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# Properties of Green's function and the existence of different types of solutions for nonlinear fractional BVP with a parameter in integral boundary conditions

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## Abstract

This paper is concerned with the impact of the parameter on the existence of different types of solutions for a class of nonlinear fractional integral boundary value problems with a parameter that causes the sign of Green's function associated with the BVP to change. By using the Guo–Krasnoselskii fixed point theorem, the Leray–Schauder nonlinear alternative, and the analytic technique, we give the range of the parameter for the existence of strong positive solutions, strong negative solutions, negative solutions, and sign-changing solutions for the boundary value problem. Some examples are given to illustrate our main results.

**MSC:** 34B18; 34B15

**Keywords:** Integral boundary value problem; Caputo fractional order derivative; Parameter; Sign-changing solution; Cone; Fixed point

## 1 Introduction and preliminaries

Fractional differential equations are *recognized* as adequate mathematical models to study some materials and processes that have memory and hereditary properties. Much effort has been devoted to this topic in the last ten years. As a result, this theory has become an important area of investigation in differential equation theories. For a small sample of such work, we refer the reader to the monographs [1–4] and the papers [5–18].

Because of the extensive application in mathematics and the applied science, fractional boundary value problems with parameters have attracted considerable attention and obtained some interesting results; see, for instance, the works of Bai [19], Song and Bai [20], Jiang [21], Sun et al. [22], Zhai and Xu [23], and Zhang and Liu [24] on the eigenvalue problems; the works of Jia and Liu [25], Wang and Liu [26], Su et al. [27], and Li et al. [28] on the problems with disturbance parameters in the boundary conditions; and the work of Wang and Guo [29] on the eigenvalue problems with a disturbance parameter in the boundary conditions.

At the same time, we also notice that another type of fractional integral boundary value problems with  $\mu$  in the boundary conditions has received much attention; see [30–33] and the references therein. Bashir Ahmad et al. [30] studied the fractional boundary value

problem given by

$$\begin{cases} D_{0+}^q x(t) = f(t, x(t)), & 0 < t < 1, 1 < q \leq 2, \\ x(0) = 0, & x(1) = \mu \int_0^\eta x(s) ds, \quad 0 < \eta < 1, \end{cases}$$

where  $\mu \in \mathbb{R}$  and  $\mu \neq \frac{2}{\eta^2}$ , and obtained the existence and uniqueness results of solutions by using Banach's fixed point theorem, Krasnoselskii's fixed point theorem, and the Leray–Schauder degree theory. Zhang et al. [31] applied fixed point index theory to investigate the existence of positive solutions for the following boundary value problem:

$$\begin{cases} D_{0+}^\alpha x(t) + h(t)f(t, x(t)) = 0, & 0 < t < 1, 3 < \alpha \leq 4, \\ x(0) = x'(0) = x''(0) = 0, \\ x(1) = \mu \int_0^\eta x(s) ds, & 0 \leq \frac{\mu \eta^\alpha}{\alpha} < 1, 0 < \eta \leq 1. \end{cases}$$

He [32] discussed the existence of positive solutions for the fractional differential equations

$$\begin{cases} {}^C D_{0+}^\alpha x(t) + f(t, x(t)) = 0, & 0 < t < 1, 3 < \alpha \leq 4, \\ x''(0) = x'''(0) = 0, \\ x'(0) = x(1) = \mu \int_0^1 x(s) ds, & 0 < \mu < 2 \end{cases}$$

by the Leray–Schauder nonlinear alternative and a fixed-point theorem in cones. Wang et al. [33] investigated the existence of positive solutions for the problem given by

$$\begin{cases} {}^C D_{0+}^\alpha x(t) + \lambda f(t, x(t)) = 0, & 0 < t < 1, n < \alpha \leq n+1, n \geq 2, n \in \mathbb{N}, \\ x(0) = x''(0) = x'''(0) = \dots = x^{(n)}(0) = 0, \\ x(1) = \mu \int_0^1 x(s) ds, & 0 < \mu < 2 \end{cases}$$

by the Guo–Krasnoselskii fixed point theorem.

We notice that the letter  $\mu$  in [30–33] is essentially treated as a constant rather than a parameter, especially it is required to guarantee the nonnegativity of corresponding Green's function in [31–33]. In fact, when the above  $\mu$  is a parameter, it is inevitable that it has great influence on the property of Green's function associated with the boundary value problem. It is well known that the property of Green's function is crucial to studying the property of solutions for boundary value problems. Thus, it is natural to ask what effect the parameter  $\mu$  has on properties of solutions. This is a very significant topic, but to the best of author's knowledge, there are no papers reported on it.

Motivated by the above-mentioned works, in this paper we will study the following fractional integral boundary value problem (BVP) with a parameter  $\mu$ :

$$\begin{cases} {}^C D_{0+}^\alpha x(t) + f(t, x(t)) = 0, & 0 < t < 1, \\ x'(0) = 0, & x(1) = \mu \int_0^1 x(s) ds, \end{cases} \quad (1)$$

where  ${}^C D_{0+}^\alpha$  is the Caputo fractional derivative of order  $\alpha$ ,  $1 < \alpha < 2$ ;  $f \in C([0, 1] \times \mathbb{R}, \mathbb{R}_+)$ ,  $\mathbb{R}_+ = [0, +\infty)$  and  $\mu > 0$ .

Different from the literature [30–33], the purpose of this paper is to divide the range of the parameter  $\mu$  for the existence of strong positive solutions, strong negative solutions, negative solutions, and sign-changing solutions for BVP (1). The main tools used in this paper are the Guo–Krasnoselskii fixed point theorem, the Leray–Schauder nonlinear alternative, and the analytic technique.

The Guo–Krasnoselskii fixed point theorem has been extensively applied to discuss the existence and multiplicity of positive solutions for boundary value problems, see for instance [26, 32], and [33]. However, to our knowledge, there are no papers to apply this theorem to study the existence of solutions for problems such as BVP (1) where the sign of its corresponding Green's function is changing. So, this fixed point theorem may be the first time to be applied to discuss the existence of various solutions for such problems.

The paper is organized as follows. In Sect. 2, we establish the integral equation and the operator equation equivalent to BVP (1). In particular, we present some properties of the corresponding Green's function. In Sect. 3, we give the range of the parameter  $\mu$  on the existence of strong positive solutions, strong negative solutions, negative solutions, and sign-changing solutions for BVP (1). These results show the impact of the parameter  $\mu$  on the existence of different types of solutions. Finally, two examples are given to illustrate our main results.

A function  $x$  is called a solution of BVP (1) if  $x \in AC^2[0, 1]$ ,  ${}^C D_{0+}^\alpha x(t) \in C[0, 1]$  and satisfies BVP (1). Let  $x$  be a solution of BVP (1),  $x$  is called a strong positive solution (strong negative solution) if  $x(t) > 0$  ( $x(t) < 0$ ) for  $t \in [0, 1]$ ;  $x$  is called a negative solution if  $x(t) \leq 0$  and  $x(t) \not\equiv 0$  for  $t \in [0, 1]$ ;  $x$  is called a non-positive solution if  $x(t) \leq 0$  for  $t \in [0, 1]$ ; and  $x$  is called a sign-changing solution if there exist  $t_1, t_2 \in [0, 1]$  such that  $x(t_1)x(t_2) < 0$ .

To be clear, we present some basic notations and results from fractional calculus theory.

**Definition 1.1** ([1–4]) Let  $x : (0, +\infty) \rightarrow \mathbb{R}$  be a function and  $\alpha > 0$ . The Riemann–Liouville fractional integral of order  $\alpha$  of  $x$  is defined by

$$I_{0+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

provided that the integral exists. The Caputo fractional derivative of order  $\alpha$  of  $x$  is defined by

$${}^C D_{0+}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds,$$

provided that the right-hand side is pointwise defined on  $(0, +\infty)$ , where  $n = [\alpha] + 1$ ,  $n-1 < \alpha < n$ , and  $\Gamma(\alpha)$  denotes the gamma function. If  $\alpha = n$ , then  ${}^C D_{0+}^\alpha x(t) = x^{(n)}(t)$ .

**Lemma 1.2** ([1–4]) If  $x \in C^n[0, 1]$ , then

$$I_{0+}^\alpha {}^C D_{0+}^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where  $n$  is the smallest integer greater than or equal to  $\alpha$ .

**Lemma 1.3** ([1]) *If  $x \in AC^n[0, 1]$ , then the Caputo fractional derivative  ${}^C D_{0^+}^\alpha x(t)$  exists almost everywhere on  $[0, 1]$ , where*

$$AC^n[0, 1] = \{x \in C^{n-1}[0, 1] | x^{(n-1)} \text{ is absolutely continuous}\}$$

and  $n$  is the smallest integer greater than or equal to  $\alpha$ .

In the rest of this section, we present some notations on cone theory and some known results on fixed points theory. For details on cone theory, see [34] and [35].

Let  $E$  be a real Banach space and  $\theta$  be the zero element of  $E$ . Recall that a nonempty closed convex set  $P \subset E$  is a cone if it satisfies (i)  $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$ ; (ii)  $x \in P, -x \in P \Rightarrow x = \theta$ . Obviously, if  $P$  is a cone in  $E$ , then  $-P = \{x \in E | -x \in P\}$  is a cone of  $E$ , also.

**Lemma 1.4** ([34, 35] (Guo–Krasnoselskii)) *Let  $P$  be a cone in a real Banach space  $E$  and  $\Omega_1, \Omega_2$  be bounded open subsets in  $E$  with  $\theta \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$ . Assume that  $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$  is a completely continuous operator such that*

- (i)  $\|Tx\| \leq \|x\|$  for  $x \in P \cap \partial\Omega_1$  and  $\|Tx\| \geq \|x\|$  for  $x \in P \cap \partial\Omega_2$ , or
- (ii)  $\|Tx\| \geq \|x\|$  for  $x \in P \cap \partial\Omega_1$ , and  $\|Tx\| \leq \|x\|$  for  $x \in P \cap \partial\Omega_2$ .

Then  $T$  has a fixed point in  $P \cap (\overline{\Omega_2} \setminus \Omega_1)$ .

**Lemma 1.5** ([36]) *Let  $E$  be a Banach space,  $X$  be a convex set of  $E$ ,  $D$  be a relatively open subset of  $X$ , and  $\theta \in D$ . Suppose that  $T : \overline{D} \rightarrow X$  is a continuous, compact map, then either (i)  $T$  has a fixed point in  $\overline{D}$ , or (ii) there exist  $u \in \partial D$  and  $\lambda \in (0, 1)$  with  $u = \lambda Tu$ .*

According to the fixed point index theory, it is easy to see the following result.

**Lemma 1.6** *Let  $P$  be a cone in a real Banach space  $E$  and  $\Omega$  be bounded open subsets in  $E$  with  $\theta \in \Omega$ . Assume that  $T : P \cap \overline{\Omega} \rightarrow P$  is a completely continuous operator such that  $\|Tx\| \leq \|x\|$  for  $x \in P \cap \partial\Omega$ , then  $T$  has a fixed point in  $P \cap \overline{\Omega}$ .*

## 2 Green's function and equivalent equation

In this section, we apply Lemmas 1.2 and 1.3 to obtain the integral equation and the operator equation equivalent to BVP (1) and present the properties of its corresponding Green's function.

**Lemma 2.1** *For any given  $\mu \neq 1$ ,  $x_\mu$  is a solution of BVP (1) if and only if  $x_\mu \in C[0, 1]$  is a solution of the following integral equation:*

$$x_\mu(t) = \int_0^1 G_\mu(t, s) f(s, x_\mu(s)) ds, \quad (2)$$

where

$$G_\mu(t, s) = G_0(t, s) + \frac{\mu}{1 - \mu} \int_0^1 G_0(\tau, s) d\tau, \quad (3)$$

$$G_0(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s < t \leq 1, \\ (1-s)^{\alpha-1}, & 0 \leq t \leq s < 1. \end{cases} \quad (4)$$

*Proof* If  $x \in C[0, 1]$  is a solution of (1), from Lemma 1.2 we get

$$\begin{aligned}x(t) &= -I_{0+}^{\alpha} f(t, x(t)) + c_0 + c_1 t, \\x'(t) &= -I_{0+}^{\alpha-1} f(t, x(t)) + c_1.\end{aligned}$$

By  $y(t)$  we denote  $f(t, x(t))$ , then  $y(t)$  is continuous on  $[0, 1]$ . Since  $\int_0^1 (1-s)^{\alpha-2} ds$  is convergent,  $x'(0) = 0$ , and  $x(1) = \mu \int_0^1 x(s) ds$ , we obtain  $c_1 = 0$  and  $c_0 = I_0^{\alpha} y(1) + \mu \int_0^1 x(s) ds$ , which means that

$$x(t) = \int_0^1 G_0(t, s) y(s) ds + \mu \int_0^1 x(s) ds. \quad (5)$$

Moreover,

$$\int_0^1 x(s) ds = \frac{1}{1-\mu} \int_0^1 \int_0^1 G_0(\tau, s) y(s) d\tau ds.$$

Substituting the value into (5), we can obtain that the solution  $x \in C[0, 1]$  satisfies (2).

On the other hand, if  $x \in C[0, 1]$  is the solution of (2), then

$$\begin{aligned}x(t) &= \frac{1}{\Gamma(\alpha)} \left( \int_0^1 (1-s)^{\alpha-1} f(s, x(s)) ds - \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \right) + \mu \int_0^1 x(s) ds, \\x'(t) &= -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(s)) ds.\end{aligned}$$

It is easy to see that  $x' \in AC[0, 1]$  and  $x \in AC^2[0, 1]$ . From Lemma 1.3 we obtain that  ${}^C D_{0+}^{\alpha} x$  exists almost everywhere on  $[0, 1]$ . Note that

$$x''(t) = -\frac{d}{dt} \left( \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(s)) ds \right) = -\frac{d}{dt} I_{0+}^{\alpha-1} f(t, x(t)),$$

we conclude that

$$\begin{aligned}{}^C D_{0+}^{\alpha} x(t) &= I_{0+}^{2-\alpha} x''(t) = -I_{0+}^{1-(\alpha-1)} \frac{d}{dt} I_{0+}^{\alpha-1} f(t, x(t)) \\&= -{}^C D_{0+}^{\alpha-1} I_{0+}^{\alpha-1} f(t, x(t)) = -f(t, x(t)),\end{aligned}$$

and  $x$  is the solution of BVP (1). The proof is complete.  $\square$

**Remark 2.2** It follows from (3) and (4) that

$$G_{\mu}(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{1}{1-\mu} [(1-s)^{\alpha-1} - \frac{\mu}{\alpha} (1-s)^{\alpha}] - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ \frac{1}{1-\mu} [(1-s)^{\alpha-1} - \frac{\mu}{\alpha} (1-s)^{\alpha}], & 0 \leq t \leq s \leq 1. \end{cases} \quad (6)$$

**Lemma 2.3**  $G_{\mu}(t, s)$  is continuous on  $[0, 1] \times [0, 1]$  for every  $\mu \neq 1$ ; and  $G_{\mu}(t, s)$  is monotone decreasing with respect to  $t$  for every  $s \in [0, 1]$  and  $\mu \neq 1$ . Moreover,

(i) for  $0 < \mu < 1$ ,

$$0 \leq \frac{\mu(\alpha-1)}{(1-\mu)\Gamma(\alpha+1)} (1-s)^{\alpha-1} \leq G_{\mu}(t, s) \leq \frac{1}{(1-\mu)\Gamma(\alpha)} (1-s)^{\alpha-1}, \quad t, s \in [0, 1];$$

(ii) for  $1 < \mu < \alpha$ ,

$$\frac{-\mu}{(\mu-1)\Gamma(\alpha)}(1-s)^{\alpha-1} \leq G_\mu(t,s) \leq \frac{-(\alpha-\mu)}{(\mu-1)\Gamma(\alpha+1)}(1-s)^{\alpha-1} \leq 0, \quad t,s \in [0,1];$$

(iii) for  $\mu = \alpha$ ,

$$\frac{-\mu}{(\mu-1)\Gamma(\alpha)}(1-s)^{\alpha-1} \leq G_\mu(t,s) \leq \frac{-\mu s}{(\mu-1)\Gamma(\alpha)}(1-s)^{\alpha-1} \leq 0, \quad t,s \in [0,1];$$

(iv) for  $\mu > \alpha$ ,  $G_\mu(0,0) > 0$ ,  $G_\mu(1,0) < 0$ , and

$$\frac{\mu(1-\alpha-s)}{\Gamma(\alpha+1)(\mu-1)}(1-s)^{\alpha-1} \leq G_\mu(t,s) \leq \frac{\mu(1-s-\frac{\alpha}{\mu})}{\Gamma(\alpha+1)(\mu-1)}(1-s)^{\alpha-1}, \quad t,s \in [0,1].$$

Moreover, there exist  $\sigma_1, \sigma_2 \in (0,1)$  with  $\sigma_1 < \sigma_2$  and  $0 < \gamma < 1$  such that  $G_\mu(t,s) \leq 0$  for  $(t,s) \in [\sigma_1, \sigma_2] \times [0,1]$ , and

$$\min_{\sigma_1 \leq t \leq \sigma_2} |G_\mu(t,s)| \geq \frac{\mu}{\Gamma(\alpha+1)(\mu-1)} \begin{cases} \gamma, & 0 \leq s \leq \sigma_2, \\ |1-s-\frac{\alpha}{\mu}|(1-s)^{\alpha-1}, & \sigma_2 < s \leq 1. \end{cases} \quad (7)$$

*Proof* For  $\mu \neq 1$ , it is obvious from (6) that

$$G_\mu(1,s) \leq G_\mu(t,s) \leq G_\mu(s,s), \quad t,s \in [0,1], \quad (8)$$

where

$$G_\mu(1,s) = \frac{\mu(\alpha-1+s)}{(1-\mu)\Gamma(\alpha+1)}(1-s)^{\alpha-1}, \quad s \in [0,1],$$

$$G_\mu(s,s) = G_\mu(0,s) = \frac{\mu(\frac{\alpha}{\mu}-1+s)}{(1-\mu)\Gamma(\alpha+1)}(1-s)^{\alpha-1}, \quad s \in [0,1].$$

In addition, note that  $G_\mu(s,s) = \frac{-s}{(\mu-1)\Gamma(\alpha)}(1-s)^{\alpha-1}$  for  $\mu = \alpha$ , then it is easy to verify conclusions (i), (ii), and (iii). Next, let us prove conclusion (iv).

When  $\mu > \alpha$ , it is easy to check that  $G_\mu(0,0) = \frac{1}{(1-\mu)\Gamma(\alpha)}(1-\frac{\mu}{\alpha}) > 0$ ,  $G_\mu(1,0) = \frac{\mu}{(1-\mu)\Gamma(\alpha)}(1-\frac{1}{\alpha}) < 0$ . Let

$$g_\mu(s) = \left(1 - \frac{\alpha}{\mu} - s\right)(1-s)^{\alpha-1} \quad \text{and} \quad g(s) = (1-\alpha-s)(1-s)^{\alpha-1}, \quad s \in [0,1].$$

From (6),  $G_\mu(t,s)$  can be written as follows:

$$G_\mu(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{\mu}{\alpha(\mu-1)}g_\mu(s) - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ \frac{\mu}{\alpha(\mu-1)}g_\mu(s), & 0 \leq t \leq s \leq 1. \end{cases} \quad (9)$$

Moreover,

$$\frac{\mu}{\Gamma(\alpha+1)(\mu-1)}g(s) \leq G_\mu(t,s) \leq \frac{\mu}{\Gamma(\alpha+1)(\mu-1)}g_\mu(s), \quad t,s \in [0,1]. \quad (10)$$

Since

$$g'_\mu(s) = (1-s)^{\alpha-2} \alpha \left( s - 1 + \frac{\alpha-1}{\mu} \right) \quad \text{and} \quad g'(s) = \alpha(1-s)^{\alpha-2}(s-2+\alpha),$$

it is easy to check that  $g(s) \leq 0$  and  $g(s) \leq g_\mu(s)$  for  $s \in [0, 1]$ , and

$$g_\mu(s) \begin{cases} > 0 & \text{and monotone decreasing on } [0, 1 - \frac{\alpha}{\mu}], \\ = 0 & \text{for } s = 1 - \frac{\alpha}{\mu}, \\ < 0 & \text{and monotone decreasing on } (1 - \frac{\alpha}{\mu}, 1 - \frac{\alpha-1}{\mu}], \\ = -\frac{1}{\mu} \left( \frac{\alpha-1}{\mu} \right)^{\alpha-1} & \text{for } s = 1 - \frac{\alpha-1}{\mu} \text{ (the minimal value of } g_\mu), \\ < 0 & \text{and monotone increasing on } [1 - \frac{\alpha-1}{\mu}, 1); \end{cases}$$

$$g(s) \begin{cases} < 0 & \text{and monotone decreasing on } [0, 2-\alpha], \\ = -(\alpha-1)^{\alpha-1} & \text{for } s = 2-\alpha \text{ (the minimal value of } g), \\ < 0 & \text{and monotone increasing on } [2-\alpha, 1). \end{cases}$$

Thus, we obtain that  $|g_\mu(s)|, |g(s)| < 1$  for  $s \in [0, 1]$  and  $|g_\mu(s)| \leq |g(s)|$  for  $s \in [1 - \frac{\alpha-1}{\mu}, 1]$ .

This, together with (10), leads to

$$|G_\mu(t, s)| \leq \frac{\mu}{\Gamma(\alpha+1)(\mu-1)} \begin{cases} 1, & 0 \leq s \leq 1 - \frac{\alpha-1}{\mu}, \\ |g(s)|, & 1 - \frac{\alpha-1}{\mu} < s \leq 1, \end{cases} \quad \forall t \in [0, 1], \quad (11)$$

and

$$|G_\mu(t, s)| \leq \frac{\mu}{\Gamma(\alpha+1)(\mu-1)}, \quad \forall (t, s) \in [0, 1] \times [0, 1]. \quad (12)$$

Set  $\sigma_1 = \frac{\mu-\alpha}{\mu-1} = 1 - \frac{\alpha-1}{\mu-1}$  and  $\sigma_2 = 1 - \frac{\alpha-1}{\mu}$ , then  $1 - \frac{\alpha}{\mu} < \sigma_1 < \sigma_2 < 1$ . For any given  $t \in [\sigma_1, \sigma_2]$ , by (9) we have

$$G_\mu(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{\mu}{\alpha(\mu-1)} g_\mu(s) - (t-s)^{\alpha-1}, & 0 \leq s \leq t, \\ \frac{\mu}{\alpha(\mu-1)} g_\mu(s), & t \leq s \leq 1. \end{cases} \quad (13)$$

In particular, we claim that  $G_\mu(\sigma_1, s) \leq 0$  for  $s \in [0, 1]$ . Indeed, it is evident that

$$G_\mu(\sigma_1, 0) = -\frac{\mu-\alpha}{\Gamma(\alpha+1)(\mu-1)} \left( \alpha \left( \frac{\mu-1}{\mu-\alpha} \right)^{2-\alpha} - 1 \right) < 0,$$

$$G_\mu(\sigma_1, \sigma_1) = -\frac{\mu-\alpha}{\Gamma(\alpha+1)(\mu-1)^2} \left( \frac{\alpha-1}{\mu-1} \right)^{\alpha-1} < 0,$$

and  $G_\mu(\sigma_1, 1) = 0$ . This, together with

$$\frac{\partial^2 G_\mu(\sigma_1, s)}{\partial s^2} > 0, \quad s \in [0, \sigma_1) \cup (\sigma_1, 1), \quad (14)$$

yields that  $G_\mu(\sigma_1, s) \leq 0$  for  $0 \leq s \leq 1$ . Moreover, for any  $\sigma_1 \leq t \leq \sigma_2$ , we have

$$G_\mu(\sigma_2, s) \leq G_\mu(t, s) \leq G_\mu(\sigma_1, s) \leq 0, \quad 0 \leq s \leq 1.$$

This, together with (10) and (13), yields that, for any  $\sigma_1 \leq t \leq \sigma_2$ ,

$$\frac{\mu}{\Gamma(\alpha+1)(\mu-1)} g(s) \leq G_\mu(t, s) \leq G_\mu(\sigma_1, s) \leq 0, \quad 0 \leq s \leq \sigma_2,$$

and

$$G_\mu(t, s) = \frac{\mu}{\Gamma(\alpha+1)(\mu-1)} g_\mu(s), \quad \sigma_2 < s \leq 1.$$

In addition, noting (14) and the following inequality

$$G_\mu(\sigma_1, \sigma_1) = \frac{\mu}{\Gamma(\alpha+1)(\mu-1)} g_\mu(\sigma_1) > \frac{\mu}{\Gamma(\alpha+1)(\mu-1)} g_\mu(\sigma_2) = G_\mu(\sigma_1, \sigma_2),$$

we obtain that

$$\min_{\sigma_1 \leq t \leq \sigma_2} |G_\mu(t, s)| = \min_{\sigma_1 \leq t \leq \sigma_2} (-G_\mu(t, s)) = \frac{\mu}{\Gamma(\alpha+1)(\mu-1)} |g_\mu(s)|, \quad \sigma_2 < s \leq 1,$$

and

$$\begin{aligned} \min_{\sigma_1 \leq t \leq \sigma_2} |G_\mu(t, s)| &= \min_{\sigma_1 \leq t \leq \sigma_2} (-G_\mu(t, s)) \\ &\geq \min\{|G_\mu(\sigma_1, 0)|, |G_\mu(\sigma_1, \sigma_1)|, |G_\mu(\sigma_1, \sigma_2)|\} \\ &= \min\{|G_\mu(\sigma_1, 0)|, |G_\mu(\sigma_1, \sigma_1)|\} \\ &= \frac{\mu\gamma}{\Gamma(\alpha+1)(\mu-1)}, \quad 0 \leq s \leq \sigma_2, \end{aligned}$$

where

$$\gamma = \min\left\{\frac{\mu-\alpha}{\mu} \left(\alpha \left(\frac{\mu-1}{\mu-\alpha}\right)^{2-\alpha} - 1\right), \frac{\mu-\alpha}{\mu(\mu-1)} \left(\frac{\alpha-1}{\mu-1}\right)^{\alpha-1}\right\}, \quad (15)$$

and  $0 < \gamma < 1$ . So, (7) holds. This completes the proof.

Set  $E = C[0, 1]$ , the Banach space of all continuous functions on  $[0, 1]$  with the norm  $\|x\| = \max\{|x(t)| : t \in [0, 1]\}$ . Let

$$P = \{x \in C[0, 1] : x(t) \geq 0, t \in [0, 1]\},$$

then  $P$  and  $-P$  are cones in  $E$ . In addition, we set

$$\begin{aligned} P_{1\mu} &= \left\{x \in P : x(t) \geq \frac{\mu(\alpha-1)}{\alpha} \|x\|, t \in [0, 1]\right\}, \quad 0 < \mu < 1, \\ P_{2\mu} &= \left\{x \in -P : -x(t) \geq \frac{\alpha-\mu}{\alpha\mu} \|x\|, t \in [0, 1]\right\}, \quad 1 < \mu < \alpha, \end{aligned}$$

and

$$K_\mu = \left\{ x \in C[0, 1] \mid -x(t) \geq \frac{\gamma}{\mu\alpha} \|x\|, \sigma_1 \leq t \leq \sigma_2 \right\}, \quad \mu > \alpha,$$

where  $\sigma_1 = 1 - \frac{\alpha-1}{\mu-1}$ ,  $\sigma_2 = 1 - \frac{\alpha-1}{\mu}$ , and  $\gamma$  is given as (15).  $\square$

**Lemma 2.4**  $P_{1\mu} \subset P$ ,  $P_{2\mu} \subset -P$ , and  $K_\mu$  are all cones in  $E$ .

*Proof* It is clear that  $P_{1\mu} \subset P$  and  $P_{2\mu} \subset -P$  are cones in  $E$ . Next we prove that  $K_\mu$  is a cone in  $E$ . Indeed, it is easy to verify that (i)  $K_\mu$  is a nonempty closed convex subset in  $E$ ; and (ii)  $\lambda \geq 0$  and  $x \in K_\mu \Rightarrow \lambda x \in K_\mu$ . So, we only need to show that if  $x \in K_\mu$  and  $-x \in K_\mu$ , then  $x = \theta$ . If otherwise, we have  $\|x\| > 0$ . This, together with  $x \in K_\mu$  and  $-x \in K_\mu$ , implies that  $-x(t) \geq \frac{\gamma}{\mu\alpha} \|x\| > 0$  and  $-(-x(t)) = x(t) \geq \frac{\gamma}{\mu\alpha} \|x\| > 0$  for  $\sigma_1 \leq t \leq \sigma_2$ , this is a contradiction. So,  $K_\mu$  is a cone in  $E$ . The proof is complete.  $\square$

Define the operator  $T_\mu : C[0, 1] \rightarrow C[0, 1]$  by

$$(T_\mu x)(t) = \int_0^1 G_\mu(t, s) f(s, x(s)) ds, \quad x \in C[0, 1]. \quad (16)$$

For any  $\mu \neq 1$ , it is clear by Lemma 2.1 that  $x_\mu$  is a solution of BVP (1)  $\Leftrightarrow x_\mu$  is a fixed point of  $T_\mu$  in  $E$ .

**Lemma 2.5** The operator  $T_\mu : E \rightarrow E$  is completely continuous.

*Proof* The proof is similar to that of Lemma 2.2 in [29].  $\square$

**Lemma 2.6** (i)  $T_\mu(E) \subset P_{1\mu}$  for  $\mu \in (0, 1)$ ; (ii)  $T_\mu(E) \subset P_{2\mu}$  for  $\mu \in (1, \alpha)$ ; and (iii)  $T_\mu(E) \subset K_\mu$  for  $\mu \in (\alpha, +\infty)$ .

*Proof* (i) Given  $\mu \in (0, 1)$ . It is clear from Lemma 2.3(i) that  $T_\mu(E) \subset P$ . Moreover, for any  $x \in P$ , we have

$$\|T_\mu x\| \leq \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s)) ds.$$

This, together with Lemma 2.3(i), yields that

$$\begin{aligned} (T_\mu x)(t) &\geq \frac{\mu(\alpha-1)}{(1-\mu)\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s)) ds \\ &\geq \frac{\mu(\alpha-1)}{\alpha} \|T_\mu x\|, \quad t \in [0, 1], \end{aligned}$$

which implies that  $T_\mu(P) \subset P_{1\mu}$ . Consequently, we have  $T_\mu(E) \subset P_{1\mu}$ .

(ii) Given  $\mu \in (1, \alpha)$ . Arguing similarly as above (i), we can obtain  $T_\mu(E) \subset P_{2\mu}$ .

(iii) Given  $\mu \in (\alpha, +\infty)$ . For any  $x \in E$ , from (11), we have

$$\|T_\mu x\| \leq \frac{\mu}{(\mu-1)\Gamma(\alpha+1)} \left( \int_0^{\sigma_2} f(s, x(s)) ds + \int_{\sigma_2}^1 |g(s)| f(s, x(s)) ds \right).$$

On the other hand, since

$$\min_{s \in [\sigma_2, 1]} \frac{|1 - s - \frac{\alpha}{\mu}|}{|1 - \alpha - s|} \geq \frac{1}{\alpha\mu},$$

we have  $|g_\mu(s)| \geq \frac{1}{\alpha\mu}|g(s)|$  for  $s \in [\sigma_2, 1]$ . This, together with Lemma 2.3(iv), yields that

$$\begin{aligned} -(T_\mu x)(t) &\geq \frac{\mu}{(\mu-1)\Gamma(\alpha+1)} \left( \gamma \int_0^{\sigma_2} f(s, x(s)) ds + \int_{\sigma_2}^1 |g_\mu(s)| f(s, x(s)) ds \right) \\ &\geq \frac{\mu}{(\mu-1)\Gamma(\alpha+1)} \left( \gamma \int_0^{\sigma_2} f(s, x(s)) ds + \frac{1}{\mu\alpha} \int_{\sigma_2}^1 |g(s)| f(s, x(s)) ds \right) \\ &\geq \frac{\gamma}{\mu\alpha} \|T_\mu x\|, \quad \forall \sigma_1 \leq t \leq \sigma_2. \end{aligned}$$

This gives that  $T_\mu(E) \subset K_\mu$ . The proof is complete.  $\square$

Throughout this paper, we always use the following denotations:

$$\Omega_r = \{x \in E \mid \|x\| < r\}, \quad \partial\Omega_r = \{x \in E \mid \|x\| = r\}, \quad \overline{\Omega}_r = \Omega_r \cup \partial\Omega_r, \quad r > 0,$$

and

$$\begin{aligned} f^0 &= \limsup_{|x| \rightarrow 0} \max_{t \in [0, 1]} \frac{f(t, x)}{|x|}, & f_0 &= \liminf_{|x| \rightarrow 0} \min_{t \in [0, 1]} \frac{f(t, x)}{|x|}, \\ f^\infty &= \limsup_{|x| \rightarrow \infty} \max_{t \in [0, 1]} \frac{f(t, x)}{|x|}, & f_\infty &= \liminf_{|x| \rightarrow \infty} \min_{t \in [0, 1]} \frac{f(t, x)}{|x|}. \end{aligned}$$

### 3 Existence results of various types of solutions

In this section, we first discuss the property of solutions for BVP (1) and then give the interval of the parameter  $\mu$  on the existence of at least one strong positive solution, strong negative solution, non-positive solution, negative solution, and sign-changing solution.

**Lemma 3.1** *If  $x_\mu$  is a solution of BVP (1) for  $\mu \neq 1$ , then  $x_\mu(t)$  is decreasing with respect to  $t$  for  $t \in [0, 1]$ .*

*Proof* It follows from (6) and (16) that

$$x_\mu(t) = \int_0^1 G_\mu(s, t) f(s, x_\mu(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_\mu(s)) ds; \quad (17)$$

moreover,

$$x'_\mu(t) = -\frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} f(s, x_\mu(s)) ds \leq 0, \quad t \in [0, 1],$$

which implies that the solution  $x_\mu(t)$  is decreasing on  $[0, 1]$ . This ends the proof.  $\square$

According to Lemma 2.6 and Lemma 2.3(iii), we can obtain the following result.

**Lemma 3.2** *If  $x_\mu$  is a solution of BVP (1) for  $\mu \neq 1$ , then  $x_\mu \in P_{1\mu}$ ,  $x_\mu \in P_{2\mu}$ ,  $x_\mu \in -P$ , and  $x_\mu \in K_\mu$  for  $\mu \in (0, 1)$ ,  $\mu \in (1, \alpha)$ ,  $\mu = \alpha$ , and  $\mu \in (\alpha, +\infty)$ , respectively.*

The following conditions will be used.

$$(L1) \quad f_0 = +\infty, f^\infty = 0;$$

$$(L2) \quad f^0 = 0, f_\infty = +\infty;$$

$$(H) \quad f(t, 0) \not\equiv 0 \text{ on } [0, 1].$$

It is interesting to point out that (H) is independent of  $f_0 = +\infty$ . For example, let  $f(t, x) = (t + 1)\sqrt{|x|}$ , then  $f_0 = \liminf_{|x| \rightarrow 0} \min_{t \in [0, 1]} \frac{f(t, x)}{|x|} = +\infty$ , but  $f(t, 0) \equiv 0$ . So  $f_0 = +\infty \not\Rightarrow (H)$ . On the other hand, let  $f(t, x) = t + x^2$ , then  $f(t, 0) = t \not\equiv 0$ , but  $f_0 = 0$ . This means that (H)  $\not\Rightarrow f_0 = +\infty$ .

For any given  $x \in C[0, 1]$ , set

$$I_{Lx} = \left\{ \mu \in (\alpha, +\infty) \mid \mu \int_0^1 (1-s)^\alpha f(s, x(s)) ds < \alpha \int_0^1 (1-s)^{\alpha-1} f(s, x(s)) ds \right\},$$

$$I_{Ex} = \left\{ \mu \in (\alpha, +\infty) \mid \mu \int_0^1 (1-s)^\alpha f(s, x(s)) ds = \alpha \int_0^1 (1-s)^{\alpha-1} f(s, x(s)) ds \right\},$$

$$I_{Gx} = \left\{ \mu \in (\alpha, +\infty) \mid \mu \int_0^1 (1-s)^\alpha f(s, x(s)) ds > \alpha \int_0^1 (1-s)^{\alpha-1} f(s, x(s)) ds \right\},$$

then  $I_{Lx} \cup I_{Ex} \cup I_{Gx} = (\alpha, +\infty)$ .

**Theorem 3.3** *Suppose that (L1) holds. Then BVP (1) has at least one solution  $x_\mu$  for any  $\mu \neq 1$ ; in particular,  $x_\mu$  is a strong positive solution, a strong negative solution, a non-positive solution, a negative solution, and a sign-changing solution for  $\mu \in (0, 1)$ ,  $\mu \in (1, \alpha) \cup I_{Lx_\mu}$ ,  $\mu = \alpha$ ,  $\mu \in I_{Ex_\mu}$ , and  $\mu \in I_{Gx_\mu}$ , respectively.*

*Proof* We prove all the statements in four steps.

(i) Given  $\mu \in (0, 1)$ . According to Lemma 2.6, we only need to find a fixed point of  $T_\mu$  in  $P_{1\mu}$ . Since  $f_0 = +\infty$ , there exists  $r_1 > 0$  such that

$$f(t, x) \geq \frac{(1-\mu)\alpha^2 \Gamma(\alpha+1)}{\mu^2(\alpha-1)^2} x, \quad \forall x \in [0, r_1], t \in [0, 1].$$

This, together with Lemma 2.3(i) and the definition of  $P_{1\mu}$ , leads to

$$(T_\mu x)(t) \geq \frac{\alpha^2}{\mu(\alpha-1)} \int_0^1 (1-s)^{\alpha-1} x(s) ds \geq \|x\|, \quad \forall x \in P_{1\mu} \cap \partial\Omega_{r_1},$$

which means that

$$\|T_\mu x\| \geq \|x\|, \quad \forall x \in P_{1\mu} \cap \partial\Omega_{r_1}.$$

On the other hand, it follows from  $f^\infty = 0$  that there exists  $l_1 > 0$  such that

$$f(t, x) \leq (1-\mu)\Gamma(\alpha+1)x, \quad x \geq l_1, t \in [0, 1]. \quad (18)$$

Set  $R_1 > \max\{\frac{\alpha}{(\alpha-1)\mu}l_1, 2r_1\}$ , then it is easy to see that

$$\min_{t \in [0,1]} x(t) \geq \frac{\mu(\alpha-1)}{\alpha} R_1 > l_1, \quad x \in P_{1\mu} \cap \partial\Omega_{R_1}.$$

Moreover, it follows from Lemma 2.3(i) and (18) that

$$(T_\mu x)(t) \leq \frac{1}{\Gamma(\alpha)(1-\mu)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s)) ds \leq \|x\|, \quad x \in P_{1\mu} \cap \partial\Omega_{R_1},$$

which implies that

$$\|T_\mu x\| \leq \|x\|, \quad x \in P_{1\mu} \cap \partial\Omega_{R_1}.$$

Therefore, it follows from Lemma 1.4 that BVP (1) has at least one solution  $x_\mu \in P_{1\mu}$  with  $r_1 \leq \|x_\mu\| \leq R_1$ . This, together with the definition of  $P_{1\mu}$ , implies that  $x_\mu$  is a strong positive solution.

(ii) Given  $\mu \in (1, \alpha)$ . From Lemma 2.6 we only need to find a fixed point of  $T_\mu$  in  $P_{2\mu}$ . By  $f_0 = +\infty$ , there exists  $r_2 > 0$  such that

$$f(t, x) \geq \frac{(\mu-1)\alpha^2\mu\Gamma(\alpha+1)}{(\alpha-\mu)^2} |x|, \quad x \in [-r_2, 0], t \in [0, 1],$$

which together with Lemma 2.3(ii) and the definition of  $P_{2\mu}$  gives

$$-(T_\mu x)(t) \geq \frac{\alpha^2\mu}{\alpha-\mu} \int_0^1 (1-s)^{\alpha-1} |x(s)| ds \geq \|x\|, \quad x \in P_{2\mu} \cap \partial\Omega_{r_2}.$$

This means that

$$\|T_\mu x\| \geq \|x\|, \quad x \in P_{2\mu} \cap \partial\Omega_{r_2}.$$

In addition, it follows from  $f^\infty = 0$  that there exists  $l_2 > 0$  such that

$$f(t, x) \leq \frac{(\mu-1)\Gamma(\alpha+1)}{\mu} |x|, \quad x \leq -l_2, t \in [0, 1]. \quad (19)$$

Set  $R_2 > \max\{\frac{\alpha\mu}{\alpha-\mu}l_2, 2r_2\}$ , then it is easy to see that

$$-x(t) \geq \frac{\alpha-\mu}{\alpha\mu} R_2 > l_2, \quad \forall t \in [0, 1], x \in P_{2\mu} \cap \partial\Omega_{R_2}.$$

This together with Lemma 2.3(ii) and (19) leads to

$$|(T_\mu x)(t)| \leq \alpha \int_0^1 (1-s)^{\alpha-1} |x(s)| ds \leq \|x\|, \quad x \in P_{2\mu} \cap \partial\Omega_{R_2},$$

which implies that

$$\|T_\mu x\| \leq \|x\|, \quad x \in P_{2\mu} \cap \partial\Omega_{R_2}.$$

Therefore, applying Lemma 1.4, we obtain that BVP (1) has at least one solution  $x_\mu \in P_{2\mu}$  with  $r_2 < \|x_\mu\| \leq R_2$ . This, together with the definition of  $P_{2\mu}$ , implies that  $x_\mu$  is a strong negative solution.

(iii) Given  $\mu = \alpha$ . It is obvious in view of Lemma 2.3(iii) that  $T_\mu(E) \subset -P$ . So, we only need to find a fixed point of  $T_\mu$  in  $-P$ . It follows from  $f^\infty = 0$  that there exists  $l_3 > 0$  such that

$$f(t, x) \leq \Gamma(\alpha)(\alpha - 1)|x|, \quad x \leq -l_3. \quad (20)$$

We assert that there exists  $R_3 > 0$  such that

$$x \neq \lambda T_\mu x, \quad \forall \lambda \in (0, 1), x \in (-P) \cap \partial\Omega_{R_3}. \quad (21)$$

In order to prove, the assertion we consider two cases.

Case 1. The function  $f$  is bounded on  $[0, 1] \times (-\infty, 0]$ , that is, there exists  $M > 0$  such that  $f(t, x) \leq M$  for  $t \in [0, 1]$  and  $x \in (-\infty, 0]$ . Take  $R_3 > \frac{M}{(\alpha-1)\Gamma(\alpha)}$ , then (21) holds. Suppose, to the contrary, that there exist  $\bar{x} \in (-P) \cap \partial\Omega_{R_3}$  and  $\bar{\lambda} \in (0, 1)$  such that  $\bar{x} = \bar{\lambda} T_\mu \bar{x}$ , that is,

$$\bar{x}(t) = \bar{\lambda} \int_0^1 G_\mu(t, s) f(s, \bar{x}(s)) ds, \quad t \in [0, 1].$$

This, together with  $\mu = \alpha$ , implies that

$$|\bar{x}(t)| \leq \frac{\bar{\lambda} M \alpha}{(\alpha - 1)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} ds < \frac{M}{(\alpha - 1)\Gamma(\alpha)} < R_3, \quad t \in [0, 1],$$

which implies that  $R_3 < R_3$ , this is a contradiction.

Case 2.  $f$  is an unbounded function on  $[0, 1] \times (-\infty, 0]$ . In this case, we can take  $R_3 > l_3$  such that

$$f(t, x) \leq f(t, -R_3), \quad t \in [0, 1], x \in [-R_3, 0]. \quad (22)$$

Moreover, (21) holds. Suppose, to the contrary, that there exist  $\bar{x} \in (-P) \cap \partial\Omega_{R_3}$  and  $\bar{\lambda} \in (0, 1)$  such that  $\bar{x} = \bar{\lambda} T_\mu \bar{x}$ . Then, by (20) and (22), we have

$$|\bar{x}(t)| \leq \frac{\bar{\lambda} \alpha}{(\alpha - 1)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(s, -R_3) ds \leq \bar{\lambda} R_3 < R_3, \quad t \in [0, 1],$$

which implies that  $R_3 < R_3$ , this is a contradiction.

Consequently, applying Lemma 1.5, we obtain that BVP (1) has at least one solution  $x_\mu \in (-P) \cap \overline{\Omega}_{R_3}$ , that is, the solution  $x_\mu(t) \leq 0$  for  $t \in [0, 1]$ . So,  $x_\mu$  is a non-positive solution.

(iv) Given  $\mu > \alpha$ . Similarly, we only need to find a fixed point of  $T_\mu$  in  $K_\mu$ . It follows from  $f_0 = \infty$  that there exists  $r_4 > 0$  such that

$$f(t, x) \geq \frac{\Gamma(\alpha + 1)\mu(\mu - 1)^2\alpha}{\gamma^2(\alpha - 1)} |x|, \quad \forall t \in [0, 1] \text{ and } |x| < r_4. \quad (23)$$

For  $x \in K_\mu \cap \partial\Omega_{r_4}$ , it follows from (23), (7), and the definition of  $K_\mu$  that

$$\begin{aligned} |(T_\mu x)(t)| &= \int_0^1 (-G_\mu(t, s))f(s, x(s)) ds \\ &\geq \frac{\Gamma(\alpha + 1)\mu(\mu - 1)^2\alpha}{\gamma^2(\alpha - 1)} \int_{\sigma_1}^{\sigma_2} \min_{t \in [\sigma_1, \sigma_2]} |G_\mu(t, s)| |x(s)| ds \\ &\geq \frac{\mu^2(\mu - 1)\alpha}{\gamma(\alpha - 1)} \int_{\sigma_1}^{\sigma_2} |x(s)| ds \geq \frac{\mu(\mu - 1)(\sigma_2 - \sigma_1)}{\alpha - 1} \|x\| \\ &= \|x\|, \quad \forall \sigma_1 \leq t \leq \sigma_2, \end{aligned}$$

which implies that

$$\|T_\mu x\| \geq \|x\|, \quad x \in K_\mu \cap \partial\Omega_{r_4}.$$

On the other hand, it follows from  $f^\infty = 0$  that there exists  $l_4 > 0$  such that

$$f(t, x) \leq \frac{\Gamma(\alpha + 1)(\mu - 1)}{\mu} |x|, \quad |x| \geq l_4, t \in [0, 1]. \quad (24)$$

In order to show that there exists  $R_4 > r_4 > 0$  such that, for any  $x \in K_\mu \cap \partial\Omega_{R_4}$ ,

$$|(T_\mu x)(t)| \leq \|x\|, \quad \forall t \in [0, 1], \quad (25)$$

there are two cases to be considered.

Case 1.  $f$  is bounded on  $[0, 1] \times (-\infty, +\infty)$ , that is, there exists  $M > 0$  such that

$$f(t, x) \leq M, \quad (t, x) \in [0, 1] \times (-\infty, +\infty).$$

Take  $R_4 \geq \max\{\frac{M\mu}{(\mu-1)\Gamma(\alpha+1)}, 2r_4\}$ , then for any  $x \in K_\mu \cap \partial\Omega_{R_4}$ , it is easy to see from (12) that

$$|(T_\mu x)(t)| \leq \frac{M\mu}{(\mu - 1)\Gamma(\alpha + 1)} \leq R_4 = \|x\|, \quad t \in [0, 1],$$

that is, (25) holds.

Case 2.  $f(t, x)$  is unbounded on  $[0, 1] \times (-\infty, +\infty)$ . We can choose  $R_4 > \max\{l_4, 2r_4\}$  such that

$$f(t, x) \leq f(t, -R_4) \quad \text{or} \quad f(t, x) \leq f(t, R_4), \quad t \in [0, 1], x \in [-R_4, R_4].$$

This, together with (24), leads to

$$f(t, x) \leq \frac{\Gamma(\alpha + 1)(\mu - 1)}{\mu} R_4, \quad t \in [0, 1], x \in [-R_4, R_4].$$

Moreover, for any  $x \in K_\mu \cap \partial\Omega_{R_4}$ , we have

$$|(T_\mu x)(t)| \leq \frac{\mu}{(\mu - 1)\Gamma(\alpha + 1)} \int_0^1 f(s, x(s)) ds \leq R_4 = \|x\|, \quad t \in [0, 1],$$

that is, (25) holds.

Noting that (25) implies that  $\|T_\mu x\| \leq \|x\|$  for any  $x \in K_\mu \cap \partial\Omega_{R_4}$ , it follows from Lemma 1.4 that BVP (1) has a solution  $x_\mu$  with  $r_4 < \|x_\mu\| < R_4$ . It is easy to check that

$$x_\mu(1) \leq x(\sigma_2) \leq -\frac{(\mu-1)\gamma}{\mu} \|x_\mu\| < 0. \quad (26)$$

In particular, when  $\mu \in I_{Lx_\mu}$ , from the definition of  $I_{Lx_\mu}$  we have

$$x_\mu(0) = \frac{\mu \int_0^1 (1-s)^\alpha f(s, x_\mu(s)) ds - \alpha \int_0^1 (1-s)^{\alpha-1} f(s, x_\mu(s)) ds}{(\mu-1)\Gamma(\alpha+1)} < 0,$$

this together with Lemma 3.1 and (26) leads to  $x_\mu(t) < 0$  for  $t \in [0, 1]$ , that is,  $x_\mu$  is a strong negative solution. Similarly, when  $\mu \in I_{Ex_\mu}$ , we can obtain that  $x_\mu(t) \leq 0$  and  $x_\mu(t) \not\equiv 0$  for  $t \in [0, 1]$ , that is,  $x_\mu$  is a negative solution; and when  $\mu \in I_{Gx_\mu}$ , we can obtain  $x_\mu(0) > 0$ , this together with (26) means that  $x_\mu$  is a sign-changing solution. The proof is complete.  $\square$

**Corollary 3.4** *Suppose that (H) holds. If  $f^\infty = 0$ , then BVP (1) has at least one non-zero solution  $x_\mu$  for any  $\mu \neq 1$ . Furthermore, this solution  $x_\mu$  is a strong positive solution, a strong negative solution, a negative solution, and a sign-changing solution for  $\mu \in (0, 1)$ ,  $\mu \in (1, \alpha] \cup I_{Lx_\mu}$ ,  $\mu \in I_{Ex_\mu}$ , and  $\mu \in I_{Rx_\mu}$ , respectively.*

*Proof* According to Lemma 1.5, Lemma 1.6, and the proof of Theorem 3.3, we obtain that BVP (1) has at least one solution  $x_\mu$  for any  $\mu \neq 1$ .

Next we show that  $x_\mu(t) \not\equiv 0$  on  $[0, 1]$ . Suppose, to the contrary, that  $x_\mu(t) \equiv 0$  on  $[0, 1]$ , then we have

$$x_\mu(t) = \int_0^1 G_\mu(t, s) f(s, 0) ds = 0, \quad \forall t \in [0, 1],$$

which implies that

$$x_\mu(0) - x_\mu(1) = \int_0^1 (G_\mu(0, s) - G_\mu(1, s)) f(s, 0) ds = 0.$$

Since  $G(0, s) - G(1, s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}$  for  $s \in [0, 1]$ , then

$$\int_0^1 (1-s)^{\alpha-1} f(s, 0) ds = 0,$$

this implies that  $f(t, 0) \equiv 0$  on  $[0, 1]$ , which contradicts condition (H).

Since  $x_\mu(t) \not\equiv 0$  on  $[0, 1]$ , it is obvious that  $\|x_\mu\| > 0$ . Moreover, from Lemma 2.6 we have

$$x_\mu(t) \geq \frac{\mu(\alpha-1)}{\alpha} \|x_\mu\| > 0, \quad t \in [0, 1], \mu \in (0, 1),$$

and

$$-x_\mu(t) \geq \frac{\alpha-\mu}{\alpha\mu} \|x_\mu\| > 0, \quad t \in [0, 1], \mu \in (1, \alpha),$$

which mean that  $x_\mu$  is a strong positive solution and a strong negative solution for  $\mu \in (0, 1)$  and  $\mu \in (1, \alpha)$ , respectively. In addition, similar to the proof of Theorem 3.3, we obtain that  $x_\mu$  is a strong negative solution, a negative solution, and a sign-changing solution for  $\mu \in I_{Lx_\mu}$ ,  $\mu \in I_{Ex_\mu}$ , and  $\mu \in I_{Gx_\mu}$ , respectively.

Finally, we shall show that  $x_\mu$  is a strong negative solution for  $\mu = \alpha$ . By view of Lemma 3.1, we only need to show that  $x_\mu(0) < 0$ . It is evident from Lemma 3.2 that  $x_\mu(0) \leq 0$ . If  $x_\mu(0) = 0$ , that is,

$$\int_0^1 G_\mu(0, s) f(s, x_\mu(s)) ds = \frac{\mu \int_0^1 ((1-s)^\alpha - (1-s)^{\alpha-1}) f(s, x_\mu(s)) ds}{(\mu-1)\Gamma(\alpha+1)} = 0,$$

which implies that  $f(t, x_\mu(t)) = 0$  for  $t \in [0, 1]$ . Moreover,

$$x_\mu(t) = \int_0^1 G_\mu(t, s) f(s, x_\mu(s)) ds = 0, \quad \forall t \in [0, 1],$$

this contradicts the fact  $x_\mu(t) \not\equiv 0$  on  $[0, 1]$ . Hence, we have  $x_\mu(0) < 0$ . The proof is complete.  $\square$

**Theorem 3.5** *Suppose that (L2) holds. Then BVP (1) has the zero solution for every  $\mu \neq 1$ . In addition, BVP (1) has at least one non-zero solution  $x_\mu$  for any  $\mu \neq 1, \alpha$ ; furthermore,  $x_\mu$  is a strong positive solution, a strong negative solution, a negative solution, and a sign-changing solution for  $\mu \in (0, 1)$ ,  $\mu \in (1, \alpha) \cup I_{Lx_\mu}$ ,  $\mu \in I_{Ex_\mu}$ , and  $\mu \in I_{Rx_\mu}$ , respectively.*

*Proof* Since  $f^0 = 0$  implies that  $f(t, 0) = 0$  for  $t \in [0, 1]$ , then  $x_\mu(t) \equiv 0$  is a solution of BVP (1) for any  $\mu \neq 1$ . In the sequel, we prove the rest of the statements in three steps.

(i) Given  $\mu \in (0, 1)$ . In this case, we only need to find a non-zero fixed point of  $T_\mu$  in  $P_{1\mu}$ . It is evident by  $f^0 = 0$  that there exists  $r_1 > 0$  such that

$$f(t, x) \leq (1-\mu)\Gamma(\alpha+1)x, \quad 0 \leq x \leq r_1, t \in [0, 1].$$

Then, for  $x \in P_{1\mu} \cap \partial\Omega_{r_1}$ , we can obtain that

$$\|T_\mu x\| \leq \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s)) ds \leq \|x\|.$$

On the other hand, by  $f_\infty = +\infty$  there exists  $l_1 > 0$  such that

$$f(t, x) \geq \frac{(1-\mu)\alpha^2\Gamma(\alpha+1)}{\mu^2(\alpha-1)^2}x, \quad x \geq l_1. \quad (27)$$

Set  $R_1 > \max\{\frac{\alpha}{\mu(\alpha-1)}l_1, 2r_1\}$ , then for  $x \in P_{1\mu} \cap \partial\Omega_{R_1}$ , we have

$$x(t) \geq \frac{\mu(\alpha-1)}{\alpha}R_1 \geq l_1, \quad \forall t \in [0, 1].$$

Moreover, it follows from Lemma 2.3(i), the definition of  $P_{1\mu}$ , and (27) that

$$\|T_\mu x\| \geq \frac{\alpha^2}{\mu(\alpha-1)} \int_0^1 (1-s)^{\alpha-1} x(s) ds \geq \|x\|.$$

Therefore, applying Lemma 1.4 we obtain that BVP (1) has a solution  $x_\mu$  with  $r_1 < \|x_\mu\| < R_1$ ; moreover,  $x_\mu$  is a strong positive solution.

(ii) Given  $1 < \mu < \alpha$ . In this case, we only need to find a non-zero fixed point of  $T_\mu$  in  $P_{2\mu}$ . By a similar argument as the above (i), there exist  $r_2 > 0$  and  $l_2 > 0$  such that

$$\begin{aligned} f(t, x) &\leq \frac{(\mu - 1)\Gamma(\alpha + 1)}{\mu} |x|, \quad 0 \leq x \leq -r_2, t \in [0, 1] \quad \text{and} \\ f(t, x) &\geq \frac{\Gamma(\alpha + 1)(\mu - 1)\mu\alpha^2}{(\alpha - \mu)^2} |x|, \quad x \leq -l_2, t \in [0, 1]. \end{aligned}$$

Moreover,

$$\|T_\mu x\| \leq \frac{\mu}{(\mu - 1)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(s, x(s)) ds \leq \|x\|, \quad \forall x \in P_{2\mu} \cap \partial\Omega_{r_2},$$

and

$$\begin{aligned} \|T_\mu x\| &\geq \frac{\alpha - \mu}{\Gamma(\alpha + 1)(\mu - 1)} \int_0^1 (1 - s)^{\alpha-1} f(s, x(s)) ds \\ &\geq \frac{\alpha^2 \mu}{\alpha - \mu} \int_0^1 (1 - s)^{\alpha-1} |x(s)| ds \\ &\geq \|x\|, \quad x \in P_{2\mu} \cap \partial\Omega_{R_2}, \end{aligned}$$

where  $R_2 > \max\{\frac{\alpha\mu l_2}{\alpha - \mu}, r_2\}$ . Therefore, BVP (1) has a solution  $x_\mu$  with  $r_2 \leq \|x_\mu\| \leq R_2$ , and  $x_\mu$  is a strong negative solution.

(iii) Given  $\mu > \alpha$ . In this case, we only need to find a non-zero fixed point of  $T_\mu$  in  $K_\mu$ . Clearly, there exists  $r_3 > 0$  such that

$$f(t, x) \leq \frac{(\mu - 1)\Gamma(\alpha + 1)}{\mu} |x|, \quad |x| \leq r_3, t \in [0, 1],$$

and  $\|T_\mu x\| \leq \|x\|$  for  $x \in K_\mu \cap \partial\Omega_{r_3}$ .

In addition, there exists  $l_3 > 0$  such that

$$f(t, x) \geq \frac{\alpha\Gamma(\alpha + 1)\mu(\mu - 1)^2}{(\alpha - 1)\gamma^2} |x|, \quad |x| \geq l_3, t \in [0, 1]. \quad (28)$$

Set  $R_3 > \max\{\frac{\mu\alpha l_2}{\gamma}, 2r_2\}$ , then for  $x \in K_\mu \cap \partial\Omega_{R_3}$ , we have

$$|x(t)| = -x(t) \geq \frac{\gamma}{\mu\alpha} R_3 > l_3, \quad \sigma_1 \leq t \leq \sigma_2;$$

moreover, it follows from (7) and (28) that, for any  $\sigma_1 \leq t \leq \sigma_2$ ,

$$|(T_\mu x)(t)| \geq \int_{\sigma_1}^{\sigma_2} |G_\mu(t, s)| f(s, x(s)) ds \geq \frac{\alpha(\mu - 1)\mu^2}{(\alpha - 1)\gamma} \int_{\sigma_1}^{\sigma_2} |x(s)| ds \geq \|x\|.$$

This implies that

$$\|T_\mu x\| \geq \|x\|, \quad x \in K \cap \partial\Omega_{R_3}.$$

Applying Lemma 1.5 we obtain that BVP (1) has a non-zero solution  $x_\mu$  with  $r_3 \leq \|x_\mu\| \leq R_3$ . Similar to the proof of Theorem 3.3, we further obtain that  $x_\mu$  is a strong negative solution, a negative solution, and a sign-changing solution for  $\mu \in I_{Lx_\mu}$ ,  $\mu \in I_{Ex_\mu}$ , and  $\mu \in I_{Gx_\mu}$ , respectively. The proof is complete.  $\square$

**Remark 3.6** In particular, let  $f(t, x) = \varphi(t)\psi(x)$ , where  $\psi : (-\infty, +\infty) \rightarrow [0, +\infty)$  is continuous, and  $\varphi : [0, 1] \rightarrow [0, +\infty)$  is continuous and satisfies  $\varphi(t) \not\equiv 0$  for  $t \in [0, 1]$ . If we replace (L1), (L2), and (H) by the following conditions:

$$(L1') \quad \psi_0 = \liminf_{|x| \rightarrow 0} \frac{\psi(x)}{|x|} = +\infty, \quad \psi^\infty = \limsup_{|x| \rightarrow \infty} \frac{\psi(x)}{|x|} = 0,$$

$$(L2') \quad \psi^0 = \limsup_{|x| \rightarrow 0} \frac{\psi(x)}{|x|} = 0, \quad \psi_\infty = \liminf_{|x| \rightarrow \infty} \frac{\psi(x)}{|x|} = +\infty,$$

$$(H') \quad \psi(0) \neq 0,$$

respectively, then the conclusions of Theorem 3.3, Corollary 3.4, and Theorem 3.5 still hold.

## 4 Examples

In the section, we give two concrete examples to illustrate our results.

**Example 4.1** In BVP (1), let  $\alpha = \frac{3}{2}$  and  $f(t, x) = th(x) + te^{-t}$  for  $t \in [0, 1]$  and  $x \in (-\infty, +\infty)$ , where

$$h(x) = \begin{cases} 1, & |x| \leq 3, \\ \frac{(1+e^3)(|x|+e^{|x|})}{(3+e^3)(1+e^{|x|})}, & |x| > 3. \end{cases}$$

Then (H) is satisfied and  $f^\infty = 0$ . Therefore, applying Corollary 3.4, we obtain that BVP (1) has at least one non-zero solution  $x_\mu$  for any  $\mu \neq 1$ . It is evident that  $|h(x)| \leq 1$  for  $x \in (-\infty, +\infty)$ . Moreover, for  $\mu > \alpha$ , from (12) we have

$$|x_\mu(t)| = \frac{\mu}{\Gamma(\frac{5}{2})(\mu-1)} \int_0^1 (sh(x_\mu(s)) + se^{-s}) ds \leq \frac{\alpha}{\Gamma(\frac{5}{2})(\alpha-1)} = \frac{4}{\sqrt{\pi}} < 3, \quad t \in [0, 1].$$

Hence,  $f(t, x_\mu(t)) = t(1 + e^{-t})$ . By a calculation we have  $\int_0^1 s(1-s)^{1.5}(1+e^{-s}) ds \doteq 0.189203$  and  $\int_0^1 s(1-s)^{0.5}(1+e^{-s}) ds \doteq 0.421468$ . So, the solution of

$$\mu \int_0^1 s(1-s)^{1.5}(1+e^{-s}) ds = \frac{3}{2} \int_0^1 s(1-s)^{0.5}(1+e^{-s}) ds$$

is  $\mu \doteq 3.341395$ , which implies that

$$I_{Lx_\mu} = (1.5, 3.341395), \quad I_{Ex_\mu} = \{3.341395\}, \quad I_{Gx_\mu} = (3.341395, +\infty).$$

Consequently, this solution  $x_\mu$  is a strong positive solution, a strong negative solution, a negative solution, and a sign-changing solution for  $\mu \in (0, 1)$ ,  $\mu \in (1, 3.341395)$ ,  $\mu = 3.341395$ , and  $\mu \in (3.341395, +\infty)$ , respectively.

**Example 4.2** In BVP (1), let  $f(t, x) = \varphi(t)\psi(x)$ ,  $\psi(x) = |x|^q$  for  $(t, x) \in [0, 1] \times (-\infty, +\infty)$ , where  $q \geq 0$ ,  $q \neq 1$ , and  $\varphi \in C[0, 1]$  with  $\varphi(t) \geq 0$  and  $\varphi(t) \not\equiv 0$  for  $t \in [0, 1]$ .

(i) If  $q = 0$ , then  $\psi(x) = 1$ . In addition, take  $\varphi(t) = (1 - t)^\beta$ ,  $\beta \geq 0$ . It is clear that (H') and (L1') hold. In combination with Remark 3.6 and Corollary 3.4, we obtain that BVP (1) has at least one non-zero solution  $x_\mu$  for any  $\mu \neq 1$ . It follows by a straightforward calculation that

$$I_{Lx_\mu} = \left( \alpha, \alpha + \frac{\alpha}{\alpha + \beta} \right), \quad I_{Ex_\mu} = \left\{ \alpha + \frac{\alpha}{\alpha + \beta} \right\}, \quad I_{Gx_\mu} = \left( \alpha + \frac{\alpha}{\alpha + \beta}, +\infty \right).$$

Thus, the solution  $x_\mu$  is a strong positive solution, a strong negative solution, a negative solution, and a sign-changing solution for  $\mu \in (0, 1)$ ,  $\mu \in (1, \alpha + \frac{\alpha}{\alpha + \beta})$ ,  $\mu = \alpha + \frac{\alpha}{\alpha + \beta}$ , and  $(\alpha + \frac{\alpha}{\alpha + \beta}, +\infty)$ , respectively.

(ii) If  $0 < q < 1$ , then (L1') holds. In combination with Remark 3.6 and Theorem 3.3, we obtain that BVP (1) has at least one solution  $x_\mu$  for every  $\mu \neq 1$ ; furthermore,  $x_\mu$  is a strong positive solution, a strong negative solution, a non-positive solution, a negative solution, and a sign-changing solution for  $\mu \in (0, 1)$ ,  $\mu \in (1, \alpha) \cup I_{Lx_\mu}$ ,  $\mu = \alpha$ ,  $\mu \in I_{Ex_\mu}$ , and  $\mu \in I_{Rx_\mu}$ , respectively.

(iii) If  $q > 1$ , then (L2') holds. In combination with Remark 3.6 and Theorem 3.5, we obtain that BVP (1) has the zero solution for every  $\mu \neq 1$ . In addition, BVP (1) has at least one non-zero solution  $x_\mu$  for every  $\mu \neq 1, \alpha$ ; furthermore,  $x_\mu$  is a strong positive solution, a strong negative solution, a negative solution, and a sign-changing solution for  $\mu \in (0, 1)$ ,  $\mu \in (1, \alpha) \cup I_{Lx_\mu}$ ,  $\mu \in I_{Ex_\mu}$ , and  $\mu \in I_{Rx_\mu}$ , respectively.

#### Acknowledgements

The author sincerely thanks the editor and reviewers for their valuable suggestions and useful comments.

#### Funding

This research was supported by the National Natural Science Foundation of China (11361047).

#### Availability of data and materials

Not applicable.

#### Competing interests

The author declares that she has no competing interests.

#### Authors' contributions

The author read and approved the final manuscript.

#### Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 7 January 2019 Accepted: 7 April 2019 Published online: 18 April 2019

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