RESEARCH

Open Access

Riemann–Hilbert problems with shift on the Lyapunov curve for null-solutions of iterated Beltrami equations

Guoan Guo^{1*} , Chenkai Jin¹, Pei Dang² and Zhihua Du³

*Correspondence:

guoguoan@njupt.edu.cn

¹School of Science, Nanjing

University of Posts and

Telecommunications, Nanjing,

P.R. China

Full list of author information is
available at the end of the article

Abstract

In this article, we study some Riemann–Hilbert problems with shift on the Lyapunov curve for generalized polyanalytic functions, which are null-solutions of a class of iterated Beltrami equations. Firstly, we obtain two integral representations of these functions by using Cauchy formula associated with the Beltrami equations and explicitly constructing two weakly singular kernels. Next, we develop a theory of matrix factorization for triangular matrix functions with shift in the frame of Beltrami equations and solve the Riemann–Hilbert problems by using them and the decomposition theorem for the iterated Beltrami equations. Finally, we get explicit formulae of solutions and conditions of solvability for the Riemann–Hilbert problems.

Keywords: Riemann–Hilbert problem; Polynomial Beltrami equation; Integral representation; Canonical matrix; Triangular matrix

1 Introduction

As an extension of the classical Riemann boundary value problems (for short, Riemann BVPs) [1–4], Riemann–Hilbert BVPs with shift for analytic functions (also called Haseman BVPs) firstly arose in the works of Hilbert and Haseman. Subsequently, since these BVPs have a wide range of applications, such as singular integral equations, operator theory, partial differential equations (PDEs), shell theory, fluid dynamics, elasticity theory, computational mechanics, and so on, they were extensively studied by many scholars [1, 5–9]. In recent years, these problems (in particular, Riemann BVPs) have been generalized to ones for some different classes of functions like Cauchy-type integrals with densities from variable exponent Lebesgue space [10], polynogenic functions in Clifford analysis [11], polyanalytic functions [12–14], polyharmonic functions [15–17], and even the families defined by other PDEs (especially polynomial Cauchy–Riemann equations) on some different curves with some suitable boundary value conditions [18–22].

Beltrami equations, as a remarkable generalization of the classical Cauchy–Riemann equations, play an important role in analysis, geometry, dynamic systems, and so on [5, 23, 24]. Consequently, the theory of Riemann–Hilbert problems for solutions of Beltrami equations (the latter are also called generalized analytic functions) was extensively developed even in some abstract settings [4, 5, 8, 25]. In recent years, Blaya and his collaborators studied some jump Riemann BVPs for solutions of a special Beltrami equation with good

properties, which are called β -analytic functions (see [26, 27]) and were firstly proposed by Tungatarov [26]. Afterwards, some variations of these BVPs and related problems aroused a great deal of interest in some fields such as harmonic analysis, infinitesimal geometry, and fractal theory [27–33]. More recently, Kats made a deep study on jump Riemann BVPs for a wider class on non-rectifiable curves in [30, 32]. They gave explicit solutions and conditions of solvability by introducing integration over non-rectifiable paths and establishing the corresponding Cauchy–Pompeiu formula.

Moreover, for some reasons of theory and practical applications, there was a great deal of research on Haseman and Riemann BVPs for polyanalytic functions and metaanalytic functions on the unit circle. For example, the Haseman BVPs, mixed or single periodic BVPs for bi-analytic, and metaanalytic functions, were intensively investigated by means of techniques using decomposition theorem, formal canonical matrix, and transformation [18, 19, 34–37]. More recently, using matrix transformation method based on analytic extension on the unit circle, Riemann BVPs for Hardy functions class defined by null-solutions of polynomial Cauchy–Riemann equations were discussed by Ku, He, and Wang in [22]. However, these methods in [18, 19, 22], which avoid constructing canonical matrix, are not applicable to BVPs for general curves without extension techniques. At a later time, Wang also discussed a Riemann BVP for automorphic polyanalytic functions connected with a rotation group from a view of applications in [38]. In fact, under suitable assumption, this problem can be regarded as a special Haseman problem for polyanalytic functions with a linear fractional shift (which is called a rotation shift there).

Combining Riemann BVP for β -analytic functions with Haseman problem for polyanalytic functions, it is a natural idea to discuss Riemann–Hilbert problem with shift on the general Lyapunov curve for certain iterated Beltrami equations in a wider context. Furthermore, the study of this type of BVPs will not only be helpful to study some other types of BVPs or integral equations with shift in a broader sense, but also provide a valuable tool to solve BVPs for some linear and nonlinear PDEs [9]. In addition, we find that this research can enrich the theoretical development of matrix functions factorization applied to the fields of mathematics and physics in the future. However, to our knowledge, very little information can be found about the theory of Riemann–Hilbert problems with shift for null-solutions of higher order iterated Beltrami equations (i.e., the generalized β -analytic functions of higher order) or β -analytic functions on the general Lyapunov curve. The main obstacles lie in two aspects: One is that integral representations of functions in this wider class have not been explicitly constructed due to the complexity of constructing weakly singular kernels. The other is that it is difficult to explicitly construct canonical matrix of coefficient matrix in the context of β -analytic functions because of lacking β -analytic extension for general Lyapunov curve instead of unit circle and thus failing to construct a similar transformation introduced in [18, 19, 22]. Motivated by these considerations, the objective of this paper is to bring together two types of BVPs, including Haseman BVPs for polyanalytic functions and Riemann BVPs for some polynomial Beltrami equations, and develop a theory of Riemann–Hilbert problems with shift on the general Lyapunov curves for generalized β -analytic functions.

The main idea of our approach is to construct two weakly singular kernels and derive integral representations of sectionally generalized β -analytic functions. In this way, we can construct a canonical function of the scalar function, and thus obtain the solution of Riemann–Hilbert problem with shift for those generalized β -analytic functions

with one order. Furthermore, we develop a theory of the novel canonical matrix in the frame of β -analytic matrix functions. Finally, by decomposition theorem for solutions of higher order iterated Beltrami equations, we transform the original problems to the vector-valued Riemann–Hilbert problems with shift for β -analytic functions, and further construct canonical matrix of triangular matrix functions explicitly, and thus obtain the explicit solutions and conditions of solvability. The main results extend those of [18, 19, 22, 27, 29, 30, 32, 34–38].

This type of BVP with shift on the non-rectifiable curve is also an interesting research subject, but it needs to construct a more complicated approximation kernel, introduce a new integral tool and develop a new matrix factorization theory. Therefore, this topic exceeds the scope of this paper.

The outline of the paper is as follows. In Sect. 2, we recall some necessary facts about generalized β -analytic functions, introduce the β -analytic functions class, and establish decomposition theorem on this class. In Sect. 3, by invoking some compact integral operators related to Cauchy integral operator with shift, we construct weakly singular kernels, and thus derive two different integral representations for sectional solutions of some Beltrami equations, which are used to solve Riemann–Hilbert problems with shift for iterated Beltrami equations on the Lyapunov curve. In the last section, the canonical factorization of β -analytic matrix functions is introduced, and the approach of constructing explicit canonical matrix with respect to triangular matrix functions is also discussed in detail. As a consequence, we finally give the explicit solutions and the conditions of solvability for Riemann–Hilbert problems with shift for iterated Beltrami equations.

2 Generalized β -analytic functions

In this section, we recall some basic notions, notations, and facts in [30, 32] which will be used in the sequel, and establish a decomposition theorem for a class of higher order iterated Beltrami equations.

Let $L = \{t \in \mathbb{C} : t = t(s), 0 \leq s \leq d\}$ be a simple closed Lyapunov curve oriented counter-clockwise, where s is the arc-length parameter. It divides the whole complex plane into two domains: a bounded simply connected domain $D^+ \ni 0$ and $D^- = \mathbb{C} \setminus \overline{D^+}$. Let $H^\mu(L)$ be the Banach space consisting of all Hölder continuous functions with exponent μ ($0 < \mu \leq 1$) equipped with the norm

$$\|\phi\|_{H^\mu} = \max_{t \in L} |\phi(t)| + h(\phi) \quad \text{with } h(\phi) = \sup_{\tau, t \in L} \frac{|\phi(\tau) - \phi(t)|}{|\tau - t|^\mu} < +\infty. \quad (2.1)$$

Throughout this paper, we always assume that $a(s) \in H^\mu(L)$, where $a(s)$ is the angle between the tangent line at $t = t(s)$ on L and the positive real axis. Let Ω be an open set such that $L \subset \overline{\Omega}$, and f be a fixed analytic function in a domain Δ satisfying $\inf_{t \in L} |f(t)| > 0$ and $\overline{\Omega} \subset \Delta$. Denote Z to be the set of all zeros of f and f' . A complex-valued function $\Phi \in C^1(\Omega)$ is called a generalized β -analytic function in Ω if it satisfies the following Beltrami equation:

$$\partial_{\bar{z}} \Phi = \beta \frac{f \overline{f'}}{f f'} \partial_z \Phi, \quad 0 \leq \beta < 1, \quad (2.2)$$

on $\Omega \setminus Z$, where $z = x + iy$ with $x = \operatorname{Re} z$, $y = \operatorname{Im} z$

$$\partial_{\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \partial_z = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right). \quad (2.3)$$

The class of all such functions in Ω is denoted by $H_f^\beta(\Omega)$. In particular, when $f(z) = z$, denote $H_f^\beta(\Omega)$ by $H_1^\beta(\Omega)$ for simplicity.

In [30, 32], when $\Omega = D^+$, the Cauchy formula for $\phi \in H_f^\beta(D^+) \cap C(\overline{D^+})$, viz.,

$$\phi(z) = \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\phi(t)f'(t) dt}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} + \frac{\beta\phi(t)f(t)\overline{f(t)}^{-1}\overline{f'(t)} d\bar{t}}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} \right], \quad (2.4)$$

holds for $z \in D^+$ with $\theta = \beta/(1-\beta)$. Obviously, one can easily get that

$$0 = \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\phi(t)f'(t) dt}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} + \frac{\beta\phi(t)f(t)\overline{f(t)}^{-1}\overline{f'(t)} d\bar{t}}{\overline{f(t)}(f(t) - f(z)|f(z)/f(t)|^{2\theta})} \right] \quad (2.5)$$

for $\phi \in H_f^\beta(D^+) \cap C(\overline{D^+})$ and $z \in D^-$.

By improving the properties of f stated in (2.2), we get a new integral representation of a generalized β -analytic function in an unbounded domain.

Theorem 2.1 *Let f be a univalent function on the extended complex plane $\overline{\mathbb{C}}$ satisfying $f(\infty) = \infty$ and $\inf_{t \in L} |f(t)| > 0$. If $\phi \in H_f^\beta(\overline{\mathbb{C}} \setminus \overline{D^+}) \cap C(\overline{\mathbb{C}} \setminus D^+)$, then*

$$\frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\phi(t)f'(t) dt}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} + \frac{\beta\phi(t)f(t)\overline{f(t)}^{-1}\overline{f'(t)} d\bar{t}}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} \right] = \phi(\infty) \quad (2.6)$$

for $z \in D^+$, and

$$\frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\phi(t)f'(t) dt}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} + \frac{\beta\phi(t)f(t)\overline{f(t)}^{-1}\overline{f'(t)} d\bar{t}}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} \right] = \phi(\infty) - \phi(z) \quad (2.7)$$

for $z \in D^-$.

Proof Here we only prove (2.7) since it is similar to (2.6). Since f is a univalent function mapping $\mathbb{C}$ to $\mathbb{C}$ and $f(\infty) = \infty$, then for any $z \in D^-$, there exists a sufficiently large positive constant R such that

$$z \in U_R = \{\zeta \in \mathbb{C} : |f(\zeta)| < R\} \supset D^+.$$

Thus, when $\phi \in H_f^\beta(\overline{\mathbb{C}} \setminus \overline{D^+}) \cap C(\overline{\mathbb{C}} \setminus D^+)$, by (2.4), we have

$$\frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\phi(t)f'(t) dt}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} + \frac{\beta\phi(t)f(t)\overline{f(t)}^{-1}\overline{f'(t)} d\bar{t}}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} \right] = A(R) - \phi(z) + \phi(\infty)$$

with

$$A(R) = \frac{1}{2\pi i(1-\beta)} \int_{\partial U_R} \left[\frac{(\phi(t) - \phi(\infty))f'(t) dt}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} + \frac{\beta(\phi(t) - \phi(\infty))f(t)\overline{f(t)}^{-1}\overline{f'(t)} d\bar{t}}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} \right].$$

Denote

$$\xi = f(z), \quad \omega = f(t), \quad \partial B_R = \{w : |w| = R\}.$$

By a change of variables, we have

$$\begin{aligned} |A(R)| &= \frac{1}{2\pi(1-\beta)} \left| \int_{\partial B_R} \frac{(\phi(f^{-1}(\omega)) - \phi(\infty)) d\omega}{\omega - \xi |\xi/\omega|^{2\theta}} + \int_{\partial B_R} \frac{\beta \omega (\phi(f^{-1}(\omega)) - \phi(\infty)) d\bar{\omega}}{\bar{\omega}(\omega - \xi |\xi/\omega|^{2\theta})} \right| \\ &\leq \frac{(1+\beta)|R|^{1+2\theta}}{(1-\beta)(|R|^{1+2\theta} - |\xi|^{1+2\theta})} \max_{|w|=R} |\phi(f^{-1}(\omega)) - \phi(\infty)|. \end{aligned}$$

Obviously, $\lim_{R \rightarrow +\infty} A(R) = 0$. So (2.7) follows immediately. $\square$

In what follows, f is always assumed to satisfy the condition of Theorem 2.1. In view of (2.4), define the Cauchy-type integral as follows:

$$(C_L^{f,\beta} \phi)(z) = \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\phi(t)f'(t) dt}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} + \frac{\beta \phi(t)f(t)\overline{(f(t))^{-1}f'(t)} d\bar{t}}{f(t) - f(z)|f(z)/f(t)|^{2\theta}} \right], \quad (2.8)$$

where $z \notin L$ and $\phi \in C(L)$. It is easy to check that $C_L^{f,\beta} \phi \in H_f^\beta(D^+ \cup D^-)$ and $(C_L^{f,\beta} \phi)(\infty) = 0$.

Let n be a positive integer. Set

$$\partial_z^{f,\beta} = \partial_z - \beta \frac{f\bar{f}'}{\bar{f}f'} \partial_{\bar{z}}, \quad \partial_{z_0}^{f,\beta} = I, \quad \partial_{z_n}^{f,\beta} = \underbrace{\partial_z^{f,\beta} \circ \partial_z^{f,\beta} \circ \cdots \circ \partial_z^{f,\beta}}_n, \quad (2.9)$$

where $\partial_{\bar{z}}$, ∂_z are given in (2.3) and I is the identity operator.

Definition 2.1 Let ϕ have continuous partial derivatives up to n in an open set $\Omega \subset \mathbb{C}$. If

$$(\partial_{z_n}^{f,\beta} \phi)(z) = 0, \quad \forall z \in \Omega, \quad (2.10)$$

then ϕ is said to be a generalized β -analytic function of order n in Ω , or simply a generalized β -analytic function. The collection of all generalized β -analytic functions in Ω is denoted by $H_{n,f}^\beta(\Omega)$.

It is easy to know that $H_{1,f}^\beta(\Omega) = H_f^\beta(\Omega)$ represents the family of generalized β -analytic functions in Ω [30, 32], and $H_{n,f}^0(\Omega) = H_n(\Omega) = \{\phi : \partial_z^n \phi(z) = 0, z \in \Omega\}$ is the class of polyanalytic functions of order n in Ω [12, 14]. Moreover, when $f(z) = z$, the class $H_{1,f}^\beta(\Omega)$ is just the one consisting of β -analytic functions in [26–29, 31, 33], and some higher order Cauchy–Pompeiu integral representations for these functions have been discussed in [39].

Next, we establish a decomposition theorem for generalized β -analytic functions.

Theorem 2.2 Let Ω be an open set in the complex plane, $\alpha_j \in H_f^\beta(\Omega)$ be a given function for every $j = 1, 2, \dots, n$, and f be defined as in Theorem 2.1. Then

$$H_{n,f}^\beta(\Omega) = \bigoplus_{j=1}^n (\bar{z} + \alpha_j(z))^{j-1} H_f^\beta(\Omega), \quad (2.11)$$

where $\bigoplus_{j=1}^n$ denotes a direct sum. More precisely, for any $U \in H_{n,f}^\beta(\Omega)$, there exists uniquely $V_j \in H_f^\beta(\Omega)$, $1 \leq j \leq n$, such that

$$U(z) = \sum_{j=1}^n (\bar{z} + \alpha_j(z))^{j-1} V_j(z), \quad z \in \Omega. \quad (2.12)$$

Proof By induction on n , it is sufficient to show that

$$H_{n,f}^\beta(\Omega) = H_{n-1,f}^\beta(\Omega) \oplus (\bar{z} + \alpha_n(z))^{n-1} H_f^\beta(\Omega). \quad (2.13)$$

Clearly, $(H_{n-1,f}^\beta(\Omega) \oplus (\bar{z} + \alpha_n(z))^{n-1} H_f^\beta(\Omega)) \subset H_{n,f}^\beta(\Omega)$.

On the other hand, for any $U \in H_{n,f}^\beta(\Omega)$, by applying the operator $\partial_{\bar{z}}^{f,\beta}$, one immediately gets

$$U(z) = U_1(z) + (\bar{z} + \alpha_n(z))^{n-1} U_2(z) \quad (2.14)$$

with

$$U_1(z) = U(z) - \frac{(\bar{z} + \alpha_n(z))^{n-1}}{(n-1)!} (\partial_{\bar{z}^{n-1}}^{f,\beta} U)(z) \in H_{n-1,f}^\beta(\Omega), \quad \text{and}$$

$$U_2(z) = \frac{(\partial_{\bar{z}^{n-1}}^{f,\beta} U)(z)}{(n-1)!} \in H_f^\beta(\Omega).$$

Moreover, for $U_1 \in H_{n-1,f}^\beta(\Omega)$ and $U_2 \in H_f^\beta(\Omega)$, if $U_1(z) + (\bar{z} + \alpha_n(z))^{n-1} U_2(z) = 0$, then it is easy to obtain that $U_1 = U_2 = 0$ by letting the operator $\partial_{\bar{z}^{n-1}}^{f,\beta}$ act on both sides of the above equality. $\square$

Remark 2.1 If $\beta = 0$ and all $\alpha_j = 0$ for $j = 1, 2, \dots, n$, then Theorem 2.2 is degenerated to the classical decomposition theorem for polyanalytic functions in [12, 14].

3 Integral representations for generalized β -analytic functions

In this section, we will discuss some properties of singular integral operators with shift, and give two integral representations for sectionally generalized β -analytic functions by weakly singular kernels (which are constructive) and the method of integral equations.

To begin with, we consider the following Cauchy singular integral operator:

$$(S_L^{f,\beta} \phi)(t) = \frac{1}{\pi i(1-\beta)} \int_L \frac{(\phi(\tau) - \phi(t))f'(\tau)}{f(\tau) - f(t)|f(t)f'(\tau)|^{2\theta}} \left[d\tau + \beta \frac{f(\tau)\overline{f'(\tau)}}{f(\tau)f'(\tau)} d\bar{\tau} \right] + \phi(t) \quad (3.1)$$

for $\phi \in H^\mu(L)$, $t \in L$, where the integral is understood in the sense of Cauchy principle value as in [27, 28] if it exists. The above integral can be rewritten as

$$(S_L^{f,\beta} \phi)(t) = 2 \int_L e_t(\tau) n_q(\tau) (\phi(\tau) - \phi(t)) ds + \phi(t), \quad t \in L, \quad (3.2)$$

with

$$e_t(\tau) = \frac{1}{2\pi i(1-\beta)} \cdot \frac{f'(\tau)}{f(\tau) - f(t)|f(t)f'(\tau)|^{2\theta}}, \quad n_q(\tau) = n(\tau) - \beta \frac{f(\tau)\overline{f'(\tau)}}{f(\tau)f'(\tau)} \bar{n}(\tau),$$

where $n(\tau)$ denotes the exterior unit normal vector to L at the point τ , and ds is the arc-length differential.

The relation between Cauchy singular integral operator and Cauchy-type integral is stated in the following theorem.

Theorem 3.1 *Let f be defined as Theorem 2.1 satisfying $f(0) = 0$. If $\phi \in H^\mu(L)$, then $(S_L^{f,\beta}\phi)(t)$ exists for all $t \in L$, and the Sokhotski–Plemelj formulas*

$$(C_L^{f,\beta}\phi)^\pm(t) = \frac{1}{2}((S_L^{f,\beta}\phi)(t) \pm \phi(t)), \quad t \in L \quad (3.3)$$

hold, where $(C_L^{f,\beta}\phi)^+(t)$ and $(C_L^{f,\beta}\phi)^-(t)$ denote the limit values of $(C_L^{f,\beta}\phi)(z)$ when z approaches t non-tangentially from D^+ and D^- respectively. Moreover, the Cauchy singular integral operator $S_L^{f,\beta}$ is bounded on $H^\mu(L)$, that is,

$$\|S_L^{f,\beta}\phi\|_{H^\mu(L)} \leq c\|\phi\|_{H^\mu(L)}, \quad \forall \phi \in H^\mu(L). \quad (3.4)$$

Proof Let S be any fixed, bounded, and simply-connected closed region ($0 \notin S$). We firstly show that

$$c_1|z_2 - z_1| \leq |f(z_2)|f(z_2)|^{2\theta} - f(z_1)|f(z_1)|^{2\theta}| \leq c_2|z_2 - z_1| \quad (3.5)$$

for all $z_1, z_2 \in S$, where the positive constants c_1, c_2 only depend on S and f .

In fact, since f is univalent on S , we have $\inf_{z \in S} |f'(z)| > 0$ and thus $\inf_{z \in S_1} |\frac{d}{dz}(f^{-1}(z))| > 0$, where f^{-1} and S_1 denote the inverse and range of f on S respectively. Therefore, noting that $\sup_{z \in S} |f'(z)| \in (0, +\infty)$, one has

$$c_3|z_2 - z_1| \leq |f(z_2) - f(z_1)| \leq c_4|z_2 - z_1|, \quad (3.6)$$

where c_3, c_4 are positive constants independent of the choice of z_1, z_2 . By virtue of (3.6) and mean value theorem, one further has

$$\begin{aligned} & |f(z_2)|f(z_2)|^{2\theta} - f(z_1)|f(z_1)|^{2\theta}| \\ &= 1/2 \left((f(z_2) - f(z_1))(|f(z_2)|^{2\theta} + |f(z_1)|^{2\theta}) \right. \\ &\quad \left. + (f(z_2) + f(z_1))(|f(z_2)|^{2\theta} - |f(z_1)|^{2\theta}) \right) \\ &\leq \left(\max_{z \in S} |f(z)|^{2\theta} + 2\theta \max_{z \in S} |f(z)| \cdot \max_{z \in S} |f(z)|^{2\theta-1} \right) \\ &\quad \times |f(z_2) - f(z_1)| \leq c_2|z_2 - z_1|, \end{aligned} \quad (3.7)$$

which completes the proof of the right-hand side in (3.5).

Note that the inverse of $\omega = f(z)|f(z)|^{2\theta}$ on S is

$$z = f^{-1}(\omega|\omega|^{-2\theta/(1+2\theta)}), \quad \omega \in S_2 = \{\omega : \omega = f(z)|f(z)|^{2\theta}, z \in S\}. \quad (3.8)$$

According to the assumption on f , S_2 is also a bounded closed region which excludes the origin. Let $z_k = f^{-1}(\omega_k|\omega_k|^{-2\theta/(1+2\theta)})$ with $\omega_k = f(z_k)|f(z_k)|^{2\theta} \in S_2$ for $k = 1, 2$. By a similar

argument to (3.7), it follows from (3.6) that

$$\begin{aligned}
 |z_2 - z_1| &\leq 1/(c_3) |f(z_2) - f(z_1)| \\
 &= 1/(2c_3) \left| (\omega_2 - \omega_1) (|\omega_2|^{-2\theta/(1+2\theta)} + |\omega_1|^{-2\theta/(1+2\theta)}) \right. \\
 &\quad \left. + (\omega_2 + \omega_1) (|\omega_2|^{-2\theta/(1+2\theta)} - |\omega_1|^{-2\theta/(1+2\theta)}) \right| \\
 &\leq 1/(c_3) \left(\max_{\omega \in S_2} |\omega|^{-2\theta/(1+2\theta)} + \frac{2\theta}{1+2\theta} \max_{\omega \in S_2} |\omega| \cdot \max_{z \in S_2} |\omega|^{(-4\theta-1)/(1+2\theta)} \right) \\
 &\quad \times |f(z_2)| |f(z_2)|^{2\theta} - f(z_1) |f(z_1)|^{2\theta}, \tag{3.9}
 \end{aligned}$$

which immediately completes the proof of the left-hand side in (3.5).

Secondly, by (3.5), (3.6), and a similar argument to Lemma 2.4.1 of [3], it is easy to get that $(S_L^{f,\beta} \phi)(t)$ exists for all $t \in L$.

Finally, it is well known that a simple closed Lyapunov curve must be a regular one. Let b be the diameter of L . For $\varepsilon \in (0, b/2)$, $L_\varepsilon(t) = \{\tau \in L : |\tau - t| \leq \varepsilon\}$, define

$$\begin{aligned}
 L_\varepsilon^{k,\beta}(\phi, t, z) &= (C_L^{f,\beta} \phi)(z) - (2-k)\phi(t) - \frac{1}{2\pi i(1-\beta)} \int_{L \setminus L_\varepsilon(t)} \left[\frac{\phi(\tau) f'(\tau) d\tau}{f(\tau) - f(t) |f(t)/f(\tau)|^{2\theta}} \right. \\
 &\quad \left. + \frac{\beta \phi(\tau) f(\tau) \overline{f'(\tau)} d\bar{\tau}}{\overline{f(\tau)} (f(\tau) - f(t) |f(t)/f(\tau)|^{2\theta})} \right], \quad \forall \phi \in H^\mu(L),
 \end{aligned}$$

where $t \in L$, $k = 1$ for $z \in D^+$ and $k = 2$ for $z \in D^-$.

In the case of Hölder continuous functions, $\varphi(\xi) = \xi^\mu$ is a majorant, and the corresponding continuous modulus $\omega_\phi(\xi)$ of ϕ in [27] is replaced with $h(\phi)\xi^\mu$, where $\xi \in (0, b]$, the constant $h(\phi)$ is defined by (2.1). Thus in this case $\omega_\phi(\xi) \leq \|\phi\|_{H^\mu(L)} \xi^\mu$. As a consequence, by applying (2.4), (2.5), (3.5), (3.6), and Theorem 2.1, it follows from a similar argument to Lemma 2 and Theorem 3 in [27] that

$$|L_\varepsilon^{k,\beta}(\phi, t, z)| \leq c \left(\frac{\varepsilon}{\text{dist}(z, L)} \varepsilon^\mu \|\phi\|_{H^\mu(L)} + \varepsilon \int_\varepsilon^b \frac{\|\phi\|_{H^\mu(L)}}{x^{2-\mu}} dx \right)$$

for $|z - t| = \varepsilon/2$, where notation c only depends on f, f' , and L . In what follows, c will be frequently used for constants which may vary from one occurrence to the next. Let z tend to t non-tangentially, then there exists a positive constant c_5 such that $|z - t| < c_5 \text{dist}(z, L)$, where c_5 depends on t . Thus from the last estimate we immediately prove the desired formula (3.3).

In view of the alternative definition (3.2) of $S_L^{f,\beta}$ and estimate (3.5), when $|t_1 - t_2| = 2\nu \leq b$ and $\phi \in H^\mu(L)$, then by adopting the same argument to Theorem 2 in [40] in the setting of general Hölder continuous Douglas algebra-valued functions with $f(z)$, $\|\phi\|_{H^\mu(L)}$ and $h(\phi)\xi^\mu$ in place of z , the norm of $\mathcal{H}_\varphi(\gamma)$ and $\omega_\phi(\xi)$ respectively, we get

$$\begin{aligned}
 &|(S_L^{f,\beta} \phi)(t_1) - (S_L^{f,\beta} \phi)(t_2)| \\
 &\leq c \left(\int_0^\nu \frac{\|\phi\|_{H^\mu(L)} \xi^\mu}{\xi} d\xi + \nu \int_\nu^b \frac{\|\phi\|_{H^\mu(L)} \xi^\mu}{\xi^2} d\xi + \|\phi\|_{H^\mu(L)} \nu^\mu \right) \\
 &\leq c \|\phi\|_{H^\mu(L)} \left[\frac{\nu^\mu}{\mu} + \frac{\nu^\mu}{1-\mu} + \nu^\mu \right] = c \|\phi\|_{H^\mu(L)} \nu^\mu,
 \end{aligned}$$

which implies that $S_L^{f,\beta}\phi \in H^\mu(L)$. Moreover, by chord-arc inequality (see [3]), it follows from (3.2) and (3.5) that

$$\begin{aligned} |(S_L^{f,\beta}\phi)(t)| &\leq (1+\beta) \frac{\sup_{\tau \in L} |f'(\tau)| |f(\tau)|^{2\theta}}{\pi(1-\beta)} \int_0^b \frac{\|\phi\|_{H^\mu(L)}}{|\tau-t|^{1-\mu}} ds + \|\phi\|_{H^\mu(L)} \\ &\leq c \|\phi\|_{H^\mu(L)}. \end{aligned}$$

Therefore, it follows that $S_L^{f,\beta}$ is bounded on $H^\mu(L)$. For much more details, we refer the reader to [27, 40]. $\square$

Next, in order to obtain integral representation formulas for generalized β -analytic functions, we study a nice property of singular integral operator with shift stated in Theorem 3.2 below.

Let α be a positive shift on L , more precisely, α is an orientation-preserving homeomorphism of L onto itself. Moreover, assume that $\alpha'(t) \neq 0$ on L and $\alpha' \in H^\mu(L)$. Define the shift operator as

$$(W\phi)(t) = \phi(\alpha(t)), \quad t \in L. \quad (3.10)$$

Obviously, W is a bounded and invertible operator on $H^\mu(L)$.

Theorem 3.2 *The operator $WS_L^{f,\beta}W^{-1} - S_L^{f,\beta}$ is a compact operator on $H^\lambda(L)$ for any $\lambda < \mu$.*

Proof It will be divided into two steps. The first one is to obtain the simplified kernel of the operator.

Let $B = WS_L^{f,\beta}W^{-1} - S_L^{f,\beta}$. For any $\varphi \in H^\lambda(L)$,

$$\begin{aligned} (B\varphi)(t) &= \frac{1}{\pi i(1-\beta)} \int_L \frac{\varphi(\hat{\alpha}(\tau))f'(\tau)}{f(\tau) - f(\alpha(t))|f(\alpha(t))/f(\tau)|^{2\theta}} \left[d\tau + \beta \frac{f(\tau)\overline{f'(\tau)}}{\overline{f(\tau)}f'(\tau)} d\bar{\tau} \right] \\ &\quad - \frac{1}{\pi i(1-\beta)} \int_L \frac{\varphi(\tau)f'(\tau)}{f(\tau) - f(t)|f(t)/f(\tau)|^{2\theta}} \left[d\tau + \beta \frac{f(\tau)\overline{f'(\tau)}}{\overline{f(\tau)}f'(\tau)} d\bar{\tau} \right] \\ &\triangleq \frac{1}{\pi i(1-\beta)} \int_L K_1(\tau, t) \varphi(\tau) d\tau \end{aligned}$$

with the kernel

$$\begin{aligned} K_1(\tau, t) &= \frac{f'(\alpha(\tau))\alpha'(\tau) + \beta f(\alpha(\tau))(\overline{f(\alpha(\tau))})^{-1} \overline{f'(\alpha(\tau))\alpha'(\tau)\tau'^2(s)}}{f(\alpha(\tau)) - f(\alpha(t))|f(\alpha(t))/f(\alpha(\tau))|^{2\theta}} \\ &\quad - \frac{|f(\tau)|^{2\theta}(f'(\tau) + \beta f(\tau)\overline{f'(\tau)})^{-1} \overline{f'(\tau)\tau'^2(s)}}{f(\tau)|f(\tau)|^{2\theta} - f(t)|f(t)|^{2\theta}}, \end{aligned} \quad (3.11)$$

where $\tau = \tau(s)$, s is the arc parameter and $\hat{\alpha}$ stands for the inverse function of α .

The second one is to prove that $K_1(\tau, t)$ is a weakly singular kernel by a transform which is constructive and some techniques concerning different inequalities. This step is crucial.

To do so, denote the curves

$$\hat{L} = \{\omega : \omega = f(\tau), \tau \in L\}, \quad \Gamma = \{\xi : \xi = f(\tau)|f(\tau)|^{2\theta}, \tau \in L\}. \quad (3.12)$$

Obviously, $\xi = \omega|\omega|^{2\theta}$ is a bijective map from $\hat{L}$ to Γ satisfying $\arg \xi = \arg \omega$. Furthermore, by conformal mapping of f , Γ is also a closed and anticlockwise curve. Let

$$\xi = f(\tau)|f(\tau)|^{2\theta}, \quad \tau \in L, \quad (3.13)$$

its inverse is $\tau = \tau(\xi) = f^{-1}(\xi|\xi|^{-2\theta/(1+2\theta)})$. Corresponding to transformation (3.13), we introduce another transform

$$\delta(\xi) = f(\alpha(\tau))|f(\alpha(\tau))|^{2\theta}, \quad \xi \in \Gamma. \quad (3.14)$$

Clearly, δ is a positive shift on Γ . Moreover, let $\eta = f(t)|f(t)|^{2\theta}$ and thus $\delta(\eta) = f(\alpha(t)) \times |f(\alpha(t))|^{2\theta}$.

Now we start to prove that $\delta'(\xi) \in H^\mu(\Gamma)$ and Γ is a Lyapunov curve.

Firstly, we show that Γ is a Lyapunov curve. Write that

$$f(\alpha(\tau)) = f(\alpha(\tau(s))) = f(\alpha(s)) = \mu(s) + i\nu(s) = f(s),$$

where $\tau = \tau(s)$ and s is the arc-length parameter. By the chain rule, one has

$$\begin{cases} f'(s) = f'(\alpha(\tau))\alpha'(\tau)\tau'(s), \\ 2\mu(s)\mu'(s) + 2\nu(s)\nu'(s) = f(s)\overline{f'(s)} + \overline{f(s)}f'(s). \end{cases}$$

Therefore,

$$\begin{aligned} \frac{d\delta}{d\tau} &= f'(\alpha(\tau))\alpha'(\tau)|f(\alpha(\tau))|^{2\theta} \\ &\quad + \theta f(\alpha(\tau))|f(\alpha(\tau))|^{2\theta-2} [2\mu(s)\mu'(s) + 2\nu(s)\nu'(s)] \frac{1}{\tau'(s)} \\ &= |f(\alpha(\tau))|^{2\theta} \\ &\quad \times [(1 + \theta)\alpha'(\tau)f'(\alpha(\tau)) + \overline{\theta\tau'^2(s)\alpha'(\tau)f'(\alpha(\tau))}f(\alpha(\tau))\overline{(f(\alpha(\tau)))}^{-1}]. \end{aligned} \quad (3.15)$$

Similarly,

$$\frac{d}{d\tau} [f(\tau)|f(\tau)|^{2\theta}] = |f(\tau)|^{2\theta} [(1 + \theta)f'(\tau) + \overline{\theta\tau'^2(s)f'(\tau)}f(\tau)\overline{(f(\tau))}^{-1}]. \quad (3.16)$$

Recall that $\xi = f(\tau)|f(\tau)|^{2\theta}$, and s, σ denote the arc-length parameters with respect to $\tau \in L, \xi \in \Gamma$ respectively. By an argument similar to the one for conformal gluing theorem in [7], it is easy to verify that $\sigma = \sigma(s) = \int_0^s |\xi'_\tau(\tau(s))| ds$, and thus $\sigma(s)$ is monotonically increasing and its inverse $s(\sigma)$ is also monotonically increasing. In addition, if $s_1 < s_2$, then

$$|\sigma_2 - \sigma_1| = |\sigma(s_2) - \sigma(s_1)| = \left| \int_{s_1}^{s_2} |\xi'_\tau(\tau(s))| ds \right| \geq \min_{s \in [0, d]} |\xi'_\tau(\tau(s))| |s_2 - s_1|.$$

By the assumption on f , it follows $f, f' \neq 0$ on L . Therefore, by (3.16), we have that $\min_{s \in [0, d]} |\xi'_\tau(\tau(s))| > 0$, which implies $s = s(\sigma) \in H^1([0, \hat{d}]) \subset H^\mu([0, \hat{d}])$, where $\hat{d}$ is the

length of Γ . This leads to $\tau(s(\sigma)), \tau'(s(\sigma)) \in H^\mu([0, \hat{d}])$ due to the fact that $|\tau'(s)| = 1$ and L is a Lyapunov curve. Moreover,

$$\frac{d\xi}{d\sigma} = \frac{d\xi}{d\tau} \frac{d\tau}{ds} \frac{ds}{d\sigma} = \frac{\xi'(\tau)\tau'(s)}{|\xi'(\tau)|} = \tau'(s)e^{i\arg \xi'(\tau)} \in H^\mu([0, \hat{d}]). \quad (3.17)$$

Hence Γ is also a Lyapunov curve.

Next, we show that $\delta'(\xi) \in H^\mu(\Gamma)$. For every $\tau, t \in L$, from (3.12), (3.13), and (3.9), it follows that there exist $\xi = f(\tau)|f(\tau)|^{2\theta} \in \Gamma$, $\eta = f(t)|f(t)|^{2\theta} \in \Gamma$ such that

$$|\tau(\xi) - \tau(\eta)| = |\tau - t| \leq C|f(\tau)|f(\tau)|^{2\theta} - f(t)|f(t)|^{2\theta}| = C|\xi - \eta|, \quad (3.18)$$

where the constant $C > 0$. This immediately leads to $\tau(\xi) \in H^1(\Gamma) \subset H^\mu(\Gamma)$. As mentioned above, when L is a Lyapunov curve, one has $\tau'(s) \in H^\mu([0, \hat{d}])$. Furthermore, combining (3.15) with (3.17), it follows that $\sigma'(\xi), \delta'(\tau(\xi)) \in H^\mu(\Gamma)$, and thus implies $\delta'(\xi) = \delta'(\tau(\xi))\tau'(\xi) \in H^\mu(\Gamma)$.

Finally, we prove that $K_1(\tau, t)$ defined by (3.11) is really a weakly singular kernel. In fact, using (3.15) together with (3.16), $K_1(\tau, t)$ can be rewritten as

$$K_1(\tau, t) = K_2(\tau, t)\hat{K}_1(\xi, \eta) \quad (3.19)$$

with

$$\begin{cases} K_2(\tau, t) = |f(\tau)|^{2\theta}[f'(\tau) + \beta\overline{\tau'^2(s)}f'(\tau)f(\tau)\overline{f(\tau)}^{-1}], \\ \hat{K}_1(\xi, \eta) = \frac{\delta'(\xi)}{\delta(\xi) - \delta(\eta)} - \frac{1}{\xi - \eta}. \end{cases}$$

Noting that $f, f' \neq 0$ on L again, it follows from (3.15) and (3.16) that

$$\inf_{\xi \in \Gamma} |\delta'(\xi)| = \inf_{\tau \in L} \left| \frac{\delta'(\tau)}{\xi'(\tau)} \right| > \frac{\inf_{\tau \in L} |\delta'(\tau)|}{\sup_{\tau \in L} |\xi'(\tau)|} > 0. \quad (3.20)$$

Therefore, by a well-known result in [3, 7], it follows from the fact that Γ is a Lyapunov curve, $\delta'(\xi) \in H^\mu(\Gamma)$, (3.18), and (3.20) that

$$\left| \frac{\delta'(\xi)}{\delta(\xi) - \delta(\eta)} - \frac{1}{\xi - \eta} \right| \leq \frac{M_4}{|\xi - \eta|^{1-\mu}} \leq \frac{M_5}{|\tau - t|^{1-\mu}} \quad (3.21)$$

with constants $M_4, M_5 > 0$. Moreover, by (3.19), $K_2(\tau, t)$ is a bounded function. Thus, the kernel $K_1(\tau, t)$ of the operator $WS_L^{f, \beta}W^{-1} - S_L^{f, \beta}$ is weakly singular. So $WS_L^{f, \beta}W^{-1} - S_L^{f, \beta}$ is a compact operator by some well-known results (for details, see [7, 41]). $\square$

Remark 3.1 When $\beta = 0$, Theorem 3.2 corresponds to Theorem 1 in [7]. Moreover, from the above argument, we see that the kernel $K_1(\tau, t)$ in (3.11) is weakly singular, that is,

$$|K_1(\tau, t)| \leq \frac{M}{|\tau - t|^{1-\mu}} \quad \text{with } 0 < \mu \leq 1.$$

Hence, the associated integral operator

$$(\mathbf{K}\phi)(t) = \phi(t) + \frac{1}{2\pi i(1 - \beta)} \int_L K_1(\tau, t)\phi(\tau) d\tau, \quad \phi \in H^\mu(L) \quad (3.22)$$

is a quasi-Fredholm one. Furthermore, for any $h \in H^\mu(L)$, Fredholm equation $(K\phi)(t) = h(t)$ has finite linearly independent solutions. (In fact, the solution is unique since the kernel $K_1(\tau, t)$ is weakly singular, see Theorem 3.3 below.)

With the above preliminaries, we can derive the integral representations for sectionally generalized β -analytic function with jump curve L , which is very crucial in solving the Riemann–Hilbert problem with shift in what follows.

Theorem 3.3 *Let α be a positive shift on L , $\alpha'(t) \neq 0$ for any $t \in L$ and $\alpha' \in H^\mu(L)$, and $\hat{\alpha}$ be the inverse function of α . Suppose that $\phi \in H_f^\beta(\overline{\mathbb{C}} \setminus L)$, its boundary values $\phi^\pm(t) \in H^\mu(L)$, and set $\phi^\pm(z) = \phi(z)|_{D^\pm}$, then there exist unique functions $\rho, \hat{\rho} \in H^\mu(L)$ such that*

$$\phi^+(z) = \frac{1}{2\pi i(1-\beta)} \int_L \frac{\hat{\alpha}'(\tau)\rho(\hat{\alpha}(\tau))f'(\tau)}{f(\tau)-f(z)|f(z)/f(\tau)|^{2\theta}} \left[d\tau + \beta \frac{f(\tau)\overline{f'(\tau)}}{\overline{f(\tau)}f'(\tau)} d\bar{\tau} \right] \quad (3.23)$$

and

$$\phi^-(z) = \frac{1}{2\pi i(1-\beta)} \int_L \frac{\rho(\tau)f'(\tau)}{f(\tau)-f(z)|f(z)/f(\tau)|^{2\theta}} \left[d\tau + \beta \frac{f(\tau)\overline{f'(\tau)}}{\overline{f(\tau)}f'(\tau)} d\bar{\tau} \right] + \phi^-(\infty), \quad (3.24)$$

as well as

$$\phi^+(z) = \frac{1}{2\pi i(1-\beta)} \int_L \frac{\hat{\rho}(\hat{\alpha}(\tau))f'(\tau)}{f(\tau)-f(z)|f(z)/f(\tau)|^{2\theta}} \left[d\tau + \beta \frac{f(\tau)\overline{f'(\tau)}}{\overline{f(\tau)}f'(\tau)} d\bar{\tau} \right] \quad (3.25)$$

and

$$\phi^-(z) = \frac{1}{2\pi i(1-\beta)} \int_L \frac{\hat{\rho}(\tau)f'(\tau)}{f(\tau)-f(z)|f(z)/f(\tau)|^{2\theta}} \left[d\tau + \beta \frac{f(\tau)\overline{f'(\tau)}}{\overline{f(\tau)}f'(\tau)} d\bar{\tau} \right] + \phi^-(\infty). \quad (3.26)$$

Proof Firstly, we prove (3.23) and (3.24). To do so, we begin to show that the following integral equation

$$\rho(t) + \frac{1}{2\pi i(1-\beta)} \int_L K_3(\tau, t)\rho(\tau) d\tau = \alpha'(t)\phi^+(\alpha(t)) - \phi^-(t) + \phi^-(\infty) \quad (3.27)$$

with

$$K_3(\tau, t) = \frac{\alpha'(t)(f'(\alpha(\tau)) + \beta\overline{\tau'^2(s)}f'(\alpha(\tau))\frac{f(\alpha(\tau))\overline{\alpha'(\tau)}}{\overline{f(\alpha(\tau))\alpha'(\tau)}})}{f(\alpha(\tau)) - f(\alpha(t))|f(\alpha(t))/f(\alpha(\tau))|^{2\theta}} - \frac{|f(\tau)|^{2\theta}(f'(\tau) + \beta\overline{\tau'^2(s)}f'(\tau)\overline{f(\tau)}\overline{f'(\tau)})^{-1}}{f(\tau)|f(\tau)|^{2\theta} - f(t)|f(t)|^{2\theta}} \quad (3.28)$$

has a unique solution $\rho \in H^\mu(L)$.

By direct calculation, we find that $K_3(\tau, t)$ can be rewritten as

$$K_3(\tau, t) = K_2(\tau, t) \left[\hat{K}_1(\xi, \eta) - \frac{(\alpha'(\tau) - \alpha'(t))\delta'(\xi)}{\alpha'(\tau)(\delta(\xi) - \delta(\eta))} \right], \quad (3.29)$$

where $\eta = f(t)|f(t)|^{2\theta}$, $\xi, K_2(\tau, t)$, and $\hat{K}_1(\xi, \eta)$ are given in (3.13) and (3.19) respectively.

By (3.15), (3.18), and (3.20), we have

$$\begin{aligned} \left| \frac{(\alpha'(\tau) - \alpha'(t))\delta'(\xi)}{\alpha'(\tau)(\delta(\xi) - \delta(\eta))} \right| &\leq \|\alpha'\|_{H^\mu} \cdot \frac{\max_{\xi \in \Gamma} |\delta'(\xi)|}{\min_{\tau \in L} |\alpha'(\tau)|} \cdot \frac{|\tau - t|^\mu}{|\delta(\xi) - \delta(\eta)|} \\ &\leq N_2 \frac{|\xi - \eta|^\mu}{|\delta(\xi) - \delta(\eta)|} \leq \frac{N_3}{|\xi - \eta|^{1-\mu}}, \end{aligned}$$

where Γ is defined in (3.12), N_2, N_3 are constants, and ξ is in some deleted neighborhood of η . Hence, by the last inequality and (3.21), we have proved that $K_3(\tau, t)$ is a weakly singular kernel. This implies that (3.27) is a Fredholm equation.

Next, we show that (3.27) is uniquely solvable and its solution is just the desired function to seek. To this end, it is sufficient to show that the corresponding homogenous equation of (3.27)

$$\rho(t) + \frac{1}{2\pi i(1-\beta)} \int_L K_3(\tau, t) \rho(\tau) d\tau = 0 \quad (3.30)$$

has only the trivial null solution.

Let ρ_0 be any solution to (3.30). Construct

$$\begin{aligned} \Psi^+(z) &= \frac{1}{2\pi i(1-\beta)} \int_L \frac{\hat{\alpha}'(\tau)\rho_0(\hat{\alpha}(\tau))f'(\tau)}{f(\tau)-f(z)|f(z)/f(\tau)|^{2\theta}} \left[d\tau + \beta \frac{f(\tau)\overline{f'(\tau)}}{f(\tau)f'(\tau)} d\bar{\tau} \right], \\ \Psi^-(z) &= \frac{(1-\beta)^{-1}}{2\pi i} \int_L \frac{\rho_0(\tau)f'(\tau)}{f(\tau)-f(z)|f(z)/f(\tau)|^{2\theta}} \left[d\tau + \beta \frac{f(\tau)\overline{f'(\tau)}}{f(\tau)f'(\tau)} d\bar{\tau} \right] + \phi^-(\infty). \end{aligned} \quad (3.31)$$

According to Theorem 3.1 and $\hat{\alpha}'(\alpha(t)) = 1/(\alpha'(t))$, by a change of variables, then we obtain

$$\alpha'(t)\Psi^+(\alpha(t)) - \Psi^-(t) + \phi^-(\infty) = 0.$$

Let

$$\hat{\Psi}(z) = \Psi^+(z), \quad z \in D^+, \quad \text{and} \quad \hat{\Psi}(z) = \Psi^-(z) - \phi^-(\infty), \quad z \in D^-. \quad (3.32)$$

Then $\hat{\Psi}$ satisfies the following Riemann–Hilbert problem with shift:

$$\begin{cases} \hat{\Psi}^+(\alpha(t)) = 1/(\alpha'(t))\hat{\Psi}^-(t), & t \in L, \\ \hat{\Psi}(\infty) = 0. \end{cases} \quad (3.33)$$

Note that $\text{Ind}_L \alpha'(t) = 0$, let

$$X(z) = \begin{cases} \exp\{(C_L^{f,\beta}(\hat{\mu} \circ \hat{\alpha}))(z)\}, & z \in D^+, \\ \exp\{(C_L^{f,\beta} \hat{\mu})(z)\}, & z \in D^-, \end{cases}$$

where $C_L^{f,\beta}$ is defined as in (2.8), and $\hat{\mu}$ is a solution of the Fredholm equation $(\mathbf{K}\hat{\mu})(t) = \ln(1/\alpha'(t))$ with $\mathbf{K}$ defined by (3.22), here $\hat{\mu} \circ \hat{\alpha}$ denotes the composite function of $\hat{\mu}$ and $\hat{\alpha}$.

It is easy to get that

$$\begin{cases} X \in H_f^\beta(D^+ \cup D^-), & X^+(\alpha(t)) = X^-(t)/(\alpha'(t)), \quad t \in L, \\ X^\pm \in H^\mu(L), & X^\pm(t) \neq 0 \quad \text{on } L, \quad X(\infty) = 1. \end{cases}$$

Hence, problem (3.33) is further reduced to

$$\begin{cases} [\hat{\Psi}/X]^+(\alpha(t)) = [\hat{\Psi}/X]^-(t), & t \in L, \\ (\hat{\Psi}/X)(\infty) = 0. \end{cases} \quad (3.34)$$

By Cauchy formulas (2.4) and (2.7) in Theorem 2.1, one has

$$\left[\frac{\hat{\Psi}}{X} \right](z) = \begin{cases} (C_L^{f,\beta}(\hat{\Psi}/X)^+)(z), & z \in D^+, \\ -(C_L^{f,\beta}(\hat{\Psi}/X)^-)(z), & z \in D^-. \end{cases}$$

Applying the Sokhotski–Plemelj formulas (3.3) in Theorem 3.1 for the function $\hat{\Psi}/X$, which is continuous up to the boundary L from $D^\pm$, we get

$$\begin{cases} [\hat{\Psi}/X]^+(t) - (S_L^{f,\beta}(\hat{\Psi}/X)^+)(t) = 0, \\ [\hat{\Psi}/X]^-(t) + (S_L^{f,\beta}(\hat{\Psi}/X)^-)(t) = 0. \end{cases}$$

Taking the substitution $t \rightarrow \alpha(t)$ and $\tau \rightarrow \alpha(\tau)$ in the first equation of the above system, and then adding it to the second one, it follows from condition (3.34) that

$$[\hat{\Psi}/X]^-(t) - \frac{1}{2\pi i(1-\beta)} \int_L K_1(\tau, t) [\hat{\Psi}/X]^-(\tau) d\tau = 0 \quad (3.35)$$

with $K_1(\tau, t)$ defined by (3.11). By Remark 3.1 or Theorem 3.2, (3.35) is a Fredholm equation, and thus it has only finite linearly independent solutions at most, so does problem (3.34). This is a contradiction because $\{\phi^m : m \in \mathbb{N}\}$ must be the set consisting of infinite linearly independent solutions to (3.34) provided that ϕ is any non-constant solution to (3.34). Since $(\hat{\Psi}/X)(\infty) = 0$, (3.34) has only the trivial null solution, i.e., $\hat{\Psi}/X = 0$. Therefore, we conclude from (3.32) that $\Psi^+(z) \equiv 0$, $\Psi^-(z) \equiv \phi^-(\infty)$. Let

$$\varphi(z) = (C_L^{f,\beta} \rho_0)(z), \quad z \in D^+, \quad \text{and} \quad \varphi(z) = -(C_L^{f,\beta}(\hat{\alpha}' \cdot (\rho_0 \circ \hat{\alpha}')))(z), \quad z \in D^-.$$

Noting that $\Psi^+(z) \equiv 0$, $\Psi^-(z) \equiv \phi^-(\infty)$ and (3.31), by (3.3), we obtain that $\varphi \in H_f^\beta(D^+ \cup D^-)$ with $\varphi(\infty) = 0$, $\hat{\alpha}'(t)\rho_0(\hat{\alpha}(t)) = \varphi^-(t)$, $\rho_0(t) = \varphi^+(t)$. This implies that φ satisfies the following Riemann–Hilbert problem with shift $\hat{\alpha}(t)$:

$$\begin{cases} \varphi^+(\hat{\alpha}(t)) = \varphi^-(t)/\hat{\alpha}'(t), & t \in L, \\ \varphi(\infty) = 0. \end{cases}$$

Observing that $\hat{\alpha}$ is still a positive shift on L , therefore, as the above argument to problem (3.33), similarly, we get $\varphi \equiv 0$, and then $\rho_0 \equiv 0$. Hence, (3.30) indeed has the unique null solution.

Now we establish the integral representations (3.23) and (3.24). Let $\rho \in H^\mu(L)$ be the unique solution to Fredholm equation (3.27). Construct

$$\hat{\varphi}(z) = \begin{cases} (C_L^{f,\beta}(\hat{\alpha}' \cdot (\rho \circ \hat{\alpha}')))(z), & z \in D^+, \\ (C_L^{f,\beta}\rho)(z) + \phi^-(\infty), & z \in D^-. \end{cases}$$

By the Sokhotski–Plemelj formulas (3.3), then

$$\begin{cases} [\phi - \hat{\varphi}]^+(\alpha(t)) = 1/(\alpha'(t))[\phi - \hat{\varphi}]^-(t), & t \in L, \\ (\phi - \hat{\varphi})(\infty) = 0. \end{cases}$$

Discussing as above, we at once have $\phi \equiv \hat{\varphi}$.

As for (3.25) and (3.26), only by considering the equation

$$\rho(t) + \frac{1}{2\pi i(1-\beta)} \int_L K_1(\tau, t) \rho(\tau) d\tau = \phi^+(\alpha(t)) - \phi^-(t) + \phi^-(\infty)$$

with $K_1(\tau, t)$ defined by (3.11), we can derive the desired formulas in the same manner. $\square$

Remark 3.2 In (3.23) and (3.24), $\rho(t)$ is the unique solution of the integral equation

$$\rho(t) + \frac{1}{2\pi i(1-\beta)} \int_L K_3(\tau, t) \rho(\tau) d\tau = \alpha'(t)\phi^+(\alpha(t)) - \phi^-(t) + \phi^-(\infty), \quad (3.36)$$

where $K_3(\tau, t)$ is given in (3.28), while $\hat{\rho}(t)$ in (3.25) and (3.26) is uniquely determined by

$$\hat{\rho}(t) + \frac{1}{2\pi i(1-\beta)} \int_L K_1(\tau, t) \hat{\rho}(\tau) d\tau = \phi^+(\alpha(t)) - \phi^-(t) + \phi^-(\infty) \quad (3.37)$$

with $K_1(\tau, t)$ given in (3.11). Moreover, by (3.29), $\hat{\rho} = \rho + \rho_*$, where ρ_* is uniquely determined by the following integral equation:

$$\rho_*(t) + \frac{1}{2\pi i(1-\beta)} \int_L K_2(\tau, t) \frac{(\alpha'(\tau) - \alpha'(t))\delta'(\xi)}{\alpha'(\tau)(\delta(\xi) - \delta(\eta))} \rho_*(\tau) d\tau = (1 - \alpha'(t))\phi^+(\alpha(t)),$$

in which $\delta(\xi)$ and $K_2(\tau, t)$ are defined by (3.14) and (3.19) respectively. Furthermore, the corresponding homogeneous equations of (3.36) and (3.37) have only the null solution.

Remark 3.3 In general, as problem (3.34), if $\Phi \in H_f^\beta(D^+ \cup D^-)$, then the homogenous Riemann–Hilbert problem with shift

$$\Phi^+(\alpha(t)) = \Phi^-(t), \quad t \in L, \quad \Phi(\infty) = 0, \quad (3.38)$$

has only the trivial solution $\Phi = 0$.

Remark 3.4 When $\alpha(t) \equiv t$, representations (3.23) and (3.24) in Theorem 3.3 are reduced to Theorem 1 in [30, 32] and Theorem 2.1 in the last section respectively. In this case, (3.23) and (3.24) are also completely consistent with (3.25) and (3.26). Moreover, in the case

of $\alpha(t) \equiv t$ and $f(z) \equiv z$, Theorem 3.3 exactly corresponds to the integral representation theorem for any β -analytic function $\phi \in H_f^\beta(D^+ \cup D^-)$, which is also Hölder continuous up to the boundary L from $D^\pm$ in [26–29]. More precisely,

$$\begin{aligned}\phi^+(z) &= \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\phi^+(t) dt}{t-z|z/t|^{2\theta}} + \frac{\beta t \phi^+(t) d\bar{t}}{\bar{t}(t-z|z/t|^{2\theta})} \right], \quad z \in D^+, \\ \phi^-(z) &= \frac{-1}{2\pi i(1-\beta)} \int_L \left[\frac{\phi^-(t) dt}{t-z|z/t|^{2\theta}} + \frac{\beta t \phi^-(t) d\bar{t}}{\bar{t}(t-z|z/t|^{2\theta})} \right] + \phi^-(\infty), \quad z \in D^-.\end{aligned}$$

When $f(z) \equiv z$, $\beta = 0$, $\phi^-(\infty) = 0$, (3.23) and (3.24) correspond to the results in [7].

In fact, Theorem 3.3 provides two different integral representations which will be alternatively used to solve the Riemann–Hilbert problem with single shift in the next section. Even for those RH problems with multiple shifts and their conjugations, these integral representations are also very useful.

4 Riemann–Hilbert problems with shift for generalized β -analytic functions

In this section, we utilize the results obtained in the previous sections and construct explicit canonical matrix of triangular matrix functions to derive the explicit solution and conditions of solvability for Riemann–Hilbert problems with shift for generalized β -analytic functions.

As a convention, X always denotes a function class, such as $H^\mu(\Omega)$, and $X^{n \times m} = \{a_{ij}\}_{n \times m}$ with $a_{ij} \in X$. Similarly, $AX^{n \times m}$ is understood as $(AX)^{n \times m}$ in the way for an operator A acting on X . In addition, throughout this section, we always choose $f(z) = z$. Denote $H_1^\beta(D^+ \cap D^-) = H_f^\beta(D^+ \cap D^-)|_{f(z)=z}$ to be the class of β -analytic functions in [27]. In this case, for any $\phi \in H^\mu(L)$, the corresponding Cauchy singular integral operator is

$$(S_L^\beta \phi)(t) = \frac{1}{\pi i(1-\beta)} \int_L \left[\frac{\phi(\tau) d\tau}{\tau-t|t/\tau|^{2\theta}} + \frac{\beta \tau \phi(\tau) d\bar{\tau}}{\bar{\tau}(\tau-t|t/\tau|^{2\theta})} \right], \quad t \notin L, \quad (4.1)$$

and the corresponding Cauchy-type integral is as follows:

$$(C_L^\beta \phi)(z) = \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\phi(t) dt}{t-z|z/t|^{2\theta}} + \frac{\beta t \phi(t) d\bar{t}}{\bar{t}(t-z|z/t|^{2\theta})} \right], \quad z \notin L. \quad (4.2)$$

Meanwhile, $K_1(\tau, t)$ defined by (3.11) with $f(z) = z$ is reduced to

$$\begin{aligned}K(\tau, t) &= \frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)|\alpha(t)/\alpha(\tau)|^{2\theta}} - \frac{1}{\tau - t|t/\tau|^{2\theta}} \\ &\quad + \frac{\beta \alpha(\tau) \overline{\alpha'(\tau) \tau^2(s)}}{\alpha(\tau)(\alpha(\tau) - \alpha(t)|\alpha(t)/\alpha(\tau)|^{2\theta})} - \frac{\beta \tau \overline{\tau^2(s)}}{\bar{\tau}(\tau - t|t/\tau|^{2\theta})}.\end{aligned} \quad (4.3)$$

Let $\kappa = \frac{1}{2\pi} [\arg G(t)]_L$, $\hat{\alpha}$ be as stated in Theorem 3.3, and

$$X(z) = \begin{cases} \exp\{(C_L^\beta(\rho \circ \hat{\alpha}))(z)\}, & z \in D^+, \\ z^{-\kappa} |z|^{-2\kappa\theta} \exp\{(C_L^\beta \rho)(z)\}, & z \in D^- \end{cases} \quad (4.4)$$

with ρ being the unique solution to the equation

$$(\mathbf{K}\rho)(t) \triangleq \rho(t) + \frac{1}{2\pi i(1-\beta)} \int_L K(\tau, t) \rho(\tau) d\tau = \ln(t^{-\kappa} |t|^{-2\kappa\theta} G(t)). \quad (4.5)$$

By Theorems 3.1 and 3.2, we see that $X(z)$ has the following basic properties:

- (1) $X \in H_1^\beta(D^+ \cup D^-)$;
- (2) $X^+(\alpha(t)) = G(t)X^-(t)$, $t \in L$, and $X^\pm \in H^\mu(L)$;
- (3) $X(z) \neq 0$, $z \in D^+ \cup D^-$, and $X^\pm(t) \neq 0$ on L ;
- (4) $\lim_{z \rightarrow \infty} |z^\kappa |z|^{2\kappa\theta} X(z)| = 1$.

As in [3, 7, 19], X given in (4.4) is called a canonical function. Moreover, when $\alpha(t) \equiv t$, X given in (4.4) is exactly the one in [29].

Definition 4.1 Let ϕ be a β -analytic function on some neighborhood of the infinity. If there exists some integer $p \in \mathbb{Z}$ such that

$$\limsup_{z \rightarrow \infty} |z^{-p} |z|^{-2p\theta} \phi(z)| = a \in (0, +\infty), \quad (4.6)$$

then ϕ is said to possess order p at infinity. As usual, such order is denoted by $\text{Ord}(\phi, \infty)$.

To start with, we consider the following Riemann–Hilbert problem with shift for generalized β -analytic functions with order one.

Problem I To find a function $\phi \in H_1^\beta(D^+ \cup D^-)$ which is Hölder continuous to the boundary L from $D^\pm$, satisfying

$$\begin{cases} \phi^+(\alpha(t)) = G(t)\phi^-(t) + g(t), & t \in L, \\ \text{Ord}(\phi, \infty) \leq m, & m \in \mathbb{Z}, \end{cases} \quad (4.7)$$

where $G, g \in H^\mu(L)$, $G(t) \neq 0$ on L .

Theorem 4.1 Let $\kappa = \frac{1}{2\pi} [\arg G(t)]_L$ and $K(\tau, t)$ be determined by (4.3). When $\kappa + m \geq 0$, problem (4.7) is solvable and the solution is given by

$$\phi(z) = X(z) [\Psi(z) + P_{\kappa+m}(W_\alpha^\beta(z))], \quad (4.8)$$

where $P_{\kappa+m}(W_\alpha^\beta(z))$ denotes a polynomial with respect to $W_\alpha^\beta(z)$ of degree not larger than $\kappa + m$, and X is given by (4.4)

$$\Psi(z) = \begin{cases} \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\rho(\hat{\alpha}(\tau)) d\tau}{\tau - z|z/\tau|^{2\theta}} + \frac{\beta\tau\rho(\hat{\alpha}(\tau)) d\bar{\tau}}{\bar{\tau}(\tau - z|z/\tau|^{2\theta})} \right], & z \in D^+, \\ \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\rho(\tau) d\tau}{\tau - z|z/\tau|^{2\theta}} + \frac{\beta\tau\rho(\tau) d\bar{\tau}}{\bar{\tau}(\tau - z|z/\tau|^{2\theta})} \right], & z \in D^-, \end{cases} \quad (4.9)$$

where ρ is the unique solution to the following equation:

$$\rho(t) + \frac{1}{2\pi i(1-\beta)} \int_L K(\tau, t) \rho(\tau) d\tau = \frac{g(t)}{X^+(\alpha(t))}. \quad (4.10)$$

In addition,

$$W_{\alpha}^{\beta}(z) = \begin{cases} \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{v(\hat{\alpha}(\tau)) d\tau}{\tau-z|z/\tau|^{2\theta}} + \frac{\beta \tau v(\hat{\alpha}(\tau)) d\bar{\tau}}{\bar{\tau}(\tau-z|z/\tau|^{2\theta})} \right], & z \in D^+, \\ \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{v(\tau) d\tau}{\tau-z|z/\tau|^{2\theta}} + \frac{\beta \tau \mu(\tau) d\bar{\tau}}{\bar{\tau}(\tau-z|z/\tau|^{2\theta})} \right] + z|z|^{2\theta}, & z \in D^-, \end{cases} \quad (4.11)$$

where v is the unique solution to the following equation:

$$v(t) + \frac{1}{2\pi i(1-\beta)} \int_L K(\tau, t) v(\tau) d\tau = t|t|^{2\theta}. \quad (4.12)$$

When $\kappa + m < 0$, if and only if

$$\int_L \rho(t) \left[t^l |t|^{2(l+1)\theta} dt + \frac{\beta}{t} t^{l+1} |t|^{2(l+1)\theta} d\bar{t} \right] = 0$$

with $l = 0, 1, \dots, -\kappa - m - 2$, (4.13)

is satisfied, the solution can be expressed as

$$\phi(z) = \begin{cases} \frac{X(z)}{2\pi i(1-\beta)} \int_L \left[\frac{\rho(\hat{\alpha}(\tau)) d\tau}{\tau-z|z/\tau|^{2\theta}} + \frac{\beta \tau \rho(\hat{\alpha}(\tau)) d\bar{\tau}}{\bar{\tau}(\tau-z|z/\tau|^{2\theta})} \right], & z \in D^+, \\ \frac{X(z)}{2\pi i(1-\beta)} \int_L \left[\frac{\rho(\tau) d\tau}{\tau-z|z/\tau|^{2\theta}} + \frac{\beta \tau \rho(\tau) d\bar{\tau}}{\bar{\tau}(\tau-z|z/\tau|^{2\theta})} \right], & z \in D^- \end{cases} \quad (4.14)$$

with ρ defined by (4.10). By convention, conditions (4.13) of solvability automatically disappear when $\kappa + m = -1$.

Proof When $\kappa + m \geq 0$, by the properties of the canonical function defined by (4.4) and the Cauchy-type integral in Theorem 3.1, problem (4.7) is equivalent to

$$\begin{cases} H^+(\alpha(t)) = H^-(t), & t \in L, \\ \text{Ord}(H, \infty) \leq \kappa + m \end{cases} \quad (4.15)$$

with

$$H(z) = \frac{\phi(z)}{X(z)} - \Psi(z), \quad z \in D^+ \cup D^-,$$

where $X(z)$ and $\Psi(z)$ are just defined by (4.4) and (4.9) respectively.

Hence, when $\kappa + m \geq 0$, the order of $H(z)$ at ∞ is not larger than $\kappa + m$. By a similar argument to Laurent series expansion for analytic functions in the deleted neighborhood of ∞ ,

$$H(z) = \sum_{j=-\infty}^{\kappa+m} c_j z^j |z|^{2j\theta}, \quad (4.16)$$

where

$$c_j = \frac{1}{2\pi i(1-\beta)} \int_{L_1} H(t) |t|^{2\theta} \left[t^{-j-1} |t|^{-2(j+1)\theta} dt + \beta \frac{t}{\bar{t}} t^{-j-1} |t|^{-2(j+1)\theta} d\bar{t} \right] \quad (4.17)$$

with $L_1 = \{t : |t| = R\}$ for sufficiently large R .

In fact, for each z in a deleted neighborhood of ∞ , there exist sufficiently large R_1, R_2 such that $R_1 < |z| < R_2$. Let $K_j = \{t : |t| = R_j\} (j = 1, 2)$ be counter-clockwise oriented and located in this neighborhood.

Since

$$\frac{1}{t^\theta |t|^{2\theta} - z^\theta |z|^{2\theta}} = - \sum_{j=0}^{+\infty} \frac{t^{j\theta} |t|^{2j\theta}}{z^{(j+1)\theta} |z|^{2(j+1)\theta}}, \quad t \in K_1,$$

$$\frac{1}{t^\theta |t|^{2\theta} - z^\theta |z|^{2\theta}} = \sum_{j=0}^{+\infty} \frac{z^{j\theta} |z|^{2j\theta}}{t^{(j+1)\theta} |t|^{2(j+1)\theta}}, \quad t \in K_2,$$

then

$$A_1(z) = \frac{1}{2\pi i(1-\beta)} \int_{K_1} \left[\frac{H(t) dt}{t - z|z/t|^{2\theta}} + \frac{\beta t H(t) d\bar{t}}{\bar{t}(t - z|z/t|^{2\theta})} \right] = - \sum_{j=1}^{+\infty} c_{1,-j} z^{-j} |z|^{-2j\theta},$$

$$A_2(z) = \frac{1}{2\pi i(1-\beta)} \int_{K_2} \left[\frac{H(t) dt}{t - z|z/t|^{2\theta}} + \frac{\beta t H(t) d\bar{t}}{\bar{t}(t - z|z/t|^{2\theta})} \right] = \sum_{j=0}^{+\infty} c_{2,j} z^j |z|^{2j\theta}$$

with $c_{1,-j}, c_{2,j}$ defined by (4.17) in which L_1 is replaced by K_1 and K_2 respectively. From Cauchy formula (2.4), we conclude that $c_{1,-j} = c_{-j}, c_{2,j} = c_j$ with c_j defined by (4.17) and $R_1 < R < R_2$. Therefore, using the Cauchy formula again, one has

$$H(z) = A_2(z) - A_1(z) = \sum_{j=-\infty}^{+\infty} c_j z^j |z|^{2j\theta}.$$

From (4.17), one has

$$|c_j| \leq \frac{1+\beta}{1-\beta} \sup_{|t|=R} |t^{-j}| |t|^{-2j\theta} |H(t)|.$$

Thus, by taking $R \rightarrow +\infty$, it follows from $\text{Ord}(H, \infty) \leq \kappa + m$ that $c_j \equiv 0$ for all $j > \kappa + m$, which implies that Laurent series of $H(z)$ in the deleted neighborhood at ∞ must be in the form of (4.16).

Denote $W_\alpha^\beta(z)$ given by (4.11), it is direct to prove that $W_\alpha^\beta \in H_1^\beta(D^+ \cup D^-)$ and $(W_\alpha^\beta)^+(\alpha(t)) = (W_\alpha^\beta)^-(t)$.

Let

$$F(z) = H(z) - Q_{\kappa+m}(z), \quad z \in D^+ \cup D^-$$

with

$$Q_{\kappa+m}(z) = \sum_{l=0}^{\kappa+m} c_l (W_\alpha^\beta(z))^l,$$

where c_l is defined by (4.17).

Observe that (4.15) is changed into

$$F^+(\alpha(t)) = F^-(t), \quad t \in L, \text{ with } F(\infty) = 0.$$

It follows from Remark 3.3 that $F(z) \equiv 0$, which leads to expression (4.8).

Similarly, when $\kappa + m < 0$, problem (4.7) is equivalent to

$$H^+(\alpha(t)) = H^-(t), \quad t \in L,$$

with zero of order not larger than $-\kappa - m$ at ∞ . In this case, if conditions (4.13) of solvability are satisfied, it follows from Remark 3.3 that $H(z) \equiv 0$, and this implies $H(z) = X(z)\Psi(z)$. $\square$

Remark 4.1 If $\alpha(t) = t$, Theorem 4.1 corresponds to Theorem 6 in [27]. If $\beta = 0$, the results in Theorem 4.1 are just some ones discussed in [5, 7, 19, 25, 36].

Next, we turn to investigating the Riemann–Hilbert problem for generalized β -analytic functions with order $n \in \mathbb{N}$, $n > 1$.

Problem II To find a function $\Phi \in H_n^\beta(D^+ \cup D^-)$ which is Hölder continuous up to the boundary L from $D^\pm$, satisfying

$$\begin{cases} [\partial_{zj-1}^\beta \Phi]^+(\alpha(t)) = G_j(t)[\partial_{zj-1}^\beta \Phi]^-(t) + g_j(t), & t \in L, j = 1, 2, \dots, n, \\ \text{Ord}(\sum_{k=j}^n \frac{(-1)^{k+j} z^{k-j}}{(k-j)!(j-1)!} \partial_{z^{k-1}}^\beta \Phi, \infty) \leq m - \lceil \frac{j-1}{1+2\theta} \rceil, \end{cases} \quad (4.18)$$

where $m \in \mathbb{Z}$, $G_j, g_j \in H^\mu(L)$, $G_j(t) \neq 0$ on L , and ∂_{zj}^β is defined by (2.9) with $f(z) = z$, i.e.,

$$\partial_z^\beta = \partial_z - \beta \frac{z}{z} \partial_z, \quad \partial_{z^0}^\beta = I, \quad \partial_{zj}^\beta = \underbrace{\partial_z^{f,\beta} \circ \partial_z^{f,\beta} \circ \dots \circ \partial_z^{f,\beta}}_j \quad (4.19)$$

for $j = 0, 1, \dots, n-1$. Moreover, throughout the remainder of this paper, $\lceil a \rceil$ always stands for the largest integer not exceeding a .

Obviously, problem (4.18) is reduced to the one discussed in [14, 18, 19, 36, 37] in the case $\beta = 0$ or $\alpha(t) = t$. In our approach, to solve problem (4.18), a matrix factorization technique in terms of canonical matrix with shift in the context of β -analytic functions is rather crucial. In the following first subsection, we will give the definition of the canonical matrix, the factorization in terms of it, and one of its applications to a homogeneous vector-valued Riemann–Hilbert problem with shift, and show how to explicitly construct the canonical matrix for some triangular matrices in the following subsection.

4.1 Constructing of canonical matrix

Let L , α be defined as in Sect. 2, and let the following two sets be

$$H_{\pm}^{\mu,\beta}(L) = \{\varphi \in H^\mu(L) : \varphi(t) = \phi^\pm(t), \phi \in H_1^\beta(D^\pm), t \in L\}, \quad (4.20)$$

where $H_1^\beta(D^\pm) = H_f^\beta(D^\pm)|_{f(z)=z}$.

Definition 4.2 Let $\theta = \beta/(1 - \beta)$ with $0 \leq \beta < 1$, and $G \in [H^\mu(L)]^{n \times n}$ satisfying $\inf_{t \in L} |\det(G(t))| > 0$. If there exists $G_\pm \in [H^{\mu, \beta}_\pm(L)]^{n \times n}$ with $\inf_{t \in L} |\det(G_\pm(t))| > 0$,

$$\Lambda(t) = \text{diag}[t^{\kappa_1} |t|^{2\kappa_1\theta}, t^{\kappa_2} |t|^{2\kappa_2\theta}, \dots, t^{\kappa_n} |t|^{2\kappa_n\theta}] \quad (4.21)$$

with $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_n$ satisfying $\kappa_j \in \mathbb{Z}$ for $j = 1, 2, \dots, n$, such that

$$G(t) = G_+(\alpha(t)) \Lambda(t) G_-(t), \quad t \in L, \quad (4.22)$$

then representation (4.22) is said to be a generalized β -analytic factorization with shift α related to $H^\mu(L)$, or simply a generalized β -analytic factorization with shift, and κ_j is called partial indices for all j . Moreover,

$$X(z) = \begin{cases} G_+(z), & z \in D^+, \\ [G_-(z)]^{-1} \Lambda^{-1}(z), & z \in D^- \end{cases} \quad (4.23)$$

is said to be a β -canonical matrix of $G(t)$ related to $H^\mu(L)$ with respect to α , or simply a canonical matrix.

Remark 4.2 If $\beta = 0$, the above definition becomes the one discussed in [1, 6]. If $\beta = 0$ and $\alpha(t) = t$, the canonical matrix in the above definition corresponds to the one of the classical canonical matrices in [1–5, 42, 43].

The following proposition provides a discriminative approach for the existence of this type of canonical matrices.

Proposition 4.1 Let $G \in [H^\mu(L)]^{n \times n}$. Then there exists the β -canonical matrix of $G(t)$ if and only if $\inf_{t \in L} |\det(G(t))| > 0$.

Proof Let $\Gamma = \{\xi : \xi = \tau |\tau|^{2\theta}, \tau \in L\}$. As stated in Sect. 3, Γ is a simply closed Lyapunov curve oriented counter-clockwise, its inner and outer domains are Ω^+ and Ω^- . In addition, $0 \in \Omega^+$ and $\infty \in \Omega^-$ respectively.

Let

$$\xi = z |z|^{2\theta}, \quad z \in \mathbb{C}. \quad (4.24)$$

Obviously, ξ is a bijection mapping $D^\pm$ onto $\Omega^\pm$. Moreover, its inverse

$$z = z(\xi) = \begin{cases} 0, & \xi = 0, \\ \xi |\xi|^{-2\theta/(1+2\theta)}, & \xi \in \mathbb{C} \setminus \{0\}. \end{cases} \quad (4.25)$$

On the one hand, observe that

$$\begin{aligned} \inf_{t \in L} |\det(G(t))| > 0 \quad \text{for } G \in [H^\mu(L)]^{n \times n} \\ \iff \inf_{\xi \in \Gamma} |\det(G(t(\xi)))| > 0 \quad \text{for } G \in [H^\mu(\Gamma)]^{n \times n}. \end{aligned}$$

Since $\inf_{\xi \in \Gamma} |\det(G(t(\xi)))| > 0$, from the classical factorization theory in [1, 6], there exists the canonical matrix

$$X(\xi) = \begin{cases} G_+(\xi), & \xi \in \Omega^+, \\ [G_-(\xi)]^{-1} \Lambda^{-1}(\xi), & \xi \in \Omega^-, \end{cases} \quad (4.26)$$

where $G_{\pm}$ are analytic in $\Omega^{\pm}$ respectively and μ -Hölder continuous up to the corresponding boundary Γ ,

$$\Lambda(\xi) = \text{diag}[\xi^{\kappa_1}, \xi^{\kappa_2}, \dots, \xi^{\kappa_n}]$$

with all integers κ_j satisfying $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_n$.

On the other hand, denote $\hat{\Phi}(\xi) = \Phi(z(\xi))$, by the chain rule, one has

$$\partial_z^{\beta} \Phi(z) = (1 + \beta) |z(\xi)|^{2\theta} \partial_{\bar{\xi}} \hat{\Phi}(\xi), \quad \forall \Phi \in C^1(D^+ \cup D^-). \quad (4.27)$$

Thus, by (4.27), pulling (4.26) back with respect to z by (4.24), we immediately obtain the desired β -canonical matrix of $G(t)$. $\square$

In the following proposition, we give an application of the factorization in terms of canonical matrix to solve a homogeneous vector-valued Riemann–Hilbert problem with shift as follows: to find a vector-valued function $\Phi = [\Phi_1, \dots, \Phi_n]^T \in [H_1^{\beta}(D^+ \cup D^-)]^n$, which is Hölder continuous up to the boundary L from $D^{\pm}$, satisfying

$$\begin{cases} \Phi^+(\alpha(t)) = G(t) \Phi^-(t), & t \in L, \\ \text{Ord}(\Phi_j, \infty) \leq -1, & j = 1, 2, \dots, n, \end{cases} \quad (4.28)$$

where $G \in [H^{\mu}(L)]^{n \times n}$, and $\det(G(t)) \neq 0$ on L .

Proposition 4.2 *Let $W_{\alpha}^{\beta}(z)$ be defined by (4.11), G have a generalized β -analytic factorization with shift in a form of (4.22), and S_{α}^{β} be the class of all solutions to the homogeneous problem (4.28). Then general solution of (4.28) can be expressed as*

$$\Phi(z) = G_+(z) \begin{bmatrix} Q_{\kappa_1-1}^1(W_{\alpha}^{\beta}(z)) \\ Q_{\kappa_2-1}^2(W_{\alpha}^{\beta}(z)) \\ \vdots \\ Q_{\kappa_n-1}^n(W_{\alpha}^{\beta}(z)) \end{bmatrix}, \quad z \in D^+ \quad (4.29)$$

and

$$\Phi(z) = (G_-(z))^{-1} \Lambda^{-1}(z) \begin{bmatrix} Q_{\kappa_1-1}^1(W_{\alpha}^{\beta}(z)) \\ Q_{\kappa_2-1}^2(W_{\alpha}^{\beta}(z)) \\ \vdots \\ Q_{\kappa_n-1}^n(W_{\alpha}^{\beta}(z)) \end{bmatrix}, \quad z \in D^-, \quad (4.30)$$

where $Q_{\kappa_p-1}^p(z)$ denotes a polynomial of degree not more than $\kappa_p - 1$, and is regarded as 0 if $\kappa_p - 1 < 0$ for $p \in \{1, 2, \dots, n\}$. Moreover,

$$\dim S_\alpha^\beta = \sum_{\kappa_j > 0} \max\{\kappa_j, 0\} = \sum_{j=1}^n \max\{\kappa_j, 0\}, \quad (4.31)$$

where the first sum is taken over $\kappa_j > 0$, $j = 1, \dots, n$.

Proof Let

$$\Psi(z) = (G_+(z))^{-1} \Phi(z), \quad z \in D^+ \quad \text{and} \quad \Psi(z) = G_-(z) \Phi(z), \quad z \in D^-. \quad (4.32)$$

By representation (4.22) of $G(t)$, Ψ satisfies the following BVP:

$$\begin{cases} \Psi_j^+(\alpha(t)) = t^{\kappa_j} |t|^{2\kappa_j\theta} \Psi_j^-(t), & t \in L, \\ \text{Ord}(\Psi_j, \infty) \leq -1, & j = 1, 2, \dots, n, \end{cases}$$

where Ψ_j denotes the j th component of Ψ .

By Theorem 4.1, the solution of the last problem is

$$\Psi_j(z) = \begin{cases} Q_{\kappa_j-1}^j(W_\alpha^\beta(z)), & z \in D^+, \\ z^{-\kappa_j} |z|^{-2\kappa_j\theta} Q_{\kappa_j-1}^j(W_\alpha^\beta(z)), & z \in D^-, \end{cases}$$

where $Q_{\kappa_j-1}^j(W_\alpha^\beta(z))$ denotes a polynomial of degree not larger than $\kappa_j - 1$ with respect to $W_\alpha^\beta(z)$ (by convention $Q_{\kappa_j-1}^j(W_\alpha^\beta(z)) \equiv 0$ when $\kappa_j - 1 < 0$). By invoking (4.32), we immediately obtain (4.29) and (4.30).

Moreover, from (4.29) and (4.30), we know that the basis of solution space S_α^β is the union of

$$\{G_+(z)(W_\alpha^\beta(z))^l \mathbf{e}_j, G_-^{-1}(z)\Lambda^{-1}(z)(W_\alpha^\beta(z))^l \mathbf{e}_j\} \quad (l = 0, 1, \dots, \kappa_j - 1)$$

for all $j \in \{1, \dots, n\}$ with $\kappa_j > 0$, where $\mathbf{e}_j$ denotes the unit column vector satisfying that its j th component is 1 and the rest are 0. This leads to conclusion (4.31). $\square$

As is well known, it is very difficult to explicitly construct the canonical matrix for general matrix, even for some special matrices in [42, 43], since many skills will be involved to accurately determine its partial indices and find explicit solutions of some related Riemann–Hilbert problems. The results here will be used in the next subsection.

In the rest of this section, we investigate how to construct the canonical matrix of the following triangular matrix:

$$G(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ 0 & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}(t) \end{bmatrix}, \quad (4.33)$$

where $a_{ij} \in H^\mu(L)$, $a_{ji}(t) \neq 0$ on L for all i, j .

Let $c_j = \frac{1}{2\pi} [\arg a_{jj}(t)]_L$, $j = 1, 2, \dots, n$. By (4.4), a generalized β -analytic factorization with shift of a_{jj} is

$$a_{jj}(t) = a_{jj}^+(\alpha(t)) t^{c_j} |t|^{2\theta c_j} a_{jj}^-(t), \quad (4.34)$$

where

$$a_{jj}^\pm(z) = \exp\{A_j(z)\}, \quad z \in D^\pm, \quad A_j(z) = \begin{cases} (C_L^\beta(\mu_j \circ \hat{\alpha}))(z), & z \in D^+, \\ -(C_L^\beta \mu_j)(z), & z \in D^- \end{cases} \quad (4.35)$$

with μ_j being the unique solution of the Fredholm equation

$$\mu_j(t) + \frac{1}{2\pi i(1-\beta)} \int_L K(\tau, t) \mu_j(\tau) d\tau = \ln(t^{-c_j} |t|^{-2\theta c_j} a_{jj}(t)).$$

Theorem 4.2 Let C_L^β , $K(\tau, t)$, and $\mathbf{K}$ be defined by (4.2), (4.3), and (4.5) respectively, and let (4.22) be a generalized β -analytic factorization with shift $\alpha(t)$ of $G(t)$. Then the partial indices κ_j of $G(t)$ satisfy

$$\max_{1 \leq i \leq n} c_i \geq \kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_n \geq \min_{1 \leq i \leq n} c_i. \quad (4.36)$$

Furthermore, if $c_1 \geq c_2 \geq \dots \geq c_n$, then an explicit generalized β -analytic factorization of G defined by (4.33) is

$$G(t) = G_+(\alpha(t)) \Lambda(t) G_-(t), \quad t \in L, \quad (4.37)$$

where

$$G_\pm^{\pm 1}(z) = \begin{bmatrix} Y_{11}(z)\varphi_{11}(z) & Y_{12}(z)\varphi_{12}(z) & \cdots & Y_{1n}(z)\varphi_{1n}(z) \\ 0 & Y_{22}(z)\varphi_{22}(z) & \cdots & Y_{2n}(z)\varphi_{2n}(z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_{nn}(z)\varphi_{nn}(z) \end{bmatrix}, \quad z \in D^\pm, \quad (4.38)$$

in which

$$Y_{pl}(z) = \begin{cases} \exp\{(C_L^\beta(\rho_p \circ \hat{\alpha}))(z)\}, & z \in D^+, \\ z^{c_l - c_p} |z|^{2(c_l - c_p)\theta} \exp\{(C_L^\beta \rho_p)(z)\}, & z \in D^- \end{cases} \quad (l \geq p, l = 1, \dots, n) \quad (4.39)$$

with ρ_p being the unique solution of the equation

$$\rho_p(t) + \frac{1}{2\pi i(1-\beta)} \int_L K(\tau, t) \rho_p(\tau) d\tau = \ln(t^{-c_p} |t|^{-2\theta c_p} a_{pp}(t)),$$

and

$$\varphi_{pp}(z) = 1, \quad z \in D^+ \cup D^-, p = 1, 2, \dots, n, \quad (4.40)$$

$$\varphi_{pl}(z) = \begin{cases} (C_L^\beta(\tilde{\rho}_{pl} \circ \hat{\alpha}))(z), & z \in D^+, \\ (C_L^\beta \tilde{\rho}_{pl})(z), & z \in D^- \end{cases} \quad (l > p, l = 2, 3, \dots, n) \quad (4.41)$$

with $\tilde{\rho}_{pl}$ being the unique solution of the equation

$$(\mathbf{K}\tilde{\rho}_{pl})(t) = \begin{cases} \frac{a_{pl}Y_{ll}^-(t)}{t^{cl}|t|^{2\theta c_l}Y_{pl}^+(\alpha(t))}, & l-p=1, \\ \frac{\sum_{j=1}^{l-p-1} \{a_{p,p+j}(t)Y_{p+j,l}^-(t)^{\frac{1}{2}}[(S_L^\beta \tilde{\rho}_{p+j,l})(t) - \tilde{\rho}_{p+j,l}(t)]\} + a_{pl}(t)Y_{ll}^-(t)}{t^{cl}|t|^{2\theta c_l}Y_{pl}^+(\alpha(t))}, & l-p \geq 2 \end{cases}$$

for $l = 2, 3, \dots, n$, and

$$\Lambda(t) = \text{diag}[t^{c_1}|t|^{2c_1\theta}, t^{c_2}|t|^{2c_2\theta}, \dots, t^{c_n}|t|^{2c_n\theta}]. \quad (4.42)$$

Moreover, $G_+^{-1}(z)$, $G_-^{-1}(z)$ stand for $G_+(z)$, the reverse of $G_-(z)$ respectively.

Proof Firstly, we prove $\kappa_1 \leq p$ with $p = \max\{c_i : 1 \leq i \leq n\}$. To this end, we seek to find a function $\Phi = [\Phi_1, \dots, \Phi_n]^T \in [H_1^\beta(D^+ \cup D^-)]^n$, which is Hölder continuous up to the boundary L from $D^\pm$, satisfying

$$\begin{cases} \Phi^+(\alpha(t)) = t^{-p}|t|^{-2p\theta}G(t)\Phi^-(t), & t \in L, \\ \text{Ord}(\Phi_j, \infty) \leq -1, & j = 1, 2, \dots, n. \end{cases} \quad (4.43)$$

Obviously, (4.43) is equivalent to

$$\begin{cases} \Phi_n^+(\alpha(t)) = a_{nn}(t)t^{-p}|t|^{-2p\theta}\Phi_n^-(t), \\ \Phi_{n-1}^+(\alpha(t)) = t^{-p}|t|^{-2p\theta}[a_{n-1,n-1}(t)\Phi_{n-1}^-(t) + a_{n-1,n}(t)\Phi_n^-(t)], \\ \vdots \\ \Phi_1^+(\alpha(t)) = a_{11}(t)t^{-p}|t|^{-2p\theta}\Phi_1^-(t) + \dots + a_{1n}(t)t^{-p}|t|^{-2p\theta}\Phi_n^-(t), \\ \text{Ord}(\Phi_j, \infty) \leq -1, & j = 1, 2, \dots, n. \end{cases} \quad (4.44)$$

Since $c_j - p - 1 < 0$ for all j , by Theorem 4.1, we get that $\Phi_n(z) \equiv 0$. Substituting it into the second equation of (4.44), we find that $\Phi_{n-1}(z) \equiv 0$. Repeating the same steps, we finally obtain that $\Phi_j(z) \equiv 0$ for all $j = 1, 2, \dots, n$. Thus (4.43) has only the trivial null solution.

Moreover, by the assumption, the partial indices of a generalized β -analytic factorization of $t^{-p}|t|^{-2p\theta}G(t)$ are $\kappa_j - p$. Thus, by Proposition 4.2, $\sum_{j=1}^n \max\{\kappa_j - p, 0\} = 0$, which implies that $\kappa_j \leq p$ for all j .

Secondly, we show that $\kappa_n \geq q$ with $q = \min\{c_i : 1 \leq i \leq n\}$. For this purpose, we consider the following problem:

$$\begin{cases} \Phi^+(\alpha(t)) = t^q|t|^{2q\theta}[G^T(t)]^{-1}\Phi^-(t), & t \in L, \\ \text{Ord}(\Phi_j, \infty) \leq -1, & j = 1, 2, \dots, n, \end{cases} \quad (4.45)$$

where A^T denotes the transposition of A . In view of $G(t)$ defined by (4.33), by simple computation, it follows that

$$[G^T(t)]^{-1} = \begin{bmatrix} a_{11}^{-1}(t) & 0 & \cdots & 0 \\ b_{21} & a_{22}^{-1}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & a_{nn}^{-1}(t) \end{bmatrix}$$

with $b_{ij} \in H^\mu(L)$, $i = 2, \dots, n$, $j = 1, \dots, n-1$.

It is easy to know that $t^q |t|^{2q\theta} [G^T(t)]^{-1}$ has the following factorization:

$$t^q |t|^{2q\theta} [G^T(t)]^{-1} = (G_+^T)^{-1} (\alpha(t)) \hat{\Lambda}(t) (G_-^T)^{-1}(t), \quad t \in L$$

with

$$\hat{\Lambda}(t) = \text{diag}[t^{q-\kappa_1} |t|^{2(q-\kappa_1)\theta}, t^{q-\kappa_2} |t|^{2(q-\kappa_2)\theta}, \dots, t^{q-\kappa_n} |t|^{2(q-\kappa_n)\theta}].$$

Note that $\frac{1}{2\pi} [\arg t^q |t|^{2q\theta} a_{jj}^{-1}]_L = q - c_j \leq 0$ for all j , by an argument similar to problem (4.43), problem (4.45) has only the zero solution. This leads to $\sum_{j=1}^n \max\{q - \kappa_j, 0\} = 0$, which implies that $\kappa_j \geq q$ for all j . The proof of (4.36) is complete.

Now we prove that if $c_1 \geq c_2 \geq \dots \geq c_n$, then there exists the upper-triangular factorization in the form of (4.37) with $\kappa_j = c_j$ for $j = 1, 2, \dots, n$ by induction on the order n of $G(t)$.

(i) If $n = 1$, the conclusion is clear by (4.34).

(ii) Suppose that the desired result has been established for all upper-triangular matrices of order less than n , then we must show that the conclusion also holds for $G(t)$ defined by (4.33) of order n . To this end, let us partition $G(t)$ in the form

$$G(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ 0 & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}(t) \end{bmatrix} \triangleq \begin{bmatrix} a_{11}(t) & \mathbf{C}(t) \\ \mathbf{O}_{(n-1) \times 1} & \tilde{G}(t) \end{bmatrix}.$$

Obviously, the upper-triangular matrix $\tilde{G}(t)$ of order $n-1$ admits $\det(\tilde{G}(t)) \neq 0$ on L and $c_2 \geq c_3 \geq \dots \geq c_n$. Consequently, by the inductive hypothesis, there exists the following generalized β -analytic factorization of $\tilde{G}(t)$:

$$\tilde{G}(t) = \tilde{G}_+(\alpha(t)) \tilde{\Lambda}(t) \tilde{G}_-(t) \quad (4.46)$$

with

$$\tilde{\Lambda}(t) = \text{diag}[t^{c_2} |t|^{2c_2\theta}, t^{c_3} |t|^{2c_3\theta}, \dots, t^{c_n} |t|^{2c_n\theta}], \quad (4.47)$$

where $c_2 \geq c_3 \geq \dots \geq c_n$, and $\tilde{G}_\pm$ are upper-triangular matrices.

Let

$$\mathbf{C}(z) = \begin{cases} a_{11}^+(z)[H_2(z), H_3(z), \dots, H_n(z)], & z \in D^+, \\ -z^{-c_1}|z|^{-2\theta c_1}[H_2(z), H_3(z), \dots, H_n(z)]\tilde{\Lambda}(z)\tilde{G}_-(z), & z \in D^-, \end{cases}$$

where

$$H_j(z) = \begin{cases} \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\mu_j(\hat{\alpha}(t)) dt}{t-z|z/t|^{2\theta}} + \frac{\beta t \mu_j(\hat{\alpha}(t)) d\bar{t}}{\bar{t}(t-z|z/t|^{2\theta})} \right], & z \in D^+, \\ \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\mu_j(t) dt}{t-z|z/t|^{2\theta}} + \frac{\beta t \mu_j(t) d\bar{t}}{\bar{t}(t-z|z/t|^{2\theta})} \right], & z \in D^- \end{cases}$$

with μ_j ($j = 2, 3, \dots, n$) being the unique solution of

$$(\mathbf{K}\mu_j)(t) = h_j(t) \quad \text{in which } [h_2(t), h_3(t), \dots, h_n(t)] \triangleq \frac{\mathbf{C}(t)\tilde{G}_-^{-1}(t)\tilde{\Lambda}^{-1}(t)}{a_{11}^+(\alpha(t))}.$$

By Theorem 4.1 and a straightforward calculation, we get

$$t^{c_1}|t|^{2\theta c_1}a_{11}^+(\alpha(t))\mathbf{C}^-(t) + \mathbf{C}^+(\alpha(t))\tilde{\Lambda}(t)\tilde{G}_-(t) = \mathbf{C}(t), \quad t \in L.$$

Therefore, the following identity

$$G(t) = \begin{bmatrix} a_{11}^+(\alpha(t)) & \mathbf{C}^+(\alpha(t)) \\ \mathbf{O}_{(n-1) \times 1} & \tilde{G}_+(\alpha(t)) \end{bmatrix} \begin{bmatrix} t^{c_1}|t|^{2c_1\theta} & \mathbf{O}_{1 \times (n-1)} \\ \mathbf{O}_{(n-1) \times 1} & \tilde{\Lambda}(t) \end{bmatrix} \begin{bmatrix} a_{11}^-(t) & \mathbf{C}^-(t) \\ \mathbf{O}_{(n-1) \times 1} & \tilde{G}_-(t) \end{bmatrix} \quad (4.48)$$

holds, where $\mathbf{C}^\pm \in [H_\pm^{\mu, \beta}(L)]^{1 \times (n-1)}$, $a_{11}^\pm$ defined by (4.35), and the upper-triangular matrix $\tilde{G}_\pm \in [H_\pm^{\mu, \beta}(L)]^{(n-1) \times (n-1)}$. Consequently, when $c_1 \geq c_2 \geq \dots \geq c_n$, we actually have proved that the right-hand side of (4.48) with $\tilde{\Lambda}(t)$ given by (4.47), is just a generalized β -analytic factorization with partial indices $\kappa_j = c_j$ for $j = 1, 2, \dots, n$.

Finally, we construct the explicit factorization of $G(t)$ by (4.48) when $c_1 \geq c_2 \geq \dots \geq c_n$. That is to say, there exist upper-triangular matrices $G_\pm \in [H_\pm^{\mu, \beta}(L)]^{n \times n}$ with $\inf_{t \in L} |\det(G_\pm(t))| > 0$ such that

$$G(t) = G_+(\alpha(t))\Lambda(t)G_-(t), \quad t \in L \quad (4.49)$$

with

$$\Lambda(t) = \text{diag}[t^{c_1}|t|^{2c_1\theta}, t^{c_2}|t|^{2c_2\theta}, \dots, t^{c_n}|t|^{2c_n\theta}].$$

To do so, it is enough to find a set of n -dimensional column vectors

$${}^{mn}\Phi(z) \triangleq \begin{bmatrix} {}^m\Psi(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{in which } {}^m\Psi(z) = \begin{bmatrix} \Psi_{1m}(z) \\ \Psi_{2m}(z) \\ \vdots \\ \Psi_{mm}(z) \end{bmatrix}, \quad m = 1, 2, \dots, n, \quad (4.50)$$

such that $\det([\overset{1n}{\Phi}, \overset{2n}{\Phi}, \dots, \overset{nn}{\Phi}](z)) \neq 0$ for $z \in D^+ \cup D^-$, $\Psi_{pm} \in H_1^\beta(D^+ \cup D^-)$, and the corresponding boundary values $\Psi_{pm}^\pm(t) \in H^\mu(L)$ satisfying

$$\begin{cases} \Psi^+_{jm}(\alpha(t)) = t^{-c_m} |t|^{-2\theta c_m} G_m(t) \Psi^-_{jm}(t), & t \in L, \\ \text{Ord}(\Psi_{jm}, \infty) \leq 0, & m = 1, 2, \dots, n, j = 1, 2, \dots, m \end{cases} \quad (4.51)$$

with

$$G_m(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1m}(t) \\ 0 & a_{22}(t) & \cdots & a_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mm}(t) \end{bmatrix}, \quad (4.52)$$

where a_{ij} are the same as the one in (4.33), $i, j = 1, 2, \dots, m$.

To avoid any misunderstanding, we often alternately use c_{ij} , c_{ij} in the sequel. Equation (4.51) is obviously equivalent to

$$\begin{cases} \Psi^+_{mm}(\alpha(t)) = a_{mm}(t) t^{-c_m} |t|^{-2c_m\theta} \Psi^-_{mm}(t), \\ \Psi^+_{m-1,m}(\alpha(t)) = t^{-c_m} |t|^{-2c_m\theta} [a_{m-1,m-1}(t) \Psi^-_{m-1,m}(t) + a_{m-1,m}(t) \Psi^-_{mm}(t)], \\ \vdots \\ \Psi^+_{1m}(\alpha(t)) = a_{11}(t) t^{-c_m} |t|^{-2c_m\theta} \Psi^-_{1m}(t) + \cdots + a_{1m}(t) t^{-c_m} |t|^{-2c_m\theta} \Psi^-_{mm}(t), \\ \text{Ord}(\Psi_{jm}, \infty) \leq 0, & m = 1, 2, \dots, n, j = 1, 2, \dots, m. \end{cases} \quad (4.53)$$

To obtain the desired particular solutions of the above problem, set

$$X_{m-k,m}(z) = \begin{cases} \exp\{(C_L^\beta(\rho_{m-k,m} \circ \hat{\alpha}))(z)\}, & z \in D^+, \\ z^{c_m - c_{m-k}} |z|^{2(c_m - c_{m-k})\theta} \exp\{(C_L^\beta \rho_{m-k,m})(z)\}, & z \in D^-, \end{cases}$$

for $k = 0, 1, \dots, m-1; m = 1, 2, \dots, n$

with $\rho_{m-k,m}$ being the unique solution of the equation

$$(\mathbf{K} \rho_{m-k,m})(t) = \ln(t^{-c_m - k} |t|^{-2c_m - k\theta} a_{m-k,m-k}(t)).$$

Furthermore, suppose that

$$\begin{aligned} \phi_{mm}(z) &= 1, \quad z \in D^+ \cup D^-, m = 1, 2, \dots, n, \\ \phi_{m-k,m}(z) &= \begin{cases} (C_L^\beta(\hat{\rho}_{m-k,m} \circ \hat{\alpha}))(z), & z \in D^+, \\ (C_L^\beta \hat{\rho}_{m-k,m})(z), & z \in D^-, \end{cases} \quad \text{for } \begin{cases} k = 1, 2, \dots, m-1, \\ m = 2, 3, \dots, n \end{cases} \end{aligned}$$

with $\hat{\rho}_{m-k,m}$ being the unique solution of

$$(\mathbf{K} \hat{\rho}_{m-k,m})(t) = \frac{\sum_{j=1}^k a_{m-k,m-k+j}(t) \Psi^-_{m-k+j,m}(t)}{t^{c_m} |t|^{2c_m\theta} X^+_{m-k,m}(\alpha(t))}.$$

By Theorem 4.1, using an iterated method, we easily get that

$$\Psi_{m-k,m}(z) = \begin{cases} X_{m-k,m}(z)\phi_{m-k,m}(z), & 1 \leq k \leq m-1, m=2,3,\dots,n, \\ X_{m,m}(z), & k=0, m=1,2,\dots,n \end{cases} \quad (4.54)$$

is a solution of (4.53) (or (4.51)).

Let

$$\begin{cases} G_+(z) \triangleq [\overset{1n}{\Psi}(z), \overset{2n}{\Psi}(z), \dots, \overset{nn}{\Psi}(z)], & z \in D^+, \\ G_-^{-1}(z) \triangleq [\overset{1n}{\Psi}(z), \overset{2n}{\Psi}(z), \dots, \overset{nn}{\Psi}(z)], & z \in D^-, \\ \Lambda(z) = \text{diag}[z^{c_1}|z|^{2c_1\theta}, z^{c_2}|z|^{2c_2\theta}, \dots, z^{c_n}|z|^{2c_n\theta}], \end{cases} \quad (4.55)$$

where $\overset{mn}{\Psi}(z)$ and $\Psi_{m-k,m}$, $k=0,1,\dots,m-1$, $m=1,2,\dots,n$, are determined by (4.50) and (4.54) respectively. It is easy to check that the factors given by (4.55) not only satisfy (4.49), but also are exactly consistent with (4.38) and (4.42). The proof is complete. $\square$

Remark 4.3 The constraint condition that $c_1 \geq c_2 \geq \dots \geq c_n$ is not essentially necessary. In fact, by using an elementary transformation and some tedious discussions, this condition can be removed automatically.

4.2 Riemann–Hilbert problem with shift

With the previous results in hand, we can solve the Riemann–Hilbert problem with shift for generalized β -analytic functions of order n with $n > 1$.

Theorem 4.3 Let $C_L^\beta, K(\tau, t), \mathbf{K}$, and W_α^β be defined by (4.2), (4.3), (4.5), and (4.11) respectively, and $q_j = \frac{1}{2\pi}[\arg G_j(t)]_L$ with $q_1 \geq q_2 \geq \dots \geq q_n$. Then problem (4.18) is solvable if and only if

$$\int_L (\rho(t))_j \left[t^l |t|^{2(l+1)\theta} dt + \frac{\beta}{t} t^{l+1} |t|^{2(l+1)\theta} d\bar{t} \right] = 0, \quad (4.56)$$

and the solution of (4.18) can be expressed as

$$\Phi(z) = \Phi_1(z) + \bar{z}\Phi_2(z) + \dots + \bar{z}^{n-1}\Phi_n(z), \quad z \in D^+ \cup D^-, \quad (4.57)$$

where $j=1,2,\dots,n$, $l=0,1,\dots,-q_j-m+\lceil \frac{j-1}{1+2\theta} \rceil -2$, $q_j \leq \lceil \frac{j-1}{1+2\theta} \rceil -m-2$, $(\rho(t))_j$ denotes the j th component of column vector $\rho(t)$, and

$$[\Phi_1(z), \Phi_2(z), \dots, \Phi_n(z)]^T \triangleq X(z)(\Psi(z) + \mathbf{Q}(z)), \quad z \in D^+ \cup D^- \quad (4.58)$$

with

$$\Psi(z) = \begin{cases} \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\rho(\hat{\alpha}(\tau))d\tau}{\tau-z|z/\tau|^{2\theta}} + \frac{\beta\tau\rho(\hat{\alpha}(\tau))d\bar{\tau}}{\bar{\tau}(\tau-z|z/\tau|^{2\theta})} \right], & z \in D^+, \\ \frac{1}{2\pi i(1-\beta)} \int_L \left[\frac{\rho(\tau)d\tau}{\tau-z|z/\tau|^{2\theta}} + \frac{\beta\tau\rho(\tau)d\bar{\tau}}{\bar{\tau}(\tau-z|z/\tau|^{2\theta})} \right], & z \in D^-, \end{cases} \quad (4.59)$$

where ρ is the unique solution of the vector-valued equation $(\mathbf{K}\rho)(t) = (X^+(\alpha(t)))^{-1}\mathbf{h}(t)$ with $\mathbf{h}(t) = [h_1(t), h_2(t), \dots, h_n(t)]^T$, in which

$$h_j(t) = \sum_{k=j}^n \frac{(-1)^{j+k}}{(k-j)!(j-1)!} \overline{\alpha(t)}^{k-j} g_k(t), \quad j = 1, 2, \dots, n, \quad (4.60)$$

and

$$X(z) = \begin{cases} G_+(z), & z \in D^+, \\ [G_-(z)]^{-1} \Lambda^{-1}(z), & z \in D^-, \end{cases} \quad (4.61)$$

$$\Lambda(z) = \text{diag}[z^{q_1}|z|^{2q_1\theta}, z^{q_2}|z|^{2q_2\theta}, \dots, z^{q_n}|z|^{2q_n\theta}], \quad (4.62)$$

$$G_{\pm}^{\pm 1}(z) = \begin{bmatrix} Y_{11}(z)\varphi_{11}(z) & Y_{12}(z)\varphi_{12}(z) & \cdots & Y_{1n}(z)\varphi_{1n}(z) \\ 0 & Y_{22}(z)\varphi_{22}(z) & \cdots & Y_{2n}(z)\varphi_{2n}(z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_{nn}(z)\varphi_{nn}(z) \end{bmatrix}, \quad z \in D^{\pm}, \quad (4.63)$$

in which

$$Y_{pl}(z) = \begin{cases} \exp\{(C_L^{\beta}(\rho_p \circ \hat{\alpha}))(z)\}, & z \in D^+, \\ z^{q_l - q_p}|z|^{2(q_l - q_p)\theta} \exp\{(C_L^{\beta}\rho_p)(z)\}, & z \in D^-, \end{cases} \quad (l \geq p, l = 1, \dots, n) \quad (4.64)$$

with ρ_p being the unique solution of the equation $(\mathbf{K}\rho_p)(t) = \ln(t^{-q_p}|t|^{-2q_p\theta}G_p(t))$ for $p = 1, 2, \dots, n$, and

$$\varphi_{pp}(z) = 1, \quad z \in D^+ \cup D^-, p = 1, 2, \dots, n, \quad (4.65)$$

$$\varphi_{pl}(z) = \begin{cases} (C_L^{\beta}(\tilde{\rho}_{pl} \circ \hat{\alpha}))(z), & z \in D^+, \\ (C_L^{\beta}\tilde{\rho}_{pl})(z), & z \in D^-, \end{cases} \quad (l > p, l = 2, 3, \dots, n) \quad (4.66)$$

with $\tilde{\rho}_{pl}$ being the unique solution of the equation

$$(\mathbf{K}\tilde{\rho}_{pl})(t) = \begin{cases} \frac{a_{pl}Y_{ll}^-(t)}{t^{q_l}|t|^{2q_l\theta}Y_{pl}^+(\alpha(t))}, & l - p = 1, \\ \frac{\sum_{j=1}^{l-p-1} \{a_{p,p+j}(t)Y_{p+j,l}^-(t)\frac{1}{2}[(S_L^{\beta}\tilde{\rho}_{p+j,l})(t) - \tilde{\rho}_{p+j,l}(t)]\} + a_{pl}(t)Y_{ll}^-(t)}{t^{q_l}|t|^{2q_l\theta}Y_{pl}^+(\alpha(t))}, & l - p \geq 2 \end{cases}$$

for $l = 2, 3, \dots, n$, in which

$$a_{pl}(t) = \begin{cases} \sum_{k=p}^l \frac{(-1)^{p+k}(l-1)!}{(k-p)!(p-1)!(l-k)!} G_k(t) \overline{\alpha(t)}^{k-p-l-k} t^{-l-k}, & l \geq p, \\ 0, & l < p \end{cases} \quad (4.67)$$

for $p, l = 1, 2, \dots, n$. Moreover,

$$\mathbf{Q}(z) = \begin{bmatrix} Q_{q_1+m-i_1}^1(W_{\alpha}^{\beta}(z)) \\ Q_{q_2+m-i_2}^1(W_{\alpha}^{\beta}(z)) \\ \vdots \\ Q_{q_n+m-i_n}^n(W_{\alpha}^{\beta}(z)) \end{bmatrix} \quad \text{with } i_j = \lceil (j-1)/(1+2\theta) \rceil, \quad (4.68)$$

where $Q_{q_k+m-i_k}^k(z)$ denotes a polynomial of degree not larger than $q_k + m - i_k$, and is regarded as 0 if $q_k + m - i_k < 0$ for $k \in \{1, 2, \dots, n\}$. When $q_k \geq i_k - m - 1$ for some k , the corresponding part of solvable conditions (4.56) automatically disappears, and in this case, its solution is still expressed by (4.58).

Proof By Theorem 2.2, there exist uniquely $\Phi_j \in H_1^\beta(D^+ \cup D^-)$ for $j = 1, 2, \dots, n$ such that

$$\Phi(z) = \Phi_1(z) + \bar{z}\Phi_2(z) + \dots + \bar{z}^{n-1}\Phi_n(z), \quad z \in D^+ \cup D^-. \quad (4.69)$$

Let

$$F(z) = \begin{bmatrix} \Phi_1(z) \\ \Phi_2(z) \\ \vdots \\ \Phi_n(z) \end{bmatrix}, \quad H(z) = \begin{bmatrix} \Phi(z) \\ \partial_{\bar{z}}^\beta \Phi(z) \\ \vdots \\ \partial_{\bar{z}^{n-1}}^\beta \Phi(z) \end{bmatrix}. \quad (4.70)$$

Then, by (4.69), we have

$$H(z) = C(z)F(z), \quad (4.71)$$

where

$$C(z) = \{c_{r,k}(z)\}_{n \times n}, \quad \text{with } c_{r,k}(z) = \begin{cases} \frac{(k-1)!}{(k-r)!} \bar{z}^{k-r}, & k \geq r, \\ 0, & k < r. \end{cases} \quad (4.72)$$

In addition, it is easy to get that

$$C^{-1}(z) = \{b_{k,j}(z)\}_{n \times n}, \quad \text{with } b_{k,j}(z) = \begin{cases} \frac{(-1)^{k+j}}{(j-k)!(k-1)!} \bar{z}^{j-k}, & j \geq k, \\ 0, & j < k. \end{cases} \quad (4.73)$$

Hence,

$$\Phi_j(z) = \sum_{k=j}^n \frac{(-1)^{k+j} \bar{z}^{k-j}}{(k-j)!(j-1)!} \partial_{\bar{z}^{k-1}}^\beta \phi(z), \quad j = 1, 2, \dots, n.$$

Substituting the last identity and (4.71) into (4.18), we obtain the following vector-valued problem:

$$\begin{cases} F^+(\alpha(t)) = G(t)F^-(t) + h(t), & t \in L, \\ \text{Ord}(\Phi_j, \infty) \leq m - \lceil \frac{j-1}{1+2\theta} \rceil, & j = 1, 2, \dots, n, \end{cases} \quad (4.74)$$

where the symbol $\lceil a \rceil$ is defined as above,

$$G(t) = \{g_{pl}(t)\}_{n \times n} = C^{-1}(\alpha(t)) \begin{bmatrix} G_1(t) & 0 & \cdots & 0 \\ 0 & G_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & G_n(t) \end{bmatrix} C(t), \quad (4.75)$$

and

$$\mathbf{h}(t) = C^{-1}(\alpha(t)) [g_1(t), g_2(t), \dots, g_n(t)]^T. \quad (4.76)$$

By simple calculation, we see from (4.72), (4.73), and (4.75) that $g_{pl}(t) = a_{pl}(t)$ with a_{pl} defined by (4.67), and $\mathbf{h}(t) = [h_1(t), h_2(t), \dots, h_n(t)]^T$ with $h_j(t) (j = 1, 2, \dots, n)$ given by (4.60).

It follows from (4.67) and (4.75) that

$$\frac{1}{2\pi} [\arg g_{jj}(t)]_L = \frac{1}{2\pi} [\arg a_{jj}(t)]_L = \frac{1}{2\pi} [\arg G_j(t)]_L = q_j.$$

Since $q_1 \geq q_2 \geq \dots \geq q_n$, by Theorem 4.2, upper-triangular matrices $G(t) = \{a_{pl}(t)\}_{n \times n}$ defined by (4.67) have a factorization in the following form:

$$G(t) = G_+(\alpha(t)) \Lambda(t) G_-(t), \quad t \in L \quad (4.77)$$

with Λ , G_+ , G_- defined by (4.62) and (4.63) respectively.

Let

$$\psi(z) = \begin{cases} (C_L^\beta(\rho \circ \hat{\alpha}))(z), & z \in D^+, \\ (C_L^\beta \rho)(z), & z \in D^- \end{cases} \quad (4.78)$$

with ρ being the unique solution of the equation $(\mathbf{K}\rho)(t) = (X^+(\alpha(t)))^{-1} \mathbf{h}(t)$, where $\mathbf{K}$, $\mathbf{h}(t)$, and $X(z)$ are defined by (4.5), (4.60), and (4.61) respectively.

Let

$$\mathbf{U}(z) = [U_1, U_1, \dots, U_n]^T(z) = X^{-1}(z) \mathbf{F}(z) - \psi(z), \quad z \in D^+ \cup D^-. \quad (4.79)$$

Denote that $G_-(z) = \{G_{jk}(z)\}_{n \times n}$, and $\psi(z) = [\psi_1(z), \dots, \psi_n(z)]^T$, where $G_-(z)$ is given by (4.63). By straightforward calculation, one has

$$U_j(z) = z^{q_j} |z|^{2q_j\theta} \sum_{k=j}^n G_{jk}(z) \Phi_k(z) - \psi_j(z), \quad z \in D^-.$$

Thus, problem (4.74) is equivalently changed into the problem as follows:

$$\begin{cases} \mathbf{U}^+(\alpha(t)) = \mathbf{U}^-(t), & t \in L, j = 1, 2, \dots, n, \\ \text{Ord}(U_j, \infty) \leq q_j + m - \lceil (j-1)/(1+2\theta) \rceil, \end{cases}$$

where $U_j \in H_1^\beta(D^+ \cup D^-)$. It follows from Proposition 4.2 that

$$\mathbf{U}(z) = \begin{bmatrix} Q_{q_1+m-i_1}^1(W_\alpha^\beta(z)) \\ Q_{q_2+m-i_2}^2(W_\alpha^\beta(z)) \\ \vdots \\ Q_{q_n+m-i_n}^n(W_\alpha^\beta(z)) \end{bmatrix}, \quad i_j = \lceil (j-1)/(1+2\theta) \rceil,$$

where $Q_{q_k+m-i_k}^k(W_\alpha^\beta(z))$ denotes a polynomial of degree not more than $q_k + m - i_k$ with respect to $W_\alpha^\beta(z)$. As a usual convention, $Q_{q_k+m-i_k}^k(W_\alpha^\beta(z)) \equiv 0$ as $q_k + m - i_k < 0$. Following from (4.79) and Theorem 4.1, the proof of this theorem is complete. $\square$

Remark 4.4 Theorem 4.3 is obtained by technically constructing weakly singular kernels and explicit canonical matrix with shift under the completely new background, i.e., β -analytic functions. Comparing with the analytic extension and transformation methods in [18, 19, 22], which are only applicable to the unit circle, our approach has some advantages for extensive applications to the general Lyapunov curve. For example, when $\beta = 0$ and L is the unit circle, Theorem 4.3 is exactly reduced to the results in [18, 19, 22, 36, 37]. Furthermore, when $\alpha(t) = t$ or $n = 1$, $\beta = 0$, $\alpha(t) = t$, Theorem 4.3 is just reduced to the corresponding results in [5, 7, 27–30]. Moreover, by developing a new theory of canonical matrix, Theorem 4.3 can be also applicable to the results in [38] under some suitable assumptions.

Acknowledgements

The authors thank Prof. H. Liu for many valuable comments and some useful discussions.

Funding

This work was supported by Macao Science and Technology Development Fund-MSAR. Ref. 045/2015/A2, and NSFC (11126065, 11401254 and 11701597).

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors completed the paper together. All authors read and approved the final manuscript.

Author details

¹School of Science, Nanjing University of Posts and Telecommunications, Nanjing, P.R. China. ²Faculty of Information Technology, Macau University of Science and Technology, Macau, P.R. China. ³Department of Mathematics, Jinan University, Guangzhou, P.R. China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 10 December 2018 Accepted: 20 May 2019 Published online: 28 May 2019

References

1. Muskhelishvili, N.I.: Singular Integral Equations. Noordhoff, Leyden (1968)
2. Gakhov, F.D.: Boundary Value Problems. Pergamon, Oxford (1966)
3. Lu, J.-K.: Boundary Value Problems for Analytic Functions. World Scientific, Singapore (1993)
4. Begehr, H.: Complex Analytic Methods for Partial Differential Equations: An Introductory Text. World Scientific, Singapore (1994)
5. Vekua, I.N.: Generalized Analytic Functions. Pergamon, Oxford (1962)
6. Vekua, N.P.: Systems of Singular Integral Equations and Some Boundary Value Problems. Noordhoff, Groningen (1967)
7. Litvinchuk, G.S.: Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shifts. Kluwer Academic, Dordrecht (2000)
8. Bantsuri, R.: Contact problems of plane elasticity theory and related boundary value problems of function theory. Mem. Differ. Equ. Math. Phys. **62**, 1–170 (2014)
9. Fokas, A.S.: A Unified Approach to Boundary Value Problems. CBMS-NSF Region Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia (2008)
10. Paataashvili, V.: Haseman's problem in classes of functions representable by the Cauchy type integral with density from $L^{p(\cdot)}(\Gamma; \rho)$. Mem. Differ. Equ. Math. Phys. **50**, 139–162 (2010)
11. He, F., Ku, M., Sommen, F., Bernstein, S.: Riemann–Hilbert problems for null-solutions to iterated generalized Cauchy–Riemann equations in axially symmetric domains. Comput. Math. Appl. **71**(10), 1990–2000 (2016)
12. Balk, M.B.: Polyanalytic Functions. Akademie Verlag, Berlin (1991)
13. Begehr, H., Schmiersau, D.: The Schwarz problem for polyanalytic functions. Z. Anal. Anwend. **24**(2), 341–351 (2005)

14. Wang, Y., Du, J.: Hilbert boundary value problem of polyanalytic functions on the unit circumference. *Complex Var. Elliptic Equ.* **51**(8–11), 923–943 (2006)
15. Begehr, H., Du, J., Wang, Y.: A Dirichlet problem for polyharmonic functions. *Ann. Mat. Pura Appl.* **187**(3), 435–457 (2008)
16. Du, Z., Guo, G., Wang, N.: Decompositions of functions and Dirichlet problems in the unit disc. *J. Math. Anal. Appl.* **362**(1), 1–16 (2010)
17. Du, Z., Qian, T., Wang, J.: L^p polyharmonic Dirichlet problems in regular domains II: the upper half plane. *J. Differ. Equ.* **252**(2), 1789–1812 (2012)
18. Wang, Y., Du, J.: Haseman boundary value problems for metaanalytic functions with different shifts on the unit circumference. *Complex Var. Elliptic Equ.* **53**(4), 325–342 (2008)
19. Wang, Y., Du, J.: Mixed boundary value problems for a pair of metaanalytic and analytic functions. *J. Math. Anal. Appl.* **369**(2), 510–524 (2010)
20. Karlovich, Y.I.: The Haseman boundary value problem with slowly oscillating coefficients and shifts. *Oper. Theory, Adv. Appl.* **259**, 463–500 (2017)
21. Wang, Y., Ku, M., Zheng, B.: Single periodic Riemann problems for null-solutions to polynomially Cauchy–Riemann equations. *Complex Var. Elliptic Equ.* **63**(6), 749–769 (2018)
22. Ku, M., He, F., Wang, Y.: Riemann–Hilbert problems for Hardy space of meta-analytic functions on the unit disc. *Complex Anal. Oper. Theory* **12**(2), 457–474 (2018)
23. Bojarski, B.: *Infinitesimal Geometry of Quasiconformal and Bi-Lipschitz Mapping in the Plane*. European Mathematical Society, Zürich (2013)
24. Baratchart, L., Leblond, J., Rigat, S., Russ, E.: Hardy space of the conjugate Beltrami equation. *J. Funct. Anal.* **259**(2), 384–427 (2010)
25. Manjavidze, G., Manjavidze, N.: Boundary-value problems for analytic and generalized analytic functions. *J. Math. Sci.* **160**(6), 745–821 (2009)
26. Tungatarov, A.: On properties of an integral operator in classes of summable functions. *Izv. AN Kazakh. SSR. Ser. Fiz.-Mat.* **132**(5), 58–62 (1985) (in Russian)
27. Blaya, R.A., Reyes, J.B., Peña, D.P.: On the jump problem for β -analytic functions. *Complex Var. Elliptic Equ.* **51**(8–11), 763–775 (2006)
28. Blaya, R.A., Reyes, J.B.: A properties of the β -Cauchy-type integral with continuous density. *Ukr. Math. J.* **60**(11), 1683–1690 (2008)
29. Blaya, R.A., Reyes, J.B., Peña, D.P., Vilair, J.M.: Riemann boundary value problem for β -analytic functions. *Int. J. Pure Appl. Math.* **42**(1), 19–37 (2008)
30. Katz, D.B., Kats, B.A.: Integral representations for solutions of some types of the Beltrami equations. *Russ. Math.* **62**(3), 18–22 (2018)
31. Blaya, R.A., Reyes, J.B., Vilair, J.M.: A jump problem for β -analytic functions in domains with fractal boundaries. *Rev. Mat. Complut.* **23**(1), 105–111 (2010)
32. Katz, D.B., Kats, B.A.: Riemann boundary value problems on nonrectifiable curves for certain Beltrami equation. *Math. Methods Appl. Sci.* **41**(8), 2507–2514 (2018)
33. Blaya, R.A., Reyes, J.B., Vilair, J.M.: The Riemann boundary problem for β -analytic functions over D-summable closed curves. *Int. J. Pure Appl. Math.* **75**(4), 441–453 (2012)
34. Fatulaev, B.F.: The main Haseman type boundary value problem for metaanalytic function in the case of circular domains. *Math. Model. Anal.* **6**(1), 68–76 (2001)
35. Rasulov, K.M., Titov, O.A.: On the solution of the second fundamental three-element boundary value problem of Carleman type in classes of functions bianalytic in the disk. *Lith. Math. J.* **46**(3), 337–346 (2006)
36. Wang, Y., Du, J.: On Haseman boundary value problem for a class of meta-analytic functions. *Acta Math. Sci.* **31B**(1), 39–48 (2011)
37. Wang, Y., Du, J.: On Haseman boundary value problems for a class of metaanalytic functions with different factors on the unit circumference. *Math. Methods Appl. Sci.* **33**(4), 576–584 (2010)
38. Wang, Y., Han, P., Wang, Y.: On Riemann problem of automorphic polyanalytic functions connected with a rotation group. *Complex Var. Elliptic Equ.* **60**(8), 1033–1057 (2015)
39. Reyes, J.B., Peña, D.P.: Some higher order Cauchy–Pompeiu integral representations. *Integral Transforms Spec. Funct.* **16**(8), 615–624 (2005)
40. Blaya, R.A., Peña, D.P., Reyes, J.B.: Conjugate hyperharmonic functions and Cauchy type integrals in Douglis analysis. *Complex Var. Theory Appl.* **48**(12), 1023–1039 (2003)
41. Krasnoselskii, M.A., Zabreiko, P.P., Pustynnik, E.I., Sobolevskii, P.E.: *Integral Operators in Spaces of Summable Functions*. Noordhoff, Netherlands (1976)
42. Ehrhardt, T., Speck, F.O.: Transformation technique towards the factorization of non-rational 2×2 matrix functions. *Linear Algebra Appl.* **353**(1–3), 53–90 (2002)
43. Clancy, K., Gohberg, I.: *Factorization of Matrix Functions and Singular Integral Operators*. Birkhäuser, Boston (1981)