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Existence and multiplicity of positive solutions for a class of singular fractional nonlocal boundary value problems

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Abstract

In this article, we consider a class of singular fractional differential equations with nonlocal boundary value conditions. The existence and multiplicity of positive solutions are derived by the fixed point index theory, and the nonlinearity $f(t,x)$ may be singular at $t = 0, 1$ and $x = 0$. The interesting point is that the existence results are closely associated with the relationship between 1 and the spectral radii corresponding to the relevant linear operators. An example is also given to demonstrate the validity of the main results.

Keywords: Fractional differential equation; Singularity; Fixed point index; Spectral radius

1 Introduction

In this paper, we consider the existence and multiplicity of positive solutions for the following fractional differential equation (FDE):

$$D_{0+}^{\alpha} u(t) + f\big(t, u(t)\big) = 0, \quad 0 < t < 1, n - 1 < \alpha \le n, \tag{1.1}$$

with conjugate type integral boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\beta} u(1) = \int_0^{\eta} a(t) D_{0+}^{\gamma} u(t)\, dV(t), \tag{1.2}$$

where D_{0+}^{α} is the standard Riemann–Liouville derivative, $n \ge 3$, $0 < \beta < 1$, $0 \le \gamma < \alpha - 1$, $\eta \in (0, 1]$, $f(t,x)$ may be singular at $t = 0, 1$ and $x = 0$, $a(t) \in L^1[0,1] \cap C(0,1)$ is nonnegative, $\int_0^{\eta} a(t) t^{\alpha-\gamma-1}\, dV(t)$ denotes the Riemann–Stieltjes integral, in which V has bounded variation.

During the last few decades, FDE have drawn more and more attention due to their numerous applications in various fields of science. Recently, many results were obtained dealing with the fractional differential equations boundary value problems (FBVP) by the use of techniques of nonlinear analysis; see [1–24] and the references therein. The nonlocal boundary value problems of fractional differential equation have particularly attracted a great deal of attention (see [25–33]). For example, a number of papers have been devoted to considering (1.1) under boundary value conditions (BC) as follows:

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{p} u(1) = \sum_{i=1}^{m} \eta_i D_{0+}^{q} u(\xi_i), \tag{1.3}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\beta} u(1) = \sum_{i=1}^{\infty} \eta_i D_{0+}^{\beta} u(\xi_i), \tag{1.4}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \int_0^1 u(t)\,dV(t), \tag{1.5}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\beta} u(1) = \int_0^{\eta} a(t) D_{0+}^{\gamma} u(t)\,dt. \tag{1.6}$$

In [12], Henderson and Luca considered the existence of positive solutions for a fractional differential equation subject to BC (1.3), where $p \in [1, n-2]$, $q \in [0, p]$. In [28], Wang and Liu considered a fractional differential equation with infinite-point boundary value conditions (1.4). In [29], by means of the fixed point index theory in cones, Wang et al. established the existence and multiplicity results of positive solutions to (1.1) with BC (1.5). When $1 \leq \beta < \alpha - 1$, Zhang and Zhong [32] established the existence of triple positive solutions for (1.1) with BC (1.6) by using the Leggett–Williams and Krasnosel'skii fixed point theorems. When $1 \leq \beta < \alpha - 1$ and f is continuous on $[0,1] \times (-\infty, +\infty)$, Zhang and Zhong [33] established the uniqueness results of solution to (1.1) with BC (1.6) by using the Banach contraction map principle.

For the case that α is an integer, Webb [34] considered the nth-order conjugate type BC (1.5). Some existence results of positive solutions have been obtained by using the fixed point index theory under the following conditions:

$$(C_1) \quad \liminf_{x \to 0+} \min_{t \in [0,1]} \frac{f(t,x)}{x} > \lambda_1; \qquad \limsup_{x \to +\infty} \max_{t \in [0,1]} \frac{f(t,x)}{x} < \lambda_1;$$

$$(C_2) \quad \limsup_{x \to 0+} \max_{t \in [0,1]} \frac{f(t,x)}{x} < \lambda_1; \qquad \liminf_{x \to +\infty} \min_{t \in [0,1]} \frac{f(t,x)}{x} > \lambda_1,$$

where λ_1 is the first eigenvalue of a linear operator.

Motivated by the above works, in this article we aim to establish the existence and multiplicity of positive solutions to problem (1.1)–(1.2). Our analysis relies on the topological degree theory on the cone derived from the properties of the Green function. This article provides some new insights. Firstly, the existence results are obtained under some conditions concerning the spectral radii with respect to the relevant linear operators, and the assumptions on f are weaker than C_1, C_2. Secondly, we consider the case that $0 < \beta < 1$ which is different from [12, 32, 33] and more general integral boundary conditions which include as special cases the multi-point problems (1.3), (1.4) and integral problems (1.5), (1.6). Finally, FBVP (1.1)–(1.2) possesses singularity, that is, $f(t,x)$ may be singular at $t = 0, 1$ and $x = 0$.

2 Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory and lemmas.

Definition 2.1 ([2]) The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds$$

provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 2.2 ([2]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Lemma 2.1 ([30]) *Let $\alpha > 0$. Then the following equality holds for $u \in L(0,1)$, $D_{0+}^{\alpha} u \in L(0,1)$:*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in R$, $i = 1, 2, \ldots, n$, $n-1 < \alpha \le n$.

Lemma 2.2 ([30]) *Assume that $g \in L(0,1)$ and $\alpha > \beta \ge 0$. Then*

$$D_{0+}^{\beta} \int_0^t (t-s)^{\alpha-1} g(s)\, ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} g(s)\, ds.$$

Lemma 2.3 *Assume that $a \in L^1[0,1] \cap C(0,1)$, V is a function of bounded variation, and*

$$\Delta := \Gamma(\alpha-\gamma) - \Gamma(\alpha-\beta) \int_0^{\eta} a(t) t^{\alpha-\gamma-1}\, dV(t) \neq 0.$$

Then, for any $y \in L[0,1] \cap C(0,1)$, the unique solution of the boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & D_{0+}^{\beta} u(1) = \int_0^{\eta} a(t) D_{0+}^{\gamma} u(t)\, dV(t), \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s) y(s)\, ds,$$

where

$$G(t,s) = G_1(t,s) + h(s) t^{\alpha-1},$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \end{cases}$$

$$\begin{aligned} G_2(t,s) &= \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \\ t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\gamma-1}, & 0 \le s \le t \le 1, \end{cases} \\ h(s) &= \frac{\Gamma(\alpha-\gamma)}{\Delta} \int_0^\eta a(t) G_2(t,s)\, dV(t). \end{aligned}$$

Proof It follows from Lemma 2.1 that the solution of (2.1) can be expressed by

$$\begin{aligned} u(t) &= -I_{0+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}. \end{aligned}$$

By $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we know that $c_2 = \cdots = c_n = 0$. Then we obtain

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1}. \tag{2.2}$$

From Lemma 2.2, we have

$$\begin{aligned} D_{0+}^{\gamma} u(t) &= -\frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} y(s)\, ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\gamma)} t^{\alpha-\gamma-1}, \\ D_{0+}^{\beta} u(t) &= -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s)\, ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}. \end{aligned} \tag{2.3}$$

Then we get

$$D_{0+}^{\beta} u(1) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)}. \tag{2.4}$$

From (2.3) we have

$$\begin{aligned} &\int_0^\eta a(t) D_{0+}^{\gamma} u(t)\, dV(t) \\ &\quad= \frac{-1}{\Gamma(\alpha-\gamma)} \int_0^\eta a(t) \int_0^t (t-s)^{\alpha-\gamma-1} y(s)\, ds\, dV(t) + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\gamma)} \int_0^\eta a(t) t^{\alpha-\gamma-1}\, ds. \end{aligned}$$

Combining (2.3) with (2.4), we get

$$c_1 = \frac{\Gamma(\alpha-\gamma) \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds - \Gamma(\alpha-\beta) \int_0^\eta a(t) \int_0^t (t-s)^{\alpha-\gamma-1} y(s)\, ds\, dV(t)}{\Gamma(\alpha)\Delta}.$$

Substituting into (2.2), we have that the unique solution of (2.1) is

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1} \\ &= t^{\alpha-1} \frac{\Gamma(\alpha-\gamma) \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds - \Gamma(\alpha-\beta) \int_0^\eta a(t) \int_0^t (t-s)^{\alpha-\gamma-1} y(s)\, ds\, dV(t)}{\Gamma(\alpha)\Delta} \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha)\Delta}\int_0^{\eta} a(t)t^{\alpha-\gamma-1}\,dV(t)\int_0^1(1-s)^{\alpha-\beta-1}y(s)\,ds \\
&\quad - \frac{\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha)\Delta}\int_0^{\eta} a(t)\int_0^t (t-s)^{\alpha-\gamma-1}y(s)\,ds\,dV(t) \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}y(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds \\
&= \frac{\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha)\Delta} \\
&\quad \times \int_0^{\eta} a(t)\left[t^{\alpha-\gamma-1}\int_0^1(1-s)^{\alpha-\beta-1}y(s)\,ds - \int_0^t(t-s)^{\alpha-\gamma-1}y(s)\,ds\right]dV(t) \\
&\quad + \int_0^1 G_1(t,s)y(s)\,ds \\
&= \frac{\Gamma(\alpha-\beta)t^{\alpha-1}}{\Delta}\int_0^{\eta} a(t)\int_0^1 G_2(t,s)y(s)\,ds\,dV(t) + \int_0^1 G_1(t,s)y(s)\,ds \\
&= \int_0^1\left[G_1(t,s)+h(s)t^{\alpha-1}\right]y(s)\,ds \\
&= \int_0^1 G(t,s)y(s)\,ds.
\end{aligned}
$$

□

Lemma 2.4 *The function $G_1(t,s)$ has the following properties*:

(1) $G_1(t,s) > 0$, $\forall t,s \in (0,1)$;

(2) $\Gamma(\alpha)G_1(t,s) \le t^{\alpha-1}(1-s)^{\alpha-\beta-1}$, $\forall t,s \in [0,1]$;

(3) $\beta s(1-s)^{\alpha-\beta-1}t^{\alpha-1} \le \Gamma(\alpha)G_1(t,s) \le s(1-s)^{\alpha-\beta-1}$, $\forall t,s \in [0,1]$.

Proof It is clear that (1), (2) hold. So we just need to prove that (3) holds.

When $0 < s \le t < 1$. Noticing $\alpha > 2$, we have

$$\frac{\partial}{\partial t}\left[t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}}\right] = (\alpha-1)t^{\alpha-2}\left[1-\left(\frac{t-s}{t(1-s)}\right)^{\alpha-2}\right] \ge 0,$$

which implies

$$t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}} \le 1-(1-s) = s.$$

Noticing $0 < \beta < 1$, we have

$$
\begin{aligned}
t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} &= (1-s)^{\alpha-\beta-1}\left[t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-\beta-1}}\right] \\
&\le (1-s)^{\alpha-\beta-1}\left[t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}}\right] \le s(1-s)^{\alpha-\beta-1}. \qquad (2.5)
\end{aligned}
$$

By

$$\frac{\partial}{\partial s}\left[\beta s + (1-s)^{\beta}\right] \le 0, \quad \forall s \in [0,1),$$

we have

$$\beta s + (1-s)^{\beta} \le 1, \quad \forall s \in [0,1].$$

Therefore,

$$\begin{aligned} &t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \\ &\quad \ge t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\beta}(t-ts)^{\alpha-\beta-1} \\ &\quad = t^{\alpha-1}\left[1-\left(1-\frac{s}{t}\right)^{\beta}\right](1-s)^{\alpha-\beta-1} \\ &\quad \ge t^{\alpha-1}\left[1-(1-s)^{\beta}\right](1-s)^{\alpha-\beta-1} \\ &\quad \ge \beta s(1-s)^{\alpha-\beta-1}t^{\alpha-1}. \end{aligned} \tag{2.6}$$

When $0 \le t \le s \le 1$. It is easy to get

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1} \le s^{\alpha-1}(1-s)^{\alpha-\beta-1} \le s(1-s)^{\alpha-\beta-1}. \tag{2.7}$$

On the other hand, we have

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1} \ge st^{\alpha-1}(1-s)^{\alpha-\beta-1} \ge \beta s(1-s)^{\alpha-\beta-1}t^{\alpha-1}. \tag{2.8}$$

It follows from (2.5)–(2.8) that (3) holds. □

We make the following assumptions throughout this paper:

(A_1) $a(t) \in L^1[0,1] \cap C(0,1)$, V is a function of bounded variation;

(A_2) $\Delta := \Gamma(\alpha-\gamma) - \Gamma(\alpha-\beta)\int_0^{\eta} a(t)t^{\alpha-\gamma-1}\,dV(t) \neq 0$, and $h(s) \ge 0$ for $s \in [0,1]$;

(A_3) $f : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous. In addition, for any $R \ge r > 0$, there exists $\Psi_{r,R} \in L^1[0,1] \cap C(0,1)$ such that

$$f(t,x) \le \Psi_{r,R}(t), \quad \forall t \in (0,1), x \in \left[\beta r t^{\alpha-1}, R\right].$$

Lemma 2.5 *The Green function $G(t,s)$ has the following properties*:

(1) $G(t,s) > 0$, $\forall t,s \in (0,1)$;

(2) $G(t,s) \le t^{\alpha-1}\Phi_1(s)$, $\forall t,s \in [0,1]$;

(3) $\beta t^{\alpha-1}\Phi_2(s) \le G(t,s) \le \Phi_2(s)$, $\forall t,s \in [0,1]$,

where

$$\Phi_1(s) = \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s), \qquad \Phi_2(s) = \frac{s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s).$$

Proof It can be directly deduced from Lemma 2.4 and the definition of $G(t,s)$, so we omit the proof. □

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0\le t\le 1}|u(t)|$, $B_r = \{u \in E : \|u\| < r\}$. Define the cone Q by

$$Q = \left\{u \in E : u(t) \ge \beta\|u\|t^{\alpha-1}, t \in [0,1]\right\}.$$

For convenience, we list here some assumptions to be used later:

(H_1) There exist $r_1 > 0$ and a nonnegative function $b_1 \in L^1[0,1]$ with $\int_0^1 b_1(s)\,ds > 0$ such that

$$f(t,x) \geq b_1(t)x, \quad \forall (t,x) \in (0,1) \times (0,r_1];$$

(H_2) There exist $r_2 > 0$ and a nonnegative function $b_2 \in L^1[0,1]$ with $\int_0^1 b_2(s)\,ds > 0$ such that

$$f(t,x) \leq b_2(t)x, \quad \forall (t,x) \in (0,1) \times [r_2,+\infty);$$

(H_3) There exist $r_3 > 0$ and a nonnegative function $b_3 \in L^1[0,1]$ with $\int_0^1 b_3(s)\,ds > 0$ such that

$$f(t,x) \leq b_3(t)x, \quad \forall (t,x) \in (0,1) \times (0,r_3];$$

(H_4) There exist $r_4 > 0$ and a nonnegative function $b_4 \in L^1[0,1]$ with $\int_0^1 b_4(s)\,ds > 0$ such that

$$f(t,x) \geq b_4(t)x, \quad \forall (t,x) \in (0,1) \times [r_4,+\infty).$$

Define operators A and L_i as follows:

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds, \\ L_iu(t) &= \int_0^1 G(t,s)b_i(s)u(s)\,ds, \quad i = 1,2,3,4. \end{aligned}$$

Lemma 2.6 *For any $r > 0$, $A : Q \setminus B_r \to Q$ is completely continuous.*

Proof For any $u \in Q \setminus B_r$, we have $\beta r t^{\alpha-1} \leq u(t) \leq \|u\|$. It follows from (A_3) that there exists $\Psi_{r,\|u\|} \in L^1[0,1] \cap C(0,1)$ such that

$$f\bigl(t,u(t)\bigr) \leq \Psi_{r,\|u\|}(t), \quad \forall t \in (0,1).$$

Therefore,

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds \leq \int_0^1 \Phi_2(s)f\bigl(s,u(s)\bigr)\,ds \\ &\leq \int_0^1 \Phi_2(s)\Psi_{r,\|u\|}(s)\,ds < +\infty. \end{aligned}$$

On the other hand,

$$Au(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds \geq \beta t^{\alpha-1} \int_0^1 \Phi_2(s)f\bigl(s,u(s)\bigr)\,ds \geq \beta t^{\alpha-1}\|u\|.$$

So, the operator $A : Q \setminus B_r \to Q$ is well defined.

For any $D \in Q \setminus B_r$ is a bounded set. There exists $R > r$ such that $r \leq \|v\| \leq R$, $\forall v \in D$. By the above proof, we have

$$Av(t) \leq \int_0^1 \Phi_2(s)\Psi_{r,R}(s)\,ds < +\infty,$$

which implies $A(D)$ is uniformly bounded.

It is clear that $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $t', t'' \in [0,1]$, $|t' - t''| < \delta$, $s \in [0,1]$, one has

$$\bigl|G(t',s) - G(t'',s)\bigr| < \frac{\varepsilon}{\int_0^1 \Psi_{r,R}(s)\,ds + 1}.$$

Consequently,

$$\begin{aligned}
\bigl|(AV)(t') - (AV)(t'')\bigr| &\leq \int_0^1 \bigl|G(t',s) - G(t'',s)\bigr| f\bigl(s,u(s)\bigr)\,ds \\
&\leq \int_0^1 \bigl|G(t',s) - G(t'',s)\bigr| \Psi_{r,R}(s)\,ds \\
&\leq \int_0^1 \frac{\varepsilon}{\int_0^1 \Psi_{r,R}(s)\,ds + 1} \Psi_{r,R}(s)\,ds < \varepsilon.
\end{aligned}$$

This means that $A(D)$ is equicontinuous. By the Arzela–Ascoli theorem, we know that $A : Q \setminus B_r \to Q$ is compact.

Next, we will prove that A is continuous. Assume that $\{u_n\} \subset Q \setminus B_r$ and $\|u_n - u_0\| \to 0$ ($n \to +\infty$). Then there exists $R > r$ such that

$$r \leq \|u_n\| \leq R, \quad n = 0, 1, 2, \ldots.$$

For any $\varepsilon > 0$, by the absolute continuity of integral, $\exists \delta \in (0, \frac{1}{2})$ such that

$$\int_0^\delta \Phi_2(s)\Psi_{r,R}(s)\,ds < \frac{\varepsilon}{6}, \qquad \int_{1-\delta}^1 \Phi_2(s)\Psi_{r,R}(s)\,ds < \frac{\varepsilon}{6}.$$

Since $f(t,x)$ is uniformly continuous on $[\delta, 1-\delta] \times [\beta r t^{\alpha-1}, R]$ and $\|u_n - u_0\| \to 0$, there exists $N > 0$ such that, for any $n > N$, we have

$$\bigl|f\bigl(t,u_n(t)\bigr) - f\bigl(t,u_0(t)\bigr)\bigr| < \frac{\varepsilon}{3\int_0^1 \Phi_2(s)\,ds}, \quad t \in [\delta, 1-\delta].$$

Then

$$\begin{aligned}
\|Au_n - Au_0\| &\leq \max_{0 \leq t \leq 1} \int_0^1 G(t,s)\bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u_0(s)\bigr)\bigr|\,ds \\
&\leq \int_0^1 \Phi_2(s)\bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u_0(s)\bigr)\bigr|\,ds \\
&\leq 2\int_0^\delta \Phi_2(s)\Psi_{r,R}(s)\,ds + \int_\delta^{1-\delta} \Phi_2(s)\bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u_0(s)\bigr)\bigr|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ 2\int_{1-\delta}^{1} \Phi_2(s)\Psi_{r,R}(s)\,ds \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

So A is continuous. The proof is completed. □

By the extension theorem of a completely continuous operator (see Theorem 2.7 of [35]), for any $r > 0$, there exists the extension operator $\widetilde{A} : Q \to Q$, which is still completely continuous. Without loss of generality, we still write it as A.

By virtue of the Krein–Rutmann theorem and Lemma 2.5, we have the following lemma.

Lemma 2.7 *Assume that $b_i \in L^1[0,1]$ $(i = 1,2,3,4)$ are nonnegative functions satisfying $\int_0^1 b_i(s)\,ds > 0$. Then $L_i : Q \to Q$ is a completely continuous linear operator. Moreover, the spectral radius $r(L_i) > 0$ and L_i has a positive eigenfunction φ_i corresponding to its first eigenvalue $(r(L_i))^{-1}$, that is, $L_i\varphi_i = r(L_i)\varphi_i$.*

Set

$$
T_n u(t) = \int_{a_n}^{1} G(t,s)b_4(s)u(s)\,ds, \tag{2.9}
$$

where $1 > a_1 > \cdots > a_n > a_{n+1} > \cdots$, and $a_n \to 0$. By [34, 36], we have the following lemma.

Lemma 2.8 *The spectral radius $\{r(T_n)\}$ is increasing and converges to $r(L_4)$.*

Lemma 2.9 ([35]) *Let P be a cone in a Banach space E and Ω be a bounded open set in E. Suppose that A: $\overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P$ with $u_0 \neq \theta$ such that*

$$
u - Au \neq \lambda u_0, \quad \forall \lambda \geq 0, u \in \partial\Omega \cap P,
$$

then $i(A, \Omega \cap P, P) = 0$.

Lemma 2.10 ([35]) *Let P be a cone in a Banach space E and Ω be a bounded open set in E. Suppose that A: $\overline{\Omega} \cap P \to P$ is a completely continuous operator. If*

$$
Au \neq \lambda u, \quad \forall \lambda \geq 1, u \in \partial\Omega \cap P,
$$

then $i(A, \Omega \cap P, P) = 1$.

3 Main results

Theorem 3.1 *Assume that there exist $r_2 > r_1 > 0$ such that (H_1) and (H_2) hold. In addition,*

$$
r(L_1) \geq 1 > r(L_2) > 0.
$$

Then FBVP (1.1)–(1.2) has at least one positive solution.

Proof It follows from (H_1) that, for any $u \in \partial B_{r_1} \cap Q$, we have

$$Au(t) = \int_0^1 G(t,s)f\big(s,u(s)\big)\,ds \geq \int_0^1 G(t,s)b_1(s)u(s)\,ds = L_1u(t).$$

We may suppose that A has no fixed points on $\partial B_{r_1} \cap Q$ (otherwise, the proof is finished). Now we show that

$$u - Au \neq \mu\varphi_1, \quad \forall u \in \partial B_{r_1} \cap Q, \mu \geq 0, \tag{3.1}$$

here φ_1 is the positive eigenfunction corresponding to the first eigenvalue of L_1, that is, $L_1\varphi_1 = r(L_1)\varphi_1$. If otherwise, there exist $u_1 \in \partial B_{r_1} \cap Q$ and $\mu_0 > 0$ such that

$$u_1 - Au_1 = \mu_0\varphi_1,$$

which implies

$$u_1 = Au_1 + \mu_0\varphi_1 \geq \mu_0\varphi_1.$$

Denote

$$\mu^* = \sup\{\mu : u_1 \geq \mu\varphi_1\}.$$

It is clear that $\mu^* \geq \mu_0$ and $u_1 \geq \mu^*\varphi_1$. Notice that L_1 is nondecreasing, we have $L_1u_1 \geq \mu^*L_1\varphi_1 = \mu^*r(L_1)\varphi_1 \geq \mu^*\varphi_1$. Then

$$u_1 = Au_1 + \mu_0\varphi_1 \geq L_1u_1 + \mu_0\varphi_1 \geq \big(\mu^* + \mu_0\big)\varphi_1,$$

which contradicts the definition of μ^*. Hence (3.1) holds and we have from Lemma 2.9 that

$$i(A, B_{r_1} \cap Q, Q) = 0. \tag{3.2}$$

Set

$$W = \{u \in Q \backslash B_{r_1} \mid u = \mu Au, 0 \leq \mu \leq 1\}.$$

In the following, we will prove that W is bounded.

For any $u \in W$, we have

$$f\big(t,u(t)\big) \leq b_2(t)u(t) + f\big(t,\tilde{u}(t)\big),$$

where $\tilde{u}(t) = \min\{u(t), r_2\}$. It is clear that $\beta r_1 t^{\alpha-1} \leq \tilde{u}(t) \leq r_2$. Then

$$u(t) = \mu Au(t) \leq Au(t) \leq L_2u(t) + A\tilde{u}(t) \leq L_2u(t) + M,$$

where

$$M=\int_0^1 \Phi_2(s)\Psi_{r_1,r_2}(s)\,ds.$$

Thus

$$(I-L_2)u(t)\le M,\quad t\in[0,1].$$

It follows from $r(L_2)<1$ that the inverse operator of $(I-L_2)$ exists and

$$(I-L_2)^{-1}=I+L_2+L_2^2+\cdots+L_2^n+\cdots.$$

So, $u(t)\le (I-L_2)^{-1}M\le M\|(I-L_2)^{-1}\|$, $t\in[0,1]$, which implies W is bounded.

Select $R>\max\{r_2, M\|(I-L_2)^{-1}\|\}$. Then, by Lemma 2.10, we have

$$i(A,B_R\cap Q,Q)=1. \tag{3.3}$$

By (3.2) and (3.3) we have that

$$i\bigl(A,(B_R\backslash\bar{B}_{r_1})\cap Q,Q\bigr)=i(A,B_R\cap Q,Q)-i(A,B_{r_1}\cap Q,Q)=1,$$

which implies that A has at least one fixed point on $(B_R\backslash\bar{B}_{r_1})\cap Q$. This means that FBVP (1.1)–(1.2) has at least one positive solution. □

Theorem 3.2 *Assume that there exist $r_4>r_3>0$ such that (H_3) and (H_4) hold. In addition,*

$$r(L_4)>1\ge r(L_3)>0.$$

Then FBVP (1.1)–(1.2) has at least one positive solution.

Proof We may suppose that A has no fixed points on $\partial B_{r_3}\cap Q$ (otherwise, the proof is finished). In the following, we prove that

$$Au\ne\mu u,\quad \forall u\in\partial B_{r_3}\cap Q, \mu>1.$$

If otherwise, there exists $u_1\in\partial B_{r_3}\cap Q$, $\mu_0>1$ such that $Au_1=\mu_0u_1$. It follows from (H_3) that

$$\mu_0u_1=Au_1\le L_3u_1.$$

Noticing L_3 is nondecreasing, we get

$$\mu_0^2u_1\le\mu_0L_3u_1\le L_3^2u_1.$$

By induction, one has

$$\mu_0^nu_1\le L_3^nu_1,$$

which implies

$$\left\|\mu_0^n u_1\right\| \leq \left\|L_3^n u_1\right\| \leq \left\|L_3^n\right\| \|u_1\|.$$

Then

$$r(L_3) = \lim_{n\to+\infty} \sqrt[n]{\left\|L_3^n\right\|} \geq \mu_0 > 1,$$

this contradicts $r(L_3) \leq 1$. We have from Lemma 2.10 that

$$i(A, B_{r_3} \cap Q, Q) = 1. \tag{3.4}$$

On the other hand, by Lemma 2.8, we can select m large enough such that

$$r(T_m) > 1.$$

Let $R_m = r_4(\beta a_m^{\alpha-1})^{-1}$. Then, for any $u \in \partial B_{R_m} \cap Q$, one has

$$u(t) \geq \beta\|u\|t^{\alpha-1} \geq r_4, \quad t \in [a_m, 1], \tag{3.5}$$

where T_m, a_m are defined by (2.9). By virtue of the Krein–Rutmann theorem, we have that there exists a positive eigenfunction ψ_m corresponding to the first eigenvalue of T_m, that is, $T_m\psi_m = r(T_m)\psi_m$.

For $u \in \partial B_{R_m} \cap Q$. It follows from (H_4) and (3.5) that

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)f\big(s,u(s)\big)\,ds \geq \int_{a_m}^1 G(t,s)f\big(s,u(s)\big)\,ds \\ &\geq \int_{a_m}^1 G(t,s)b_4(s)u(s)\,ds = (T_m u)(t), \quad t \in [0,1]. \end{aligned}$$

We may suppose that A has no fixed points on $\partial B_{R_m} \cap Q$ (otherwise, the proof is finished). Now we will prove that

$$u - Au \neq \mu\psi_m, \quad \forall u \in \partial B_{R_m} \cap Q, \mu > 0. \tag{3.6}$$

If otherwise, there exist $u_1 \in \partial B_{R_m} \cap Q$ and $\mu_0 > 0$ such that

$$u_1 - Au_1 = \mu_0\psi_m.$$

Denote

$$\mu^* = \sup\{\mu : u_1 \geq \mu\psi_m\}.$$

It is clear that $\mu^* \geq \mu_0$ and $u_1 \geq \mu^*\psi_m$. Then

$$\begin{aligned} u_1 &= Au_1 + \mu_0\psi_m \geq T_m u_1 + \mu_0\psi_m \\ &\geq \mu^* T_m\psi_m + \mu_0\psi_m = \mu^* r(T_m)\psi_m + \mu_0\psi_m \\ &\geq \big(\mu^* + \mu_0\big)\varphi_m, \end{aligned}$$

which contradicts the definition of μ^*. Hence (3.6) holds, and we have from Lemma 2.9 that

$$i(A, B_{R_m} \cap Q, Q) = 0. \tag{3.7}$$

Equations (3.4) and (3.7) yield

$$i\big(A, (B_{R_m} \backslash \bar{B}_{r_3}) \cap Q, Q\big) = i(A, B_{R_m} \cap Q, Q) - i(A, B_{r_3} \cap Q, Q) = -1,$$

which implies that FBVP (1.1)–(1.2) has at least one positive solution on $(B_{R_m} \backslash \bar{B}_{r_3}) \cap Q$. □

Theorem 3.3 *Assume that there exist* $r_4 > r_5 > r_1 > 0$ *such that* (H_1), (H_4) *and*

(H_5) *There exist* $r_5 > 0$ *and a nonnegative function* $b_5 \in L^1[0,1]$ *such that*

$$f(t,x) \leq b_5(t) r_5, \quad \forall (t,x) \in (0,1) \times \big[\beta r_1 t^{\alpha-1}, r_5\big]$$

hold. Moreover, $r(L_1) \geq 1$, $r(L_4) > 1$, *and* $\|L_5\| < 1$. *Then FBVP* (1.1)–(1.2) *has at least two positive solutions* u_1 *and* u_2 *with* $r_1 < \|u_1\| < r_5 < \|u_2\|$.

Proof For any $u \in \partial B_{r_5} \cap Q$, we will prove that

$$Au \neq \lambda u, \quad \forall \lambda \geq 1.$$

If otherwise, there exist $u_1 \in \partial B_{r_5} \cap Q$ and $\lambda_0 \geq 1$ such that $Au_1 = \lambda_0 u_1$. Then we have

$$\lambda_0 u_1 = Au_1 = \int_0^1 G(t,s) f\big(s, u_1(s)\big)\, ds \leq \int_0^1 G(t,s) r_5 b_5(s)\, ds \leq \|L_5\| r_5 < r_5,$$

which implies that $\|u_1\| < r_5$, this contradicts $u_1 \in \partial B_{r_5} \cap Q$. Then, by Lemma 2.10, we have

$$i(A, B_{r_5} \cap Q, Q) = 1. \tag{3.8}$$

By the proof of Theorem 3.1 and Theorem 3.2, we have that (3.2) and (3.7) hold. Combining with (3.8), we have

$$i\big(A, (B_{R_m} \backslash \bar{B}_{r_5}) \cap Q, Q\big) = -1,$$
$$i\big(A, (B_{r_5} \backslash \bar{B}_{r_1}) \cap Q, Q\big) = 1,$$

which implies that FBVP (1.1)–(1.2) has at least two positive solutions u_1 and u_2 with $r_1 < \|u_1\| < r_5 < \|u_2\|$. □

Theorem 3.4 *Assume that there exist* $r_2 > r_6 > r_3 > 0$ *such that* (H_2), (H_3) *and*

(H_6) *There exist* $r_6 > 0$, $\rho \in (0,1)$, *and a nonnegative function* $b_6 \in L^1[0,1]$ *such that*

$$f(t,x) \geq b_6(t) r_6, \quad \forall (t,x) \in [\rho, 1] \times \big[\beta \rho^{\alpha-1} r_6, r_6\big]$$

hold. Moreover, $r(L_2) < 1$, $r(L_3) \leq 1$, *and*

$$\int_\rho^1 \Phi_2(s)b_6(s)\,ds > \beta^{-1}.$$

Then FBVP (1.1)–(1.2) *has at least two positive solutions* u_1 *and* u_2 *with* $r_3 < \|u_1\| < r_6 < \|u_2\|$.

Proof For any $u \in \partial B_{r_6} \cap Q$, we have $u(t) \geq \beta t^{\alpha-1} r_6 \geq \beta\rho^{\alpha-1} r_6$, $\forall t \in [\rho, 1]$. Then

$$\begin{aligned}\|Au\| &\geq \max_{t\in[0,1]} \beta t^{\alpha-1}\int_\rho^1 \Phi_2(s) f\bigl(s, u(s)\bigr)\,ds \\ &= \beta \int_\rho^1 \Phi_2(s) f\bigl(s, u(s)\bigr)\,ds \\ &\geq \beta r_6 \int_\rho^1 \Phi_2(s) b_6(s)\,ds > \|u\|.\end{aligned}$$

Then, for any $u_0 > \theta$, we have

$$u - Au \neq \lambda u_0, \quad \forall\lambda \geq 0, u \in \partial B_R \cap Q.$$

It follows from Lemma 2.9 that

$$i(A, B_{r_6} \cap Q, Q) = 0. \tag{3.9}$$

By (H_2) and (H_3), similar to the proof of Theorem 3.1 and Theorem 3.2, we can choose $r_3 < r_6 < r_2 < R$ such that (3.3) and (3.4) hold. Combining with (3.9), we have

$$\begin{aligned}&i\bigl(A, (B_R \backslash \bar{B}_{r_6}) \cap Q, Q\bigr) = 1, \\ &i\bigl(A, (B_{r_6} \backslash \bar{B}_{r_3}) \cap Q, Q\bigr) = -1,\end{aligned}$$

which implies that FBVP (1.1)–(1.2) has at least two positive solutions u_1 and u_2 with $r_3 < \|u_1\| < r_6 < \|u_2\|$. This completes the proof. □

4 Example

Example 4.1 Consider the following singular boundary value problem:

$$\begin{cases} D_{0+}^{\frac{7}{2}} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & D_{0+}^{\frac{1}{2}} u(1) = \int_0^1 a(t) D_{0+}^{\frac{3}{2}} u(t)\,dV(t), \end{cases} \tag{4.1}$$

where

$$\begin{aligned}&a(t) \equiv 0.95, \\ &V(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq t \leq 1, \end{cases}\end{aligned}$$

$$f(t,x)=\begin{cases}120(t-\frac{1}{8})^2t^{-\frac{1}{2}}x^{-\frac{1}{6}}+(t-\frac{1}{4})^2(1-t)^{-\frac{1}{2}}x, & (0,1)\times(0,1],\\ [120(t-\frac{1}{8})^2t^{-\frac{1}{2}}+(t-\frac{1}{4})^2(1-t)^{-\frac{1}{2}}]\cos^2(\frac{3\pi x-3\pi}{13{,}120}), & (0,1)\times(1,6561],\\ \frac{40}{27}(t-\frac{1}{8})^2t^{-\frac{1}{2}}x^{\frac{1}{2}}+\frac{1}{9}(t-\frac{1}{4})^2(1-t)^{-\frac{1}{2}}x^{\frac{1}{4}}, & (0,1)\times(6561,+\infty).\end{cases}$$

It is clear that

$$G_1(t,s)=\frac{8}{15\sqrt{\pi}}\begin{cases}t^{\frac{5}{2}}(1-s)^2, & 0\le t\le s\le 1,\\ t^{\frac{5}{2}}(1-s)^2-(t-s)^{\frac{5}{2}}, & 0\le s\le t\le 1,\end{cases}$$

$$G_2(t,s)=\frac{8}{15\sqrt{\pi}}\begin{cases}t(1-s)^2, & 0\le t\le s\le 1,\\ t(1-s)^2-(t-s), & 0\le s\le t\le 1,\end{cases}$$

$$h(s)=\frac{76}{15\sqrt{\pi}}\begin{cases}(1-s)^2, & \frac{1}{2}\le s\le 1,\\ s^2, & 0\le s<\frac{1}{2},\end{cases}$$

$$\Phi_1(s)=\frac{8(1-s)^2}{15\sqrt{\pi}}+h(s),$$

$$\Phi_2(s)=\frac{8s(1-s)^2}{15\sqrt{\pi}}+h(s),$$

$$G(t,s)=G_1(t,s)+h(s)t^{\frac{5}{2}},$$

$$\Delta:=\Gamma(2)-\Gamma(3)\int_0^1 a(t)t\,dV(t)=0.05.$$

Denote

$$b_1(t)=120\left(t-\frac{1}{8}\right)^2t^{-\frac{1}{2}}+\left(t-\frac{1}{4}\right)^2(1-t)^{-\frac{1}{2}},\quad t\in(0,1),$$

$$b_2(t)=\frac{b_1(t)}{9}.$$

It is clear that (A_1), (A_2), (A_3), (H_1), and (H_2) hold.

Define operators L_1 and L_2 as follows:

$$L_1u(t)=\int_0^1 G(t,s)b_1(s)u(s)\,ds,$$

$$L_2u(t)=\int_0^1 G(t,s)b_2(s)u(s)\,ds=\frac{1}{9}L_1u(t).$$

Denote

$$I(t)\equiv 1,\qquad e(t)=t^{\frac{5}{2}},\quad t\in[0,1].$$

By Lemma 2.5, we have

$$(L_1e)(t)=\int_0^1 G(t,s)b_1(s)s^{\frac{5}{2}}\,ds\ge\left[\frac{1}{2}\int_0^1\Phi_2(s)b_1(s)s^{\frac{5}{2}}\,ds\right]e(t).$$

Then we can obtain

$$\left(L_1^n e\right)(t) = L_1\left(L_1^{n-1} e\right)(t) \geq \left[\frac{1}{2}\int_0^1 \Phi_2(s) b_1(s) s^{\frac{5}{2}}\,ds\right]^n e(t),$$

which implies that

$$r(L_1) \geq \frac{1}{2}\int_0^1 \Phi_2(s) b_1(s) s^{\frac{5}{2}}\,ds.$$

Notice that

$$\begin{aligned}
\int_0^1 \Phi_2(s) b_1(s) s^{\frac{5}{2}}\,ds &> \int_0^1 120\left(s-\frac{1}{8}\right)^2 s^{-\frac{1}{2}} \Phi_2(s) s^{\frac{5}{2}}\,ds \\
&= 120\int_0^1 \left(s-\frac{1}{8}\right)^2 s^2\left[\frac{8s(1-s)^2}{15\sqrt{\pi}} + h(s)\right] ds \\
&= \frac{64}{\sqrt{\pi}}\int_0^1 \left(s-\frac{1}{8}\right)^2 s^3(1-s)^2\,ds \\
&\quad + \frac{608}{\sqrt{\pi}}\left[\int_0^{\frac{1}{2}} \left(s-\frac{1}{8}\right)^2 s^4\,ds + \int_{\frac{1}{2}}^1 \left(s-\frac{1}{8}\right)^2 s^2(1-s)^2\,ds\right] \\
&\approx 0.13836 + 1.87114 = 2.0095.
\end{aligned}$$

Therefore

$$r(L_1) > 1.$$

On the other hand,

$$\begin{aligned}
(L_1 I)(t) &= \int_0^1 G(t,s) b_1(s)\,ds \leq \int_0^1 \Phi_2(s) b_1(s)\,ds \\
&= \frac{8}{15\sqrt{\pi}}\int_0^1 s(1-s)^2 b_1(s)\,ds + \frac{76}{15\sqrt{\pi}}\int_0^1 s^2 b_1(s)\,ds \\
&\quad - \frac{76}{15\sqrt{\pi}}\int_{\frac{1}{2}}^1 (2s-1) b_1(s)\,ds \\
&< \frac{8}{15\sqrt{\pi}}\int_0^1 s(1-s)^2 b_1(s)\,ds + \frac{76}{15\sqrt{\pi}}\int_0^1 s^2 b_1(s)\,ds \\
&\quad - \frac{76}{15\sqrt{\pi}}\int_{\frac{1}{2}}^1 (2s-1)\times 120\left(s-\frac{1}{8}\right)^2 s^{-\frac{1}{2}}\,ds \\
&\approx 8.3.
\end{aligned}$$

From

$$L_2 u(t) = \frac{1}{9} L_1 u(t),$$

we have

$$r(L_2) = \frac{1}{9} r(L_1) \leq \frac{1}{9}\|L_1\| = \frac{1}{9}(L_1 I)(t) < \frac{8.3}{9} < 1.$$

Then

$$0 < \frac{1}{9} < r(L_2) < 1 < r(L_1).$$

By Theorem 3.1, we know that FBVP (4.1) has at least one positive solution.

Remark 4.1 It is clear that

$$\liminf_{x\to 0+} \min_{t\in(0,1)} \frac{f(t,x)}{x} = 0; \qquad \limsup_{x\to +\infty} \max_{t\in(0,1)} \frac{f(t,x)}{x} = +\infty;$$
$$\limsup_{x\to 0+} \max_{t\in(0,1)} \frac{f(t,x)}{x} = +\infty; \qquad \liminf_{x\to +\infty} \min_{t\in(0,1)} \frac{f(t,x)}{x} = 0,$$

which implies that neither (C_1) nor (C_2) holds.

5 Conclusions

In this paper, we consider the existence of positive solution for fractional differential equations with conjugate type integral conditions. Both the existence and multiplicity of positive solutions are considered. The interesting point lies in that the nonlinearity $f(t,x)$ may be singular at $t = 0, 1$ and $x = 0$, and the existence results are closely associated with the relationship between 1 and the spectral radii corresponding to the relevant linear operators.

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Abbreviations
FDE, Fractional differential equations; FBVP, Fractional differential equations boundary value problems; BC, Boundary value conditions.

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