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A posteriori error estimates for fourth order hyperbolic control problems by mixed finite element methods

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Abstract
In this paper, we consider the a posteriori error estimates of the mixed finite element method for the optimal control problems governed by fourth order hyperbolic equations. The state is discretized by the order k Raviart–Thomas mixed elements and control is discretized by piecewise polynomials of degree k. We adopt the mixed elliptic reconstruction to derive the a posteriori error estimates for both the state and the control approximations.

MSC: 49J20; 65N30

Keywords: A posteriori error estimates; Optimal control problems; Fourth order hyperbolic equations; Mixed finite element methods

1 Introduction

The finite element approximation for optimal control problems has an enormously important function in the numerical approach for these problems. Scientists have studied extensively this area; see, for example, [4, 12, 13, 21, 25]. They discussed the a priori error estimates using finite element approximations, such as [1, 16, 23], in which elliptic or parabolic problems are considered by optimal control theory. They studied adaptivity for many optimal control problems; for example, see [4, 11, 17, 20–22].

In some optimal control problems, for the objective function containing a gradient of the state variable, we use mixed finite element methods to discretize the state equation, so that the scalar variable and its flux variable can be approximated in the same accuracy; for example, see [3]. Many scientists have addressed the mixed finite element methods for elliptic problems [6–8, 14], for the first bi-harmonic equation [5], for parabolic problems [26] and for hyperbolic problems [9, 15].

The purpose of this work is to discuss the a posteriori error estimates of the semidiscrete mixed finite element approximation for fourth order hyperbolic optimal control problems. Considering the fourth order hyperbolic equations by the idea of a mixed elliptic reconstruction [24], we obtain the error estimates for the state and the control approximations. The following is the model we considered:

$$\min_{u\in K\in U}\left\{\frac{1}{2}\int_0^T\left(\|\Delta y\|^2+\|y-y_d\|^2+\|u\|^2\right)dt\right\}, \tag{1.1}$$

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$$y_{tt}(x,t) + \Delta^2 y(x,t) = f(x,t) + u(x,t), \quad x \in \Omega, t \in J, \tag{1.2}$$

$$y(x,t) = \frac{\partial y}{\partial n} = 0, \quad x \in \partial\Omega, t \in J, \tag{1.3}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{1.4}$$

$$y_t(x,0) = y_1(x), \quad x \in \Omega, \tag{1.5}$$

where $\Omega \subset \mathbf{R}^2$ is an open set of polygon with $\partial\Omega$. K is in $U = L^2(J; L^2(\Omega))$, a closed convex set, $J = [0,T]$, $f, y_d \in L^2(J; L^2(\Omega))$ and $y_0, y_1 \in H^4(\Omega)$. K is defined as follows:

$$K = \left\{ u \in U : \int_0^T \int_\Omega u \, dx \, dt \geq 0 \right\}. \tag{1.6}$$

Let $\tilde{y} = -\Delta y$, $\tilde{\boldsymbol{p}} = -\nabla y$ and $\boldsymbol{p} = -\nabla \tilde{y}$, then (1.1)–(1.5) can be written as

$$\min_{u \in K \in U} \left\{ \frac{1}{2} \int_0^T \left(\|\tilde{y}\|^2 + \|y - y_d\|^2 + \|u\|^2 \right) dt \right\}, \tag{1.7}$$

$$y_{tt}(x,t) + \operatorname{div} \boldsymbol{p}(x,t) = f(x,t) + u(x,t), \quad x \in \Omega, t \in J, \tag{1.8}$$

$$\boldsymbol{p}(x,t) = -\nabla \tilde{y}(x,t), \quad x \in \Omega, t \in J, \tag{1.9}$$

$$\operatorname{div} \tilde{\boldsymbol{p}}(x,t) = \tilde{y}(x,t), \quad x \in \Omega, t \in J, \tag{1.10}$$

$$\tilde{\boldsymbol{p}}(x,t) = -\nabla y(x,t), \quad x \in \Omega, t \in J, \tag{1.11}$$

$$y(x,t) = -\tilde{p} \cdot n = 0, \quad x \in \partial\Omega, t \in J, \tag{1.12}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{1.13}$$

$$y_t(x,0) = y_1(x), \quad x \in \Omega. \tag{1.14}$$

In the paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev space on Ω with a norm $\|v\|_{m,p}$ given by $\|v\|_{m,p}^p := \sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(\Omega)}^p$, and a seminorm $|v|_{m,p}$ given by $|v|_{m,p}^p := \sum_{|\alpha| = m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : \gamma(D^\alpha v)|_{\partial\Omega} = 0, |\alpha| = m\}$, where γ is the trace operator. We denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)(H_0^m(\Omega))$.

We denote by $L^s(0,T; W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = (\int_0^T \|v\|_{W^{m,p}(\Omega)}^s \, dt)^{\frac{1}{s}}$ for $s \in [1,\infty)$, and the standard modification for $s = \infty$. For simplicity of presentation, we denote $\|v\|_{L^s(J;W^{m,p}(\Omega))}$ by $\|v\|_{L^s(W^{m,p})}$. Similarly, one can define the spaces $H^1(J; W^{m,p}(\Omega))$ and $C^k(J; W^{m,p}(\Omega))$. We can find details in [19]. C is a general positive constant independent of h.

The rest of this paper is as follows: In Sect. 2, we introduce the optimal control problems and its mixed finite element scheme. Section 2 ends with the definition of the mixed elliptic reconstructions, which is useful in deriving the a posteriori estimates for the fourth order hyperbolic optimal control problems in Sect. 3. Finally, we make some concluding remarks in Sect. 4.

2 Optimal control problems for mixed methods

A semidiscrete approximation of a mixed finite element for the optimal control problems (1.7)–(1.14) will be constructed. We set the state spaces $\boldsymbol{L} = L^2(J; \boldsymbol{V})$, $\boldsymbol{L}_0 = L^2(J; \boldsymbol{V}_0)$ and

$Q = L^2(J; W)$, $W = L^2(\Omega)$, where $\boldsymbol{V}$, and $\boldsymbol{V}_0$ are defined as follows:

$$\boldsymbol{V} = H(\mathrm{div}; \Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2, \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\},$$
$$\boldsymbol{V}_0 = H_0(\mathrm{div}; \Omega) = \{\boldsymbol{v} \in H(\mathrm{div}, \Omega), \boldsymbol{v} \cdot n|_{\partial\Omega} = 0\}.$$

The space $\boldsymbol{V}$ is a Hilbert space, its norm is defined as follows:

$$\|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)} = \left(\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathrm{div}\,\boldsymbol{v}\|_{0,\Omega}^2\right)^{1/2}.$$

Now we introduce operators: div, ∇, curl and Curl. For any $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \in (H^1(\Omega))^2$ or $w \in H^1(\Omega)$,

$$\mathrm{div}\,\boldsymbol{v} = \partial_1\boldsymbol{v}_1 + \partial_2\boldsymbol{v}_2, \qquad \nabla w = (\partial_1 w, \partial_2 w), \tag{2.1}$$
$$\mathrm{curl}\,\boldsymbol{v} = \partial_1\boldsymbol{v}_2 - \partial_2\boldsymbol{v}_1, \qquad \mathrm{Curl}\,w = (-\partial_2 w, \partial_1 w). \tag{2.2}$$

Next, (1.7)–(1.14) can be rewritten into weak form as follows: find $(\tilde{\boldsymbol{p}}, y, \boldsymbol{p}, y, u) \in (\boldsymbol{L}_0 \times Q \times \boldsymbol{L} \times Q \times K)$, such that

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_0^T \left(\|\tilde{y}\|^2 + \|y - y_d\|^2 + \|u\|^2\right) dt\right\}, \tag{2.3}$$
$$(y_{tt}, w) + (\mathrm{div}\,\boldsymbol{p}, w) = (f + u, w), \quad \forall w \in W, t \in J, \tag{2.4}$$
$$(\boldsymbol{p}, \boldsymbol{v}) - (\tilde{y}, \mathrm{div}\,\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_0, t \in J, \tag{2.5}$$
$$(\mathrm{div}\,\tilde{\boldsymbol{p}}, w) = (\tilde{y}, w), \quad \forall w \in W, t \in J, \tag{2.6}$$
$$(\tilde{\boldsymbol{p}}, \boldsymbol{v}) - (y, \mathrm{div}\,\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.7}$$
$$y(x, 0) = y_0(x), \quad \forall x \in \Omega, \tag{2.8}$$
$$y_t(x, 0) = y_1(x), \quad \forall x \in \Omega. \tag{2.9}$$

From [18], we know that the above optimal control problem has a unique solution $(\tilde{\boldsymbol{p}}, y, \boldsymbol{p}, \tilde{y}, u)$, and that $(\tilde{\boldsymbol{p}}, y, \boldsymbol{p}, \tilde{y}, u)$ is the solution of (2.3)–(2.9) if and only if there is a co-state $(\tilde{\boldsymbol{q}}, z, \boldsymbol{q}, \tilde{z}) \in (\boldsymbol{L}_0 \times Q \times \boldsymbol{L} \times Q)$ such that $(\tilde{\boldsymbol{p}}, y, \boldsymbol{p}, \tilde{y}, \tilde{\boldsymbol{q}}, z, \boldsymbol{q}, \tilde{z}, u)$ satisfies the following optimality conditions:

$$(y_{tt}, w) + (\mathrm{div}\,\boldsymbol{p}, w) = (f + u, w), \quad \forall w \in W, t \in J, \tag{2.10}$$
$$(\boldsymbol{p}, \boldsymbol{v}) - (\tilde{y}, \mathrm{div}\,\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.11}$$
$$(\mathrm{div}\,\tilde{\boldsymbol{p}}, w) = (\tilde{y}, w), \quad \forall w \in W, t \in J, \tag{2.12}$$
$$(\tilde{\boldsymbol{p}}, \boldsymbol{v}) - (y, \mathrm{div}\,\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.13}$$
$$y(x, 0) = y_0(x), \quad \forall x \in \Omega, \tag{2.14}$$
$$y_t(x, 0) = y_1(x), \quad \forall x \in \Omega, \tag{2.15}$$
$$(z_{tt}, w) + (\mathrm{div}\,\boldsymbol{q}, w) = (y - y_d, w), \quad \forall w \in W, t \in J, \tag{2.16}$$
$$(\boldsymbol{q}, \boldsymbol{v}) - (\tilde{z}, \mathrm{div}\,\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.17}$$

$$(\operatorname{div} \tilde{\boldsymbol{q}}, w) = (\tilde{y} + \tilde{z}, w), \quad \forall w \in W, t \in J, \tag{2.18}$$

$$(\tilde{\boldsymbol{q}}, \boldsymbol{v}) - (z, \operatorname{div} \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.19}$$

$$z(x, T) = 0, \quad \forall x \in \Omega, \tag{2.20}$$

$$z_t(x, T) = 0, \quad \forall x \in \Omega, \tag{2.21}$$

$$\int_0^T (u + z, \tilde{u} - u)\, dt \geq 0, \quad \forall \tilde{u} \in K, \tag{2.22}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

K is a control constraint, so we can get a relationship between u and z. This relationship is important for our result.

Lemma 2.1 *Let* (z, u) *be the solution of* (2.10)–(2.22). *Then we have* $u = \max\{0, \check{z}\} - z$, *where*

$$\check{z} = \frac{\int_0^T \int_\Omega z\, dx\, dt}{\int_0^T \int_\Omega 1\, dx\, dt}$$

denotes the integral average on $\Omega \times J$ *of the function* z.

Let $\mathcal{T}_h$ be regular triangulations of Ω, h_τ is the diameter of τ and $h = \max h_\tau$. Furthermore, let $\mathcal{E}_h$ be the set of element sides of the triangulation $\mathcal{T}_h$ with $\Gamma_h = \bigcup \mathcal{E}_h$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denote the Raviart–Thomas space [3] associated with the triangulations $\mathcal{T}_h$ of Ω. P_k denotes the space of polynomials of total degree no greater than k ($k \geq 0$). Let $\boldsymbol{V}(\tau) = \{\boldsymbol{v} \in P_k^2(\tau) + x \cdot P_k(\tau)\}$, $W(\tau) = P_k(\tau)$. We set

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \forall \tau \in \mathcal{T}_h, \boldsymbol{v}_h|_\tau \in \boldsymbol{V}(\tau)\},$$

$$W_h := \{w_h \in W : \forall \tau \in \mathcal{T}_h, w_h|_\tau \in W(\tau)\},$$

$$K_h := L^2(J; W_h) \cap K.$$

We now discretize (2.3)–(2.9). We calculate $(\tilde{\boldsymbol{p}}_h, y_h, \boldsymbol{p}_h, \tilde{y}_h, u_h)$ such that

$$\min_{u_h \in K_h} \left\{ \frac{1}{2} \int_0^T \left(\|\tilde{y}_h\|^2 + \|y_h - y_d\|^2 + \|u_h\|^2 \right) dt \right\}, \tag{2.23}$$

$$(y_{htt}, w_h) + (\operatorname{div} \boldsymbol{p}_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.24}$$

$$(\boldsymbol{p}_h, \boldsymbol{v}_h) - (\tilde{y}_h, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \tag{2.25}$$

$$(\operatorname{div} \tilde{\boldsymbol{p}}_h, w_h) = (\tilde{y}_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.26}$$

$$(\tilde{\boldsymbol{p}}_h, \boldsymbol{v}_h) - (y_h, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \tag{2.27}$$

$$y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.28}$$

$$y_{ht}(x, 0) = y_1^h(x), \quad \forall x \in \Omega, \tag{2.29}$$

where $y_0^h(x) \in W_h$ and $y_1^h(x) \in W_h$ are the mixed elliptic projections of y_0 and y_1. The optimal control problem (2.23)–(2.29) again has an unique solution $(\tilde{\boldsymbol{p}}_h, y_h, \boldsymbol{p}_h, \tilde{y}_h, u_h)$, and $(\tilde{\boldsymbol{p}}_h, y_h, \boldsymbol{p}_h, \tilde{y}_h, u_h)$ is the solution of (2.23)–(2.29) if and only if there is a co-state $(\tilde{\boldsymbol{q}}_h, z_h, \boldsymbol{q}_h, \tilde{z}_h)$ such that the following optimality conditions hold:

$$(y_{htt}, w_h) + (\operatorname{div} \boldsymbol{p}_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.30}$$

$$(\boldsymbol{p}_h, \boldsymbol{v}_h) - (\tilde{y}_h, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \tag{2.31}$$

$$(\operatorname{div} \tilde{\boldsymbol{p}}_h, w_h) = (\tilde{y}_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.32}$$

$$(\tilde{\boldsymbol{p}}_h, \boldsymbol{v}_h) - (y_h, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \tag{2.33}$$

$$y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.34}$$

$$y_{ht}(x, 0) = y_1^h(x), \quad \forall x \in \Omega, \tag{2.35}$$

$$(z_{htt}, w_h) + (\operatorname{div} \boldsymbol{q}_h, w_h) = (y_h - y_d, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.36}$$

$$(\boldsymbol{q}_h, \boldsymbol{v}_h) - (\tilde{z}_h, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \tag{2.37}$$

$$(\operatorname{div} \tilde{\boldsymbol{q}}_h, w_h) = (\tilde{y}_h + \tilde{z}_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.38}$$

$$(\tilde{\boldsymbol{q}}_h, \boldsymbol{v}_h) - (z_h, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \tag{2.39}$$

$$z_h(x, T) = 0, \quad \forall x \in \Omega, \tag{2.40}$$

$$z_{ht}(x, T) = 0, \quad \forall x \in \Omega, \tag{2.41}$$

$$\int_0^T (u_h + z_h, \tilde{u}_h - u_h)\, dt \geq 0, \quad \forall \tilde{u}_h \in K_h. \tag{2.42}$$

For Lemma 2.1, the relationship between u_h and z_h is given as follows:

$$u_h = \max\{0, \check{z}_h\} - z_h, \tag{2.43}$$

where $\check{z}_h = \frac{\int_0^T \int_\Omega z_h\, dx\, dt}{\int_0^T \int_\Omega 1\, dx\, dt}$ is the integral average on $\Omega \times J$ of the function z_h.

Now, we give the local definition of div_h, $\operatorname{curl}_h : H^1(\mathcal{T}_h)^2 \to L^2(\Omega)$ and ∇_h, $\operatorname{Curl}_h : H^1(\mathcal{T}_h) \to L^2(\Omega)^2$, such that for any $T \in \mathcal{T}_h$

$$\operatorname{div}_h \boldsymbol{v}|_T := \operatorname{div}(\boldsymbol{v}|_T), \qquad \operatorname{curl}_h \boldsymbol{v}|_T := \operatorname{curl}(\boldsymbol{v}|_T), \tag{2.44}$$

$$\nabla_h \boldsymbol{v}|_T := \nabla(\boldsymbol{v}|_T), \qquad \operatorname{Curl}_h \boldsymbol{v}|_T := \operatorname{Curl}(\boldsymbol{v}|_T). \tag{2.45}$$

Set $P_h : W \to W_h$ to be the orthogonal $L^2(\Omega)$-projection into W_h [2], which satisfies

$$(P_h w - w, \chi) = 0, \quad w \in W, \chi \in W_h, \tag{2.46}$$

$$\|P_h w - w\|_{0,q} \leq Ch^t \|w\|_{t,q}, \quad 0 \leq t \leq k+1, \text{if } w \in W \cap W^{t,q}(\Omega), \tag{2.47}$$

$$\|P_h w - w\|_{-r} \leq Ch^{r+t} \|w\|_t, \quad 0 \leq r, t \leq k+1, \text{if } w \in H^t(\Omega). \tag{2.48}$$

Next, introduce the Fortin projection (see [3] and [10]) $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$\big(\operatorname{div}(\Pi_h \boldsymbol{q} - \boldsymbol{q}), w_h\big) = 0, \quad \forall \boldsymbol{q} \in \boldsymbol{V}, w_h \in W_h, \tag{2.49}$$

$$\|\boldsymbol{q} - \Pi_h \boldsymbol{q}\|_{0,q} \leq Ch^r \|\boldsymbol{q}\|_{r,q}, \quad 1/q < r \leq k+1, \forall \boldsymbol{q} \in \boldsymbol{V} \cap \big(W^{r,q}(\Omega)\big)^2, \tag{2.50}$$

$$\big\|\operatorname{div}(\boldsymbol{q} - \Pi_h \boldsymbol{q})\big\|_0 \leq Ch^r \|\operatorname{div} \boldsymbol{q}\|_r, \quad 0 \leq r \leq k+1, \forall \operatorname{div} \boldsymbol{q} \in H^r(\Omega). \tag{2.51}$$

The commuting diagram property reads

$$\operatorname{div}\circ\Pi_h = P_h\circ\operatorname{div}: \boldsymbol{V}\to W_h \quad \text{and} \quad \operatorname{div}(I-\Pi_h)\boldsymbol{V}\perp W_h, \tag{2.52}$$

where I denotes the identity operator.

Next, the intermediate variable $\tilde{u}\in K$ is introduced as follows:

$$\big(y_{tt}(\tilde{u}),w\big)+\big(\operatorname{div}\boldsymbol{p}(\tilde{u}),w\big)=(f+\tilde{u},w),\quad \forall w\in W, t\in J, \tag{2.53}$$

$$\big(\boldsymbol{p}(\tilde{u}),\boldsymbol{v}\big)-\big(\tilde{y}(\tilde{u}),\operatorname{div}\boldsymbol{v}\big)=0,\quad \forall \boldsymbol{v}\in\boldsymbol{V}, t\in J, \tag{2.54}$$

$$\big(\operatorname{div}\tilde{\boldsymbol{p}}(\tilde{u}),w\big)=\big(\tilde{y}(\tilde{u}),w\big),\quad \forall w\in W, t\in J, \tag{2.55}$$

$$\big(\tilde{\boldsymbol{p}}(\tilde{u}),\boldsymbol{v}\big)-\big(y(\tilde{u}),\operatorname{div}\boldsymbol{v}\big)=0,\quad \forall \boldsymbol{v}\in\boldsymbol{V}, t\in J, \tag{2.56}$$

$$y(\tilde{u})(x,0)=y_0(x),\quad \forall x\in\Omega, \tag{2.57}$$

$$y_t(\tilde{u})(x,0)=y_1(x),\quad \forall x\in\Omega, \tag{2.58}$$

$$\big(z_{tt}(\tilde{u}),w\big)+\big(\operatorname{div}\boldsymbol{q}(\tilde{u}),w\big)=\big(y(\tilde{u})-y_d,w\big),\quad \forall w\in W, t\in J, \tag{2.59}$$

$$\big(\boldsymbol{q}(\tilde{u}),\boldsymbol{v}\big)-\big(\tilde{z}(\tilde{u}),\operatorname{div}\boldsymbol{v}\big)=0,\quad \forall \boldsymbol{v}\in\boldsymbol{V}, t\in J, \tag{2.60}$$

$$\big(\operatorname{div}\tilde{\boldsymbol{q}}(\tilde{u}),w\big)=\big(\tilde{y}(\tilde{u})+\tilde{z}(\tilde{u}),w\big),\quad \forall w\in W, t\in J, \tag{2.61}$$

$$\big(\tilde{\boldsymbol{q}}(\tilde{u}),\boldsymbol{v}\big)-\big(z(\tilde{u}),\operatorname{div}\boldsymbol{v}\big)=0,\quad \forall \boldsymbol{v}\in\boldsymbol{V}, t\in J, \tag{2.62}$$

$$z(\tilde{u})(x,T)=0,\quad \forall x\in\Omega, \tag{2.63}$$

$$z_t(\tilde{u})(x,T)=0,\quad \forall x\in\Omega. \tag{2.64}$$

Next, we present mixed elliptic constructions $(\tilde{\bar{\boldsymbol{p}}},\bar{y},\bar{\boldsymbol{p}},\tilde{\bar{y}},\tilde{\bar{\boldsymbol{q}}},\bar{z},\bar{\boldsymbol{q}},\tilde{\bar{z}})\in(\boldsymbol{V}\times W)^4$:

$$(\tilde{\bar{\boldsymbol{p}}},\boldsymbol{v})-(\bar{y},\operatorname{div}\boldsymbol{v})=0,\quad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{2.65}$$

$$(\operatorname{div}\tilde{\bar{\boldsymbol{p}}},w)=(\tilde{y}_h,w),\quad \forall w\in W, \tag{2.66}$$

$$(\bar{\boldsymbol{p}},\boldsymbol{v})-(\tilde{\bar{y}},\operatorname{div}\boldsymbol{v})=0,\quad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{2.67}$$

$$(\operatorname{div}\bar{\boldsymbol{p}},w)=(f+u_h-y_{htt},w),\quad \forall w\in W, \tag{2.68}$$

$$(\tilde{\bar{\boldsymbol{q}}},\boldsymbol{v})-(\bar{z},\operatorname{div}\boldsymbol{v})=0,\quad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{2.69}$$

$$(\operatorname{div}\tilde{\bar{\boldsymbol{q}}},w)=(\tilde{z}_h+\tilde{y}_h,w),\quad \forall w\in W, \tag{2.70}$$

$$(\bar{\boldsymbol{q}},\boldsymbol{v})-(\tilde{\bar{z}},\operatorname{div}\boldsymbol{v})=0,\quad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{2.71}$$

$$(\operatorname{div}\bar{\boldsymbol{q}},w)=(y_h-y_d-z_{htt},w),\quad \forall w\in W. \tag{2.72}$$

For simplicity of presentation, we resolve the errors in following forms:

$$\tilde{\boldsymbol{p}}-\tilde{\boldsymbol{p}}_h=\tilde{\boldsymbol{p}}-\tilde{\boldsymbol{p}}(u_h)+\tilde{\boldsymbol{p}}(u_h)-\tilde{\bar{\boldsymbol{p}}}+\tilde{\bar{\boldsymbol{p}}}-\tilde{\boldsymbol{p}}_h=r_1+e_1+\eta_1, \tag{2.73}$$

$$y-y_h=y-y(u_h)+y(u_h)-\bar{y}+\bar{y}-y_h=r_2+e_2+\eta_2, \tag{2.74}$$

$$\boldsymbol{p}-\boldsymbol{p}_h=\boldsymbol{p}-\boldsymbol{p}(u_h)+\boldsymbol{p}(u_h)-\bar{\boldsymbol{p}}+\bar{\boldsymbol{p}}-\boldsymbol{p}_h=r_3+e_3+\eta_3, \tag{2.75}$$

$$\tilde{y}-\tilde{y}_h=\tilde{y}-\tilde{y}(u_h)+\tilde{y}(u_h)-\tilde{\bar{y}}+\tilde{\bar{y}}-\tilde{y}_h=r_4+e_4+\eta_4, \tag{2.76}$$

$$\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_h = \tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}(u_h) + \tilde{\boldsymbol{q}}(u_h) - \bar{\tilde{\boldsymbol{q}}} + \bar{\tilde{\boldsymbol{q}}} - \tilde{\boldsymbol{q}}_h = r_5 + e_5 + \eta_5, \tag{2.77}$$

$$z - z_h = z - z(u_h) + z(u_h) - \bar{z} + \bar{z} - z_h = r_6 + e_6 + \eta_6, \tag{2.78}$$

$$\boldsymbol{q} - \boldsymbol{q}_h = \boldsymbol{q} - \boldsymbol{q}(u_h) + \boldsymbol{q}(u_h) - \bar{\boldsymbol{q}} + \bar{\boldsymbol{q}} - \boldsymbol{q}_h = r_7 + e_7 + \eta_7, \tag{2.79}$$

$$\tilde{z} - \tilde{z}_h = \tilde{z} - \tilde{z}(u_h) + \tilde{z}(u_h) - \bar{\tilde{z}} + \bar{\tilde{z}} - \tilde{z}_h = r_8 + e_8 + \eta_8. \tag{2.80}$$

From mixed elliptic reconstructions [24], we derive the error estimates as below.

Lemma 2.2 ([8, 14]) *For Raviart–Thomas elements, there exists a positive constant C which depends the domain Ω, the shape regularity of the elements and polynomial degree k such that*

$$\|\eta_2\| \le C\Big(\big\|h^{1+\min\{1,k\}}(\operatorname{div}\tilde{\boldsymbol{p}}_h - \tilde{y}_h)\big\| + \min_{w_h\in W_h}\big\|h(\tilde{\boldsymbol{p}}_h - \nabla_h w_h)\big\|\Big), \tag{2.81}$$

$$\|\eta_{2t}\| \le C\Big(\big\|h^{1+\min\{1,k\}}(\operatorname{div}\tilde{\boldsymbol{p}}_{ht} - \tilde{y}_{ht})\big\| + \min_{w_h\in W_h}\big\|h(\tilde{\boldsymbol{p}}_{ht} - \nabla_h w_h)\big\|\Big), \tag{2.82}$$

$$\|\eta_{2tt}\| \le C\Big(\big\|h^{1+\min\{1,k\}}(\operatorname{div}\tilde{\boldsymbol{p}}_{htt} - \tilde{y}_{htt})\big\| + \min_{w_h\in W_h}\big\|h(\tilde{\boldsymbol{p}}_{htt} - \nabla_h w_h)\big\|\Big), \tag{2.83}$$

$$\|\eta_{2ttt}\| \le C\Big(\big\|h^{1+\min\{1,k\}}(\operatorname{div}\tilde{\boldsymbol{p}}_{httt} - \tilde{y}_{httt})\big\| + \min_{w_h\in W_h}\big\|h(\tilde{\boldsymbol{p}}_{httt} - \nabla_h w_h)\big\|\Big), \tag{2.84}$$

$$\|\eta_4\| \le C\Big(\big\|h^{1+\min\{1,k\}}(y_{htt} + \operatorname{div}\boldsymbol{p}_h - f - u_h)\big\| + \min_{w_h\in W_h}\big\|h(\boldsymbol{p}_h - \nabla_h w_h)\big\|\Big), \tag{2.85}$$

$$\|\eta_{4t}\| \le C\Big(\big\|h^{1+\min\{1,k\}}(y_{htt} + \operatorname{div}\boldsymbol{p}_h - f - u_h)_t\big\| + \min_{w_h\in W_h}\big\|h(\boldsymbol{p}_{ht} - \nabla_h w_h)\big\|\Big), \tag{2.86}$$

$$\|\eta_{4tt}\| \le C\Big(\big\|h^{1+\min\{1,k\}}(y_{htt} + \operatorname{div}\boldsymbol{p}_h - f - u_h)_{tt}\big\| + \min_{w_h\in W_h}\big\|h(\boldsymbol{p}_{htt} - \nabla_h w_h)\big\|\Big), \tag{2.87}$$

$$\|\eta_6\| \le C\Big(\big\|h^{1+\min\{1,k\}}(\operatorname{div}\tilde{\boldsymbol{q}}_h - \tilde{y}_h - \tilde{z}_h)\big\| + \min_{w_h\in W_h}\big\|h(\tilde{\boldsymbol{q}}_h - \nabla_h w_h)\big\|\Big), \tag{2.88}$$

$$\|\eta_{6t}\| \le C\Big(\big\|h^{1+\min\{1,k\}}(\operatorname{div}\tilde{\boldsymbol{q}}_{ht} - \tilde{y}_{ht} - \tilde{z}_{ht})\big\| + \min_{w_h\in W_h}\big\|h(\tilde{\boldsymbol{q}}_{ht} - \nabla_h w_h)\big\|\Big), \tag{2.89}$$

$$\|\eta_{6tt}\| \le C\Big(\big\|h^{1+\min\{1,k\}}(\operatorname{div}\tilde{\boldsymbol{q}}_{htt} - \tilde{y}_{htt} - \tilde{z}_{htt})\big\| + \min_{w_h\in W_h}\big\|h(\tilde{\boldsymbol{q}}_{htt} - \nabla_h w_h)\big\|\Big), \tag{2.90}$$

$$\|\eta_8\| \le C\Big(\big\|h^{1+\min\{1,k\}}(z_{htt} + \operatorname{div}\boldsymbol{q}_h - y_h + y_d)\big\| + \min_{w_h\in W_h}\big\|h(\boldsymbol{q}_h - \nabla_h w_h)\big\|\Big), \tag{2.91}$$

$$\|\eta_{8t}\| \le C\Big(\big\|h^{1+\min\{1,k\}}(z_{htt} + \operatorname{div}\boldsymbol{q}_h - y_h + y_d)_t\big\| + \min_{w_h\in W_h}\big\|h(\boldsymbol{q}_{ht} - \nabla_h w_h)\big\|\Big), \tag{2.92}$$

$$\|\eta_1\| \le C\big(\big\|h^{\frac{1}{2}}J(\tilde{\boldsymbol{p}}_h\cdot\boldsymbol{t})\big\|_{0,\Gamma_h} + \|h\operatorname{curl}_h\tilde{\boldsymbol{p}}_h\| + \big\|h(\operatorname{div}\tilde{\boldsymbol{p}}_h - \tilde{y}_h)\big\|\big), \tag{2.93}$$

$$\|\eta_3\| \le C\big(\big\|h^{\frac{1}{2}}J(\boldsymbol{p}_h\cdot\boldsymbol{t})\big\|_{0,\Gamma_h} + \|h\operatorname{curl}_h\boldsymbol{p}_h\| + \big\|h(y_{htt} + \operatorname{div}\boldsymbol{p}_h - f - u_h)\big\|\big), \tag{2.94}$$

$$\|\eta_5\| \le C\big(\big\|h^{\frac{1}{2}}J(\tilde{\boldsymbol{q}}_h\cdot\boldsymbol{t})\big\|_{0,\Gamma_h} + \|h\operatorname{curl}_h\tilde{\boldsymbol{q}}_h\| + \big\|h(\operatorname{div}\tilde{\boldsymbol{q}}_h - \tilde{y}_h - \tilde{y}_h)\big\|\big), \tag{2.95}$$

$$\|\eta_7\| \le C\big(\big\|h^{\frac{1}{2}}J(\boldsymbol{q}_h\cdot\boldsymbol{t})\big\|_{0,\Gamma_h} + \|h\operatorname{curl}_h\boldsymbol{q}_h\| + \big\|h(z_{htt} + \operatorname{div}\boldsymbol{q}_h - y_h + y_d)\big\|\big), \tag{2.96}$$

$$\|\operatorname{div}\eta_1\| \le C\|\operatorname{div}\tilde{\boldsymbol{p}}_h - \tilde{y}_h\|, \|\operatorname{div}\eta_3\| \le C\|y_{htt} + \operatorname{div}\boldsymbol{p}_h - f - u_h\|, \tag{2.97}$$

$$\|\operatorname{div}\eta_5\| \le C\|\operatorname{div}\tilde{\boldsymbol{q}}_h - \tilde{y}_h - \tilde{z}_h\|, \|\operatorname{div}\eta_7\| \le C\|z_{htt} + \operatorname{div}\boldsymbol{q}_h - y_h + y_d\|, \tag{2.98}$$

where ∇_h and curl_h have been defined in (2.44)–(2.45), $J(\boldsymbol{v}\cdot\boldsymbol{t})$ denotes the jump of $\boldsymbol{v}\cdot\boldsymbol{t}$ across the element edge E for all $\boldsymbol{v}\in\boldsymbol{V}$ with $\boldsymbol{t}$ being the tangential unit vector along the edge $E\in\Gamma_h$.

3 Error estimation of optimal control

In this part, a posteriori error estimation of optimal control problems shall be given. From (2.53)–(2.56) and (2.65)–(2.68), we obtain the error equations

$$(e_1,\boldsymbol{v})-(e_2,\operatorname{div}\boldsymbol{v})=0,\quad \forall \boldsymbol{v}\in\boldsymbol{V}, \tag{3.1}$$

$$(\operatorname{div}e_1,w)=(e_4+\eta_4,w),\quad \forall w\in W, \tag{3.2}$$

$$(e_3,\boldsymbol{v})-(e_4,\operatorname{div}\boldsymbol{v})=0,\quad \forall \boldsymbol{v}\in\boldsymbol{V}, \tag{3.3}$$

$$(e_{2tt},w)+(\operatorname{div}e_3,w)=-(\eta_{2tt},w),\quad \forall w\in W. \tag{3.4}$$

Lemma 3.1 *Let e_1–e_4 satisfy (3.1)–(3.4). Then we have*

$$\begin{aligned}
&\|e_2\|_{L^\infty(J;L^2(\Omega))}+\|e_1\|_{L^\infty(J;H(\operatorname{div};\Omega))}+\|e_4\|_{L^\infty(J;L^2(\Omega))}+\|e_3\|_{L^\infty(J;H(\operatorname{div};\Omega))}\\
&\quad\le C\big(\|\eta_{4t}\|_{L^2(J;L^2(\Omega))}+\|y_1-y_1^h\|+\|\eta_{2t}(0)\|+\|\Delta y_0+\tilde{y}_h(0)\|+\|\eta_4\|_{L^\infty(J;L^2(\Omega))}\\
&\qquad+\|\eta_{2tt}\|_{L^\infty(J;L^2(\Omega))}+\|\eta_{2ttt}\|_{L^2(J;L^2(\Omega))}+\|\eta_{4tt}\|_{L^2(J;L^2(\Omega))}+\|y_0-y_0^h\|\\
&\qquad+\|\operatorname{div}\eta_3(0)\|+\|\Delta^2 y_0-\operatorname{div}p_h(0)\|\big)\\
&\qquad+\|\Delta y_1+\tilde{y}_{ht}(0)\|+\|\eta_{4t}(0)\|\big)+\|\eta_2(0)\|\big).
\end{aligned} \tag{3.5}$$

Proof Differentiating (3.1)–(3.2) with respect to t, we get

$$(e_{1t},\boldsymbol{v})-(e_{2t},\operatorname{div}\boldsymbol{v})=0,\quad \forall \boldsymbol{v}\in\boldsymbol{V}, \tag{3.6}$$

$$(\operatorname{div}e_{1t},w)=(e_{4t}+\eta_{4t},w),\quad \forall w\in W. \tag{3.7}$$

We choose $\boldsymbol{v}=e_3$ in (3.6), $w=-e_4$ in (3.7), $\boldsymbol{v}=-e_{1t}$ in (3.3) and $w=e_{2t}$ in (3.4), separately, then add up the four equations and obtain

$$(e_{2t},e_{2tt})+(e_{4t},e_4)=-(\eta_{4t},e_4)-(\eta_{2tt},e_{2t}). \tag{3.8}$$

We integrate (3.8) from 0 to t, use Gronwall's inequality and the Cauchy inequality, then we obtain

$$\begin{aligned}
&\|e_4\|_{L^\infty(J;L^2(\Omega))}+\|e_{2t}\|_{L^\infty(J;L^2(\Omega))}\\
&\quad\le C\big(\|\eta_{4t}\|_{L^2(J;L^2(\Omega))}+\|\eta_{2tt}\|_{L^2(J;L^2(\Omega))}+\|e_{2t}(0)\|+\|e_4(0)\|\big),
\end{aligned} \tag{3.9}$$

where

$$\|e_{2t}(0)\|\le\|y_1-y_1^h\|+\|\eta_{2t}(0)\|, \tag{3.10}$$

$$\|e_4(0)\|\le\|\Delta y_0+\tilde{y}_h(0)\|+\|\eta_4(0)\|. \tag{3.11}$$

Note that

$$\int_0^t e_{2s}\,dt=e_2(t)-e_2(0),$$

then we have

$$\begin{aligned}\|e_2\| &\le C\big(\|e_{2t}\|_{L^\infty(J;L^2(\Omega))} + \big\|e_2(0)\big\|\big)\\ &\le C\big(\|e_{2t}\|_{L^\infty(J;L^2(\Omega))} + \big\|y_0 - y_0^h\big\| + \big\|\eta_2(0)\big\|\big).\end{aligned} \tag{3.12}$$

Letting $\boldsymbol{v} = e_1$ in (3.1), $w = e_2$ in (3.2), $\boldsymbol{v} = e_3$ in (3.3) and $w = e_4$ in (3.4), respectively. We get

$$\|e_1\|_{L^\infty(J;L^2(\Omega))} \le \|\eta_4\|_{L^\infty(J;L^2(\Omega))} + \|e_4\|_{L^\infty(J;L^2(\Omega))} + \|e_2\|_{L^\infty(J;L^2(\Omega))}, \tag{3.13}$$

$$\|e_3\|_{L^\infty(J;L^2(\Omega))} \le \|e_{2tt}\|_{L^\infty(J;L^2(\Omega))} + \|\eta_{2tt}\|_{L^\infty(J;L^2(\Omega))} + \|e_4\|_{L^\infty(J;L^2(\Omega))}. \tag{3.14}$$

Differentiating (3.3)–(3.4) and (3.6)–(3.7) with respect to t, we get

$$(e_{3t}, \boldsymbol{v}) - (e_{4t}, \operatorname{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.15}$$

$$(e_{2ttt}, w) + (\operatorname{div} e_{3t}, w) = -(\eta_{2ttt}, w), \quad \forall w \in W, \tag{3.16}$$

$$(e_{1tt}, \boldsymbol{v}) - (e_{2tt}, \operatorname{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.17}$$

$$(\operatorname{div} e_{1tt}, w) = (e_{4tt} + \eta_{4tt}, w), \quad \forall w \in W. \tag{3.18}$$

We choose $v = -e_{1tt}$ in (3.15), $w = e_{2tt}$ in (3.16), $v = e_{3t}$ in (3.17), $w = -e_{4t}$ in (3.18), separately. We derive the following after addition for four equations:

$$(e_{2ttt}, e_{2tt}) + (e_{4tt}, e_{4t}) = -(\eta_{4tt}, e_{4t}) - (\eta_{2ttt}, e_{2tt}). \tag{3.19}$$

Similar to (3.9), we derive

$$\begin{aligned}&\|e_{2tt}\|_{L^\infty(J;L^2(\Omega))} + \|e_{4t}\|_{L^\infty(J;L^2(\Omega))}\\ &\quad\le C\big(\|\eta_{2ttt}\|_{L^2(J;L^2(\Omega))} + \|\eta_{4tt}\|_{L^2(J;L^2(\Omega))} + \big\|e_{2tt}(0)\big\| + \big\|e_{4t}(0)\big\|\big).\end{aligned} \tag{3.20}$$

Taking $t = 0$ and $w = e_{2tt}(0)$ in (3.4) leads to

$$\begin{aligned}\big\|e_{2tt}(0)\big\| &\le \big\|\operatorname{div} e_3(0)\big\| + \big\|\eta_{2tt}(0)\big\|\\ &\le \big\|\operatorname{div} \eta_3(0)\big\| + \big\|\eta_{2tt}(0)\big\| + \big\|\Delta^2 y_0 - \operatorname{div} p_h(0)\big\|.\end{aligned} \tag{3.21}$$

Note that

$$\begin{aligned}\big\|e_{4t}(0)\big\| &\le \big\|\tilde{y}_t(u_h)(0) - \tilde{y}_{ht}(0)\big\| + \big\|\eta_{4t}(0)\big\|\\ &\le \big\|\Delta y_1 + \tilde{y}_{ht}(0)\big\| + \big\|\eta_{4t}(0)\big\|.\end{aligned} \tag{3.22}$$

At last, setting $w = \operatorname{div} e_1$ and $w = \operatorname{div} e_3$ in (3.2) and (3.4), we get

$$\|\operatorname{div} e_1\|_{L^\infty(J;L^2(\Omega))} \le \|\eta_4\|_{L^\infty(J;L^2(\Omega))} + \|e_4\|_{L^\infty(J;L^2(\Omega))}, \tag{3.23}$$

$$\|\operatorname{div} e_3\|_{L^\infty(J;L^\infty(\Omega))} \le \|e_{2tt}\|_{L^\infty(J;L^2(\Omega))} + \|\eta_{2tt}\|_{L^\infty(J;L^2(\Omega))}. \tag{3.24}$$

Thus, using (3.9)–(3.14) and (3.20)–(3.24), we complete the proof. □

By (2.59)–(2.62) and (2.69)–(2.72), we obtain the error equations

$$(e_5, \boldsymbol{v}) - (e_6, \operatorname{div} \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.25}$$

$$(\operatorname{div} e_5, w) = (e_4 + e_8 + \eta_4 + \eta_8, w), \quad \forall w \in W, \tag{3.26}$$

$$(e_7, \boldsymbol{v}) - (e_8, \operatorname{div} \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.27}$$

$$(e_{6tt}, w) + (\operatorname{div} e_7, w) = (\eta_{6tt} + \eta_2 + e_2, w), \quad \forall w \in W. \tag{3.28}$$

Lemma 3.2 *Let e_5–e_8 satisfy* (3.25)–(3.28). *Then we get*

$$\begin{aligned}
&\|e_6\|_{L^\infty(J;L^2(\Omega))} + \|e_5\|_{L^\infty(J;H(\mathrm{div};\Omega))} + \|e_8\|_{L^\infty(J;L^2(\Omega))} + \|e_7\|_{L^2(J;H(\mathrm{div};\Omega))} \\
&\quad \le C\big(\|\eta_4\|_{L^\infty(J;L^2(\Omega))} + \|\eta_{4t}\|_{L^2(J;L^2(\Omega))} + \|\eta_{4tt}\|_{L^2(J;L^2(\Omega))} + \|\eta_2\|_{L^2(J;L^2(\Omega))} \\
&\qquad + \|\eta_{6tt}\|_{L^2(J;L^2(\Omega))} + \|\eta_8\|_{L^\infty(J;L^2(\Omega))} + \|\eta_{8t}\|_{L^2(J;L^2(\Omega))} + \|\eta_{2ttt}\|_{L^2(J;L^2(\Omega))} \\
&\qquad + \|e_4\|_{L^\infty(J;L^2(\Omega))} + \|e_2\|_{L^2(J;L^2(\Omega))} + \big\|\Delta y_1 + \tilde{y}_{ht}(0)\big\| + \big\|\eta_{4t}(0)\big\| \\
&\qquad + \big\|\operatorname{div} \eta_3(0)\big\| + \big\|\eta_{2tt}(0)\big\| + \big\|\Delta^2 y_0 - \operatorname{div} \boldsymbol{p}_h(0)\big\|\big).
\end{aligned} \tag{3.29}$$

Proof At first, setting $t = T$ in (2.69)–(2.70) and (3.25)–(3.26), we derive

$$e_6(T) = e_{6t}(T) = 0 \tag{3.30}$$

and

$$\big\|e_8(T)\big\| \le C\big(\big\|e_4(T)\big\| + \big\|\eta_4(T)\big\| + \big\|\eta_8(T)\big\|\big). \tag{3.31}$$

Differentiating (3.25)–(3.26) with respect to t, we obtain

$$(e_{5t}, \boldsymbol{v}) - (e_{6t}, \operatorname{div} \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.32}$$

$$(\operatorname{div} e_{5t}, w) = (e_{4t} + e_{8t} + \eta_{4t} + \eta_{8t}, w), \quad \forall w \in W. \tag{3.33}$$

Let $\boldsymbol{v} = -e_7$ in (3.32), $w = e_8$ in (3.33), $\boldsymbol{v} = e_{5t}$ in (3.27) and $w = -e_{6t}$ in (3.28), separately. After adding up the new equations, we have

$$-(e_{6tt}, e_{6t}) - (e_{8t}, e_8) = (e_{4t} + \eta_{4t} + \eta_{8t}, e_8) - (\eta_{6tt} + \eta_2 + e_2, e_{6t}). \tag{3.34}$$

Integrating (3.34) from t to T, from (3.30), Gronwall's inequality and the Cauchy inequality, it is easy to see that

$$\begin{aligned}
&\|e_{6t}\|_{L^\infty(J;L^2(\Omega))} + \|e_8\|_{L^\infty(J;L^2(\Omega))} \\
&\quad \le C\big(\|\eta_{4t}\|_{L^2(J;L^2(\Omega))} + \|e_{4t}\|_{L^2(J;L^2(\Omega))} + \|\eta_{8t}\|_{L^2(J;L^2(\Omega))} \\
&\qquad + \|e_2\|_{L^2(J;L^2(\Omega))} + \|\eta_2\|_{L^2(J;L^2(\Omega))} + \|\eta_{6tt}\|_{L^2(J;L^2(\Omega))}\big) + \big\|e_8(T)\big\|.
\end{aligned} \tag{3.35}$$

Letting $\boldsymbol{v} = e_7$ in (3.27), we get

$$(e_7, e_7) = (e_8, \operatorname{div} e_7). \tag{3.36}$$

Next, for (3.25), we differentiate two times with respect to t, and set $\boldsymbol{v} = e_7$. for (3.26), we also differentiate two times with respect to t, and set $w = e_8$. For (3.27), we set $\boldsymbol{v} = e_{5tt}$. For (3.28), we set $w = \operatorname{div} e_7$. Combining the new four equalities, we derive

$$\begin{aligned}\|\operatorname{div} e_7\|_{L^2(J;L^2(\Omega))} \leq{}& C\big(\|e_2\|_{L^2(J;L^2(\Omega))} + \|\eta_{6tt}\|_{L^2(J;L^2(\Omega))} + \|\eta_2\|_{L^2(J;L^2(\Omega))} + \|\eta_{4t}\|_{L^\infty(J;L^2(\Omega))} \\ &+ \|\eta_{8t}\|_{L^\infty(J;L^2(\Omega))}\big) + \|e_4\|_{L^2(J;L^2(\Omega))}\big) + \|e_{8t}\|_{L^2(J;L^2(\Omega))}\big).\end{aligned} \tag{3.37}$$

At last, similar to (3.13)–(3.14), (3.23)–(3.24) and (3.36), we have

$$\begin{aligned}\|e_5\|_{L^\infty(J;H(\mathrm{div};\Omega))} \leq{}& C\big(\|e_4\|_{L^\infty(J;L^2(\Omega))} + \|e_8\|_{L^\infty(J;L^2(\Omega))} \\ &+ \|\eta_4\|_{L^\infty(J;L^2(\Omega))} + \|\eta_8\|_{L^\infty(J;L^2(\Omega))}\big) + \|e_6\|_{L^\infty(J;L^2(\Omega))}\big),\end{aligned} \tag{3.38}$$

$$\begin{aligned}&\|e_7\|_{L^2(J;H(\mathrm{div};\Omega))} \\ &\quad\leq C\big(\|e_2\|_{L^2(J;L^2(\Omega))} + \|\eta_{6tt}\|_{L^2(J;L^2(\Omega))} + \|\eta_2\|_{L^2(J;L^2(\Omega))} + \|\eta_{4t}\|_{L^\infty(J;L^2(\Omega))} \\ &\qquad + \|\eta_{8t}\|_{L^\infty(J;L^2(\Omega))} + \|e_4\|_{L^2(J;L^2(\Omega))}\big) + \|e_{8t}\|_{L^2(J;L^2(\Omega))} + \|e_8\|_{L^2(J;L^2(\Omega))}\big).\end{aligned} \tag{3.39}$$

Thus, combining Lemma 3.1, (3.31), (3.35)–(3.40) and (3.5), we complete the proof. □

Choosing $\tilde{u} = u$ and $\tilde{u} = u_h$ in (2.53)–(2.64), respectively, we have the error equations

$$(r_1, \boldsymbol{v}) - (r_2, \operatorname{div} \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.40}$$

$$(\operatorname{div} r_1, w) = (r_4, w), \quad \forall w \in W, \tag{3.41}$$

$$(r_3, \boldsymbol{v}) - (r_4, \operatorname{div} \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.42}$$

$$(r_{2tt}, w) + (\operatorname{div} r_3, w) = (u - u_h, w), \quad \forall w \in W, \tag{3.43}$$

$$(r_5, \boldsymbol{v}) - (r_6, \operatorname{div} \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.44}$$

$$(\operatorname{div} r_5, w) = (r_4 + r_8, w), \quad \forall w \in W, \tag{3.45}$$

$$(r_7, \boldsymbol{v}) - (r_8, \operatorname{div} \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.46}$$

$$(r_{6tt}, w) + (\operatorname{div} r_7, w) = (r_2, w), \quad \forall w \in W. \tag{3.47}$$

Similar to Lemmas 3.1 and 3.2, Lemma 3.3 is given below.

Lemma 3.3 *Let r_1–r_8 satisfy* (3.40)–(3.47). *Then we have*

$$\begin{aligned}&\|r_{2t}\|_{L^\infty(J;L^2(\Omega))} + \|r_4\|_{L^\infty(J;L^2(\Omega))} + \|r_1\|_{L^\infty(J;H(\mathrm{div};\Omega))} + \|r_3\|_{L^2(J;H(\mathrm{div};\Omega))} \\ &\quad\leq C\|u - u_h\|_{L^2(J;L^2(\Omega))},\end{aligned} \tag{3.48}$$

$$\begin{aligned}&\|r_{4t}\|_{L^\infty(J;L^2\Omega))} + \|r_{6t}\|_{L^\infty(J;L^2\Omega))} + \|r_8\|_{L^\infty(J;L^2(\Omega))} \\ &\quad\leq \epsilon\big\|u(0) - u_h(0)\big\| \\ &\qquad + C\|u - u_h\|_{L^2(J;L^2(\Omega))} + \epsilon\big\|(u - u_h)_t\big\|_{L^2(J;L^2(\Omega))},\end{aligned} \tag{3.49}$$

$$\begin{aligned}
&\|r_5\|_{L^\infty(J;H(\mathrm{div};\Omega))} + \|r_7\|_{L^2(J;H(\mathrm{div};\Omega))} \\
&\quad \le \epsilon \big\|u(0) - u_h(0)\big\| \\
&\qquad + C\|u - u_h\|_{L^2(J;L^2(\Omega))} + \epsilon \big\|(u-u_h)_t\big\|_{L^2(J;L^2(\Omega))},
\end{aligned} \tag{3.50}$$

where ϵ is an arbitrary small positive constant.

Lemma 3.4 [15] *Let $(\tilde{\boldsymbol{p}}, y, \boldsymbol{p}, \tilde{y}, \tilde{\boldsymbol{q}}, z, \boldsymbol{q}, \tilde{z}, u)$ and $(\tilde{\boldsymbol{p}}_h, y_h, \boldsymbol{p}_h, \tilde{y}_h, \tilde{\boldsymbol{q}}_h, z_h, \boldsymbol{q}_h, u_h)$ be the solutions of* (2.10)–(2.22) *and* (2.30)–(2.42), *respectively. Suppose that $(u_h + z_h)|_\tau \in H^1(\tau)$ and that there exists $w \in K_h$ such that*

$$\|u - u_h\|_{L^2(J;L^2(\Omega))} \le C(\theta + \big\|z_h - z(u_h)\big\|_{L^2(J;L^2(\Omega))}, \tag{3.51}$$

where

$$\theta = \left(\int_0^T \sum_\tau h_\tau^2 |u_h + z_h|^2_{H^1(\tau)}\,dt\right)^{\frac{1}{2}}.$$

Now, by Lemmas 3.1–3.3, the important result of this paper is given as follows.

Theorem 3.1 *Let $(y, \tilde{\boldsymbol{p}}, \tilde{y}, \boldsymbol{p}, z, \tilde{\boldsymbol{q}}, \tilde{z}, \boldsymbol{q}, u)$ and $(y_h, \tilde{\boldsymbol{p}}_h, \tilde{y}_h, \boldsymbol{p}_h, z_h, \tilde{\boldsymbol{q}}_h, \tilde{z}_h, \boldsymbol{q}_h, u_h)$ be the solutions of* (2.9)–(2.19) *and* (2.26)–(2.36), *respectively. Then we have*

$$\begin{aligned}
&\|u - u_h\|_{L^\infty(J;L^2(\Omega))} + \|y - y_h\|_{L^\infty(J;L^2(\Omega))} + \|\tilde{y} - \tilde{y}_h\|_{L^\infty(J;L^2(\Omega))} \\
&\qquad + \|\tilde{\boldsymbol{p}} - \tilde{\boldsymbol{p}}_h\|_{L^\infty(J;H(\mathrm{div};\Omega))} + \|\boldsymbol{p} - \boldsymbol{p}_h\|_{L^\infty(J;H(\mathrm{div};\Omega))} + \|z - z_h\|_{L^\infty(J;L^2(\Omega))} \\
&\qquad + \|\tilde{z} - \tilde{z}_h\|_{L^\infty(J;L^2(\Omega))} + \|\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_h\|_{L^\infty(J;H(\mathrm{div};\Omega))} + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(J;H(\mathrm{div};\Omega))} \\
&\quad \le C\big(\|\eta_1\|_{L^\infty(J;H(\mathrm{div};\Omega))} + \|\eta_3\|_{L^\infty(J;H(\mathrm{div};\Omega))} + \|\eta_5\|_{L^\infty(J;H(\mathrm{div};\Omega))} + \|\eta_7\|_{L^2(J;H(\mathrm{div};\Omega))} \\
&\qquad + \|\eta_4\|_{L^\infty(J;L^2(\Omega))} + \|\eta_{4t}\|_{L^2(J;L^2(\Omega))} + \|\eta_{4tt}\|_{L^2(J;L^2(\Omega))} + \|\eta_2\|_{L^2(J;L^2(\Omega))} \\
&\qquad + \|\eta_{2tt}\|_{L^\infty(J;L^2(\Omega))} + \|\eta_{2ttt}\|_{L^2(J;L^2(\Omega))} + \|\eta_6\|_{L^\infty(J;L^2(\Omega))} + \|\eta_{6t}\|_{L^2(J;L^2(\Omega))} \\
&\qquad + \|\eta_{6tt}\|_{L^2(J;L^2(\Omega))} + \|\eta_8\|_{L^\infty(J;L^2(\Omega))} + \|\eta_{8t}\|_{L^2(J;L^2(\Omega))} + \big\|y_0 - y_0^h\big\| \\
&\qquad + \big\|\Delta y_0 + \tilde{y}_h(0)\big\| + \big\|y_1 - y_1^h\big\| + \big\|\mathrm{div}\,\eta_3(0)\big\| + \big\|\Delta^2 y_0 - \mathrm{div}\,\boldsymbol{p}_h(0)\big\|\big) \\
&\qquad + \big\|\Delta y_1 + \tilde{y}_{ht}(0)\big\| + \big\|\eta_2(0)\big\| + \big\|\eta_{2t}(0)\big\| + \big\|\eta_{4t}(0)\big\|.
\end{aligned} \tag{3.52}$$

Proof From Lemma 2.1 and (2.43), we have

$$\big\|u(0) - u_h(0)\big\| \le \|u - u_h\|_{L^\infty(J;L^2(\Omega))} \le C\|z - z_h\|_{L^\infty(J;L^2(\Omega))}, \tag{3.53}$$

$$\begin{aligned}
\big\|(u-u_h)_t\big\|_{L^2(J;L^2(\Omega))} &= \big\|(z-z_h)_t\big\|_{L^2(J;L^2(\Omega))} \\
&\le \|e_{6t}\|_{L^2(J;L^2(\Omega))} + \|\eta_{6t}\|_{L^2(J;L^2(\Omega))} + \|r_{6t}\|_{L^2(J;L^2(\Omega))}.
\end{aligned} \tag{3.54}$$

For sufficiently small ϵ, using Lemmas 3.1–3.3 and (3.35), (3.53)–(3.54), we complete the proof. □

4 Conclusion and future work

In the article, using semidiscrete Raviart–Thomas mixed finite element methods, we studied fourth order hyperbolic equations of quadratic problems for optimal control, and then got the posteriori error estimates. In subsequent work, an a posteriori estimation will be considered by a fully discrete approximation of the mixed finite element. Of course, the error estimates of the same problems certainly also can be discussed with nonlinear fourth order hyperbolic equations.

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Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CH, ZG and LG participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

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References

1. Arada, N., Casas, E., Tröltzsch, F.: Error estimates for the numerical approximation of a semilinear elliptic control problem. Comput. Optim. Appl. **23**, 201–229 (2002)
2. Babuška, I., Strouboulis, T.: The Finite Element Method and Its Reliability. Oxford University Press, Oxford (2001)
3. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer, Berlin (1991)
4. Brunner, H., Yan, N.: Finite element methods for optimal control problems governed by integral equations and integro-differential equations. Numer. Math. **101**, 1–27 (2005)
5. Cao, W., Yang, D.: Ciarlet–Raviart mixed finite element approximation for an optimal control problem governed by the first bi-harmonic equation. J. Comput. Appl. Math. **233**(2), 372–388 (2009)
6. Chen, Y.: Superconvergence of quadratic optimal control problems by triangular mixed finite elements. Int. J. Numer. Methods Eng. **75**, 881–898 (2008)
7. Chen, Y., Huang, Y., Liu, W.B., Yan, N.N.: Error estimates and superconvergence of mixed finite element methods for convex optimal control problems. J. Sci. Comput. **42**, 382–403 (2009)
8. Chen, Y., Liu, W.B.: A posteriori error estimates for mixed finite element solutions of convex optimal control problems. J. Comput. Appl. Math. **211**, 76–89 (2008)
9. Chen, Y., Sun, C.M.: Error estimates and superconvergence of mixed finite element methods for fourth order hyperbolic control problems. Appl. Math. Comput. **244**, 642–653 (2014)
10. Douglas, J., Roberts, J.E.: Global estimates for mixed finite element methods for second order elliptic equations. Math. Comput. **44**, 39–52 (1985)
11. Gong, W., Yan, N.: A posteriori error estimate for boundary control problems governed by the parabolic partial differential equations. J. Comput. Math. **27**, 68–88 (2009)
12. Haslinger, J., Neittaanmaki, P.: Finite Element Approximation for Optimal Shape Design. Wiley, Chichester (1989)
13. Hou, L., Turner, J.C.: Analysis and finite element approximation of an optimal control problem in electrochemistry with current density controls. Numer. Math. **71**, 289–315 (1995)
14. Hou, T.: Error estimates of mixed finite element approximations for a class of fourth order elliptic control problems. Bull. Korean Math. Soc. **4**(50), 1127–1144 (2013)
15. Hou, T.: A posteriori $L^\infty(L^2)$-error estimates of semidiscrete mixed finite element methods for hyperbolic optimal control problems. Bull. Korean Math. Soc. **50**, 321–341 (2013)
16. Knowles, G.: Finite element approximation of parabolic time optimal control problems. SIAM J. Control Optim. **20**, 414–427 (1982)
17. Li, R., Liu, W., Ma, H., Tang, T.: Adaptive finite element approximation of elliptic control problems. SIAM J. Control Optim. **41**, 1321–1349 (2002)
18. Lions, J.: Optimal Control of Systems Governed by Partial Differential Equations. Springer, Berlin (1971)

19. Lions, J., Magenes, E.: Non Homogeneous Boundary Value Problems and Applications. Grandlehre, vol. 181. Springer, Berlin (1972)
20. Liu, W., Ma, H., Tang, T., Yan, N.: A posteriori error estimates for discontinuous Galerkin time-stepping method for optimal control problems governed by parabolic equations. SIAM J. Numer. Anal. **42**, 1032–1061 (2004)
21. Liu, W., Yan, N.: A posteriori error estimates for convex boundary control problems. SIAM J. Numer. Anal. **39**, 73–99 (2001)
22. Liu, W., Yan, N.: A posteriori error estimates for optimal control problems governed by Stokes equations. SIAM J. Numer. Anal. **40**, 1850–1869 (2003)
23. Mcknight, R., Bosarge, Jr., W.: The Ritz–Galerkin procedure for parabolic control problems. SIAM J. Control Optim. **11**, 510–524 (1973)
24. Memon, S., Nataraj, N., Pani, A.K.: An a posteriori error analysis of mixed finite element Galerkin approximations to second order linear parabolic problems. SIAM J. Numer. Anal. **50**, 1367–1393 (2012)
25. Neittaanmaki, P., Tiba, D.: Optimal Control of Nonlinear Parabolic Systems: Theory, Algorithms and Applications. Dekker, New York (1994)
26. Xing, X., Chen, Y.: Error estimates of mixed methods for optimal control problems governed by parabolic equations. Int. J. Numer. Methods Eng. **75**, 735–754 (2008)

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