left|\frac{1}{\ln(\omega|x|)} G_D(x, y) - \frac{1}{\ln(\omega|x_0|)} G_D(x_0, y)\right| \ln\bigl(\omega|y|\bigr)\bigl|q(y)\bigr|\,dy = 0.$$

That is, there exists $\delta > 0$ with $\delta < \frac{r}{2}$ such that if $x \in B(x_0, \delta) \cap D$ then

$$\int_{D_0} \left|\frac{1}{\ln(\omega|x|)} G_D(x, y) - \frac{1}{\ln(\omega|x_0|)} G_D(x_0, y)\right| \ln\bigl(\omega|y|\bigr)\bigl|q(y)\bigr|\,dy \le \frac{\varepsilon}{2}$$

and

$$\bigl|\vartheta(x) - \vartheta(x_0)\bigr| \le \varepsilon.$$

This implies that

$$\lim_{x \to x_0} \vartheta(x) = \vartheta(x_0).$$

If $x_0 \in \partial D$ and $x \in B(x_0, \frac{r}{2}) \cap D$, then we have

$$\begin{aligned}
\bigl|\vartheta(x)\bigr| &\le \sup_{\xi \in D} \frac{1}{\ln(\omega|\xi|)} \int_{B(x_0,r)\cap D} G_D(\xi, y) \ln\bigl(\omega|y|\bigr)\bigl|q(y)\bigr|\,dy \\
&\quad + \sup_{\xi \in D} \frac{1}{\ln(\omega|\xi|)} \int_{(|y|\ge M)\cap D} G_D(\xi, y) \ln\bigl(\omega|y|\bigr)\bigl|q(y)\bigr|\,dy \\
&\quad + \int_{D_0} \frac{1}{\ln(\omega|x|)} G_D(x, y) \ln\bigl(\omega|y|\bigr)\bigl|q(y)\bigr|\,dy.
\end{aligned}$$

Now, since $\lim_{x \to x_0} \frac{\ln(\omega|y|)}{\ln(\omega|x|)} G_D(x, y) = 0$, for all $y \in D_0$, we deduce by similar arguments as above that

$$\lim_{x \to x_0} \vartheta(x) = 0.$$

It remains to prove that $\lim_{|x|\to\infty}\vartheta(x)=0$. Indeed, let $M>0$ and $x\in D$ be such that $|x|\geq M+1$. Then we have

$$\begin{aligned}\left|\vartheta(x)\right| &\leq \sup_{\xi\in D}\frac{1}{\ln(\omega|\xi|)}\int_{(|y|\geq M)\cap D}G_D(\xi,y)\ln\bigl(\omega|y|\bigr)\left|q(y)\right|dy\\ &\quad+\frac{1}{\ln(\omega|x|)}\int_{(|y|\leq M)\cap D}G_D(x,y)\ln\bigl(\omega|y|\bigr)\left|q(y)\right|dy.\end{aligned}$$

Since $\lim_{|x|\to\infty}\frac{\ln(\omega|y|)}{\ln(\omega|x|)}G_D(x,y)=0$ uniformly for $|y|\leq M$, then from (2.2), Lemma 2.2, Proposition 2.1(ii) and Lebesgue's dominated convergence theorem, we deduce that $\lim_{|x|\to\infty}\vartheta(x)=0$.

Hence $\vartheta\in C_0(\overline{D})$. □

2.2 Karamata class

In this section, we collect some properties of the Karamata functions, which will be used later.

Lemma 2.4 (See [47])

(i) *If $\mathcal{M}_1,\mathcal{M}_2\in\mathcal{K}_0$ (resp. $\mathcal{K}_\infty$) and $\tau\in\mathbb{R}$, then*

$$\mathcal{M}_1^\tau,\mathcal{M}_1\mathcal{M}_2 \text{ and } \mathcal{M}_1+\mathcal{M}_2 \text{ belong to } \mathcal{K}_0 \text{ (resp. } \mathcal{K}_\infty).$$

(ii) *Let $\mathcal{M}\in\mathcal{K}_0$ and $\varepsilon>0$. Then*

$$\lim_{t\to0^+}t^\varepsilon\mathcal{M}(t)=0\quad\text{and}\quad\lim_{t\to0^+}t^{-\varepsilon}\mathcal{M}(t)=\infty.$$

(iii) *Let $\mathcal{M}\in\mathcal{K}_\infty$ and $\varepsilon>0$. Then*

$$\lim_{t\to\infty}t^{-\varepsilon}\mathcal{M}(t)=0\quad\text{and}\quad\lim_{t\to\infty}t^{\varepsilon}\mathcal{M}(t)=\infty.$$

Lemma 2.5 (See [40, 47]) *Let $\gamma\in\mathbb{R}$, $\mathcal{M}\in\mathcal{K}_0$ and $\mathcal{L}\in\mathcal{K}_\infty$. Then*

(i) *$\int_0^\eta s^\gamma\mathcal{M}(s)\,ds$ converges for $\gamma>-1$, and*

$$\int_0^t s^\gamma\mathcal{M}(s)\,ds\underset{t\to0^+}{\sim}\frac{t^{\gamma+1}\mathcal{M}(t)}{\gamma+1}.$$

(ii) *$\int_0^\eta s^\gamma\mathcal{M}(s)\,ds$ diverges for $\gamma<-1$, and*

$$\int_t^\eta s^\gamma\mathcal{M}(s)\,ds\underset{t\to0^+}{\sim}-\frac{t^{\gamma+1}\mathcal{M}(t)}{\gamma+1}.$$

(iii) *$\int_1^\infty s^\gamma\mathcal{L}(s)\,ds$ converges for $\gamma<-1$, and*

$$\int_t^\infty s^\gamma\mathcal{L}(s)\,ds\underset{t\to\infty}{\sim}-\frac{t^{\gamma+1}\mathcal{L}(t)}{\gamma+1}.$$

(iv) $\int_1^\infty s^\gamma \mathcal{L}(s)\,ds$ *diverges for* $\gamma > -1$, *and*

$$\int_1^t s^\gamma \mathcal{L}(s)\,ds \underset{t\to\infty}{\sim} \frac{t^{\gamma+1}\mathcal{L}(t)}{\gamma+1}.$$

Lemma 2.6 (See [47]) *Let* $\mathcal{M} \in \mathcal{K}_0$. *Then* $\lim_{t\to 0^+} \frac{\mathcal{M}(t)}{\int_t^\eta \frac{\mathcal{M}(s)}{s}\,ds} = 0$.

In particular $t \to \int_t^\eta \frac{\mathcal{M}(s)}{s}\,ds \in \mathcal{K}_0$.

If further $\int_0^\eta \frac{\mathcal{M}(s)}{s}\,ds$ *converges, then* $\lim_{t\to 0^+} \frac{\mathcal{M}(t)}{\int_0^t \frac{\mathcal{M}(s)}{s}\,ds} = 0$.

In particular, $t \to \int_0^t \frac{\mathcal{M}(s)}{s}\,ds \in \mathcal{K}_0$.

We have the following similar properties related to the class $\mathcal{K}_\infty$.

Lemma 2.7 (See [9]) *Let* $\mathcal{L} \in \mathcal{K}_\infty$. *Then* $\lim_{t\to\infty} \frac{\mathcal{L}(t)}{\int_1^t \frac{\mathcal{L}(s)}{s}\,ds} = 0$.

In particular, $t \to \int_1^{t+1} \frac{\mathcal{L}(s)}{s}\,ds \in \mathcal{K}_\infty$.

If, furthermore, $\int_1^\infty \frac{\mathcal{L}(s)}{s}\,ds$ *converges, then* $\lim_{t\to\infty} \frac{\mathcal{L}(t)}{\int_t^\infty \frac{\mathcal{L}(s)}{s}\,ds} = 0$.

In particular, $t \to \int_t^\infty \frac{\mathcal{L}(s)}{s}\,ds \in \mathcal{K}_\infty$.

An important step in the proof of Theorem 1.5 uses the following

Proposition 2.8 ([2]) *Let* Ω *be a bounded regular domain in* $\mathbb{R}^2$ *containing* 0.

Let $\gamma, \nu \le 2$ *and* $L_3, L_4 \in \mathcal{K}_0$ *be such that*

$$\int_0^\eta s^{1-\gamma} L_3(s)\,ds < \infty \quad \text{and} \quad \int_0^\eta s^{1-\nu} L_4(s)\,ds < \infty, \quad \text{for } \eta > \operatorname{diam}(\Omega). \tag{2.3}$$

Put

$$b(x) = |x|^{-\gamma} L_3\big(|x|\big)\big(\delta_\Omega(x)\big)^{-\nu} L_4\big(\delta_\Omega(x)\big), \quad \text{for } x \in \Omega\backslash\{0\}.$$

Then for $x \in \Omega\backslash\{0\}$,

$$Vb(x) \approx \widetilde{L}_3\big(|x|\big)\big(\delta_\Omega(x)\big)^{\min(1,2-\nu)} \widetilde{L}_4\big(\delta_\Omega(x)\big),$$

where for $t \in (0,\eta)$

$$\widetilde{L}_3(t) := \begin{cases} 1 & \text{if } \gamma < 2, \\ \ln(\frac{\omega}{t}) \int_0^t \frac{L_3(s)}{s}\,ds + \int_t^\eta \ln(\frac{\omega}{s}) \frac{L_3(s)}{s}\,ds, & \text{if } \gamma = 2, \end{cases} \tag{2.4}$$

and

$$\widetilde{L}_4(t) := \begin{cases} 1, & \text{if } \nu < 1, \\ \int_t^\eta \frac{L_4(s)}{s}\,ds, & \text{if } \nu = 1, \\ L_4(t), & \text{if } 1 < \nu < 2, \\ \int_0^t \frac{L_4(s)}{s}\,ds, & \text{if } \nu = 2. \end{cases}$$

Proposition 2.9 *Let φ be a positive continuous function in D such that*

$$\varphi(x) \approx \left(\rho_D(x)\right)^{-\alpha} \mathcal{M}_0\left(\rho_D(x)\right)|x|^{-\beta} \mathcal{N}_\infty\left(|x|\right), \quad \text{for } x \in D, \tag{2.5}$$

where $\alpha \le 2 \le \beta$, $\mathcal{M}_0 \in \mathcal{K}_0$ defined on $(0,\eta)$ ($\eta > 1$) is such that

$$\int_0^\eta s^{1-\alpha} \mathcal{M}_0(s)\, ds < \infty,$$

and $\mathcal{N}_\infty(t) := \prod_{k=1}^m (\ln_k(\omega t))^{-\beta_k}$, with $\beta_k \in \mathbb{R}$ satisfying either $\beta_1 > 1$, or ($\beta_1 = 1$ and there exists $p \ge 2$ such that $\beta_2 = \cdots = \beta_{p-1} = 1$ and $\beta_p > 1$). Then

$$V\varphi(x) \approx \left(\rho_D(x)\right)^{\min(1,2-\alpha)} \left(\widetilde{\mathcal{M}}_0\left(\rho_D(x)\right)\right)\left(\widetilde{\mathcal{N}}_\infty\left(|x|\right)\right), \quad \text{for } x \in D,$$

where $\widetilde{\mathcal{M}}_0$ is defined on $(0,1)$ by

$$\widetilde{\mathcal{M}}_0(t) := \begin{cases} 1, & \text{if } \alpha < 1, \\ \int_t^\eta \frac{\mathcal{M}_0(s)}{s}\, ds, & \text{if } \alpha = 1, \\ \mathcal{M}_0(t), & \text{if } 1 < \alpha < 2, \\ \int_0^t \frac{\mathcal{M}_0(s)}{s}\, ds, & \text{if } \alpha = 2, \end{cases}$$

and $\widetilde{\mathcal{N}}_\infty(t) = 1$, if $\beta > 2$ and for $\beta = 2$, $\widetilde{\mathcal{N}}_\infty$ is defined on $[1,\infty)$ as follows:

(i) *If $\beta_1 = 1$, $\beta_2 = \cdots = \beta_{p-1} = 1$ and $\beta_p > 1$,*

$$\widetilde{\mathcal{N}}_\infty(t) := \left(\ln(\omega t)\right)\left(\ln_p(\omega t)\right)^{1-\beta_p} \prod_{k=p+1}^m \left(\ln_k(\omega t)\right)^{-\beta_k}.$$

(ii) *If $1 < \beta_1 < 2$,*

$$\widetilde{\mathcal{N}}_\infty(t) := \left(\ln(\omega t)\right)^{2-\beta_1} \prod_{k=2}^m \left(\ln_k(\omega t)\right)^{-\beta_k}.$$

(iii) *If $\beta_1 = 2$ and $\beta_i = 1$, for $2 \le i \le m$,*

$$\widetilde{\mathcal{N}}_\infty(t) := \ln_{m+1}(\omega t).$$

(iv) *If $\beta_1 = 2$, $\beta_2 = \cdots = \beta_{l-1} = 1$ and $\beta_l < 1$,*

$$\widetilde{\mathcal{N}}_\infty(t) := \left(\ln_l(\omega t)\right)^{1-\beta_l} \prod_{k=l+1}^m \left(\ln_k(\omega t)\right)^{-\mu_k}.$$

(v) *If ($\beta_1 = 2$, $\beta_2 = \cdots = \beta_{l-1} = 1$ and $\beta_l > 1$) or $\beta_1 > 2$,*

$$\widetilde{\mathcal{N}}_\infty(t) := 1.$$

Proof From (2.5), we have

$$V\varphi(x) \approx \int_D G_D(x,y)\big(\rho_D(y)\big)^{-\alpha}\mathcal{M}_0\big(\rho_D(y)\big)|y|^{-\beta}\mathcal{N}_\infty\big(|y|\big)\,dy, \quad \text{for } x \in D.$$

Using (1.3), (1.4) and the fact that $\mathcal{M}_0(\rho_D(y)) \approx \mathcal{M}_0(\delta_{D^*}(y^*))$, we obtain

$$V\varphi(x) \approx \int_D G_{D^*}\big(x^*,y^*\big)\big(\delta_{D^*}\big(y^*\big)\big)^{-\alpha}\mathcal{M}_0\big(\delta_{D^*}\big(y^*\big)\big)|y|^{-\beta}\mathcal{N}_\infty\big(|y|\big)\,dy, \quad \text{for } x \in D.$$

By letting $\xi = y^*$, we obtain

$$V\varphi(x) \approx \int_{D^*} G_{D^*}\big(x^*,\xi\big)\big(\delta_{D^*}(\xi)\big)^{-\alpha}\mathcal{M}_0\big(\delta_{D^*}(\xi)\big)|\xi|^{\beta-4}\mathcal{N}_\infty\left(\frac{1}{|\xi|}\right)d\xi, \quad \text{for } x \in D.$$

From Remark 1.4(ii), the function $t \to \mathcal{N}_\infty(\frac{1}{t})$ belongs to $\mathcal{K}_0$. Using this fact, and applying Proposition 2.8 with $\gamma = 4 - \beta \leq 2$, $\nu = \alpha \leq 2$, $L_3(t) = \mathcal{N}_\infty(\frac{1}{t})$ and $L_4 = \mathcal{M}_0$, we deduce that

$$V\varphi(x) \approx \big(\delta_{D^*}\big(x^*\big)\big)^{\min(1,2-\alpha)}\widetilde{\mathcal{M}}_0\big(\delta_{D^*}\big(x^*\big)\big)\widetilde{L}_3\big(\big|x^*\big|\big), \tag{2.6}$$

where for $r \in (0,\eta)$

$$\widetilde{\mathcal{M}}_0(r) := \begin{cases} 1, & \text{if } \alpha < 1, \\ \int_r^\eta \frac{\mathcal{M}_0(s)}{s}\,ds, & \text{if } \alpha = 1, \\ \mathcal{M}_0(r), & \text{if } 1 < \alpha < 2, \\ \int_0^r \frac{\mathcal{M}_0(s)}{s}\,ds, & \text{if } \alpha = 2, \end{cases}$$

and

$$\widetilde{L}_3(r) := \begin{cases} 1 & \text{if } \beta > 2, \\ \ln(\frac{\omega}{r})\int_0^r \frac{\mathcal{N}_\infty(\frac{1}{s})}{s}\,ds + \int_r^\eta \ln(\frac{\omega}{s})\frac{\mathcal{N}_\infty(\frac{1}{s})}{s}\,ds, & \text{if } \beta = 2. \end{cases}$$

For $\beta = 2$, by direct computation, we obtain the following:

(i) If $\beta_1 = 1$, $\beta_2 = \cdots = \beta_{p-1} = 1$ and $\beta_p > 1$,

$$\widetilde{L}_3(r) \approx \ln\left(\frac{\omega}{r}\right)\left(\ln_p\left(\frac{\omega}{r}\right)\right)^{1-\beta_p}\prod_{k=p+1}^m\left(\ln_k\left(\frac{\omega}{r}\right)\right)^{-\beta_k}.$$

(ii) If $1 < \beta_1 < 2$,

$$\widetilde{L}_3(r) \approx \left(\ln\left(\frac{\omega}{r}\right)\right)^{2-\beta_1}\prod_{k=2}^m\left(\ln_k\left(\frac{\omega}{r}\right)\right)^{-\beta_k}.$$

(iii) If $\beta_1 = 2$ and $\beta_i = 1$, for $2 \leq i \leq m$,

$$\widetilde{L}_3(r) \approx \ln_{m+1}\left(\frac{\omega}{r}\right).$$

(iv) If $\beta_1 = 2$, $\beta_2 = \cdots = \beta_{l-1} = 1$ and $\beta_l < 1$,

$$\widetilde{L}_3(r) \approx \left(\ln_l\left(\frac{\omega}{r}\right)\right)^{1-\beta_l} \prod_{k=l+1}^{m} \left(\ln_k\left(\frac{\omega}{r}\right)\right)^{-\mu_k}.$$

(v) If ($\beta_1 = 2$, $\beta_2 = \cdots = \beta_{l-1} = 1$ and $\beta_l > 1$) or $\beta_1 > 2$,

$$\widetilde{L}_3(r) = 1.$$

We let $\widetilde{\mathcal{N}}_\infty(t) := \widetilde{L}_3(\frac{1}{t})$, for $t \in [1,\infty)$. The required result follows from (2.6) and the fact that $|x^*| = \frac{1}{|x|}$, for $x \in D$.

The proof is completed. □

Proposition 2.10 *Under condition* (H), *we have*

$$Vp(x) \approx \theta(x), \quad \textit{for } x \in D,$$

where $p(x) := a(x)\theta^\sigma(x)$, $\sigma < 1$ *and* θ *is defined in* (1.7).

Proof Let a be a function satisfying condition (H). Using (1.6) and (1.7), we obtain

$$p(x) = a(x)\theta^\sigma(x) \approx \big(\rho_D(x)\big)^{-\alpha}\mathcal{M}_0\big(\rho_D(x)\big)|x|^{-\beta}\mathcal{N}_\infty\big(|x|\big)$$

where $\alpha = \lambda - \min(1, \frac{2-\lambda}{1-\sigma})\sigma$, $\beta = \mu \geq 2$ and for $t \in (0,\eta)$

$$\mathcal{M}_0(t) := \begin{cases} \mathcal{L}_0(t), & \text{if } \lambda < 1, \\ \mathcal{L}_0(t)(\int_t^\eta \frac{\mathcal{L}_0(s)}{s}\,ds)^{\frac{\sigma}{1-\sigma}}, & \text{if } \lambda = 1, \\ (\mathcal{L}_0(t))^{\frac{1}{1-\sigma}}, & \text{if } 1 < \lambda < 2, \\ \mathcal{L}_0(t)(\int_0^t \frac{\mathcal{L}_0(s)}{s}\,ds)^{\frac{\sigma}{1-\sigma}}, & \text{if } \lambda = 2, \end{cases}$$

and for $s \geq 1$, $\mathcal{N}_\infty(s) = \mathcal{L}_\infty(s)$, if $\mu > 2$ while for $\mu = 2$, $\mathcal{N}_\infty$ is defined as follows:

– If $\mu_1 = 1 + \sigma$, $\mu_2 = \cdots = \mu_{p-1} = 1$ and $\mu_p > 1$,

$$\mathcal{N}_\infty(s) = \big(\ln(\omega s)\big)^{-1}\big(\ln_p(\omega s)\big)^{\frac{\sigma-\mu_p}{1-\sigma}} \prod_{k=2}^{p-1}\big(\ln_k(\omega s)\big)^{-1} \prod_{k=p+1}^{m}\big(\ln_k(\omega s)\big)^{-\frac{\mu_k}{1-\sigma}}.$$

– If $1 + \sigma < \mu_1 < 2$

$$\mathcal{N}_\infty(s) = \big(\ln(\omega s)\big)^{\frac{2\sigma-\mu_1}{1-\sigma}} \prod_{k=2}^{m}\big(\ln_k(\omega s)\big)^{-\frac{\mu_k}{1-\sigma}}.$$

– If $\mu_1 = 2$ and $\mu_i = 1$, for all $2 \leq i \leq m$,

$$\mathcal{N}_\infty(s) = \big(\ln_{m+1}(\omega s)\big)^{\frac{\sigma}{1-\sigma}} \prod_{k=1}^{m}\big(\ln_k(\omega s)\big)^{-\mu_k}.$$

– If $\mu_1 = 2$, $\mu_2 = \cdots = \mu_{l-1} = 1$ and $\mu_l < 1$,

$$\mathcal{N}_\infty(s) = \big(\ln(\omega s)\big)^{-2}\big(\ln_l(\omega s)\big)^{\frac{\sigma-\mu_l}{1-\sigma}}\prod_{k=2}^{l-1}\big(\ln_k(\omega s)\big)^{-1}\prod_{k=l+1}^{m}\left(\ln_k\left(\frac{\omega s}{t}\right)\right)^{-\frac{\mu_k}{1-\sigma}}.$$

– If ($\mu_1 = 2$, $\mu_2 = \cdots = \mu_{l-1} = 1$ and $\mu_l > 1$) or $\mu_1 > 2$,

$$\mathcal{N}_\infty(s) = \prod_{k=1}^{m}\big(\ln_k(\omega s)\big)^{-\mu_k},$$

Since $\lambda \le 2$, then it follows that $\alpha \le 2$.

By applying Proposition 2.9, we obtain

$$Vp(x) \approx \big(\rho_D(x)\big)^{\min(1,2-\alpha)}\big(\widetilde{\mathcal{M}}_0\big(\rho_D(x)\big)\big)\big(\widetilde{\mathcal{N}}_\infty\big(|x|\big)\big).$$

Since $\min(1, 2-\nu) = \min(1, \frac{2-\lambda}{1-\sigma})$, we deduce for $x \in D$,

$$Vp(x) \approx \big(\rho_D(x)\big)^{\min(1,\frac{2-\lambda}{1-\sigma})}\big(\widetilde{\mathcal{M}}_0\big(\rho_D(x)\big)\big)\big(\widetilde{\mathcal{N}}_\infty\big(|x|\big)\big) \approx \theta(x).$$

This completes the proof. □

3 Proof of Theorem 1.5

In order to prove Theorem 1.5, we need first to establish some preliminary results related to the following problem (P_γ) with $\gamma > 0$:

$$(P_\gamma)\begin{cases} -\Delta u(x) = a(x)u^\sigma(x), x \in D \text{ (in the distributional sense)},\\ u > 0 \text{ in } D,\\ \lim_{x\to\partial D}\frac{u(x)}{\ln(\omega|x|)} = \gamma,\\ \lim_{|x|\to\infty}\frac{u(x)}{\ln(\omega|x|)} = \gamma. \end{cases}$$

The next lemma will be used in what follows.

Lemma 3.1 (See [2]) *Let Ω be a bounded regular domain in $\mathbb{R}^2$ with $0 \in \Omega$. Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{K}_0$ and set*

$$\psi(\xi) := |\xi|^{-\lambda_1}\mathcal{M}_1\big(|\xi|\big)\big(\delta_\Omega(\xi)\big)^{-\lambda_2}\mathcal{M}_2\big(\delta_\Omega(\xi)\big), \quad \textit{for } \xi \in \Omega.$$

The following properties are equivalent:

(i)

$$\lim_{\alpha\to 0}\sup_{\zeta\in\Omega}\int_{B(\zeta,\alpha)\cap\Omega}\frac{\delta_\Omega(\xi)}{\delta_\Omega(\zeta)}G_\Omega(\zeta,\xi)\big|\psi(\xi)\big|\,d\xi = 0,$$

where $G_\Omega(\zeta,\xi)$ the Green's function of the Laplacian in Ω.

(ii)

$$\int_0^\eta s^{1-\lambda_1}\ln\left(\frac{\omega}{s}\right)\mathcal{M}_1(s)\,ds<\infty \quad \textit{and} \quad \int_0^\eta s^{1-\lambda_2}\mathcal{M}_2(s)\,ds<\infty$$

with $\lambda_1 \le 2$ *and* $\lambda_2 \le 2$.

Proposition 3.2 *Assume that hypothesis* (H) *is fulfilled. Then the function* $q(y) := (\ln(\omega|y|))^{\sigma-1}a(y)$ *belongs to the class* $K(D)$.

Proof Let $\alpha > 0$. Since $|x^*-y^*| = \frac{|x-y|}{|x||y|}$, $|x|>1$ and $|y|>1$, it follows that if $y\in B(x,\alpha)$ then $y^*\in B(x^*,\alpha)$. Therefore by (1.6), (1.3) and (1.4), we have

$$\begin{aligned}
&\int_{B(x,\alpha)\cap D}\frac{\rho_D(y)}{\rho_D(x)}G_D(x,y)\bigl(\ln(\omega|y|)\bigr)^{\sigma-1}a(y)\,dy\\
&\quad\le C\int_{B(x,\alpha)\cap D}\frac{(\delta_{D^*}(y^*))^{1-\lambda}}{\delta_{D^*}(x^*)}G_{D^*}\bigl(x^*,y^*\bigr)\bigl(\ln(\omega|y|)\bigr)^{\sigma-1}\mathcal{L}_0\bigl(\delta_{D^*}(y^*)\bigr)|y|^{-\mu}\mathcal{L}_\infty\bigl(|y|\bigr)\,dy\\
&\quad\le C\int_{B(x^*,\alpha)\cap D^*}\frac{(\delta_{D^*}(\xi))^{1-\lambda}}{\delta_{D^*}(x^*)}G_{D^*}\bigl(x^*,\xi\bigr)\mathcal{M}_2\bigl(\delta_{D^*}(\xi)\bigr)|\xi|^{\mu-4}\mathcal{M}_1\bigl(|\xi|\bigr)\,d\xi\\
&\quad\le C\sup_{\zeta\in D^*}\int_{B(\zeta,\alpha)\cap D^*}\frac{(\delta_{D^*}(\xi))^{1-\lambda}}{\delta_{D^*}(\zeta)}G_{D^*}(\zeta,\xi)\mathcal{M}_2\bigl(\delta_{D^*}(\xi)\bigr)|\xi|^{\mu-4}\mathcal{M}_1\bigl(|\xi|\bigr)\,d\xi,
\end{aligned}$$

where $\mathcal{M}_1(s) := (\ln(\frac{\omega}{s}))^{(\sigma-1)}\mathcal{L}_\infty(\frac{1}{s})$ and $\mathcal{M}_2(s) := \mathcal{L}_0(s)$.

By hypothesis (H), Remark 1.4(ii) and Lemma 2.4, we have $\mathcal{M}_1,\mathcal{M}_2\in\mathcal{K}_0$ and condition (ii) in Lemma 3.1 is satisfied.

Hence

$$\lim_{\alpha\to 0}\sup_{x\in D}\int_{B(x,\alpha)\cap D}\frac{\rho_D(y)}{\rho_D(x)}G_D(x,y)\bigl(\ln(\omega|y|)\bigr)^{\sigma-1}a(y)\,dy=0.$$

Next, we claim that

$$\lim_{M\to+\infty}\sup_{x\in D}\int_{(|y|\ge M)\cap D}\frac{\rho_D(y)}{\rho_D(x)}G_D(x,y)\bigl(\ln(\omega|y|)\bigr)^{\sigma-1}a(y)\,dy=0.$$

Indeed, by the above argument, for $\varepsilon>0$, there exists $\alpha>0$ such that

$$\sup_{\zeta\in D^*}\int_{B(\zeta,\alpha)\cap D^*}\frac{(\delta_{D^*}(\xi))^{1-\lambda}}{\delta_{D^*}(\zeta)}G_{D^*}(\zeta,\xi)\mathcal{M}_2\bigl(\delta_{D^*}(\xi)\bigr)|\xi|^{\mu-4}\mathcal{M}_1\bigl(|\xi|\bigr)\,d\xi\le\varepsilon.$$

Fix this α and let $M>0$. Using (1.3), we obtain

$$\begin{aligned}
&\int_{(|y|\ge M)\cap D}\frac{\rho_D(y)}{\rho_D(x)}G_D(x,y)\bigl(\ln(\omega|y|)\bigr)^{\sigma-1}a(y)\,dy\\
&\quad\le C\int_{(|\xi|\le\frac{1}{M})\cap D^*}\frac{(\delta_{D^*}(\xi))^{1-\lambda}}{\delta_{D^*}(x^*)}G_{D^*}\bigl(x^*,\xi\bigr)\mathcal{M}_2\bigl(\delta_{D^*}(\xi)\bigr)|\xi|^{\mu-4}\mathcal{M}_1\bigl(|\xi|\bigr)\,d\xi\\
&\quad\le C\varepsilon+C\int_{(|x^*-\xi|\ge\alpha)\cap(|\xi|\le\frac{1}{M})\cap D^*}\frac{(\delta_{D^*}(\xi))^{1-\lambda}}{\delta_{D^*}(x^*)}G_{D^*}\bigl(x^*,\xi\bigr)\mathcal{M}_2\bigl(\delta_{D^*}(\xi)\bigr)|\xi|^{\mu-4}\mathcal{M}_1\bigl(|\xi|\bigr)\,d\xi
\end{aligned}$$

$$\leq C\varepsilon + C\int_{(|\xi|\leq\frac{1}{M})\cap D^*}\big(\delta_{D^*}(\xi)\big)^{2-\lambda}\mathcal{M}_2\big(\delta_{D^*}(\xi)\big)|\xi|^{\mu-4}\mathcal{M}_1\big(|\xi|\big)\,d\xi$$
$$\leq C\varepsilon + C\int_{(|\xi|\leq\frac{1}{M})\cap D^*}|\xi|^{\mu-4}\mathcal{M}_1\big(|\xi|\big)\,d\xi$$
$$\leq C\varepsilon + C\int_0^{\frac{1}{M}} s^{\mu-3}\left(\ln\left(\frac{\omega}{s}\right)\right)^{\sigma-1}\mathcal{L}_\infty\left(\frac{1}{s}\right)ds.$$

Now by using hypothesis (H), Lemmas 2.4 and 2.5, we have

$$\int_0^\eta s^{\mu-3}\left(\ln\left(\frac{\omega}{s}\right)\right)^{\sigma-1}\mathcal{L}_\infty\left(\frac{1}{s}\right)ds < \infty.$$

Therefore $\lim_{M\to+\infty}\int_0^{\frac{1}{M}} s^{\mu-3}(\ln(\frac{\omega}{s}))^{\sigma-1}\mathcal{L}_\infty(\frac{1}{s})\,ds = 0$, which gives the required result. □

Proposition 3.3 *Let $\sigma < 0$, and assume that hypothesis* (H) *is satisfied. Then for each $\gamma > 0$, problem (P_γ) has at least one positive solution $u_\gamma \in C(\overline{D})$ such that for $x \in \overline{D}$,*

$$u_\gamma(x) = \gamma\ln\big(\omega|x|\big) + \int_D G_D(x,y)a(y)u_\gamma^\sigma(y)\,dy. \tag{3.1}$$

Proof Let $\sigma < 0$ and $\gamma > 0$. By Propositions 3.2 and 2.3, we have

$$x \mapsto h(x) := \frac{1}{\ln(\omega|x|)}\int_D G_D(x,y)a(y)\big(\ln\big(\omega|y|\big)\big)^\sigma\,dy \in C_0(\overline{D}). \tag{3.2}$$

Let $\beta := \gamma + \gamma^\sigma\|h\|_\infty$ and consider the convex set Λ given by

$$\Lambda = \big\{v \in C\big(\overline{D}\cup\{\infty\}\big) : \gamma \leq v \leq \beta\big\}.$$

Define the operator T on Λ by

$$Tv(x) = \gamma + \frac{1}{\ln(\omega|x|)}\int_D G_D(x,y)a(y)(\ln\big(\omega|y|\big)^\sigma v^\sigma(y)\,dy.$$

We aim at proving that $T\Lambda$ is equicontinuous at each point of $\overline{D}$.

Indeed, let $x_0 \in D$. Since $\sigma < 0$, we have for each $v \in \Lambda$ and all $x \in D$,

$$\big|Tv(x) - Tv(x_0)\big| \leq \gamma^\sigma\int_D\left|\frac{1}{\ln(\omega|x|)}G_D(x,y) - \frac{1}{\ln(\omega|x_0|)}G_D(x_0,y)\right|\ln\big(\omega|y|\big)q(y)\,dy,$$

where $q(y) = (\ln(\omega|y|))^{\sigma-1}a(y) \in K(D)$.

Now, by following the proof of Proposition 2.3, we have for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in B(x_0,\delta)\cap D$, then

$$\gamma^\sigma\int_D\left|\frac{1}{\ln(\omega|x|)}G_D(x,y) - \frac{1}{\ln(\omega|x_0|)}G_D(x_0,y)\right|\ln\big(\omega|y|\big)q(y)\,dy \leq \varepsilon.$$

This implies that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } x \in B(x_0,\delta)\cap D, \text{ then } \big|Tv(x) - Tv(x_0)\big| \leq \varepsilon, \quad \text{for all } v \in \Lambda.$$

On the other hand, for all $v \in \Lambda$ and $x \in \overline{D}$, we have

$$\left|Tv(x) - \gamma\right| \le \gamma^{\sigma} h(x),$$

where the function h is given by (3.2). Since $h \in C_0(\overline{D})$, we deduce that

$$\lim_{x \to x_0 \in \partial D} Tv(x) = \lim_{|x| \to \infty} Tv(x) = \gamma, \quad \text{uniformly for all } v \in \Lambda.$$

So, the family $T\Lambda$ is equicontinuous in $C(\overline{D} \cup \{\infty\})$. In particular, for all $v \in \Lambda$, $Tv \in C(\overline{D} \cup \{\infty\})$ and therefore $T\Lambda \subset \Lambda$.

Moreover, since the family $\{Tv(x), v \in \Lambda\}$ is uniformly bounded in $\overline{D} \cup \{\infty\}$, then it follows from Arzelà–Ascoli theorem (see [19, p. 62] and [6, Theorem 2.3]) that $T(\Lambda)$ is relatively compact in $C(\overline{D} \cup \{\infty\})$.

Next, we prove the continuity of T in Λ. Let $(v_k)_k \subset \Lambda$ and $v \in \Lambda$ be such that $\|v_k - v\|_\infty \to 0$ as $k \to \infty$. Then we have

$$\left|Tv_k(x) - Tv(x)\right| \le \frac{1}{\ln(\omega|x|)} \int_D G_D(x,y) a(y) \bigl(\ln\bigl(\omega|y|\bigr)\bigr)^{\sigma} \left|v_k^{\sigma}(y) - v^{\sigma}(y)\right| dr.$$

Now, since

$$\left|v_k^{\sigma}(y) - v^{\sigma}(y)\right| \le 2\gamma^{\sigma},$$

we deduce by (3.2) and the dominated convergence theorem that

$$\forall x \in \overline{D}, \quad Tv_k(x) \to Tv(x) \quad \text{as } k \to \infty.$$

Since $T(\Lambda)$ is relatively compact in $C(\overline{D} \cup \{\infty\})$, we obtain

$$\|Tv_k - Tv\|_\infty \to 0 \quad \text{as } k \to \infty.$$

So, T is a compact mapping of Λ to itself. Therefore, by the Schauder fixed point theorem, there exists $v_\gamma \in \Lambda$ such that for each $x \in \overline{D}$

$$v_\gamma(x) = \gamma + \frac{1}{\ln(\omega|x|)} \int_D G_D(x,y) a(y) \bigl(\ln\bigl(\omega|y|\bigr)\bigr)^{\sigma} v_\gamma^{\sigma}(y)\, dy. \tag{3.3}$$

Since $v_\gamma^{\sigma} \le \gamma^{\sigma}$, we deduce from (3.3) and (3.2) that

$$\lim_{x \to \partial D} v_\gamma(x) = \lim_{|x| \to \infty} v_\gamma(x) = \gamma. \tag{3.4}$$

Put $u_\gamma(x) = \ln(\omega|x|) v_\gamma(x)$, for $x \in \overline{D}$. Then $u_\gamma \in C(\overline{D})$ and we have

$$u_\gamma(x) = \gamma \ln\bigl(\omega|x|\bigr) + \int_D G_D(x,y) a(y) u_\gamma^{\sigma}(y)\, dy, \tag{3.5}$$

as well as

$$\gamma \ln\bigl(\omega|x|\bigr) \le u_\gamma(x) \le \beta \ln\bigl(\omega|x|\bigr). \tag{3.6}$$

Now, since the function $y \mapsto a(y)u_\gamma^\sigma(y) \in L^1_{\mathrm{loc}}(D)$ and from (3.5) the function $x \mapsto \int_D G_D(x,y)a(y)u_\gamma^\sigma(y)\,dy \in L^1_{\mathrm{loc}}(D)$, we deduce by (1.5) that u_γ satisfies

$$-\Delta u_\gamma(x) = a(x)u_\gamma^\sigma(x), \quad x \in D \text{ (in the distributional sense).}$$

By (3.4), we have

$$\lim_{x\to\partial D}\frac{u_\gamma(x)}{\ln(\omega|x|)} = \lim_{|x|\to\infty}\frac{u_\gamma(x)}{\ln(\omega|x|)} = \gamma.$$

This completes the proof. □

The next result based on the complete maximum principle is established in [51, Lemma 3.1].

Lemma 3.4 *Let $g \in \mathcal{B}^+(D)$ and v be a nonnegative superharmonic function on D. Then for any $w \in \mathcal{B}(D)$ such that $V(g|w|) < \infty$ and $w + V(gw) = v$, we have*

$$0 \le w \le v.$$

Corollary 3.5 *Let $\sigma < 0$, and assume that hypothesis* (H) *is satisfied. For $0 < \gamma_1 \le \gamma_2$, we denote by $u_{\gamma_i} \in C(\overline{D})$ the solution of problem (P_γ) satisfying* (3.1). *Then we have*

$$0 \le u_{\gamma_2}(x) - u_{\gamma_1}(x) \le (\gamma_2 - \gamma_1)\ln\bigl(\omega|x|\bigr), \quad \textit{for } x \in \overline{D}. \tag{3.7}$$

Proof Let g be the function defined on D by

$$g(x) = \begin{cases} a(x)\frac{u_{\gamma_2}^\sigma(x)-u_{\gamma_1}^\sigma(x)}{u_{\gamma_1}(x)-u_{\gamma_2}(x)}, & \text{if } u_{\gamma_1}(x) \ne u_{\gamma_2}(x), \\ 0, & \text{if } u_{\gamma_1}(x) = u_{\gamma_2}(x). \end{cases}$$

Since $\sigma < 0$, then $g \in \mathcal{B}^+(D)$, and we have

$$u_{\gamma_2} - u_{\gamma_1} + V\bigl(g(u_{\gamma_2} - u_{\gamma_1})\bigr) = (\gamma_2 - \gamma_1)\ln\bigl(\omega|x|\bigr). \tag{3.8}$$

On the other hand, by using (3.1), (3.2) and (3.6), we obtain, for $x \in \overline{D}$,

$$V\bigl(g|u_{\gamma_2} - u_{\gamma_1}|\bigr)(x) \le \bigl(\gamma_1^\sigma + \gamma_2^\sigma\bigr)\int_D G_D(x,y)a(y)\bigl(\ln\bigl(\omega|y|\bigr)\bigr)^\sigma\,dy < \infty.$$

Hence the required result follows from (3.8) and Lemma 2.4 with $v(x) = \ln(\omega|x|)$. □

Proposition 3.6 *Let $\sigma < 0$. Under hypothesis* (H), *problem* (1.2) *has at least one positive solution $u \in C(\overline{D})$ such that for $x \in \overline{D}$,*

$$u(x) = \int_D G_D(x,y)a(y)u^\sigma(y)\,dy. \tag{3.9}$$

Proof Let $(\gamma_k)_k$ be a positive sequence decreasing to zero. Let $u_k \in C(\overline{D})$ be the solution of problem (P_{γ_k}) satisfying (3.1). By Corollary 3.5, the sequence $(u_k)_k$ decreases to a function u, and since $\sigma < 0$ the sequence $(u_k - \gamma_k \ln(\omega|x|))_k$ increases to u. Therefore, by using (3.1), (3.6) and the fact that $\sigma < 0$, we obtain for each $x \in \overline{D}\backslash\{0\}$,

$$\begin{aligned} u(x) \geq u_k(x) - \gamma_k \ln\bigl(\omega|x|\bigr) &= \int_D G_D(x,y)a(y)u_k^\sigma(y)\,dy \\ &\geq \beta_k^\sigma \int_D G_D(x,y)a(y)\bigl(\ln\bigl(\omega|y|\bigr)\bigr)^\sigma\,dy > 0, \end{aligned}$$

where $\beta_k := \gamma_k + \gamma_k^\sigma \|h\|_\infty$ and h is given by (3.2).

By the monotone convergence theorem, we obtain

$$u(x) = \int_D G_D(x,y)a(y)u^\sigma(y)\,dy.$$

Since for each $x \in \overline{D}$, $u(x) = \inf_k u_k(x) = \sup_k(u_k(x) - \gamma_k \ln(\omega|x|))$, u is an upper and lower semi-continuous function on $\overline{D}$ and so $u \in C(\overline{D})$.

Since the function $y \mapsto a(y)u^\sigma(y)$ is in $L^1_{\mathrm{loc}}(D)$ and from (3.9) the function $x \mapsto \int_D G_D(x,y)a(y)u^\sigma(y)\,dy$ is also in $L^1_{\mathrm{loc}}(D)$, we deduce by (1.5) that

$$-\Delta u(x) = a(x)u^\sigma(x), \quad x \in D\backslash\{0\} \text{ (in the distributional sense).}$$

Finally, using the fact that for all $x \in D$, $0 < u(x) \leq u_k(x)$ and that u_k is a solution of problem (P_{γ_k}), we deduce that

$$\lim_{x \to \partial D} u(x) = 0 \quad \text{and} \quad \lim_{|x| \to \infty} \frac{u(x)}{\ln(\omega|x|)} = 0.$$

Hence u is a solution of problem (1.2). ☐

Proof of Theorem 1.5 Assume that function a satisfies hypothesis (H). By Proposition 2.10, there exists $m \geq 1$ such that for each D,

$$\frac{1}{m}\theta(x) \leq Vp(x) \leq m\theta(x), \tag{3.10}$$

where θ is the function defined in (1.7) and $p(y) := a(y)\theta^\sigma(y)$.

We split the proof into the following two cases:

Case 1: $\sigma < 0$.

By Proposition 3.6, problem (1.2) has a positive continuous solution u satisfying (3.9). We claim that u satisfies (1.8).

By (3.10), we have

$$m^\sigma (Vp)^\sigma(x) \leq \theta^\sigma(x) \leq m^{-\sigma}(Vp)^\sigma(x). \tag{3.11}$$

Let $c = m^{-\frac{\sigma}{1-\sigma}}$. Then by elementary calculus we have

$$cVp = V\bigl(a(cVp)^\sigma\bigr) + Vf, \tag{3.12}$$

where $f(x) := ca(x)[\theta^\sigma(x) - m^\sigma(Vp)^\sigma(x)]$, for $x \in \overline{D}$.

Clearly, we have $f \in \mathcal{B}^+(D)$ and by using (3.9) and (3.12), we obtain

$$cVp - u + V\bigl(a\bigl(u^\sigma - (cVp)^\sigma\bigr)\bigr) = Vf. \tag{3.13}$$

Let g be the function defined on D by

$$g(x) = \begin{cases} a(x)\frac{u^\sigma(x)-(cVp)^\sigma(x)}{(cVp)(x)-u(x)}, & \text{if } u(x) \neq (cVp)(x), \\ 0, & \text{if } u(x) = (cVp)(x). \end{cases}$$

Then $g \in \mathcal{B}^+(D)$ and since $\sigma < 0$, we have

$$a\bigl(u^\sigma - (cVp)^\sigma\bigr) = g(cVp - u). \tag{3.14}$$

Therefore the relation (3.13) becomes

$$(cVp - u) + V\bigl(g(cVp - u)\bigr) = Vf.$$

Now since $f \in \mathcal{B}^+(D)$ by using (3.14), (3.9), (3.12) and (3.10), we obtain

$$\begin{aligned} V\bigl(g|cVp - u|\bigr) &\le V\bigl(au^\sigma\bigr) + V\bigl(a(cVp)^\sigma\bigr) \\ &\le u + cVp \\ &\le u + cm\theta < \infty. \end{aligned}$$

Hence by Lemma 3.4, we obtain

$$u \le cVp.$$

Similarly, we prove that

$$\frac{1}{c}Vp \le u.$$

Thus, by (3.10), u satisfies (1.7).

Case 2: $0 \le \sigma < 1$.

Let $\rho(x) = \frac{1}{\ln(\omega|x|)}\theta(x)$, for $x \in D$. By (3.10), we have

$$\frac{1}{m}\rho(x) \le \frac{1}{\ln(\omega|x|)}Vp(x) \le m\rho(x). \tag{3.15}$$

Put $c = m^{\frac{1}{1-\sigma}}$ and consider the closed convex set given by

$$A = \left\{v \in C_0(\overline{D}), \frac{1}{c}\rho \le v \le c\rho\right\}.$$

Clearly $\rho \in A$.

Define the operator S on A by

$$Sv(x) := \frac{1}{\ln(\omega|x|)} \int_D G_D(x,y)a(y)\big(\ln\big(\omega|y|\big)\big)^{\sigma} v^{\sigma}(y)\,dy, \quad x \in D.$$

By using (3.15), we obtain for all $v \in A$,

$$\frac{1}{c}\rho \le Sv \le c\rho.$$

Since for all $v \in A$, we have

$$\big|v^{\sigma}(y)\big| \le c^{\sigma}\big\|\rho^{\sigma}\big\|_{\infty}, \quad \text{for all } y \in D,$$

we deduce as in the proof of Proposition 3.3 that

$$Sv \in C_0(\overline{D}), \quad \text{for all } v \in A.$$

So, $S(A) \subset A$.

Let $(v_k)_k \subset C_0(\overline{D})$ defined by

$$v_0 = \frac{1}{c}\rho \quad \text{and} \quad v_{k+1} = Sv_k, \quad \text{for } k \in \mathbb{N}.$$

Since the operator S is nondecreasing and $S(A) \subset A$, we deduce that

$$\frac{1}{c}\rho = v_0 \le v_1 \le v_2 \le \cdots \le v_k \le v_{k+1} \le c\rho.$$

Therefore, by the monotone convergence theorem, the sequence $(v_k)_k$ converges to a function v such that for each $x \in \overline{D}$,

$$v(x) = \frac{1}{\ln(\omega|x|)} \int_D G_D(x,y)a(y)\big(\ln\big(\omega|y|\big)\big)^{\sigma} v^{\sigma}(y)\,dy$$

and

$$\frac{1}{c}\rho(x) \le v(x) \le c\rho(x).$$

Since v is bounded, we prove by similar arguments as in the proof of Proposition 3.3 that $v \in C_0(\overline{D})$.

Put $u(x) = \ln(\omega|x|)v(x)$. Then $u \in C(\overline{D})$ satisfies the equation

$$u(x) = V\big(au^{\sigma}\big)(x), \quad \text{for } x \in D. \tag{3.16}$$

Finally, since the function $y \mapsto a(y)u^{\sigma}(y)$ is in $L^1_{\mathrm{loc}}(D)$ and from (3.16) the function $x \mapsto V(au^{\sigma})(x)$ is also in $L^1_{\mathrm{loc}}(D)$, we deduce by (1.5) that u is a solution of problem (1.2). The proof of Theorem 1.5 is completed. $\square$

Example 3.7 Let $\sigma < 1$ and $a \in C(D)$ be such that

$$a(x) \approx |x|^{-2}\bigl(\ln(\omega|x|)\bigr)^{-2}\bigl(\ln_2(\omega|x|)\bigr)^{-\beta}\bigl(\rho_D(x)\bigr)^{-2}\left(\ln\left(\frac{\omega}{\rho_D(x)}\right)\right)^{-\gamma},$$

where $\beta \in \mathbb{R}$ and $\gamma > 1$. Then, by Theorem 1.5, problem (1.2) has at least one positive solution $u \in C(\overline{D})$ satisfying the following estimates:

(i) If $\beta = 1$, then for $x \in D$,

$$u(x) \approx \bigl(\ln_3(\omega|x|)\bigr)^{\frac{1}{1-\sigma}}\left(\ln\left(\frac{\omega}{\rho_D(x)}\right)\right)^{\frac{1-\gamma}{1-\sigma}}.$$

(ii) If $\beta < 1$, then for $x \in D$,

$$u(x) \approx \bigl(\ln_2(\omega|x|)\bigr)^{\frac{1-\beta}{1-\sigma}}\left(\ln\left(\frac{\omega}{\rho_D(x)}\right)\right)^{\frac{1-\gamma}{1-\sigma}}.$$

(iii) If $\beta > 1$, then for $x \in D$,

$$u(x) \approx \left(\ln\left(\frac{\omega}{\rho_D(x)}\right)\right)^{\frac{1-\gamma}{1-\sigma}}.$$

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