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# A generalization of the compression cone method for integral operators with changing sign kernel functions

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**Abstract**

In this paper, a new class of order cones in the space of continuous functions is introduced. The result unifies some previous work in studying the existence of solutions for differential equations using the compression cone techniques and fixed point theorems. It is shown that the method is more adaptable, particularly in dealing with changing sign Green's functions. Applications are illustrated by examples. Limitations of such a new method are also discussed.

**Keywords:** Boundary value problem; Cone; Existence of solutions; Fixed point index; Green's function; Integral equation; Kernel function

## 1 Introduction

Recently, it has been shown that the following Hammerstein integral equation has important applications in the rapidly developing field of machine learning [2]:

$$Nu(s) := \int_0^T g(t,s)f\bigl(s,u(s)\bigr)\,ds. \tag{1}$$

In fact, existence of fixed points for (1) has interesting applications in computing systems. As shown in Fig. 1, $g$ is a continuous impulse response, $u$ is the continuous output, and $f$ is a controller that generates continuous input from the previous feedback. Convergence of the system is governed by fixed points of the corresponding integral operator. Other applications of the integral equation include models of a chemical reactor [7], a thermostat [23], and circuit design [3].

It is known that equation (1) can be seen as an inverse of a differential equation subject to certain boundary conditions. The Green's function of the boundary value problem (BVP) becomes the kernel of the integral operator. The so-called "compression cone" principle can be used to study existence of fixed points for the integral equation, and therefore the conclusion of existence of solutions for the BVP. For some recent work in higher-order BVPs, we refer to [5, 16, 20, 27] and the references therein. First, the definition of order cone in an abstract Banach space is given below.

**Definition 1.1** ([25], p. 276) Let $X$ be a Banach space and $K$ be a subset of $X$. Then $K$ is called an order cone iff:

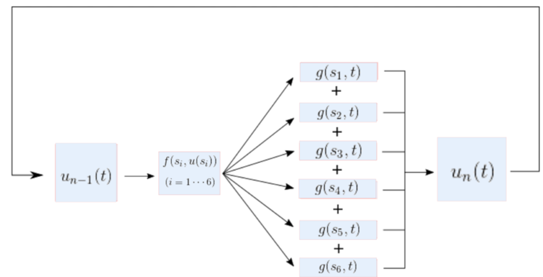



**Figure 1** Iterative machine learning $u_n(t) = \int_\Omega g(s,t) f(s, u_{n-1}(s))\, ds$

(i) $K$ is closed, nonempty, and $K \neq \{0\}$;
(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in K \Rightarrow ax + by \in K$;
(iii) $x \in K$ and $-x \in K \Rightarrow x = 0$.

As a typical example, the following well-known Guo–Krasnoselskii's fixed point theorem is a result of cone compression and expansion.

**Theorem 1.2** ([10]) *Let $K \subset X$ be a cone of the real Banach space $X$. Suppose that $\Omega_1$ and $\Omega_2$ are two bounded open sets in $X$ such that $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ be completely continuous. If either*
(a) $\|Tx\| \leq \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|$ *for* $x \in K \cap \partial\Omega_2$, *or*
(b) $\|Tx\| \leq \|x\|$ *for* $x \in K \cap \partial\Omega_2$ *and* $\|Tx\| \geq \|x\|$ *for* $x \in K \cap \partial\Omega_1$
*holds, then $T$ has at least one fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.*

To construct the cone in a space such as $C[0, T]$, usually a positive Green's function for the BVP is required to ensure a positive kernel for the integral operator. Consequently, it leads to existence of positive solutions for the original BVP [12, 17, 21, 23, 24, 26]. Changing sign solutions have drawn relatively less attention. In the literature [13, 14, 22], Webb and Infante proved the existence of changing sign solutions when the Green's function is only positive in a subinterval so that the cone can be defined as follows:

$$K_{\mathrm{WI}} := \Bigl\{ u \in C[0, T] : \min_{t \in [a,b]} u(t) \geq \delta \|u\| \Bigr\},$$

where $\delta > 0$ is obtained from the Green's function. In [18], Ma used the following cone for changing sign Green's functions:

$$K_M := \biggl\{ u \in C[0, T] : u(t) \geq 0 \ \forall t \in [0, T], \int_0^T u(t)\, dt \geq \delta \|u\| \biggr\}.$$

Generally speaking, comparing to positive Green's functions, it is more difficult to construct a suitable cone when the kernel of the integral operator is not positive. In this paper, a bounded linear functional $L$ is used to define a new type of cones in dealing with changing sign Green's functions for differential equations:

$$K := \bigl\{ u \in C[0, T] : L(u) \geq \delta \|u\| \bigr\}. \tag{2}$$

The idea of construction is a generalization of the previous work. For example, $K_M$ can be directly obtained by taking $L(u) = \int_0^T u(t)\,dt$, while $K_{\mathrm{WI}}$ can be written as union of a family of cones defined as (2)

$$K_{\mathrm{WI}} = \bigcap_{\tau\in[a,b]} K_\tau,$$

where $K_\tau = \{u \in C[0,T] : L_\tau(u) \geq \delta\|u\|\}$ and $L_\tau(u) = u(\tau)$.

The new class of cones allows us to deal with differential equations with broader types of Green's functions. Roughly speaking, instead of requiring an upper bound and a lower bound for a given Green's function, such $L$ introduces a different measurement with partial order. We only require Green's functions to be 'positive' in the sense of the new measurement.

Applying the generalized cone and fixed point index theory, we obtain new existence results of (1). The sub-linear and super-linear cases are also discussed along with examples to illustrate their applications.

## 2 Main result

Consider the existence of a fixed point for the integral equation

$$Nu(s) := \int_0^T g(t,s)f\bigl(s,u(s)\bigr)\,ds,$$

where $u \in C[0,T]$ with standard norm $\|u\| = \max_{t\in[0,T]}\{|u(t)|\}$.

Let $\mathbb{L}$ be a collection of bounded linear functionals $L : C[0,T] \to \mathbb{R}$ with norm $\|L\|_* = \max_{\{u\in C[0,1],\|u\|=1\}}\{|Lu|\}$. We will use the following two sets of assumptions for the kernel $g$ and the nonlinear function $f$ respectively.

- $(H_1)$ Assume that $g : [0,T] \times [0,T] \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition with respect to $s$. There exist a non-trivial $L \in \mathbb{L}$, positive measurable function $\Omega$ with $\int_0^T \Omega(s)\,ds < \infty$, and a constant $\delta > 0$ such that, for any given $s^* \in [0,T]$,
  - $(H_{1a})$ $\max_{t\in[0,T]}\{|g(t,s^*)|\} \leq \Omega(s^*)$;
  - $(H_{1b})$ $\delta\Omega(s^*) \leq k(s^*)$, where $k$ is defined as $k(s^*) = Lg(t,s^*)$.
- $(H_2)$ Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ is continuous and $M = \frac{1}{\int_0^T k(s)\,ds}$.
  - $(H_{2a})$ There exists $0 < r < \infty$ such that $\inf_{\substack{0\leq t\leq T\\ -r\leq x\leq r}}\{f(t,x)\} \geq \|L\|_* rM$.
  - $(H_{2b})$ There exists $0 < R < \infty$, $R \neq r$, such that $\sup_{\substack{0\leq t\leq T\\ -R\leq x\leq R}}\{f(t,x)\} < \delta RM$.

Denote

$$K := \bigl\{u \in C[0,T] : L(u) \geq \delta\|u\|\bigr\}, \qquad K_r := \bigl\{u \in C[0,T] : L(u) \geq \delta\|u\|, \|u\| \leq r\bigr\}.$$

It is clear that $K_r \subseteq K \subseteq C[0,T]$. We will prove that $K$ is a convex cone of $C[0,T]$. Furthermore, choose $Q > \max\{r,R\}$ to be large enough, define $\tilde{f} : [0,T] \times \mathbb{R} \to \mathbb{R}_+$

$$\tilde{f}(t,x) = \begin{cases} f(t,x) & |x| \leq Q, \\ f(t,Q) & x > Q, \\ f(t,-Q) & x < -Q. \end{cases}$$

Since $f$ is continuous, $\tilde{f}$ is clearly bounded. Define $\tilde{N} = \int_0^T g(t,s)\tilde{f}(s,u(s))\,ds$. The main result is given below.

**Theorem 2.1** *If* $(H_1)$, $(H_2)$ *are satisfied, and* $\int_0^T Lg(t,s)\,ds > 0$, *then* $N$ *has at least one nontrivial fixed point.*

To prove Theorem 2.1, we have the following Lemmas 2.2, 2.3, and 2.4 from the properties of fixed point index [8]. Let $K$ be a cone in a Banach space $X$.

**Lemma 2.2** *If there exists* $e \in K\backslash\{0\}$ *s.t.* $x \neq Nx + \lambda e$ *for all* $x \in \partial K_R$ *and all* $\lambda > 0$, *then the fixed point index* $i(N, K_R, K) = 0$ [13].

**Lemma 2.3** *Let* $N : K \to K$ *be a completely continuous mapping. If* $Nu \neq \mu u$ *for all* $u \in \partial K_r$ *and all* $\mu \geq 1$, *then the fixed point index* $i(N, K_r, K) = 1$ [6].

**Lemma 2.4** *Let* $P$ *be an open set and* $N : P \to S$ *be a compact mapping. If* $i(N, P, S) \neq 0$, *then* $N$ *has at least one fixed point in* $P$ [15].

In addition, the new Lemma 2.5 shows that $\tilde{f}$ carries out the same fixed point properties as those of $f$.

**Lemma 2.5** *If* $f$ *satisfies the conditions of* $(H_2)$, *then the corresponding* $\tilde{f}$ *also satisfies* $(H_2)$. *Moreover, for any* $u \in C[0,1]$ *such that* $\tilde{N}(u) = u$, *if* $\|u\| \leq Q$, *it also satisfies* $N(u) = u$.

*Proof* From $(H_2)$, $\tilde{f}$ clearly is continuous on $[0,T] \times \mathbb{R}$. For $(H_{2a})$, we have

$$\inf_{\substack{0\leq t\leq T\\ -r\leq x\leq r}} \{\tilde{f}(t,x)\} = \inf_{\substack{0\leq t\leq T\\ -r\leq x\leq r}} \{f(t,x)\} \geq \|L\|_* rM.$$

Therefore $(H_{2a})$ is also satisfied by $\tilde{f}$. The same argument can be made on $(H_{2b})$. □

**Lemma 2.6** *If* $(H_1)$ *is satisfied,* $k$ *defined in* $(H_1)$ *is Lipschitz continuous.*

*Proof* For any given $s, s^* \in [0,T]$,

$$\left|k(s) - k\left(s^*\right)\right| = \left|L\left(g(t,s) - g\left(t,s^*\right)\right)\right| \leq \|L\|_* \left|g(t,s) - g\left(t,s^*\right)\right| \leq \|L\|_* c\left|s - s^*\right|.$$

Thus $k(s)$ is also Lipschitz continuous. □

The proof of Theorem 2.1 relies on interchanging two bounded linear operators. We first give Fubini's theorem and the Riesz representation theorem for the proof of Lemma 2.9.

**Theorem 2.7** (Fubini's theorem [1]) *Let* $g \in \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be a measurable function such that*

$$\int_{\Omega_1\times\Omega_2} \left|g(\omega_1,\omega_2)\right| d\mathbb{P} < \infty,$$

*where* $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$. *Then*

(a) *For almost all* $\omega_1 \in \Omega_1$, $g(\omega_1,\omega_2)$ *is an integrable function of* $\omega_2$.

(b) *For almost all* $\omega_2 \in \Omega_2$, $g(\omega_1,\omega_2)$ *is an integrable function of* $\omega_1$.
(c) *There exists an integrable function* $h:\Omega_1 \to \mathbb{R}$ *such that* $\int_{\Omega_2} g(\omega_1,\omega_2)\,d\mathbb{P}_2 = h(\omega_1)$ *a.s.* (*i.e., except for a set of* $\omega_1$ *of zero* $\mathbb{P}_1$*-measure for which* $\int_{\Omega_2} g(\omega_1,\omega_2)\,d\mathbb{P}_2$ *is undefined or finite*).
(d) *There exists an integrable function* $h:\Omega_2 \to \mathbb{R}$ *such that* $\int_{\Omega_1} g(\omega_1,\omega_2)\,d\mathbb{P}_2 = h(\omega_2)$ *a.s.* (*i.e., except for a set of* $\omega_1$ *of zero* $\mathbb{P}_2$*-measure for which* $\int_{\Omega_1} g(\omega_1,\omega_2)\,d\mathbb{P}_2$ *is undefined or finite*).
(e) *We have*

$$\begin{aligned}\int_{\Omega_1}\left[\int_{\Omega_2} g(\omega_1,\omega_2)\,d\mathbb{P}_2\right]d\mathbb{P}_1 &= \int_{\Omega_2}\left[\int_{\Omega_1} g(\omega_1,\omega_2)\,d\mathbb{P}_1\right]d\mathbb{P}_2 \\ &= \int_{\Omega_1\times\Omega_2} g(\omega_1,\omega_2)\,d\mathbb{P}.\end{aligned}$$

**Theorem 2.8** (Riesz representation theorem [19]) *A functional F defined on* $C[a,b]$ *is linear and continuous if and only if there exists a function* $g \in \mathrm{BV}$ (*bounded variation function*) *such that*

$$F(f) = \int_a^b f\,dg \quad \text{for } f \in C[a,b].$$

**Lemma 2.9** *Assume that* $(H_1)$ *and* $(H_2)$ *are satisfied. Let* $L, k$ *be defined as in* $(H_1)$, *then*

$$L\left(\int_0^T g(t,s)\tilde{f}\big(s,u(s)\big)\,ds\right) = \int_0^T L\big(g(t,s)\big)\tilde{f}\big(s,u(s)\big)\,ds = \int_0^T k(s)\tilde{f}\big(s,u(s)\big)\,ds.$$

*Proof* Since $L$ is a continuous linear functional defined on $C[0,T]$, by the Riesz representation theorem, there exists unique $\omega \in \mathrm{BV}$ such that $\int_0^T h(t)\,d\omega(t) = L(h)$. We know that $g(t,s)$ and $\tilde{f}(s,u(s))$ are absolutely bounded for all $s,t \in [0,T]$. Thus

$$\int_{[0,T]\times[0,T]} \big|g(s,t)\tilde{f}\big(s,u(s)\big)\big|\,d\big(s,\omega(t)\big) < \infty.$$

By Fubini's theorem,

$$\begin{aligned}L\left(\int_0^T g(t,s)\tilde{f}\big(s,u(s)\big)\,ds\right) &= \int_0^T\int_0^T g(t,s)\tilde{f}\big(s,u(s)\big)\,ds\,d\omega(t) \\ &= \int_0^T\int_0^T g(t,s)\,d\omega(t)\tilde{f}\big(s,u(s)\big)\,ds \\ &= \int_0^T L\big(g(t,s)\big)\tilde{f}\big(s,u(s)\big)\,ds \qquad (*) \\ &= \int_0^T k(s)\tilde{f}\big(s,u(s)\big)\,ds.\end{aligned}$$

For (*), since for all $s^* \in [0,T]$, $g(\cdot,s^*) \in C[0,1]$, and so $\int_0^T g(t,s^*)\,d\Omega(t) = L(g(t,s^*))$. Therefore

$$\max_{s\in[0,T]}\left\{\left|\int_0^T g(t,s)\,d\omega(t) - L\big(g(t,s)\big)\right|\right\} = 0,$$

and then (*) follows. □

**Lemma 2.10** *If $(H_1)$ and $(H_2)$ are satisfied, $K$ defined in $(H_2)$ is a cone and $\tilde{N}(K) \subseteq K$.*

*Proof* We first show that $K$ is a cone. Let $u_1, u_2 \in K$ and $0 \le a, b \in \mathbb{R}$.

$$L(au_1 + bu_2) = aL(u_1) + bL(u_2) \ge a\delta\|u_1\| + b\delta\|u_2\| = \delta\|au_1\| + \delta\|bu_2\| \ge \delta\|au_1 + bu_2\|.$$

If $u_1$ and $-u_1 \in K$, then $0 = L(u_1) + (-L(u_1)) \ge 2\delta\|u_1\|$, which means $u_1 = 0$. Clearly, for all given $s^* \in [0, T]$,

$$Lg(t, s^*) = k(s^*) \ge \delta\Omega(s^*) \ge \delta \max_{t\in[0,T]} \{g(t, s^*)|\} = \delta\|g\|.$$

So $K$ is not empty. Let $\{u_i\} \to u$ be an arbitrary convergent sequence in $K$. Since $C[0, T]$ is a Banach space, thus $u \in C[0, T]$. Also, because sequences $\{L(u_i)\} \to Lu$ and $\{\delta\|u_i\|\} \to \delta\|u\|$, we obtain

$$L(u) - \delta\|u\| = \lim_{i\to\infty} L(u_i) - \delta\|u_i\| \ge \lim_{i\to\infty} 0 = 0.$$

This shows that $K$ is a well-defined cone. We next show that $\tilde{N}(K) \subseteq K$. Let $u \in K$, clearly $\tilde{N}u \in C[0, T]$ and

$$\begin{aligned} L(\tilde{N}u) &= \int_0^T Lg(t, s)\tilde{f}(s, u(s))\, ds \quad \text{(By Lemma 2.9)} \\ &= \int_0^T k(s)\tilde{f}(s, u(s))\, ds \\ &\ge \int_0^T \delta\Omega(s)\tilde{f}(s, u(s))\, ds \\ &\ge \delta \int_0^T \max_{t\in[0,T]} \{|g(t, s)|\}\tilde{f}(s, u(s))\, ds = \delta\|\tilde{N}u\|. \end{aligned}$$

So $\tilde{N}(K) \subseteq K$. □

For compactness. The idea is similar to [13]. We see $\tilde{N}(u) = P \circ h(u)$ as a composition of a compact operator $P(u) = \int_0^T g(t, s)u(s)\, ds$ and a continuous operator $h(u) = \tilde{f}(s, u(s))$. It can be shown that $P(u)$ is compact using Arzela–Ascoli theorem.

*Proof of Theorem 2.1* We will prove it in two steps.

(1) With Lemma 2.2, we will find a subset with index 0.

(2) With Lemma 2.3, we will find a subset with index 1.

Assume that $(H_1)$, $(H_{2a})$ are satisfied. Consider $u \in \partial K_r$ if there exists $u = \tilde{N}u$ that is already a fixed point of $u = \tilde{N}(u)$ with $\|u\| = r$. Otherwise, by $(H_{2a})$, we have

$$\tilde{f}(s, u(s))|_{s\in[0,T]} \ge \inf_{\substack{0\le t\le T \\ -r\le u\le r}} (\tilde{f}(t, u)) \ge \|L\|_* rM.$$

So

$$L(\tilde{N}u) = L\left(\int_0^T g(t, s)\tilde{f}(s, u(s))\, ds\right)$$

$$
\begin{aligned}
&= \int_0^T Lg(t,s)\tilde{f}\big(s,u(s)\big)\,ds \\
&= \int_0^T k(s)\tilde{f}\big(s,u(s)\big)\,ds \\
&\geq \int_0^T k(s)\|L\|_* rM\,ds \\
&= \|L\|_* r = \|L\|_*\|u\| \geq L(u).
\end{aligned}
$$

Let $e = g(t,s^*)$ for some $s^* \in [0,T]$ such that $g(t,s^*) \not\equiv 0$. Clearly $e \in K$ and if there exist $u \in \partial K_r$, $\lambda > 0$ such that $\tilde{N}u + \lambda e = u$, then

$$
\begin{aligned}
L(u) &= L(\tilde{N}u) + \lambda L(e) \\
&= L(\tilde{N}u) + \lambda Lg\big(t,s^*\big) \\
&\geq L(\tilde{N}u) + \lambda\delta\|g\| \\
&> L(\tilde{N}u),
\end{aligned}
$$

which is a contradiction. By Lemma 2.2, $i(\tilde{N},K_r,K) = 0$. Assume that $(H_1)$, $(H_{2b})$ are satisfied. Consider $u \in \partial K_R$. If there exists $u = \tilde{N}u$, then that is already a fixed point of $u = \tilde{N}(u)$ with $\|u\| = R$. Otherwise, by $(H_{2b})$ we have

$$
\tilde{f}\big(s,u(s)\big)|_{s\in[0,1]} \leq \sup_{\substack{0\leq t\leq T \\ -R\leq u\leq R}} \big(\tilde{f}(t,u)\big) < \delta RM.
$$

Therefore,

$$
\begin{aligned}
L(\tilde{N}u) &= \int_0^T k(s)\tilde{f}\big(s,u(s)\big)\,ds \\
&< \int_0^T k(s)\delta RM\,ds \\
&= \delta R = \delta\|u\| \leq L(u).
\end{aligned}
$$

If there exist $u \in \partial K_R$, $\mu \geq 1$ such that $\tilde{N}u = \mu u$, then $L(\tilde{N}u) = L(\mu u) \geq L(u)$, which is a contradiction. By Lemma 2.3, $i(\tilde{N},K_R,K) = 1$.

Since $\tilde{N}$ has no fixed point at $\partial K_r$ and $\partial K_R$, using the properties of fixed point index, if $r > R$,

$$
i(\tilde{N},\bar{K}_r\backslash K_R,K) = i(\tilde{N},K_r,K) - i(\tilde{N},K_R,K) = -1,
$$

by Lemma 2.4, $\tilde{N}$ has at least one non-trivial solution in $\overline{K_r\backslash K_R}$. On the other hand, if $R > r$,

$$
i(\tilde{N},\bar{K}_R\backslash K_r,K) = i(\tilde{N},K_R,K) - i(\tilde{N},K_r,K) = 1.
$$

Again by Lemma 2.4, $\tilde{N}$ has at least one non-trivial solution in $\overline{K_R\backslash K_r}$.

In either case, we obtain at least one non-trivial solution for $u = \tilde{N}(u)$. Since $u$ also satisfies $\|u\| \leq \max\{r,R\} < Q$, by Lemma 2.5, $u$ is a non-trivial solution for $N$ as well. □

## 3 Sub-linear and super-linear case

Assumptions $(H_{2a})$ and $(H_{2b})$ of Theorem 2.1 can be simplified when the nonlinear part is in sub-linear or super-linear cases [4]. We introduce the following new conditions for Theorem 3.1.

$(H_3)$ Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ is continuous, $M = \frac{1}{\int_0^T k(s)\,ds}$, $K$ and $K_r$ as defined in $(H_2)$.

$(H_{3a})$ $0 < \lim_{|x|\to 0} \inf_{t\in[0,T]} f(t,x)$,

$(H_{3b})$ $0 \le f^\infty < \delta M$, where

$$f^\infty = \lim_{x\to\infty} \left( \sup_{t\in[0,T]} \frac{f(t,x)}{|x|} \right).$$

**Theorem 3.1** *If* $(H_1)$, $(H_3)$ *are satisfied, and* $\int_0^T Lg(t,s)\,ds > 0$, *then the integral equation* $N$ *has at least one non-trivial fixed point.*

*Proof* Let $(H_{3a})$ be satisfied, we have $0 < \lim_{|x|\to 0} \inf_{t\in[0,T]} f(t,x)$. Then there exist $m_1 > 0$ and $r_0 > 0$ small enough such that

$$\inf_{\substack{0\le t\le T\\ |x|\le r_0}} f(t,x) \ge \|L\|_* M m_1.$$

Let $r = \min\{m_1, r_0\}$, therefore $r \le m_1$ and $r \le r_0$. And we can have

$$\begin{aligned} \inf_{\substack{0\le t\le T\\ |x|\le r}} f(t,x) &\ge \inf_{\substack{0\le t\le T\\ |x|\le r_0}} f(t,x) \\ &\ge \|L\|_* M m_1 \\ &= \|L\|_* M r, \end{aligned}$$

which satisfy $(H_{2a})$.

For the other part, let $(H_{3b})$ be satisfied, we have $0 \le f^\infty < \delta M$. There exists $R_0 > 0$ large enough such that $\sup_{\substack{0\le t\le T\\ |x|\ge R_0}} \frac{f(t,x)}{|x|} < \delta M$.

Since $f(t,x)$ is continuous, so $f(t,x)|_{|x|<R_0}$ is bounded. Let $\sup_{\substack{0\le t\le T\\ |x|\le R_0}} f(t,x) \le \bar{B}$. Assuming $(H_{3a})$ is not true, then for any $R > 0$,

$$\sup_{\substack{0\le t\le T\\ -R\le x\le R}} \{f(t,x)\} > \delta RM.$$

Choose $R > \max\{R_0, \frac{\bar{B}}{\delta M}\}$, therefore $R > R_0$ and $R > \frac{\bar{B}}{\delta M}$. Since

$$\sup_{\substack{0\le t\le T\\ |x|\le R_0}} f(t,x) \le \bar{B} \le \delta MR,$$

we have

$$\sup_{\substack{0\le t\le T\\ R_0\le |x|\le R}} f(t,x) = \sup_{\substack{0\le t\le T\\ |x|\le R}} f(t,x) \ge \delta RM,$$

and so

$$\sup_{\substack{0\le t\le T\\|x|\ge R_0}} \frac{f(t,x)}{|x|} \ge \sup_{\substack{0\le t\le T\\R_0\le|x|\le R}} \frac{f(t,x)}{|x|} \ge \sup_{\substack{0\le t\le T\\R_0\le|x|\le R}} \frac{f(t,x)}{R} = \delta M,$$

which is a contradiction. Thus $(H_{3b})$ has to be satisfied. Since $(H_1)$, $(H_2)$ are satisfied and $\int_0^T Lg(t,s)\,ds > 0$, by Theorem 2.1, $N$ has at least one non-trivial fixed point. □

## 4 Examples

Consider the periodic boundary value problem [9, 11, 18, 28]:

$$\begin{cases} -u''(t) - \rho^2 u(t) = f(t,u(t)) & t\in[0,T],\\ u(0) = u(T), \\ u'(0) = u'(T). \end{cases}$$

When $\rho \notin \mathbb{N}$ is a positive constant, this BVP is equivalent to

$$u(t) = \int_0^T G(t,s) f\bigl(s,u(s)\bigr)\,ds,$$

where a Green's function is given as

$$G(t,s) = \begin{cases} -\frac{\sin(\rho(t-s))+\sin\rho(T-(t-s))}{2\rho(1-\cos\rho T)} & 0\le s\le t\le T,\\ -\frac{\sin(\rho(s-t))+\sin\rho(T-(s-t))}{2\rho(1-\cos\rho T)} & 0\le t\le s\le T. \end{cases}$$

*Example* 4.1 Let $\rho = \frac{2}{3}$, $T = 2\pi$, we have the following BVP:

$$\begin{cases} -u''(t) - \frac{4}{9} u(t) = f(t,u(t)) & t\in[0,2\pi],\\ u(0) = u(2\pi), \\ u'(0) = u'(2\pi), \end{cases}$$

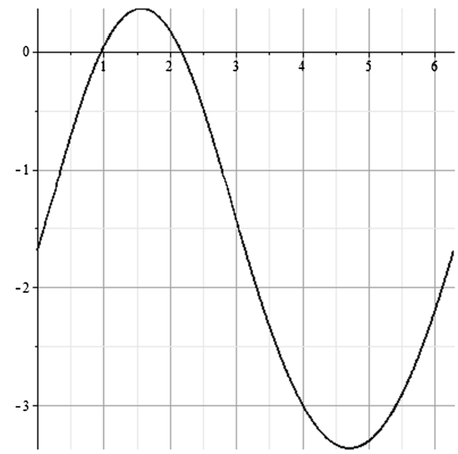
where $f(t,x) = e^{-\frac{x^2}{2}}(1+\sin(t))$ and the Green's function changes sign. If we let $Lg(t,s) = \int_0^{2\pi} -g(t,s)\,dt$, clearly $Lg(t,s) = 2.25$ for all $s\in[0,2\pi]$. Also $\max_{s,t\in[0,2\pi]}\{|g(t,s)|\} = \frac{\sqrt{3}}{2}$. Select $\Omega(s) = \frac{\sqrt{3}}{2}$ and $\delta = 1.5\sqrt{3}$ which satisfy all conditions $(H_1)$. The corresponding cone is defined as

$$K_1 = u\in C[0,2\pi] : \int_0^{2\pi} -u(t)\,dt \ge 1.5\sqrt{3}\|u\|\}.$$

As for the nonlinear part, $f$ is clearly a continuous positive function. We can calculate that $\lim_{|x|\to 0}\inf_{t\in[0,2\pi]} f(t,x) = 3$, so $(H_{3a})$ is satisfied. Moreover, since $M = \frac{2}{9\pi}$, $f^\infty = 0 < \delta M$, $(H_{3b})$ is satisfied. Therefore, by Theorem 3.1, there exists a non-trivial solution in $K_1$.

Figure 2 shows the approximation (with 1000 sample points) of the fixed point which was directly obtained from the differential equation. By direct computation we have $Lu - \delta\|u\| \approx 1.464 > 0$, which suggests that it is a fixed point in $K_1$.

**Figure 2** Solution in Example 4.1




*Example* 4.2 Let $\rho = \frac{1}{4}$, $T = 2\pi$, the period boundary value problem has the form

$$\begin{cases} -u''(t) - \frac{1}{16}u(t) = f(t,u(t)) & t \in [0,2\pi], \\ u(0) = u(2\pi), \\ u'(0) = u'(2\pi), \end{cases}$$

where

$$f(t,x) = \frac{2\cos(2t)}{x^2+1} + \pi.$$

In this case, the Green's function is negative. Let $Lg(t,s) = -g(\pi,s)$, we can show that $Lg(t,s) \in [2, 2\sqrt{2}]$ for all $s \in [0,2\pi]$. We also have $\max_{s\in[0,2\pi]}\{|g(t,s)|\} = 2\sqrt{2}$. Let $\Omega(s) = 2\sqrt{2}$ and $\delta = \frac{\sqrt{2}}{2}$. All conditions of $(H_1)$ are satisfied. Define the corresponding cone

$$K_2 = u \in C[0,2\pi] : -u(\pi) \geq 2\sqrt{2}\|u\|\}.$$

As for the nonlinear part, $f$ is clearly a continuous positive function. We can see $f(t,x) \geq \frac{23}{32}$, so $(H_{3a})$ is satisfied. We also have $M = \frac{1}{16}\pi$, $f^\infty = \frac{1}{32} < \delta M$, so $(H_{3b})$ is satisfied. By Theorem 3.1, there exists a non-trivial solution in $K_2$.

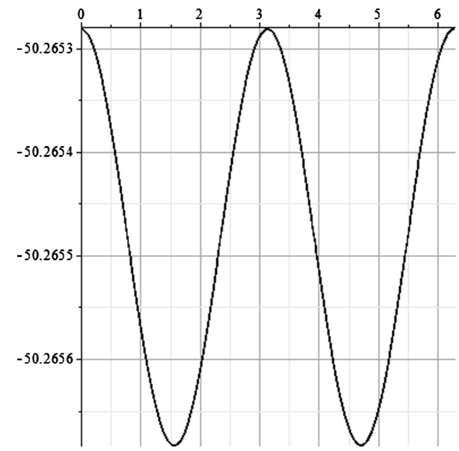
Figure 3 shows the approximation (with 1000 sample points) of the fixed point which was directly obtained from the differential equation. By direct computation we have $Lu - \delta\|u\| \approx 1.464 > 0$, which suggests that it is a fixed point in $K_2$.

We point out that results from [6, 13, 14, 18, 22, 23] are not applicable to the Green's functions in the above two examples. The cone applied in [9] and [28]

$$K = \left\{ u \in C[0,T] : u \geq 0, \int_0^T u(s)\,ds \geq \delta\|u\| \right\}$$

cannot capture the solutions that we found in the above two examples.

**Figure 3** Solution in Example 4.2




*Remark* 4.3 Consider the following example:

$$g(t,s) = \begin{cases} \sin(t-s) & 0 \le s \le t \le 2\pi, \\ \sin(s-t) & 0 \le t \le s \le 2\pi. \end{cases}$$

Since $g(t,0) + g(t,2\pi) = 0$ for all $t$, our method has failed to apply when the Green's function is reflexive.

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