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Solvability for boundary value problem of the general Schrödinger equation with general superlinear nonlinearity

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Abstract

This paper is mainly concerned with the boundary value problems for the general Schrödinger equation with general superlinear nonlinearity introduced in (Sun et al. in J. Inequal. Appl. 2018:100, 2018). We firstly study a new algorithm for finding the meromorphic solution for the mentioned equation via meromorphic inequalities presented in (Xu in J. Math. Study 38(1):71–86, 2015). Then we deal with the necessary and sufficient conditions of convergence and obtain the general solutions and the conditions of solvability for the mentioned equation by means of the meromorphic inequalities for the classical boundary value problems developed in (Guillot in J. Nonlinear Math. Phys. 25(3):497–508, 2018). These results generalize some previous results concerning the asymptotic behavior of solutions of non-delay systems of Schrödinger equations by applying the maximum principle approach with respect to the Schrödinger operator in (Wan in J. Inequal. Appl. 2017:104, 2017).

Keywords: Boundary value problem; Maximum principle approach; General Schrödinger equation

1 Introduction

This article is devoted to the study of the general Schrödinger equation with general superlinear nonlinearity in $\mathbb{R}^n$. To clarify our aim, we will first introduce the following quasilinear Schrödinger equation (see [26, 29, 42, 44]):

$$\begin{aligned} &(-\Delta)^\gamma \chi + \mathcal{V}(x)\chi = f(x,\chi), \\ &\chi \in \Lambda^\gamma(\mathbb{R}^n,\mathbb{R}), \end{aligned} \tag{1.1}$$

where $n \geq 2$, $\gamma \in (0,1)$, $x \in \mathbb{R}^n$, $\mathcal{V} \in C(\mathbb{R}^n,\mathbb{R})$ satisfying some new conditions, and f is only locally defined near the origin with respect to χ.

Problem (1.1) is related to the existence of nontrivial meromorphic solutions for the following general Schrödinger equations (see [2, 12, 20] for more details):

$$i\frac{\partial\psi}{\partial t} = (-\Delta)^\gamma\psi + \big(\mathcal{V}(x)+\omega\big)\psi - f(x,\psi), \tag{1.2}$$

where i is the imaginary unit, $\gamma \in (0,1)$, ω is a constant, $(-\Delta)^\gamma$ is the fractional Laplacian operator of order γ and $\psi : \mathbb{R}^3 \times [0,+\infty) \to \mathbb{C}$. Using variational methods, Bahrouni, Ounaies and Radulescu [1] proved the existence of two solutions of bound state solutions of sublinear Schrödinger equations with lack of compactness. Díaz, Gómez-Castro and Vázquez [11] considered the Dirichlet problem for the stationary Schrödinger fractional Laplacian equation posed in a bounded domain with zero outside conditions. Fiscella, Pucci and Saldi [13] dealt with the existence of nontrivial nonnegative solutions of Schrödinger–Hardy systems driven by two possibly different fractional Laplacian operators, via various variational methods. Bisci and Radulescu [5] studied the existence of multiple ground state solutions for a class of parametric fractional Schrödinger equations. Rybalko [28] studied an initial value problem for the one-dimensional non-stationary linear Schrödinger equation with a point singular potential. Wen and Zhao [39] presented a medium-shifted splitting iteration method to solve the discretized linear system, in which the fast algorithm can be utilized to solve the Toeplitz linear system. Chen et al. [6] investigated the existence of nontrivial solutions and multiple solutions for nonlinear Schrödinger equations with unbounded potentials. Covei [10] investigated the existence and symmetry of positive solutions for a modified Schrödinger system under the Keller–Osserman type conditions. The existence of a Green function and a uniqueness result for the Cauchy–Dirichlet problem were obtained by Polidoro and Ragusa [25]. Xue, Lv and Tang [41] employed the mountain pass theorem to obtain the existence of a positive ground state solution for quasilinear Schrödinger equations with a general nonlinear term. Wen and Chen [38] used the non-Nehari manifold method to deal with the ground state solutions for an asymptotically periodic Schrödinger–Poisson systems involving Hartree-type nonlinearities.

For Eq. (1.1) it has been proved that it possesses wildly application fields in Hilbert spaces [18, 43], uniformly convex [12] and uniformly smooth Banach spaces [34, 36]. At present, there exist many effective algorithms working in it, such as the traditional Newton method [4, 21, 31, 32, 48], the wave method [45, 46], the BFGS method [16, 19], the Levenberg–Marquardt method [3, 42], the trust region method [7, 8, 44], the conjugate gradient algorithm [9, 27], the limited BFGS method [22], etc.

As in [30, 40], we set $\varphi(x) := \frac{1}{2}\|\chi(x)\|^2$, which is the meromorphic identity for Eq. (1.1). It is equivalent to the optimization problem defined by (see [40, Lemma 2.3] for more details)

$$\min \varphi(x), \tag{1.3}$$

where $x \in \Re^n$.

The meromorphic identity methods have a main objective is to solve the so-called meromorphic identity subproblem model to get the trial step ς_l,

$$\begin{aligned} &\operatorname{Min} \mathfrak{Tp}_l(\varsigma) = \frac{1}{2}\left\|\chi(x_l) + \nabla\chi(x_l)\varsigma\right\|^2 \\ &\quad \|\varsigma\| \le \triangle. \end{aligned}$$

In 2014, an adaptive meromorphic identity model sharing a set with their derivatives was designed as follows (see [35]):

$$\operatorname{Min} \phi_l(\varsigma) = \frac{1}{2}\left\|\chi(x_l) + \nabla\chi(x_l)\varsigma\right\|^2$$

$$\|\varsigma\| \le c^p \|\chi(x_l)\|^\gamma,$$

where p is a positive integer.

In 2015, another adaptive meromorphic identity subproblem sharing certain meromorphic functions was defined by (see [47])

$$\begin{aligned} &\operatorname{Min} \mathfrak{T}\mathfrak{q}_l(\varsigma) = \frac{1}{2}\|\chi(x_l) + \mathfrak{B}_l\varsigma\|^2 \\ &\|\varsigma\| \le c^p \|\chi(x_l)\|, \end{aligned} \tag{1.4}$$

where $\mathfrak{B}_l$ is defined by

$$\mathfrak{B}_{l+1} = \mathfrak{B}_l - \frac{\mathfrak{B}_l s_l s_l^{\mathfrak{T}} \mathfrak{B}_l}{s_l^{\mathfrak{T}} \mathfrak{B}_l s_l} + \frac{y_l {y_l}^{\mathfrak{T}}}{{y_l}^{\mathfrak{T}} s_l}, \tag{1.5}$$

where

$$y_l = \chi(x_{l+1}) - \chi(x_l)$$

and

$$s_l = x_{l+1} - x_l.$$

Recently, Guillot also considered the value distribution of meromorphic solutions for the nonlinear system $\chi(x)$ at x_k (see [17]),

$$\vartheta(x_k + \varsigma) = \chi(x_k) + \nabla\chi(x_k)^{\mathfrak{T}}\varsigma + \frac{1}{2}\mathfrak{T}_k \varsigma^2, \tag{1.6}$$

where $\nabla\chi(x_k)$ is the Jacobian matrix of $\chi(x)$ at x_k and $\mathfrak{T}_k$ is three dimensional symmetric tensor. It is not difficult to see that the above meromorphic identity model (1.6) has more approximation than the normal quadratical meromorphic identity model. It has been proved that the tensor is significantly simpler when only information from one past iterate is used (see [24, 47] for more details), which obviously decreases the complex computation of the three dimensional symmetric tensor $\mathfrak{T}_k$. Then the model (1.6) can be written as the following extension:

$$\vartheta(x_l + \varsigma) = \chi(x_l) + \nabla\chi(x_l)^{\mathfrak{T}}\varsigma + \frac{3}{2}\left(s_{l-1}^{\mathfrak{T}}\varsigma\right)^2 s_{l-1}. \tag{1.7}$$

Here our meromorphic identity subproblem model is defined as follows (see [14]):

$$\begin{aligned} &\operatorname{Min} \mathfrak{N}_l(\varsigma) = \frac{1}{2}\left\|\chi(x_l) + \mathfrak{B}_l\varsigma + \frac{3}{2}\left(s_{l-1}^{\mathfrak{T}}\varsigma\right)^2 s_{l-1}\right\|^2 \\ &\|\varsigma\| \le c^p \|\chi(x_l)\|^\gamma, \end{aligned} \tag{1.8}$$

where $\mathfrak{B}_l = \mathfrak{H}_l^{-1}$ and

$$\begin{aligned} \mathfrak{H}_{l+1} &= \mathfrak{V}_l^{\mathfrak{T}} \mathfrak{H}_l \mathfrak{V}_l + \rho_l s_l s_l^{\mathfrak{T}} \\ &= \mathfrak{V}_l^{\mathfrak{T}}\left[\mathfrak{V}_{l-1}^{\mathfrak{T}} \mathfrak{H}_{l-1} \mathfrak{V}_{l-1} + \rho_{l-1} s_{l-1} s_{l-1}^{\mathfrak{T}}\right]\mathfrak{V}_l + \rho_l s_l s_l^{\mathfrak{T}} \end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= \left[\mathfrak{V}_l^{\mathfrak{T}} \cdots \mathfrak{V}_{l-m+1}^{\mathfrak{T}}\right]\mathfrak{H}_{l-m+1}[\mathfrak{V}_{l-m+1} \cdots \mathfrak{V}_l] \\
&\quad + \rho_{l-m+1}\left[\mathfrak{V}_{l-1}^{\mathfrak{T}} \cdots \mathfrak{V}_{l-m+2}^{\mathfrak{T}}\right]s_{l-m+1}s_{l-m+1}^{\mathfrak{T}}[\mathfrak{V}_{l-m+2} \cdots \mathfrak{V}_{l-1}] \\
&\quad + \cdots \\
&\quad + \rho_l s_l s_l^{\mathfrak{T}},
\end{aligned}
\tag{1.9}
$$

where (see, e.g., [20, 33, 37])

$$
\rho_l = \frac{1}{s_l^{\mathfrak{T}} y_l}, \qquad \mathfrak{V}_l = I - \rho_l y_l s_l^{\mathfrak{T}}.
$$

Let ς_l^p be the solution of (1.8). Define

$$
A\varsigma_l\left(\varsigma_l^p\right) = \varphi\left(x_l + \varsigma_l^p\right) - \varphi(x_l), \tag{1.10}
$$

and the predict reduction by

$$
P\varsigma_l\left(\varsigma_l^p\right) = \mathfrak{N}_l\left(\varsigma_l^p\right) - \mathfrak{N}_l(0). \tag{1.11}
$$

Based on the definitions of $A\varsigma_l(\varsigma_l^p)$ and $P\varsigma_l(\varsigma_l^p)$, their radio can be defined by

$$
r_l^p = \frac{A\varsigma_l(\varsigma_l^p)}{P\varsigma_l(\varsigma_l^p)}. \tag{1.12}
$$

Therefore, the meromorphic identity model algorithm for solve (1.1) is stated as follows.

Algorithm

Initialization: Let $\mathfrak{B}_0 = \mathfrak{H}_0^{-1} \in \Re^n \times \Re^n$ is a symmetric and positive definite matrix. $x_0 \in \Re^n$ and $\varrho = 0$. ρ, c and ϵ are three positive constants. Let $l := 0$;

Step 1: Stops if $\|\chi(x_l)\| < \epsilon$ holds;

Step 2: Solve (1.8) with $\triangle = \triangle_l$ to obtain ς_l^{ϱ};

Step 3: Compute $A\varsigma_l(\varsigma_l^{\varrho})$, $P\varsigma_l(\varsigma_l^{\varrho})$, and the radio r_l^{ϱ}. If $r_l^{\varrho} < \rho$, let $\varrho = \varrho + 1$, go to Step 2. If $r_l^{\varrho} \geq \rho$, go to the next step;

Step 4: Set $x_{l+1} = x_l + \varsigma_l^{\varrho}$, $y_l = \chi(x_{l+1}) - \chi(x_l)$, update $\mathfrak{B}_{l+1} = \mathfrak{H}_{l+1}^{-1}$ by (1.9) if $y_l^{\mathfrak{T}}\varsigma_l^p > 0$, otherwise set $\mathfrak{B}_{l+1} = \mathfrak{B}_l$;

Step 5: Let $l := l + 1$ and $\varrho = 0$. Go to Step 1.

In this paper, we shall focus on convergence results of the above algorithm under the following assumptions.

Assumptions

(A) The level set Ω defined by

$$
\Omega = \left\{x \mid \varphi(x) \leq \varphi(x_0)\right\} \tag{1.13}
$$

is bounded.

(B) On an open convex set Ω_1 containing Ω, the nonlinear system $\chi(x)$ is twice continuously differentiable in Ω_1.

(C) The approximation relation

$$\left\| \left[\nabla\chi(x_l) - \mathfrak{B}_l\right]\chi(x_l)\right\| = O\left(\left\|\varsigma_l^p\right\|\right) \tag{1.14}$$

is true, where ς_l^p is the solution of the model (1.8).

(D) On Ω_1, the sequence matrices $\{\mathfrak{B}_l\}$ are uniformly bounded, namely there exist constants $0 < M_0 \leq M$ satisfying

$$M_s \leq \|\mathfrak{B}_l\| \leq M_l. \tag{1.15}$$

Assumption (B) means that there exists a positive real number M_L satisfying (see [15])

$$\left\|\nabla\chi(x_l)^{\mathfrak{T}}\nabla\chi(x_l)\right\| \leq M_L. \tag{1.16}$$

2 Preliminaries

In this section, we recall some preliminary results. For any $0 < \gamma < 1$, we define the space $\Lambda^\gamma(\mathbb{R}^n)$ by

$$\Lambda^\gamma\left(\mathbb{R}^n\right) = \left\{\chi \in L^2\left(\mathbb{R}^n\right) : \frac{|\chi(x) - \chi(z)|}{|x-z|^{\frac{n+2\gamma}{2}}} \in L^2\left(\mathbb{R}^n \times \mathbb{R}^n\right)\right\}$$

with

$$\|\chi\|_\gamma^2 = \int_{\mathbb{R}^n}\left|\chi(x)\right|^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\chi(x)-\chi(z)|^2}{|x-z|^{n+2\gamma}}\,dz\,dx.$$

Remark 2.1 Consider

$$\begin{aligned} &(-\Delta)^\gamma\chi + \widehat{\mathcal{V}}(x)\chi = \widehat{f}(x,\chi), \\ &\chi \in \Lambda^\gamma\left(\mathbb{R}^n, \mathbb{R}\right), \end{aligned} \tag{2.1}$$

where $\widehat{\mathcal{V}}(x) = \mathcal{V}(x) + a_0$ and $\widehat{\mathfrak{F}}(x,\chi) = \mathfrak{F}(x,\chi) + \frac{a_0}{2}\chi^2$.

Then (2.1) is equivalent to (1.8) and it easy to check that Assumptions (A) and (B), (C) still hold for $\widehat{\mathcal{V}}$ and $\widehat{\mathfrak{F}}$ provided that those hold for $\mathcal{V}$ and $\mathfrak{F}$. Hence, in what follows, we always assume that we have

$$\int_{\mathbb{R}^n}\frac{1}{\mathcal{V}(x)}\,dx < \infty.$$

Meanwhile, we consider the following space:

$$\begin{aligned} \Lambda_{\mathcal{V}}^\gamma\left(\mathbb{R}^n\right) = \Bigg\{u \in \Lambda^\gamma\left(\mathbb{R}^n\right) : &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\chi(x)-\chi(z)|^2}{|x-z|^{n+2\gamma}}\,dz\,dx \\ &+ \int_{\mathbb{R}^n}\mathcal{V}(x)\left|\chi(x)\right|^2 dx < +\infty\Bigg\} \end{aligned}$$

equipped with

$$\|\chi\|_{\mathcal{V}}^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\chi(x)-\chi(z)|^2}{|x-z|^{n+2\gamma}}\,dz\,dx + \int_{\mathbb{R}^n} \mathcal{V}(x)\big|\chi(x)\big|^2\,dx < +\infty.$$

Lemma 2.1 *Let $\mathcal{V}$ satisfy* (A). *Then the following properties hold*:

(I) $\Lambda_{\mathcal{V}}^{\gamma}$ *is continuously embedded in* $\Lambda^{\gamma}(\mathbb{R})$.

(II) $\Lambda_{\mathcal{V}}^{\gamma}$ *is continuously embedded in* L^1.

(III) $\Lambda_{\mathcal{V}}^{\gamma}$ *is compactly embedded in* L^1.

Proof It follows from (A) and Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{R}^n}|\chi|\,dt &= \int_{\mathbb{R}^n}\big|\big(\mathcal{V}(x)\big)^{-1/2}\big(\mathcal{V}(x)\big)^{1/2}u\big|\,dx \\ &\le \int_{\mathbb{R}^n}\big(\mathcal{V}(x)\big)^{-1/2}\big|\big(\mathcal{V}(x)\big)^{1/2}u\big|\,dx \\ &\le \left(\int_{\mathbb{R}^n}\big(\mathcal{V}(x)\big)^{-1}\,dt\right)^{1/2}\left(\int_{\mathbb{R}^n}\mathcal{V}(x)\chi^2\,dx\right)^{1/2} \\ &\le \left(\int_{\mathbb{R}^n}\big(\mathcal{V}(x)\big)^{-1}\,dx\right)^{1/2}\|\chi\|_{\mathcal{V}}^2 \end{aligned}\tag{2.2}$$

for all $u\in\Lambda_{\mathcal{V}}^{\gamma}$, which means (I) and (II) hold.

Let $(\chi_n)\subset\Lambda_{\mathcal{V}}^{\gamma}$ be a bounded sequence such that $\chi_n\rightharpoonup u$ in $\Lambda_{\mathcal{V}}^{\gamma}$. We will show that $\chi_n\to u$ in L^1. It follows from Hölder's inequality that

$$\begin{aligned}&\int_{\mathbb{R}^n}|\chi_n-\chi|\,dx \\ &\le 2R\left(\int_{|x|\le R}|\chi_n-\chi|^2\,dx\right)^{1/2} + \int_{|x|>R}\big|\big(\mathcal{V}(x)\big)^{-1/2}\big(\mathcal{V}(x)\big)^{1/2}(\chi_n-\chi)\big|\,dx \\ &\le 2R\left(\int_{|x|\le R}|\chi_n-\chi|^2\,dx\right)^{1/2} + \int_{|\chi|>R}\big(\mathcal{V}(x)\big)^{-\frac{1}{2}}\big|\big(\mathcal{V}(x)\big)^{1/2}(\chi_n-\chi)\big|\,dx \\ &\le 2R\left(\int_{|x|\le R}|\chi_n-\chi|^2\,dx\right)^{1/2} \\ &\quad + \left(\int_{|x|>R}\big(\mathcal{V}(x)\big)^{-1}\,dx\right)^{1/2}\left(\int_{|x|>R}\mathcal{V}(x)(\chi_n-\chi)^2\,dx\right)^{1/2} \\ &\le 2R\left(\int_{|x|\le R}|\chi_n-\chi|^2\,dx\right)^{1/2} + \left(\int_{|x|>R}\big(\mathcal{V}(x)\big)^{-1}\,dx\right)^{1/2}\|\chi_n-u\|_{\mathcal{V}}, \end{aligned}\tag{2.3}$$

where $R>0$. Since the embedding is compact on bounded domain then, by Assumption (A) and (2.3), we have $\chi_n\to u$ in L^1. Thus (III) holds. □

3 Convergence results

To obtain the existence of an infinite sequence for the algorithm, we give some lemmas and propositions.

Lemma 3.1 *Let Assumptions* (A), (B), (C) *and* (D) *hold. We conclude that the algorithm does not infinitely circle in the inner cycle.*

Proof It is easy to see that the algorithm infinitely circles in the inner cycle, which implies that $\|g_l\| \geq \epsilon$, or the algorithm stops. Namely, the conclusion

$$\left\| \varsigma_l^p \right\| \leq \triangle_l = c^p \|g_l\| \to 0$$

is true.

So

$$\left| r_l^p - 1 \right| = \frac{|A\varsigma_l(\varsigma_l^p) - P\varsigma_l(\varsigma_l^p)|}{|P\varsigma_l(\varsigma_l^p)|} \leq \frac{2O(\|\varsigma_l^p\|^2)}{\triangle_l \|\mathfrak{B}_l \chi(x_l)\| + O(\triangle_l^2)} \to 0$$

and

$$r_l^p \geq \rho \tag{3.1}$$

for p sufficiently large, which yields a contradiction. □

The following result follows from the definition of the model (1.8).

Lemma 3.2 *Under the conditions of Lemma 3.1, $\{x_l\} \subset \Omega$ is true and $\{\varphi(x_l)\}$ converges.*

Now we can state our result.

Theorem 3.1 *Let the conditions of Lemma 3.1 hold and $\{x_l\}$ be defined as in the algorithm. Then there exists an infinite sequence $\{x_l\}$ such that*

$$\lim_{l\to\infty} \left\| \chi(x_l) \right\| = 0 \tag{3.2}$$

for this algorithm.

Proof Suppose that

$$\lim_{l\to\infty} \left\| \mathfrak{B}_l \chi(x_l) \right\| = 0 \tag{3.3}$$

holds.

By applying (1.15), we know that (3.2) holds from Lemma 2.1(I). So

$$\left\| \mathfrak{B}_{l_j} \chi(x_{l_j}) \right\| \geq \varepsilon. \tag{3.4}$$

Set

$$\mathfrak{K} = \left\{ l \mid \left\| \mathfrak{B}_l \chi(x_l) \right\| \geq \varepsilon \right\}.$$

Using Assumption (D) and the case $\|\mathfrak{B}_l \chi(x_l)\| \geq \varepsilon (l \in \mathfrak{K})$, $\|\chi(x_l)\| (l \in \mathfrak{K})$ is bounded away from 0, we assume that

$$\left\| \chi(x_l) \right\| \geq \varepsilon$$

holds for any $l \in \mathfrak{K}$.

It follows from Assumption (B), Lemma 2.1(II), (2.2) and the Hölder inequality that

$$\begin{aligned} \mathfrak{I}(\chi) &\geq \frac{1}{2}\|\chi\|_{\mathcal{V}}^2 - c_3\int_{\mathbb{R}^n}|\chi|\,dx \\ &\geq \frac{1}{2}\|\chi\|_{\mathcal{V}}^2 - c_3\biggl(\int_{\mathbb{R}^n}\bigl(\mathcal{V}(x)\bigr)^{-1}\,dx\biggr)^{1/2}\|\chi\|_{\mathcal{V}} \end{aligned} \tag{3.5}$$

for all $\chi \in \Lambda_{\mathcal{V}}^{\gamma}$.

Then it follows that $\mathfrak{I}$ is bounded from below. Moreover, if we take $(\chi_n) \subset \Lambda_{\mathcal{V}}^{\gamma}$ to be a (PS)-sequence, then we have

$$c_4 \geq \frac{1}{2}\|\chi_n\|_{\mathcal{V}}^2 - c_5\biggl(\int_{\mathbb{R}^n}\bigl(\mathcal{V}(x)\bigr)^{-1}\,dx\biggr)^{1/2}\|\chi\|_{\mathcal{V}}$$

from (3.4) and (3.5), which implies that (χ_n) is bounded in $\Lambda_{\mathcal{V}}^{\gamma}$.

So there exists a subsequence (χ_{n_l}) such that $\chi_{n_l} \rightharpoonup \chi_0$ as $l \to \infty$ for some $\chi_0 \in \Lambda_{\mathcal{V}}^{\gamma}$. It follows from Lemma 2.1(III) that

$$\chi_{n_l} \to \chi_0,$$

in L^1 as $l \to \infty$, which together with (3.3) yields

$$\biggl|\int_{\mathbb{R}^n}\bigl(\widetilde{f}(x,\chi_{n_l}) - \widetilde{f}(x,\chi_0)\bigr)(\chi_{n_l}-\chi_0)\,dx\biggr| \leq c_6\int_{\mathbb{R}^n}|\chi_{n_l}-\chi_0|\,dx \to 0 \tag{3.6}$$

as $l \to \infty$.

Noting that the sequence (χ_n) is bounded, we know that

$$\bigl(\mathfrak{I}'(\chi_{n_l}) - \mathfrak{I}'(\chi_0)\bigr)(\chi_{n_l}-\chi_0) \to 0 \tag{3.7}$$

as $l \to \infty$.

It follows from (3.6) and (3.7) that

$$\begin{aligned} \|\chi_{n_l}-\chi_0\|_{\mathcal{V}}^2 &= \bigl(\mathfrak{I}'(\chi_{n_l}) - \mathfrak{I}'(\chi_0)\bigr)(\chi_{n_l}-\chi_0) \\ &\quad + \int_{\mathbb{R}^n}\bigl(\widetilde{f}(x,\chi_{n_l}) - \widetilde{f}(x,\chi_0)\bigr).(\chi_{n_l}-\chi_0)\,dx \to 0. \end{aligned}$$

Define

$$\zeta(x) = \xi(x_1)\xi(x_2)\xi(x_3)\cdots\xi(x_n),$$

where $(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$.

Then $\operatorname{supp}\zeta \subset [0,a]^n$. Now for each $1 \leq i \leq l$, we can choose a suitable $y_i \in \mathbb{R}^n$ and define

$$\zeta_i(x) = \zeta(x - y_i), \quad \text{for all } x \in \mathbb{R}^n;$$

such that

$$\begin{aligned} & \operatorname{supp}\zeta_i \subset \varsigma_i, \qquad \operatorname{supp}\zeta_i \cap \operatorname{supp}\zeta_j = \emptyset \quad (i \neq j), \\ & \zeta_i(x) = 1 \end{aligned} \tag{3.8}$$

for any $x \in E_i$ and

$$0 \le \zeta_i(x) \le 1$$

for any $x \in \mathbb{R}^n$.

Set

$$\begin{aligned} \Theta_l &\equiv \Bigl\{ (l_1, l_2, \ldots, l_l) \in \mathbb{R}^l; \max_{1\le i\le l} |l_i| = 1 \Bigr\}, \\ \mathfrak{S}_l &\equiv \Biggl\{ \sum_{i=1}^{l} l_i\zeta_i; (l_1, l_2, \ldots, l_l) \in \Theta_l \Biggr\}. \end{aligned} \tag{3.9}$$

Then Θ_l is homeomorphic to the unit sphere in $\mathbb{R}^l$ by an odd mapping. Thus $\kappa(\Theta_l) = l$. If we define the odd and homeomorphic mapping $\psi : \Theta_l \to \mathfrak{S}_l$ by

$$\psi(l_1, l_2, \ldots, l_l) = \sum_{i=1}^{l} l_i\zeta_i,$$

where $(l_1, l_2, \ldots, l_l) \in \Theta_l$, then $\kappa(\mathfrak{S}_l) = \kappa(\Theta_l) = l$. Moreover, it is obvious that $\mathfrak{S}_l$ is compact and hence

$$\|\chi\|_{\mathcal{V}} \le \lambda_l \tag{3.10}$$

for any $u \in \mathfrak{S}_l$, where $\lambda_l > 0$.

It follows from (3.8) and (3.10) that

$$\begin{aligned} \mathfrak{I}(s\chi) &\le \frac{s}{2}\|x\|_{\mathcal{V}}^2 - \int_{\mathbb{R}^n} \mathfrak{F}\Biggl(x, s\sum_{i=1}^{l} l_i\zeta_i\Biggr)\,dx \\ &\le \frac{s^2\lambda_l^2}{2} - \sum_{i=1}^{l}\int_{\varsigma_i} \mathfrak{F}(x, sl_i\zeta_i)\,dx \end{aligned} \tag{3.11}$$

for any $s \in (0, \varepsilon)$ and

$$u = \sum_{i=1}^{l} l_i\zeta_i \in \mathfrak{S}_l.$$

So there exists an integer $i_0 \in [1, k]$ such that $|l_{i_0}| = 1$. It follows from (3.7) and (3.11) that

$$\begin{aligned} \sum_{i=1}^{l}\int_{\varsigma_i} \mathfrak{F}(x, sl_i\zeta_i)\,dx &= \int_{\mathfrak{E}_{i_0}} \mathfrak{F}(x, sl_{i_0}\zeta_{i_0})\,dx + \int_{\varsigma_{i_0}\setminus\mathfrak{E}_{i_0}} \mathfrak{F}(x, sl_{i_0}\zeta_{i_0})\,dx \\ &\quad + \sum_{i\neq i_0}\int_{\varsigma_i} \mathfrak{F}(x, sl_i\zeta_i)\,dx. \end{aligned} \tag{3.12}$$

Noting that $|l_{i_0}| = 1$, $\zeta_{i_0} \equiv 1$ on $\mathfrak{E}_{i_0}$ and $\mathfrak{F}(x,\chi)$ is even in χ, we have

$$\int_{\mathfrak{E}_{i_0}} \mathfrak{F}(x, sl_{i_0}\zeta_{i_0})\,dx = \int_{\mathfrak{E}_{i_0}} \mathfrak{F}(x,s)\,dx. \tag{3.13}$$

By Assumption (B), we have

$$\int_{\varsigma_{i_0}\setminus\mathfrak{E}_{i_0}} \mathfrak{F}(x, sl_{i_0}\zeta_{i_0})\,dx + \sum_{i\neq i_0}\int_{\varsigma_i} \mathfrak{F}(x, sl_i\zeta_i)\,dx \geq -c_l s^2, \tag{3.14}$$

where $c_l > 0$ depends only on l.

It follows from (3.10)–(3.14) that

$$\mathfrak{I}(s\chi) \leq \frac{s^2\lambda_l^2}{2} + c_l s^2 - \int_{\mathfrak{E}_{i_0}} \mathfrak{F}(x,s)\,dx.$$

Substituting $s = \varepsilon_n$ and using Assumption (D), we obtain

$$\mathfrak{I}(\varepsilon_n\chi) \leq \varepsilon_n^2\left(\frac{s^2\lambda_l^2}{2} + c_l - \left(\frac{a}{2}\right)^2 M_n\right).$$

Since $\varepsilon_n \to 0^+$ and $M_n \to \infty$, we can choose n_0 large enough such that the right side of the last inequality is negative.

Put

$$\mathfrak{A}_l = \{\varepsilon_{n_0}u; \chi \in \mathfrak{S}_l\}.$$

Then

$$\kappa(\mathfrak{A}_l) = \kappa(\mathfrak{S}_l) = l$$

and

$$\sup_{x\in\mathfrak{A}_l} \mathfrak{I}(x) < 0.$$

By Lemma 3.2, there exists a sequence of nontrivial critical points (χ_l) of $\mathfrak{I}$ such that $\mathfrak{I}(\chi_l) \leq 0$ for all $l \in \mathbb{N}$ and $\chi_l \to 0$ in $\Lambda_{\mathcal{V}}^{\gamma}$ as $l \to \infty$. Hence, (χ_l) is a sequence of solutions of (1.1). So they are also the solutions of (1.1) for large enough l.

It follows from Lemma 3.1 and the definitions of the algorithm that

$$\sum_{l\in\mathfrak{K}}\left[\varphi(x_l) - \varphi(x_{l+1})\right] \geq -\sum_{l\in\mathfrak{K}} \rho P\varsigma_l\left(\varsigma_l^{p_l}\right) \geq \sum_{l\in\mathfrak{K}} \rho\frac{1}{2}\min\left\{c^{p_l}\varepsilon, \frac{\varepsilon}{M_l^2}\right\}\varepsilon.$$

Meanwhile, Lemma 3.2 also shows that the sequence $\{\varphi(x_l)\}$ is convergent, from which one deduces that

$$\sum_{l\in\mathfrak{K}} \rho\frac{1}{2}\min\left\{c^{p_l}\varepsilon, \frac{\varepsilon}{M_l^2}\right\}\varepsilon < +\infty.$$

So $p_l \to +\infty$ as $l \to +\infty$ and $l \in \mathfrak{K}$. So it is reasonable for us to assume $p_l \geq 1$ for all $l \in \mathfrak{K}$.

So

$$\begin{aligned} &\min q_l(\varsigma) = \frac{1}{2}\left\| \chi(x_l) + \mathfrak{B}_l\varsigma + \frac{3}{2}\left(s_{l-1}^{\mathfrak{T}}\varsigma\right)^2 s_{l-1}\right\|^2 \\ &\text{s.t. } \|\varsigma\| \leq c^{p_l-1}\left\|\chi(x_l)\right\|, \end{aligned} \tag{3.15}$$

is unacceptable.

Setting $x'_{l+1} = x_l + \varsigma'_l$ one has

$$\frac{\varphi(x_l) - \varphi(x'_{l+1})}{-P\varsigma_l(\varsigma'_l)} < \rho. \tag{3.16}$$

By applying Lemma 3.1 and the definition $\triangle_l$, we know that

$$-P\varsigma_l(\varsigma'_l) \geq \frac{1}{2}\min\left\{c^{p_l-1}\varepsilon, \frac{\varepsilon}{M_l^2}\right\}\varepsilon.$$

By applying Lemma 3.2, we know that

$$\varphi(x'_{l+1}) - \varphi(x_l) - P\varsigma_l(\varsigma'_l) = O\left(\|\varsigma'_l\|^2\right) = O\left(c^{2(p_l-1)}\right).$$

So

$$\left|\frac{\varphi(x'_{l+1}) - \varphi(x_l)}{P\varsigma_l(\varsigma'_l)} - 1\right| \leq \frac{O(c^{2(p_l-1)})}{0.5\min\{c^{p_l-1}\varepsilon, \frac{\varepsilon}{M_l^2}\}\varepsilon + O(c^{2(p_l-1)}\varepsilon^2)}.$$

By applying $p_l \to +\infty$ when $l \to +\infty$ and $l \in \mathfrak{K}$, we get

$$\frac{\varphi(x_l) - \varphi(x'_{l+1})}{-P\varsigma_l(\varsigma'_l)} \to 1,$$

which yields a contradiction to (3.16), where $l \in \mathfrak{K}$. □

Remark 3.1 Our algorithm extends and improves Algorithm YL in [44] in the following ways.

(I) The iterative scheme in Algorithm YL is extended for Problem (1.1). The iterative scheme in our algorithm is more advantageous and more flexible than the iterative scheme in Algorithm YL because it involves solving four problems: a finite family of BFGSs, a finite family of Schrödinger inclusions, a general system of Schrödinger inequalities and the fixed point problem of a countable family of Schrödinger mappings.

(II) The iterative scheme in our algorithm is very different from the iterative scheme in Algorithm YL because the iterative scheme in Theorem 3.1 involves modified subgradient extragradient method and projection method. In addition, the iterative scheme in Algorithm YL is an iterative one involving neither modified subgradient extragradient method nor projection method but the iterative scheme in

Theorem 3.1 is an iterative one involving both modified subgradient extragradient method and projection method.

(III) The convergence analysis of Theorem 3.1 is based on modified subgradient extragradient method, projection method, viscosity approximation method, and Schrödinger mapping and strongly positive bounded Schrödinger operator approaches to solving a finite family of BFGSs, a finite family of Schrödinger inclusions and the fixed point problem of a countable family of Schrödinger mappings.

(IV) The argument and technique in Theorem 3.1 are different from the argument ones in Algorithm YL because we make use of the properties of the Schrödinger mappings, the properties of strongly positive boundedness Schrödinger operators and the maximum principle approach with respect to the Schrödinger operator.

4 Existence of nontrivial meromorphic solutions for the problem (1.2)

Putting

$$\tau(x) = \sup\bigl\{\tau > 0 : \mathcal{B}(x,\tau) \subseteq \mathfrak{K}\bigr\}$$

for all $x \in \mathfrak{K}$, we can show that there exists $x_0 \in \mathfrak{K}$ such that $\mathcal{B}(x_0, \mathfrak{I}) \subseteq \mathfrak{K}$, where

$$\mathfrak{I} = \sup_{x\in\mathfrak{K}} \tau(x). \tag{4.1}$$

We introduce the following condition:

(E) There exist $\mu > 0$ and $\tau > 0$ with $|\tau|^2\omega_n^2\mathfrak{I}^{n-2s}\mathcal{M} < 1$ such that (see [23])

$$aC_1\sqrt{2\mu} + \frac{aC_q^q(2\mu)^{q/2}}{q} < \frac{\mathfrak{I}^{2s}\inf_{x\in\mathfrak{K}} F(x,\tau)}{2^N\tau^2\omega_n\mathcal{M}},$$

where $\mathfrak{I}$ is given in (4.1), ω_n is the volume of

$$\mathcal{B}(x_0,\mathfrak{I}) := \bigl\{x \in \mathbb{R}^n : |x - x_0| < \mathfrak{I}\bigr\}$$

in $\mathbb{R}^n$,

$$\mathcal{M} = \frac{2^{2+n-2s}}{(1-s)(n-2s+2)} + \frac{1}{2^{n-2s}s(n-2s+2)} + \frac{1}{2s(n-2s)}.$$

Before proceeding to the proof of the main result, we give some nonlinear examples. Functions listed in Examples 4.1, 4.2 and 4.3 satisfy all Assumptions (A), (B), (C), (D) and (E), which shows that the interval in the following result is not empty.

Example 4.1 $f(x,s) = \mathcal{V}_\infty(x)\min\{|s|^\nu, 1\}s$, where $\nu \in (0, 2^* - 2)$, $\mathcal{V}_\infty \in C(\mathbb{R}^n)$ is 1-periodic in each of $x_1, x_2, \ldots, x_n$ and $\inf \mathcal{V}_\infty > \overline{\varLambda}$.

Example 4.2 $f(x,s) = \mathcal{V}_\infty(x)s[1 - \frac{1}{\ln(e+|s|^\nu)}]s$, where $\nu \in (0, 2^* - 2)$, $\mathcal{V}_\infty \in C(\mathbb{R}^n)$ is 1-periodic in each of $x_1, x_2, \ldots, x_n$ and $\inf \mathcal{V}_\infty > \overline{\varLambda}$.

Example 4.3 $f(x,s) = h(x)\min\{\frac{1}{\varrho_1}|s|^{\varrho_1-2}, \frac{1}{\varrho_2}|s|^{\varrho_2-2}\}s$, where $2 < \varrho_1 < \varrho_2 < 2^*$ and $h \in C(\mathbb{R}^n)$ is 1-periodic in each of $x_1, x_2, \ldots, x_n$ with $\inf h > 0$.

Theorem 4.1 *Let Assumptions* (A), (B), (C) *and* (D) *hold. Assume also that the condition* (E) *is satisfied. Then*

$$\lambda \in \tilde{\Psi} := \left(\frac{2^N\tau^2\omega_n\mathcal{M}}{\Im^{2s}\inf_{x\in\mathfrak{K}}F(x,\tau)}, \frac{q}{qaC_1\sqrt{2\mu}+aC_q^q(2\mu)^{q/2}}\right),$$

and the problem (1.2) *admits at least one meromorphic solutions.*

Proof Let

$$\tilde{\chi}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus \mathcal{B}(x_0,\Im), \\ \tau & \text{if } x \in \mathcal{B}(x_0,\frac{\Im}{2}), \\ \frac{2\tau}{\Im}(\Im - |x-x_0|) & \text{if } x \in \mathcal{B}(x_0,\Im)\setminus\mathcal{B}(x_0,\frac{\Im}{2}), \end{cases} \tag{4.2}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Then it is clear that $\tilde{\chi} \in X_s(\mathfrak{K})$ and $0 \le \tilde{\chi}(x) \le \tau$ for all $x \in \mathfrak{K}$, and so $\tilde{\chi} \in X_s(\mathfrak{K})$. Denote $\mathcal{B}_\Im := \mathcal{B}(x_0,\Im)$.

Then it follows that

$$\begin{aligned}
\Phi_s(\tilde{\chi}) &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\tilde{\chi}(x)-\tilde{\chi}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= \frac{1}{2}\int_{\mathcal{B}_\Im\setminus\mathcal{B}_{\frac{\Im}{2}}}\int_{\mathcal{B}_\Im\setminus\mathcal{B}_{\frac{\Im}{2}}}\frac{|\tilde{\chi}(x)-\tilde{\chi}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_{\mathcal{B}_\Im\setminus\mathcal{B}_{\frac{\Im}{2}}}\int_{\mathbb{R}^n\setminus\mathcal{B}_\Im}\frac{|\tilde{\chi}(x)-\tilde{\chi}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_{\mathcal{B}_{\frac{\Im}{2}}}\int_{\mathcal{B}_\Im\setminus\mathcal{B}_{\frac{\Im}{2}}}\frac{|\tilde{\chi}(x)-\tilde{\chi}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^n\setminus\mathcal{B}_\Im}\int_{\mathcal{B}_{\frac{\Im}{2}}}\frac{|\tilde{\chi}(x)-\tilde{\chi}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&=: \frac{1}{2}\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4.
\end{aligned}$$

Next we estimate $\mathcal{I}_1$–$\mathcal{I}_4$, by direct calculations.

• Estimate of $\mathcal{I}_1$: For any positive constant ε small enough,

$$\begin{aligned}
\mathcal{I}_1 &= \int_{\mathcal{B}_\Im\setminus\mathcal{B}_{\frac{\Im}{2}}}\int_{\mathcal{B}_\Im\setminus\mathcal{B}_{\frac{\Im}{2}}}\frac{|\tilde{\chi}(x)-\tilde{\chi}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\le \frac{2^2|\tau|^2}{\Im^2}\int_{\mathcal{B}_\Im\setminus\mathcal{B}_{\frac{\Im}{2}}}\int_{\mathcal{B}_\Im\setminus\mathcal{B}_{\frac{\Im}{2}}}\frac{|x-y|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\le \frac{2^2|\tau|^2\omega_n}{\Im^2}\int_{\mathcal{B}_\Im\setminus\mathcal{B}_{\frac{\Im}{2}}}\int_\varepsilon^{\Im+|y|} r^{2-2s-1}\,dr\,dy
\end{aligned}$$

$$\begin{aligned}
&\le \frac{2^2|\tau|^2\omega_n}{\mathfrak{I}^2}\int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\frac{(\mathfrak{I}+|y|)^{2-2s}}{2-2s}\,dy \\
&= \frac{2^2|\tau|^2\omega_n^2}{(2-2s)\mathfrak{I}^2}\int_{\frac{3}{2}\mathfrak{I}}^{2\mathfrak{I}} r^{2+n-2s-1}\,dr \\
&= \frac{2|\tau|^2\omega_n^2\mathfrak{I}^{n-2s}}{(1-s)(2+n-2s)}\left(2^{2+n-2s}-\left(\frac{3}{2}\right)^{2+n-2s}\right).
\end{aligned}$$

• Estimate of $\mathcal{I}_2$:

$$\begin{aligned}
\mathcal{I}_2 &= \int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\int_{\mathbb{R}^n\setminus\mathcal{B}_{\mathfrak{I}}}\frac{|\tilde{\chi}(x)-\tilde{\chi}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\le \frac{2^2|\tau|^2}{\mathfrak{I}^2}\int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\int_{\mathbb{R}^n\setminus\mathcal{B}_{\mathfrak{I}}}\frac{|\mathfrak{I}-|y-x_0||^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= \frac{2^2|\tau|^2\omega_n}{\mathfrak{I}^2}\int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\int_{\mathfrak{I}-|y-x_0|}^{\infty}\frac{|\mathfrak{I}-|y-x_0||^2}{r^{2s+1}}\,dr\,dy \\
&= \frac{2^2|\tau|^2\omega_n}{\mathfrak{I}^2 2s}\int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\bigl|\mathfrak{I}-|y-x_0|\bigr|^{2-2s}\,dy \\
&= \frac{2|\tau|^2\omega_n^2}{\mathfrak{I}^2 s}\int_0^{\frac{\mathfrak{I}}{2}} r^{n+2-2s-1}\,dr \\
&= \frac{|\tau|^2\omega_n^2\mathfrak{I}^{n-2s}}{2^{n-2s+1}s(n-2s+2)}.
\end{aligned}$$

• Estimate of $\mathcal{I}_3$:

$$\begin{aligned}
\mathcal{I}_3 &= \int_{\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\frac{|\tilde{\chi}(x)-\tilde{\chi}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= \frac{2^2|\tau|^2}{\mathfrak{I}^2}\int_{\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\frac{|-\frac{\mathfrak{I}}{2}+|x-x_0||^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= \frac{2^2|\tau|^2}{\mathfrak{I}^2}\int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\int_{\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\frac{|-\frac{\mathfrak{I}}{2}+|x-x_0||^2}{|x-y|^{n+2s}}\,dy\,dx \\
&= \frac{2^2|\tau|^2\omega_n}{\mathfrak{I}^2}\int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\left|-\frac{\mathfrak{I}}{2}+|x-x_0|\right|^2\int_{|x-x_0|-\frac{\mathfrak{I}}{2}}^{|x-x_0|+\frac{\mathfrak{I}}{2}}\frac{1}{r^{2s+1}}\,dr\,dx \\
&\le \frac{2|\tau|^2\omega_n}{\mathfrak{I}^2 s}\int_{\mathcal{B}_{\mathfrak{I}}\setminus\mathcal{B}_{\frac{\mathfrak{I}}{2}}}\left|-\frac{\mathfrak{I}}{2}+|x-x_0|\right|^{2-2s}\,dx \\
&= \frac{2|\tau|^2\omega_n^2}{\mathfrak{I}^2 s}\int_0^{\frac{\mathfrak{I}}{2}} t^{n-2s+1}\,dt \\
&= \frac{|\tau|^2\omega_n^2\mathfrak{I}^{n-2s}}{2^{n-2s+1}s(n-2s+2)}.
\end{aligned}$$

• Estimate of $\mathcal{I}_4$:

$$
\begin{aligned}
\mathcal{I}_4 &= \int_{\mathcal{B}_{\frac{\Im}{2}}} \int_{\mathbb{R}^n\setminus\mathcal{B}_{\Im}} \frac{|\tilde{\chi}(x)-\tilde{\chi}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= |\tau|^2 \int_{\mathcal{B}_{\frac{\Im}{2}}} \int_{\mathbb{R}^n\setminus\mathcal{B}_{\Im}} \frac{1}{|x-y|^{n+2s}}\,dx\,dy \\
&= |\tau|^2\omega_n \int_{\mathcal{B}_{\frac{\Im}{2}}} \int_{\Im-|y-x_0|}^{\infty} r^{-2s-1}\,dr\,dy \\
&= |\tau|^2\omega_n \int_{\mathcal{B}_{\frac{\Im}{2}}} \frac{1}{2s(\Im-|y-x_0|)^{2s}}\,dy \\
&= \frac{|\tau|^2\omega_n^2}{2s} \int_{\frac{\Im}{2}}^{\Im} t^{n-2s-1}\,dt \\
&= \frac{|\tau|^2\omega_n^2\Im^{n-2s}}{2s(n-2s)}\left(1-\frac{1}{2^{n-2s}}\right) \\
&= \frac{|\tau|^2\omega_n^2\Im^{n-2s}}{2s(n-2s)}.
\end{aligned}
$$

Hence, it follows from Assumption (A) that

$$
\Phi_s(\tilde{\chi}) \le |\tau|^2\omega_n^2\Im^{n-2s}\mathcal{M} < 1.
$$

Owing to Assumption (C) and the definition (4.2), we deduce that

$$
\Upsilon(\tilde{\chi}) \ge \int_{\mathcal{B}_{\frac{\Im}{2}}} F(x,\tilde{\chi})\,dx \ge \inf_{x\in\mathfrak{K}} F(x,\tau)\left(\frac{\omega_N\Im^n}{2^n}\right)
$$

and thus

$$
\frac{\Upsilon(\tilde{\chi})}{\Phi_s(\tilde{\chi})} \ge \frac{\Im^{2s}\inf_{x\in\mathfrak{K}} F(x,\tau)}{2^n\tau^2\omega_n\mathcal{M}}. \tag{4.3}
$$

Also by Assumption (B), Theorem 3.1 and the best constants C_1, C_q, we have

$$
\begin{aligned}
\Upsilon(\chi) &= \int_{\mathfrak{K}} F(x,\chi)\,dx \\
&\le a\int_{\mathfrak{K}}\left\{\left|\chi(x)\right| + \frac{1}{q}\left|\chi(x)\right|^q\right\}dx \\
&= a\|\chi\|_{L^1(\mathfrak{K})} + \frac{a}{q}\|\chi\|_{L^q(\mathfrak{K})}^q \\
&\le aC_1\|\chi\|_{X_s(\mathfrak{K})} + \frac{a}{q}C_q^q\|\chi\|_{X_s(\mathfrak{K})}^q.
\end{aligned}
$$

For each $\chi \in \Phi_s^{-1}((-\infty,\mu])$, it follows that

$$
\Upsilon(\chi) \le aC_1\sqrt{2\mu} + \frac{aC_q^q(2\mu)^{q/2}}{q}
$$

and hence

$$\sup_{u\in\Phi_s^{-1}((-\infty,\mu])} \Upsilon(\chi) \le aC_1\sqrt{2\mu} + \frac{aC_q^q(2\mu)^{q/2}}{q}.$$

From inequality (4.3) and Assumption (C), we have

$$\sup_{u\in\Phi_s^{-1}((-\infty,1])} \Upsilon(\chi) < \frac{\Upsilon(\tilde{\chi})}{\Phi_s(\tilde{\chi})}.$$

So

$$\tilde{\Psi} \subseteq \left(\frac{\Phi_s(\tilde{\chi})}{\Upsilon(\tilde{\chi})}, \frac{1}{\sup_{\Phi_s(\chi)\le 1}\Upsilon(\chi)}\right).$$

Since condition (E) is easily verified and $J_\lambda = \Phi_s - \lambda\Upsilon$ is coercive by (B), all conditions of Theorem 4.1 are satisfied for every $\lambda \in \tilde{\Psi}$. Hence, by applying Theorem 4.1 and Lemma 3.2, we know that J_λ is the critical points which is the meromorphic solution for the problem (1.2). □

5 Conclusions

In this paper we were concerned with nonlinear boundary value problem for a class of quasilinear Schrödinger equation $(-\Delta)^\gamma\chi + \mathcal{V}(x)\chi = f(x,\chi)$ with nonlinear boundary condition in $\mathbb{R}^n$. We firstly studied a new algorithm for finding the meromorphic solution for the mentioned equation via meromorphic inequalities. Then we dealt with the necessary and sufficient conditions of convergence and obtain the general solutions and the conditions of solvability for the mentioned equation by means of the meromorphic inequalities for the classical boundary value problems. These results generalized some previous results concerning the asymptotic behavior of solutions of non-delay systems of Schrödinger equations by applying the maximum principle approach with respect to the Schrödinger operator.

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