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Solvability for a class of nonlinear Hadamard fractional differential equations with parameters

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Abstract

In this paper, we investigate a class of boundary value problem of nonlinear Hadamard fractional differential equations with p-Laplacian operator. By means of the properties of the Green's functions and Guo–Krasnosel'skii fixed point theorem on cones, various existence results for positive solutions are derived in terms of different values of parameters.

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1 Introduction

Fractional calculus has emerged as an interesting field of investigation in the last few decades. The subject has been extensively developed and the literature on the topic is much enriched now, covering theoretical as well as widespread applications of this branch of mathematical analysis. In fact, the fractional differential equations have attracted more and more attention for their useful applications in various fields, such as economics, science, and engineering; see [1–5]. In the last few decades, much attention has been focused on the study of the existence of positive solutions for boundary value problems of Riemann–Liouville type or Caputo type fractional differential equations; see [6–23].

On the other hand, p-Laplacian operator is extensively applied in the mathematical modeling of several real world phenomena in physics, mechanics, dynamical systems, etc. While studying the fundamental problem of turbulent flow in a porous medium, Leibenson [24] introduced the p-Laplacian operator $\phi_p(x(t))$ in 1945. Also there has been shown much interest in obtaining the existence and multiplicity of solutions of this class of problems by employing different fixed point theorems. Recently, many scholars have paid more attention to the fractional order differential equation boundary value problems with p-Laplacian operator; see [25–32].

The system of fractional differential equations boundary value problems with p-Laplacian operator has also received much attention and has developed very rapidly; see [33–39]. In [33], He and Song discussed the following fractional order differential system with p-Laplacian operator:

$$\begin{cases} D_{0^+}^{\alpha_1}(\phi_{p_1}(D_{0^+}^{\beta_1}u(t))) = \lambda f(t,u(t)), & t \in (0,1), \\ D_{0^+}^{\alpha_2}(\phi_{p_2}(D_{0^+}^{\beta_2}u(t))) = \mu g(t,u(t)), & t \in (0,1), \\ u(0) = 0, \qquad u(1) = a_1 u(\xi_1), \qquad D_{0^+}^{\beta_1}u(0) = 0, \qquad D_{0^+}^{\beta_1}u(1) = b_1 D_{0^+}^{\beta_1}u(\eta_1), \\ v(0) = 0, \qquad v(1) = a_1 v(\xi_2), \qquad D_{0^+}^{\beta_2}v(0) = 0, \qquad D_{0^+}^{\beta_2}v(1) = b_1 D_{0^+}^{\beta_2}v(\eta_2), \end{cases}$$

where $\alpha_i, \beta_i \in (1,2]$, $D_{0^+}^{\alpha_i}$ and $D_{0^+}^{\beta_i}$ are the standard Riemann–Liouville derivatives, $\xi_i, \eta_i \in (0,1)$, $a_i, b_i \in [0,1]$, $i = 1,2$, λ and μ are positive parameters. The uniqueness of solution was established by using the Banach contraction mapping principle. Hao et al. [34] investigated the existence of positive solutions for a system of nonlinear fractional differential equations nonlocal boundary value problems with parameters and p-Laplacian operator,

$$\begin{cases} -D_{0^+}^{\alpha_1}(\phi_p(D_{0^+}^{\beta_1}u(t))) = \lambda f(t,u(t),v(t)), & t \in (0,1), \\ -D_{0^+}^{\alpha_2}(\phi_p(D_{0^+}^{\beta_2}u(t))) = \mu g(t,u(t),v(t)), & t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \qquad D_{0^+}^{\beta_1}u(0) = 0, \qquad D_{0^+}^{\beta_1}u(1) = b_1 D_{0^+}^{\beta_1}u(\eta_1), \\ v(0) = v(1) = v'(0) = v'(1) = 0, \qquad D_{0^+}^{\beta_2}v(0) = 0, \qquad D_{0^+}^{\beta_2}v(1) = b_1 D_{0^+}^{\beta_2}v(\eta_2), \end{cases}$$

where $\alpha_i \in (1,2]$, $\beta_i \in (3,4]$ $D_{0^+}^{\alpha_i}$ and $D_{0^+}^{\beta_i}$ are the standard Riemann–Liouville derivatives, $\eta_i \in (0,1)$, $b_i \in (0, \eta_i^{\frac{1-\alpha_i}{p_i-1}})$, $i = 1,2$, λ and μ are positive parameters.

It has been noticed that most of the above-mentioned work on the topic is based on Riemann–Liouville or Caputo fractional derivatives. In 1892, Hadamard [40] introduced another fractional derivative, which differs from the above-mentioned ones because its definition involves logarithmic function of arbitrary exponent and named the Hadamard derivative. Although many researchers are paying more and more attention to Hadamard type fractional differential equation, the study of the topic is still in its primary stage. For details and recent developments on Hadamard fractional differential equations, see [41–48].

From the above review of the literature concerning fractional differential equations, most of the authors investigated only the existence of solutions or positive solutions for Hadamard fractional differential equations without considering the p-Laplacian operator. A very few authors established results along with p-Laplacian operator, in [47], Wang considered the nonlinear Hadamard fractional differential equation with integral boundary condition and p-Laplacian operator

$$\begin{cases} D^{\beta}(\phi_p(D^{\alpha}u(t))) = f(t,u(t)), & t \in (1,T), \\ u(T) = \lambda I^{\alpha}u(\eta), \qquad D^{\alpha}u(1) = 0, \qquad u(1) = 0, \end{cases}$$

where f grows $p-1$ sublinearly at $+\infty$, and by using the Schauder fixed point theorem, a solution existence result is obtain. In [46], Li and Lin used the Guo–Krasnosel'skii fixed point theorem to obtain the existence and uniqueness of positive solutions. We have

$$\begin{cases} D^{\beta}(\phi_p(D^{\alpha}u(t))) = f(t,u(t)), & 1 < t < e, \\ u(1) = u'(1) = u'(e) = 0, \qquad D^{\alpha}u(1) = D^{\alpha}u(e) = 0, \end{cases}$$

where the continuous function $f : [1, e] \times [0, +\infty) \to [0, +\infty)$, $2 < \alpha \leq 3$, $1 < \beta \leq 2$, D^α denotes the standard Hadamard fractional derivative of order α. Zhang et al. [48] established some existence of positive(nontrivial) solutions for integral boundary conditions of nonlinear Hadamard fractional differential equations with p-Laplacian operator.

$$\begin{cases} D^\beta(\phi_p(D^\alpha u(t))) = f(t, u(t)), & 1 < t < e, \\ u(1) = u'(1) = u'(e) = 0, & D^\alpha u(1) = 0, \\ \phi p(D^\alpha u(e)) = \mu \int_1^e \phi_p(D^\alpha u(t)) \frac{dt}{t}, \end{cases}$$

where α, β, and μ are three real numbers with $\alpha \in (2, 3]$, $\beta \in (1, 2]$, and $\mu \in [0, \beta)$, and f is a continuous function on $[1, e] \times \mathbb{R}$.

Motivated by the aforementioned work, we investigate in this paper the existence of positive solutions for the following nonlinear Hadamard fractional differential equation with p-Laplacian operator:

$$\begin{cases} {}^H D_{1^+}^{\beta_1}(\phi_p({}^H D_{1^+}^{\alpha_1} u(t))) = \lambda^{p-1} f(t, u(t), v(t), w(t)), & t \in (1, e), \\ {}^H D_{1^+}^{\beta_2}(\phi_p({}^H D_{1^+}^{\alpha_2} v(t))) = \mu^{p-1} g(t, u(t), v(t), w(t)), & t \in (1, e), \\ {}^H D_{1^+}^{\beta_3}(\phi_p({}^H D_{1^+}^{\alpha_3} w(t))) = \upsilon^{p-1} h(t, u(t), v(t), w(t)), & t \in (1, e), \end{cases} \tag{1}$$

subject to the three-point boundary conditions

$$\begin{cases} u^{(j)}(1) = 0, \quad 0 \leq j \leq n-2, & \mu_1 u^{(p_1)}(e) = \lambda_1 u^{(p_1)}(\xi), \\ \phi_p({}^H D_{1^+}^{\alpha_1} u(1)) = 0 = ({}^H D_{1^+}^{p_2}(\phi_p({}^H D_{1^+}^{\alpha_1} u(e)))), \\ v^{(j)}(1) = 0, \quad 0 \leq j \leq m-2, & \mu_1 v^{(q_1)}(e) = \lambda_1 v^{(q_1)}(\xi), \\ \phi_p({}^H D_{1^+}^{\alpha_2} v(1)) = 0 = ({}^H D_{1^+}^{q_2}(\phi_p({}^H D_{1^+}^{\alpha_2} v(e)))), \\ w^{(j)}(1) = 0, \quad 0 \leq j \leq l-2, & \mu_1 w^{(r_1)}(e) = \lambda_1 w^{(r_1)}(\xi), \\ \phi_p({}^H D_{1^+}^{\alpha_3} w(1)) = 0 = ({}^H D_{1^+}^{r_2}(\phi_p({}^H D_{1^+}^{\alpha_3} w(e)))), \end{cases} \tag{2}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2, 3$, $\alpha_1 \in (n-1, n]$, $\alpha_2 \in (m-1, m]$, $\alpha_3 \in (l-1, l]$, $n, m, l \in \mathbb{N}$ for $n, m, l \geq 3$, $\beta_i \in (1, 2]$, $i = 1, 2, 3$, $p_1 \in [1, \alpha_1 - 1)$, $q_1 \in [1, \alpha_2 - 1)$, $\gamma_1 \in [1, \alpha_3 - 1)$, $p_2.q_2, \gamma_2 \in (0, 1]$ and $p_2, q_2, \gamma_2 \leq \beta_i - 1$, $i = 1, 2, 3$, $\mu_1, \lambda_1 \in (0, \infty)$, $\xi \in (1, e)$ are constants. ${}^H D_{1^+}^k$ denotes the Hadamard fractional derivative of order k. $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g, h \in C([1, e] \times [0, +\infty)^3, [0, +\infty))$, λ, μ and $\upsilon > 0$ are positive parameters.

Under some assumptions on f, g and h, we give intervals for the parameters λ, μ and υ such that positive solutions of (1)–(2) exist. By a positive solution of the problem (1)–(2), we mean a triplet of functions $(u, v, w) \in (C([1, e], [0, \infty)))^3$ satisfying (1)–(2) with $u(t) > 0$ for all $t \in [1, e]$, or $v(t) > 0$ for all $t \in [1, e]$, or $w(t) > 0$ for all $t \in [1, e]$ and $(u, v, w) \neq (0, 0, 0)$.

The main aim of this paper is to investigate the above Hadamard fractional differential equation with p-Laplacian operator boundary value problem (1)–(2). With the help of the properties of the Green's functions and the Guo–Krasnosel'skii fixed point theorem on cones, we established the various existence results for positive solutions were derived in terms of different values of λ, μ and υ, under different combinations of superlinearity and sublinearity of the functions f, g and h. At the end, we give an example to illustrate the feasibility of our proposed theoretical result.

2 Preliminaries

For convenience of the reader, we present some necessary definitions and lemmas from Hadamard fractional calculus theory in this section.

Definition 2.1 ([1]) The left-sided Hadamard fractional integrals of order $\alpha \in \mathbb{R}^+$ of the function $h(t)$ are defined by

$$\left({}^H I^\alpha h\right)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s} \quad (1 \le t \le e),$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 ([1]) The left-sided Hadamard fractional derivatives of order $\alpha \in (n-1, n]$, $n \in Z^+$ of the function $h(t)$ are defined by

$$\left({}^H D^\alpha h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\ln \frac{t}{s}\right)^{n-\alpha+1} h(s) \frac{ds}{s} \quad (1 \le t \le e),$$

where $\Gamma(\cdot)$ is the Gamma function.

Lemma 2.1 ([1]) *If $a, \alpha, \beta > 0$ then*

$$\left({}^H D_a^\alpha \left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{x}{a}\right)^{\beta-\alpha-1}.$$

Lemma 2.2 ([1]) *Let $q > 0$ and $u \in C[1,\infty) \cap L^1[1,\infty)$. Then the Hadamard fractional differential equation ${}^H D^q u(t) = 0$ has the solution*

$$u(t) = \sum_{i=1}^n c_i (\ln t)^{q-1}$$

and the following formula holds:

$${}^H I^{q\,H} D^q u(t) = u(t) + \sum_{i=1}^n c_i (\ln t)^{q-i}$$

where $c_i \in R$, $i = 1, 2, \ldots, n$ and $n - 1 < q < n$.

3 Green's function and bounds

In this section we present the expression and properties of Green's function associated with boundary value problem (1)–(2). In order to prove our main results, we need some preliminary results.

Lemma 3.1 *Let $\Delta_1 = \mu_1 - \lambda_1 (\ln \xi)^{\alpha_1 - p_1 - 1} > 0$, $h \in C[1, e]$ and $n - 1 < \alpha_1 \le n$, for $n \ge 3$. Then the unique solution of*

$$\begin{cases} {}^H D_{1^+}^{\alpha_1} u(t) + h(t) = 0, & 1 < t < e, \\ u^{(j)}(1) = 0, \quad 0 \le j \le n-2, & \mu_1 u^{(p_1)}(e) = \lambda_1 u^{(p_1)}(\xi), \end{cases} \tag{3}$$

is $u(t)=\int_1^e G_1(t,s)h(s)\frac{ds}{s}$, *where*

$$G_1(t,s)=G_{11}(t,s)+\frac{\lambda_1(\ln t)^{\alpha_1-1}}{\mu_1-\lambda_1(\ln\xi)^{\alpha_1-p_1-1}}G_{12}(\xi,s),$$

$$G_{11}(t,s)=\frac{1}{\Gamma(\alpha_1)}\begin{cases}(\ln t)^{\alpha_1-1}(1-\ln s)^{\alpha_1-p_1-1}-(\ln\frac{t}{s})^{\alpha_1-1}, & 1\le s\le t\le e,\\ (\ln t)^{\alpha_1-1}(1-\ln s)^{\alpha_1-p_1-1}, & 1\le t\le s\le e,\end{cases}\tag{4}$$

$$G_{12}(t,s)=\frac{1}{\Gamma(\alpha_1)}\begin{cases}(\ln t)^{\alpha_1-p_1-1}(1-\ln s)^{\alpha_1-p_1-1}-(\ln\frac{t}{s})^{\alpha_1-p_1-1}, & 1\le s\le t\le e,\\ (\ln t)^{\alpha_1-p_1-1}(1-\ln s)^{\alpha_1-p_1-1}, & 1\le t\le s\le e.\end{cases}$$

Proof It is enough to consider the case when u is a solution of (3). From Lemma 2.1 we have ${}^H I_{1^+}^{\alpha_1}\,{}^H D_{1^+}^{\alpha_1}u(t)=-{}^H I_{1^+}^{\alpha_1}h(t)$, so that

$$u(t)=-\frac{1}{\Gamma(\alpha_1)}\int_1^e\left(\ln\frac{t}{s}\right)^{\alpha_1-1}h(s)\frac{ds}{s}+c_1(\ln t)^{\alpha_1-1}+c_2(\ln t)^{\alpha_1-2}+\cdots+c_n(\ln t)^{\alpha_1-n},$$

for some $c_i\in\mathbb{R}$, $i=1,2,\ldots,n$. From $u^{(j)}(1)=0$, $j=0,1,2,\ldots,n-2$, we have $c_2=c_3=\cdots=c_n=0$. Hence

$$u^{(j)}(t)=c_1\prod_{j=1}^{p_1}(\alpha_1-j)(\ln t)^{\alpha_1-p_1-1}-\prod_{j=1}^{p_1}(\alpha_1-j)\frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha_1-p_1-1}h(s)\frac{ds}{s}.$$

Then $\mu_1u^{(p_1)}(e)=\lambda_1u^{(p_1)}(\xi)$ implies that

$$c_1=\frac{1}{\Delta_1}\left[\frac{\mu_1}{\Gamma(\alpha_1)}\int_1^e(1-\ln s)^{\alpha_1-p_1-1}h(s)\frac{ds}{s}-\frac{\lambda_1}{\Gamma(\alpha_1)}\int_1^\xi\left(\ln\frac{\xi}{s}\right)^{\alpha_1-p_1-1}h(s)\frac{ds}{s}\right].$$

As a result,

$$\begin{aligned}u(t)&=\frac{-1}{\Gamma(\alpha_1)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha_1-1}h(s)\frac{ds}{s}+\frac{\mu_1(\ln t)^{\alpha_1-1}}{\Delta_1}\int_1^e\frac{(1-\ln s)^{\alpha_1-p_1-1}}{\Gamma(\alpha_1)}h(s)\frac{ds}{s}\\&\quad-\frac{\lambda_1(\ln t)^{\alpha_1-1}}{\Delta_1}\int_1^\xi\frac{(\ln\frac{\xi}{s})^{\alpha_1-p_1-1}}{\Gamma(\alpha_1)}h(s)\frac{ds}{s}\\&=\frac{-1}{\Gamma(\alpha_1)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha_1-1}h(s)\frac{ds}{s}+\frac{(\ln t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e(1-\ln s)^{\alpha_1-p_1-1}h(s)\frac{ds}{s}\\&\quad-\frac{\lambda_1(\ln t)^{\alpha_1-1}}{\Delta_1}\int_1^\xi\frac{(\ln\frac{\xi}{s})^{\alpha_1-p_1-1}}{\Gamma(\alpha_1)}h(s)\frac{ds}{s}\\&\quad+\frac{\lambda_1(\ln t)^{\alpha_1-1}}{\Delta_1}\int_1^e\frac{(\ln\xi)^{\alpha_1-p_1-1}(1-\ln s)^{\alpha_1-p_1-1}}{\Gamma(\alpha_1)}h(s)\frac{ds}{s}\\&=\int_1^e G_1(t,s)h(s)\frac{ds}{s}.\end{aligned}$$

□

Lemma 3.2 *Let* $n-1<\alpha_1\le n$, $1<\beta_1\le 2$ *and* $y\in C[1,e]$. *Then the unique solution of*

$$\begin{cases}{}^H D_{1^+}^{\beta_1}(\phi_p({}^H D_{1^+}^{\alpha_1}u(t)))=y(t), & 1<t<e,\\ u^{(j)}(1)=0,\quad 0\le j\le n-2, & \mu_1u^{(p_1)}(e)=\lambda_1u^{(p_1)}(\xi),\\ \phi_p({}^H D_{1^+}^{\alpha_1}u(1))=0=({}^H D_{1^+}^{p_2}(\phi_p({}^H D_{1^+}^{\alpha_1}u(e)))),\end{cases}\tag{5}$$

is $u(t) = \int_1^e G_1(t,s)\phi_q(\int_1^e H_1(s,\tau)y(\tau)\frac{d\tau}{\tau})\frac{ds}{s}$, *where* $G_1(t,s)$ *is defined as* (4).

$$H_1(t,s) = \frac{1}{\Gamma(\beta_1)}\begin{cases}(\ln t)^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1} - (\ln\frac{t}{s})^{\beta_1-1}, & 1 \le s \le t \le e,\\ (\ln t)^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1}, & 1 \le t \le s \le e.\end{cases} \tag{6}$$

Proof It is enough to consider the case when u is a solution of (5). From Lemma 2.1 we have

$${}^{H}I_{1^+}^{\beta_1}{}^{H}D_{1^+}^{\beta_1}\bigl(\phi_p\bigl({}^{H}D_{1^+}^{\alpha_1}u(t)\bigr)\bigr) = \phi_p\bigl({}^{H}D_{1^+}^{\alpha_1}u(t)\bigr) + c_1(\ln t)^{\beta_1-1} + c_2(\ln t)^{\beta_1-2}$$

for some constants $c_i \in \mathbb{R}$, $i = 1,2$. In view of (5), we obtain

$${}^{H}I_{1^+}^{\beta_1}{}^{H}D_{1^+}^{\beta_1}\bigl(\phi_p\bigl(D_{1^+}^{\alpha_1}u(t)\bigr)\bigr) = I_{1^+}^{\beta_1}y(t).$$

Also we find

$$\phi_p\bigl({}^{H}D_{1^+}^{\alpha_1}u(t)\bigr) = {}^{H}I_{1^+}^{\beta_1}y(t) + c_1(\ln t)^{\beta_1-1} + c_2(\ln t)^{\beta_1-2}.$$

Note that $\phi_p({}^{H}D_{1^+}^{\alpha_1}u(1)) = 0$, we have $c_2 = 0$, then

$$\begin{aligned}
&{}^{H}D_{1^+}^{p_2}\bigl(\phi_p\bigl({}^{H}D_{1^+}^{\alpha_1}u(t)\bigr)\bigr)\\
&\quad = {}^{H}D_{1^+}^{p_2}{}^{H}I_{1^+}^{\beta_1}y(t) + c_1{}^{H}D_{1^+}^{p_2}(\ln t)^{\beta_1-1}\\
&\quad = {}^{H}I_{1^+}^{(\beta_1-p_2)}y(t) + c_1\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-p_2)}(\ln t)^{\beta_1-p_2-1}\\
&\quad = \frac{1}{\Gamma(\beta_1-p_2)}\int_1^t\left(\ln\frac{t}{s}\right)^{\beta_1-p_2-1}y(s)\frac{ds}{s} + c_1\frac{\Gamma(\beta)}{\Gamma(\beta_1-p_2)}(\ln t)^{\beta_1-p_2-1}.
\end{aligned}$$

Consequently, $({}^{H}D_{1^+}^{p_2}(\phi_p({}^{H}D_{1^+}^{\alpha_1}u(e))) = 0$, implies that

$$c_1 = \frac{-1}{\Gamma(\beta_1)}\int_1^e(1-\ln s)^{\beta_1-p_2-1}y(s)\frac{ds}{s}.$$

Therefore,

$$\begin{aligned}
\phi_p\bigl({}^{H}D_{1^+}^{\alpha_1}u(t)\bigr) &= \frac{1}{\Gamma(\beta_1)}\int_1^t\left(\ln\frac{t}{s}\right)^{\beta_1-1}y(s)\frac{ds}{s} - \frac{(\ln t)^{\beta_1-1}}{\Gamma(\beta_1)}\int_1^e(1-\ln s)^{\beta_1-p_2-1}y(s)\frac{ds}{s}\\
&= -\int_1^e H_1(t,s)y(s)\frac{ds}{s}.
\end{aligned}$$

Also we have

$${}^{H}D_{1^+}^{\alpha_1}u(t) + \phi_q\left(\int_1^e H_1(t,s)y(s)\frac{ds}{s}\right) = 0.$$

Noting Lemma 3.1 and the conditions $u^{(j)}(1) = 0$, $0 \le j \le n-2$, $\mu_1 u^{(p_1)}(e) = \lambda_1 u^{(p_1)}(\xi)$, we have $u(t) = \int_1^e G_1(t,s)\phi_q(\int_1^e H_1(t,\tau)y(\tau)\frac{d\tau}{\tau})\frac{ds}{s}$. □

Lemma 3.3 *Let $\Delta_1 > 0$. Then the Green's function $G_1(t,s)$ given by* (4) *satisfies the following inequalities*:

(i) $G_1(t,s) \geq 0$, *for all* $(t,s) \in [1,e] \times [1,e]$,

(ii) $G_1(t,s) \leq G_1(e,s)$, *for all* $(t,s) \in [1,e] \times [1,e]$,

(iii) $G_1(t,s) \geq (\frac{1}{4})^{\alpha_1-1} G_1(e,s)$, *for all* $(t,s) \in [e^{1/4}, e^{3/4}] \times (1,e)$.

Proof Consider the Green's function $G_{11}(t,s)$ given by (4).

(i) For $1 \leq t \leq s \leq e$. It is easy to see that $G_{11}(t,s) \geq 0$.

Let $1 \leq s \leq t \leq e$. Then

$$\begin{aligned}
G_{11}(t,s) &= \frac{1}{\Gamma(\alpha_1)}\left[(\ln t)^{\alpha_1-1}(1-\ln s)^{\alpha_1-p_1-1} - \left(\ln\frac{t}{s}\right)^{\alpha_1-1}\right] \\
&= \frac{1}{\Gamma(\alpha_1)}\left[(\ln t)^{\alpha_1-1}(1-\ln s)^{\alpha_1-p_1-1} - \left(1-\frac{\ln s}{\ln t}\right)^{\alpha_1-1}(\ln t)^{\alpha_1-1}\right] \\
&\geq \frac{(\ln t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\left[(1-\ln s)^{\alpha_1-p_1-1} - (1-\ln s)^{\alpha_1-1}\right] \\
&= \frac{(\ln t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\left[p_1 \ln s + \frac{p_1(p_1+1)}{2}(\ln s)^2 + \cdots\right](1-\ln s)^{\alpha_1-1} \geq 0 \\
&= \frac{(\ln t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\left[p_1 \ln s + O(\ln s)^2\right](1-\ln s)^{\alpha_1-1} \geq 0.
\end{aligned}$$

On the other hand, if $1 \leq s \leq \xi \leq e$,

$$\begin{aligned}
G_{12}(\xi,s) &= \frac{1}{\Gamma(\alpha_1)}\left[(\ln \xi)^{\alpha_1-p_1-1}(1-\ln s)^{\alpha_1-p_1-1} - \left(\ln\frac{\xi}{s}\right)^{\alpha_1-p_1-1}\right] \\
&= \frac{1}{\Gamma(\alpha_1)}\left[(\ln \xi)^{\alpha_1-p_1-1}(1-\ln s)^{\alpha_1-p_1-1} - \left(1-\frac{\ln s}{\ln \xi}\right)^{\alpha_1-p_1-1}(\ln \xi)^{\alpha_1-p_1-1}\right] \\
&\geq \frac{(\ln \xi)^{\alpha_1-p_1-1}}{\Gamma(\alpha_1)}\left[(1-\ln s)^{\alpha_1-p_1-1} - (1-\ln s)^{\alpha_1-p_1-1}\right] = 0,
\end{aligned}$$

which implies $G_1(t,s) \geq 0$. Hence the inequality (i) is proved.

(ii) For $1 \leq t \leq s \leq e$. It is easy to see that $\frac{dG_{11}(t,s)}{dt} \geq 0$.

Let $1 \leq s \leq \xi \leq e$. Then we have

$$\begin{aligned}
\frac{dG_{11}(t,s)}{dt} &= \frac{(\alpha_1-1)}{\Gamma(\alpha_1)}\left[(\ln t)^{\alpha_1-2}(1-\ln s)^{\alpha_1-p_1-1} - \left(\ln\frac{t}{s}\right)^{\alpha_1-2}\right] \\
&\geq \frac{(\alpha_1-1)(\ln t)^{\alpha_1-2}}{\Gamma(\alpha_1)}\left[(1-\ln s)^{\alpha_1-p_1-1} - (1-\ln s)^{\alpha_1-2}\right] \\
&= \frac{(\alpha_1-1)(\ln t)^{\alpha_1-2}}{\Gamma(\alpha_1)}\left[1-(1-(p_1-1)\ln s + \frac{(p_1-1)(p_1-2)}{2}(\ln s)^2\right. \\
&\quad \left. + \cdots\right](1-\ln s)^{\alpha_1-p_1-1} \\
&= \frac{(\alpha_1-1)(\ln t)^{\alpha_1-2}}{\Gamma(\alpha_1)}\left[(p_1-1)\ln s + O(\ln s)^2\right](1-\ln s)^{\alpha_1-p_1-1} \geq 0.
\end{aligned}$$

On the other hand, consider $\xi \leq s$. It is easy to see that $G_{12}(\xi,s) \geq 0$.

For $s \leq \xi$ one has

$$\begin{aligned}
&(\ln\xi)^{\alpha_1-p_1-1}(1-\ln s)^{\alpha_1-p_1-1}-\left(\ln\frac{\xi}{s}\right)^{\alpha_1-p_1-1} \\
&\quad\geq (\ln\xi)^{\alpha_1-p_1-1}(1-\ln s)^{\alpha_1-p_1-1}-(\ln\xi-\ln\xi\ln s)^{\alpha_1-p_1-1}\geq 0.
\end{aligned}$$

Therefore

$$\frac{dG_1(t,s)}{dt}=\frac{dG_{11}(t,s)}{dt}+\frac{\lambda_1(\alpha_1-1)(\ln t)^{\alpha_1-2}}{\mu_1-\lambda_1(\ln\xi)^{\alpha_1-p_1-1}}G_{12}(\xi,s)\geq 0,$$

which implies that $G_1(t,s)$ is the monotone nondecreasing function, so

$$G_1(t,s)\leq G_1(e,s)\quad \text{for all } (t,s)\in[1,e]\times[1,e].$$

Hence the inequality (ii) is proved.

(iii) For $1\leq t\leq s\leq e$,

$$\frac{G_{11}(t,s)}{G_{11}(e,s)}=\frac{(\ln t)^{\alpha-1}(1-\ln s)^{\alpha-\beta-1}}{(1-\ln s)^{\alpha-\beta-1}}=(\ln t)^{\alpha-1}.$$

For $1\leq s\leq t\leq e$,

$$\begin{aligned}
\frac{G_{11}(t,s)}{G_{11}(e,s)}&=\frac{(\ln t)^{\alpha_1-1}(1-\ln s)^{\alpha_1-p_1-1}-(\ln\frac{t}{s})^{\alpha_1-1}}{(1-\ln s)^{\alpha_1-p_1-1}-(1-\ln s)^{\alpha_1-1}} \\
&\geq\frac{(\ln t)^{\alpha_1-1}(1-\ln s)^{\alpha_1-p_1-1}-(\ln t-\ln s\ln t)^{\alpha_1-1}}{(1-\ln s)^{\alpha_1-p_1-1}-(1-\ln s)^{\alpha_1-1}} \\
&=\frac{(\ln t)^{\alpha_1-1}[(1-\ln s)^{\alpha_1-p_1-1}-(1-\ln s)^{\alpha_1-1}]}{(1-\ln s)^{\alpha_1-p_1-1}-(1-\ln s)^{\alpha_1-1}}=(\ln t)^{\alpha_1-1}.
\end{aligned}$$

Therefore

$$G_{11}(t,s)\geq(\ln t)^{\alpha_1-1}G_{11}(e,s)\quad\text{for all } (t,s)\in[1,e]\times(1,e). \tag{7}$$

From (4) and (7) we have

$$\begin{aligned}
G_1(t,s)&=G_{11}(t,s)+\frac{\lambda_1(\ln t)^{\alpha_1-1}}{\mu_1-\lambda_1(\ln\xi)^{\alpha_1-p_1-1}}G_{12}(\xi,s) \\
&\geq(\ln t)^{\alpha_1-1}G_{11}(e,s)+\frac{\lambda_1(\ln t)^{\alpha_1-1}}{\mu_1-\lambda_1(\ln\xi)^{\alpha_1-p_1-1}}G_{12}(\xi,s) \\
&=(\ln t)^{\alpha_1-1}G_1(e,s)\geq\left(\frac{1}{4}\right)^{\alpha_1-1}G_1(e,s)\quad\text{for all } (t,s)\in\left[e^{1/4},e^{3/4}\right]\times(1,e).
\end{aligned}$$

Therefore $G_1(t,s)\geq(\frac{1}{4})^{\alpha_1-1}G_1(e,s)$. Hence the inequality (iii) is proved. □

Lemma 3.4 *Let $\Delta_1>0$. Then the Green's function $H_1(t,s)$ given by (6) satisfies the following inequalities*:

(i) $0\leq H_1(t,s)\leq H_1(s,s)$, *for all* $(t,s)\in[1,e]\times[1,e]$,

(ii) $H_1(t,s) \geq \delta_1(s) H_1(s,s)$, *for all* $(t,s) \in [e^{1/4}, e^{3/4}] \times (1,e)$,

$$\delta_1(s) = \begin{cases} \frac{(\frac{3}{4})^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1}-(\frac{3}{4}-\ln s)^{\beta_1-1}}{(\ln s)^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1}} H_1(s,s), & s \in (1,r], \\ \frac{1}{(4\ln s)^{\beta-1}} H_1(s,s), & s \in [r,e). \end{cases}$$

Proof (i) For $1 \leq t \leq s \leq e$, it is easy to show that $\frac{d}{dt} H_1(t,s) \geq 0$ for all $(t,s) \in [1,e] \times [1,e]$, then

$$0 = H_1(1,s) \leq H_1(t,s) \leq H_1(s,s) \quad \text{for all } t \leq s.$$

Let $s \leq t$. Then

$$\begin{aligned} \frac{d}{dt} H_1(t,s) &= \frac{1}{\Gamma(\beta_1)} \left[(\beta_1 - 1)(\ln t)^{\beta_1-2}(1-\ln s)^{\beta_1-p_2-1} - (\beta_1-1)\left(\ln \frac{t}{s}\right)^{\beta_1-2} \right] \\ &\leq \frac{(\beta_1-1)}{\Gamma(\beta_1)} \left[(1-\ln s)^{\beta_1-p_2-1} - (1-\ln s)^{\beta_1-2} \right] \\ &= \frac{(\beta_1-1)}{\Gamma(\beta_1)} \left[(1-\ln s)^{-p_2+1} - 1 \right] (1-\ln s)^{\beta_1-2} \\ &= \frac{(\beta_1-1)}{\Gamma(\beta_1)} \left[(p_2-1)\ln s + \frac{(p_2-1)(p_2)}{2!} (\ln s)^2 + \cdots \right] (1-\ln s)^{\beta_1-2} \\ &= \frac{(\beta_1-1)}{\Gamma(\beta_1)} \left[(p_2-1)\ln s + O(\ln s)^2 \right] (1-\ln s)^{\beta_1-2} \leq 0. \end{aligned}$$

Therefore, $H_1(t,s)$ is decreasing in t for $s \in [1,e]$, which implies that $H_1(t,s) \leq H_1(s,s)$. Hence, the inequality (i) is proved.

(ii) For $s \in (1,e)$. $H_1(t,s)$ is increasing in t for $t \leq s$ and decreasing in t for $s \leq t$. We define

$$\begin{aligned} h_{11}(t,s) &= \frac{1}{\Gamma(\beta_1)} \left[(\ln t)^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1} - \left(\ln \frac{t}{s}\right)^{\beta_1-1} \right], \\ h_{12}(t,s) &= \frac{1}{\Gamma(\beta_1)} \left[(\ln t)^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1} \right], \end{aligned}$$

and applying the monotonicity of $H_1(t,s)$, we have

$$\begin{aligned} \min_{e^{1/4} \leq t \leq e^{3/4}} H_1(t,s) &= \begin{cases} h_{11}(e^{3/4},s), & s \in (1,e^{1/4}], \\ \min\{h_1(e^{3/4},s), h_{12}(e^{1/4},s)\}, & s \in [e^{1/4},e^{3/4}], \\ h_{12}(e^{1/4},s), & s \in [e^{3/4},e), \end{cases} \\ &= \begin{cases} h_{11}(e^{3/4},s), & s \in (1,r), \\ h_{12}(e^{1/4},s), & s \in [r,e) \end{cases} \\ &= \begin{cases} \frac{1}{\Gamma(\beta_1)} [(\frac{3}{4})^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1} - (\frac{3}{4}-\ln s)^{\beta_1-1}], & s \in (1,r], \\ \frac{1}{\Gamma(\beta_1)} [(\frac{1}{4})^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1}], & s \in [r,e), \end{cases} \end{aligned}$$

$$\geq \begin{cases} \frac{(\frac{3}{4})^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1}-(\frac{3}{4}-\ln s)^{\beta_1-1}}{(\ln s)^{\beta_1-1}(1-\ln s)^{\beta_1-p_2-1}} H_1(s,s), & s \in (1,r], \\ \frac{1}{(4\ln s)^{\beta_1-1}} H_1(s,s), & s \in [r,e), \end{cases}$$
$$= \delta_1(s) H_1(s,s) \quad \text{for } s \in (1,e).$$

Hence, the inequality (ii) is proved. □

We can also formulate similar results to Lemmas 3.1–3.4 for the Hadamard fractional boundary value problems

$$\begin{cases} {}^H D_{1^+}^{\beta_2}(\phi_p({}^H D_{1^+}^{\alpha_2} v(t))) = \mu^{p-1} g(t,u(t),v(t),w(t)), & t \in (1,e), \\ v^{(j)}(1) = 0, \quad 0 \leq j \leq m-2, & \mu_1 v^{(q_1)}(e) = \lambda_1 v^{(q_1)}(\xi), \\ \phi_p({}^H D_{1^+}^{\alpha_2} v(1)) = 0 = ({}^H D_{1^+}^{q_2}(\phi_p({}^H D_{1^+}^{\alpha_2} v(e)))), \end{cases} \tag{8}$$

and

$$\begin{cases} {}^H D_{1^+}^{\beta_3}(\phi_p({}^H D_{1^+}^{\alpha_3} w(t))) = \upsilon^{p-1} h(t,u(t),v(t),w(t)), & t \in (1,e), \\ w^{(j)}(1) = 0, \quad 0 \leq j \leq l-2, & \mu_1 w^{(r_1)}(e) = \lambda_1 w^{(r_1)}(\xi), \\ \phi_p({}^H D_{1^+}^{\alpha_3} w(1)) = 0 = ({}^H D_{1^+}^{r_2}(\phi_p({}^H D_{1^+}^{\alpha_3} w(e)))). \end{cases} \tag{9}$$

Remark In a similar manner, the results of the Green's functions $G_2(t,s)$, $G_3(t,s)$, $H_2(t,s)$ and $H_3(t,s)$ for the homogeneous BVP corresponding to the Hadamard fractional differential equations (8) and (9) are obtained.

Consider the following conditions:

(i) $G_i(t,s) \geq m G_i(e,s)$, for all $(t,s) \in [1,e] \times [1,e]$, $i = 1.2.3$,

(ii) $H_i(t,s) \geq \delta(s) H_i(e,s)$, for all $(t,s) \in I \times (1,e)$, $i = 1,2,3$,

where $I = [e^{1/4}, e^{3/4}]$, $m = \min\{(\frac{1}{4})^{\alpha_1-1}, (\frac{1}{4})^{\alpha_2-1}, (\frac{1}{4})^{\alpha_3-1}\}$, $\delta(s) = \min\{\delta_1(s), \delta_2(s), \delta_3(s)\}$.

Our main results are based on the following Guo–Krasnosel'skii fixed point theorem on cones.

Theorem 3.5 (Krasnosel'skii [49, 50]) *Let X be a Banach space, $K \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose furthermore that $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is a completely continuous operator such that either*

(i) *$\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, or*

(ii) *$\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$,*

holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

4 Positive solutions in a cone

In this section, we shall give sufficient conditions on λ, μ, υ, f, g and h such that positive solutions with respect to a cone for our problem (1)–(2) exist.

We present the assumptions that we shall use in the sequel:

(A1) The functions $f, g, h : [1,e] \times [0,+\infty)^3 \to [0,+\infty)$ are continuous.

(A2) $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1,2,3$, $\alpha_1 \in (n-1,n]$, $\alpha_2 \in (m-1,m]$, $\alpha_3 \in (l-1,l]$, $n,m,l \in \mathbb{N}$ for $n,m,l \geq 3$, $\beta_i \in (1,2]$, $i = 1,2,3$, $p_1 \in [1,\alpha_1-1)$, $q_1 \in [1,\alpha_2-1)$, $\gamma_1 \in [1,\alpha_3-1)$,

$p_2.q_2, \gamma_2 \in (0,1]$, $p_2, q_2, \gamma_2 \leq \beta_i - 1$, $i = 1,2,3$, $\mu_1, \lambda_1 \in (0,\infty)$, $\xi \in (1,e)$ are constants and $\Delta_1 = \mu_1 - \lambda_1(\ln\xi)^{\alpha_1-p_1-1} > 0$, $\Delta_2 = \mu_1 - \lambda_1(\ln\xi)^{\alpha_2-q_1-1} > 0$, $\Delta_3 = \mu_1 - \lambda_1(\ln\xi)^{\alpha_3-r_1-1} > 0$.

(A3) For $I = [e^{1/4}, e^{3/4}] \subset (1,e)$, we introduce the following extreme limits:

$$f_0^s = \limsup_{u+v+w\to 0} \max_{t\in[1,e]} \frac{f(t,u,v,w)}{(u+v+w)^{p-1}}, \qquad g_0^s = \limsup_{u+v+w\to 0} \max_{t\in[1,e]} \frac{g(t,u,v,w)}{(u+v+w)^{p-1}},$$

$$h_0^s = \limsup_{u+v+w\to 0} \max_{t\in[1,e]} \frac{h(t,u,v,w)}{(u+v+w)^{p-1}}, \qquad f_0^i = \liminf_{u+v+w\to 0} \min_{t\in I} \frac{f(t,u,v,w)}{(u+v+w)^{p-1}},$$

$$g_0^i = \liminf_{u+v+w\to 0} \min_{t\in I} \frac{g(t,u,v,w)}{(u+v+w)^{p-1}}, \qquad h_0^i = \liminf_{u+v+w\to 0} \min_{t\in I} \frac{h(t,u,v,w)}{(u+v+w)^{p-1}},$$

$$f_\infty^s = \limsup_{u+v+w\to \infty} \max_{t\in[1,e]} \frac{f(t,u,v,w)}{(u+v+w)^{p-1}}, \qquad g_\infty^s = \limsup_{u+v+w\to \infty} \max_{t\in[1,e]} \frac{g(t,u,v,w)}{(u+v+w)^{p-1}},$$

$$h_\infty^s = \limsup_{u+v+w\to \infty} \max_{t\in[1,e]} \frac{h(t,u,v,w)}{(u+v+w)^{p-1}}, \qquad f_\infty^i = \liminf_{u+v+w\to \infty} \min_{t\in I} \frac{f(t,u,v,w)}{(u+v+w)^{p-1}},$$

$$g_\infty^i = \liminf_{u+v+w\to \infty} \max_{t\in I} \frac{g(t,u,v,w)}{(u+v+w)^{p-1}}, \qquad h_\infty^i = \liminf_{u+v+w\to \infty} \min_{t\in I} \frac{h(t,u,v,w)}{(u+v+w)^{p-1}}.$$

Let $X = C[1,e]$, then X is a Banach space with the norm $\|u\| = \sup_{t\in[1,e]} |u(t)|$. Let $Y = X \times X \times X$, then Y is a Banach space with the norm $\|(u,v,w)\|_Y = \|u\| + \|v\| + \|w\|$.

Define a cone $P \subset Y$ by

$$\begin{aligned} P = \Big\{ &(u,v,w) \in Y : u(t) \geq 0, v(t) \geq 0, w(t) \geq 0, \forall t \in [1,e], \\ &\min_{t\in I}\big\{u(t) + v(t) + w(t)\big\} \geq m\big\|(u,v,w)\big\|_Y \Big\}, \end{aligned}$$

where $I = [e^{1/4}, e^{3/4}]$. For $\lambda, \mu, \upsilon > 0$, we define now the operator $Q : P \to Y$ by $Q(u,v,w) = (Q_1(u,v,w), Q_2(u,v,w), Q_3(u,v,w))$ with

$$Q_1(u,v,w)(t) = \lambda \int_1^e G_1(t,s)\left(\int_1^e H_1(s,\tau) f\big(\tau, u(\tau), v(\tau), w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}, \quad t \in [1,e],$$

$$Q_2(u,v,w)(t) = \mu \int_1^e G_2(t,s)\left(\int_1^e H_2(s,\tau) g\big(\tau, u(\tau), v(\tau), w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}, \quad t \in [1,e],$$

$$Q_3(u,v,w)(t) = \upsilon \int_1^e G_3(t,s)\left(\int_1^e H_3(s,\tau) h\big(\tau, u(\tau), v(\tau), w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}, \quad t \in [1,e].$$

Lemma 4.1 *$Q : P \to P$ is a completely continuous operator.*

Proof The continuity of functions $G_i(t,s), H_i(t,s)$, $i = 1,2,3$ and f, g, h implies that $Q : P \to P$ is continuous. For all $(t,s) \in I \times [1,e]$, where $I = [e^{1/4}, e^{3/4}]$, we have

$$\begin{aligned} &\min_{t\in I}\big\{Q_1(u,v,w)(t) + Q_2(u,v,w)(t) + Q_3(u,v,w)(t)\big\} \\ &= \min_{t\in I}\bigg\{\lambda \int_1^e G_1(t,s)\left(\int_1^e H_1(s,\tau) f\big(\tau, u(\tau), v(\tau), w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\quad + \mu \int_1^e G_2(t,s)\left(\int_1^e H_2(s,\tau) g\big(\tau, u(\tau), v(\tau), w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&+ \upsilon \int_1^e G_3(t,s)\left(\int_1^e H_3(s,\tau)h\big(\tau,u(\tau),v(\tau),w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\bigg\} \\
&\geq m\bigg\{\lambda \int_1^e G_1(e,s)\left(\int_1^e H_1(s,\tau)f\big(\tau,u(\tau),v(\tau),w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&+ \mu \int_1^e G_2(e,s)\left(\int_1^e H_2(s,\tau)g\big(\tau,u(\tau),v(\tau),w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&+ \upsilon \int_1^e G_3(e,s)\left(\int_1^e H_3(s,\tau)h\big(\tau,u(\tau),v(\tau),w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\bigg\} \\
&\geq m\big(\|Q_1(u,v,w)\| + \|Q_2(u,v,w)\| + \|Q_3(u,v,w)\|\big) \\
&= m\big\|\big(Q_1(u,v,w),Q_2(u,v,w),Q_3(u,v,w)\big)\big\| \\
&= m\|Q(u,v,w)\|.
\end{aligned}
$$

Thus $Q(P) \subset P$. So, we can easily show that $Q : P \to P$ is a completely continuous operator by the Arzela–Ascoli theorem. □

If $(u,v,w) \in P$ is a fixed point of operator Q, then (u,v,w) is a solution of problem (1)–(2). So, we will investigate the existence of fixed points of operator Q.

First, for f_0^s, $g_0^s, h_0^s f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$ and positive numbers $\sigma_1,\sigma_2,\sigma_3 > 0$ such that $\sigma_1 + \sigma_2 + \sigma_3 = 1$, we define the positive numbers L_1, L_2, L_3, L_4, L_5 and L_6 by

$$
\begin{aligned}
L_1 &= \sigma_1\left[m^2\int_{t\in I} G_1(e,s)\left(\int_{t\in I}\delta(\tau)H_1(\tau,\tau)f_\infty^i\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1}, \\
L_2 &= \sigma_1\left[\int_1^e G_1(e,s)\left(\int_1^e H_1(s,\tau)f_0^s\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1}, \\
L_3 &= \sigma_2\left[m^2\int_{t\in I} G_2(e,s)\left(\int_{t\in I}\delta(\tau)H_2(\tau,\tau)g_\infty^i\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1}, \\
L_4 &= \sigma_2\left[\int_1^e G_2(e,s)\left(\int_1^e H_2(s,\tau)g_0^s\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1}, \\
L_5 &= \sigma_3\left[m^2\int_{t\in I} G_3(e,s)\left(\int_{t\in I}\delta(\tau)H_3(\tau,\tau)h_\infty^i\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1}, \\
L_6 &= \sigma_3\left[\int_1^e G_3(e,s)\left(\int_1^e H_3(s,\tau)h_0^s\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1}.
\end{aligned}
$$

Theorem 4.2 *Assume that* (A1)–(A3) *hold*, $\sigma_1,\sigma_2,\sigma_3 > 0$ *with* $\sigma_1 + \sigma_2 + \sigma_3 = 1$.

(a) *If* $f_0^s, g_0^s, h_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, $L_1 < L_2$, $L_3 < L_4$ *and* $L_5 < L_6$, *then for each* $\lambda \in (L_1,L_2)$, $\mu \in (L_3,L_4)$ *and* $\upsilon \in (L_5,L_6)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [1,e]$, *for problem* (1)–(2).

(b) *If* $f_0^s = g_0^s = h_0^s = 0$, $f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, *then for each* $\lambda \in (L_1,\infty)$, $\mu \in (L_3,\infty)$ *and* $\upsilon \in (L_5,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [1,e]$, *for problem* (1)–(2).

(c) *If* $f_0^s, g_0^s, h_0^s \in (0,\infty)$, $f_\infty^i = g_\infty^i = h_\infty^i = \infty$ *then for each* $\lambda \in (0, L_2)$, $\mu \in (0, L_4)$ *and* $\upsilon \in (0, L_6)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [1, e]$, *for problem* (1)–(2).

(d) *If* $f_0^s = g_0^s = h_0^s = 0$, $f_\infty^i = g_\infty^i = h_\infty^i = \infty$, *then for each* $\lambda \in (0,\infty)$, $\mu \in (0,\infty)$ *and* $\upsilon \in (0,\infty)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [1, e]$, *for problem* (1)–(2).

Proof Because the proofs of these cases are similar, in what follows we will prove two of them, namely cases (a) and (b).

(a) For any $\lambda \in (L_1, L_2)$, $\mu \in (L_3, L_4)$, $\upsilon \in (L_5, L_6)$ let $\epsilon > 0$ be a positive number such that $0 < \epsilon < \min\{f_\infty^i, g_\infty^i, h_\infty^i\}$ and

$$\sigma_1\left[m^2\left(f_\infty^i - \epsilon\right)^{\frac{1}{p-1}} \int_{t\in I} G_1(e,s)\left(\int_{t\in I} \delta(\tau) H_1(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \le \lambda,$$

$$\sigma_1\left[\left(f_0^s + \epsilon\right)^{\frac{1}{p-1}} \int_1^e G_1(e,s)\left(\int_1^e H_1(s,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \ge \lambda,$$

$$\sigma_2\left[m^2\left(g_\infty^i - \epsilon\right)^{\frac{1}{p-1}} \int_{t\in I} G_2(e,s)\left(\int_{t\in I} \delta(\tau) H_2(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \le \mu,$$

$$\sigma_2\left[\left(g_0^s + \epsilon\right)^{\frac{1}{p-1}} \int_1^e G_2(e,s)\left(\int_1^e H_2(s,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \ge \mu,$$

$$\sigma_3\left[m^2\left(h_\infty^i - \epsilon\right)^{\frac{1}{p-1}} \int_{t\in I} G_3(e,s)\left(\int_{t\in I} \delta(\tau) H_3(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \le \upsilon,$$

$$\sigma_3\left[\left(h_0^s + \epsilon\right)^{\frac{1}{p-1}} \int_1^e G_3(e,s)\left(\int_1^e H_3(s,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \ge \upsilon.$$

By the definitions of f_0^s, g_0^s and h_0^s there exists $R_1 > 0$ such that

$$f(t,u,v,w) \le \left(f_0^s + \epsilon\right)(u+v+w)^{p-1}, \quad t \in [1,e], 0 \le u+v+w \le R_1,$$

$$g(t,u,v,w) \le \left(g_0^s + \epsilon\right)(u+v+w)^{p-1}, \quad t \in [1,e], 0 \le u+v+w \le R_1,$$

$$h(t,u,v,w) \le \left(h_0^s + \epsilon\right)(u+v+w)^{p-1}, \quad t \in [1,e], 0 \le u+v+w \le R_1.$$

Let $(u,v,w) \in P$ with $\|(u,v,w)\|_Y = R_1$ i.e. $\|u\| + \|v\| + \|w\| = R_1$, then we have

$$\begin{aligned} Q_1(u,v,w)(t) &= \lambda \int_1^e G_1(t,s)\left(\int_1^e H_1(s,\tau) f\left(\tau, u(\tau), v(\tau), w(\tau)\right)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\le \lambda \int_1^e G_1(e,s)\left(\int_1^e H_1(s,\tau)\left(f_0^s + \epsilon\right)\left(u(\tau) + v(\tau) + w(\tau)\right)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\le \lambda\left(f_0^s + \epsilon\right)^{\frac{1}{p-1}} \int_1^e G_1(e,s)\left(\int_1^e H_1(s,\tau)\left(\|u\| + \|v\| + \|w\|\right)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\le \sigma_1 \left\|(u,v,w)\right\|_Y, \quad \text{for all } t \in [1,e], \end{aligned}$$

$$
\begin{aligned}
Q_2(u,v,w)(t) &= \mu \int_1^e G_2(t,s)\left(\int_1^e H_2(s,\tau) g\big(\tau,u(\tau),v(\tau),w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&\leq \mu \int_1^e G_2(e,s)\left(\int_1^e H_2(s,\tau)\big(g_0^s+\epsilon\big)\big(u(\tau)+v(\tau)+w(\tau)\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&\leq \mu\big(g_0^s+\epsilon\big)^{\frac{1}{p-1}} \int_1^e G_2(e,s)\left(\int_1^e H_2(s,\tau)\big(\|u\|+\|v\|+\|w\|\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&\leq \sigma_2\big\|(u,v,w)\big\|_Y, \quad \text{for all } t\in[1,e], \\
Q_3(u,v,w)(t) &= \upsilon \int_1^e G_3(t,s)\left(\int_1^e H_3(s,\tau) h\big(\tau,u(\tau),v(\tau),w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&\leq \upsilon \int_1^e G_3(e,s)\left(\int_1^e H_3(s,\tau)\big(h_0^s+\epsilon\big)\big(u(\tau)+v(\tau)+w(\tau)\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&\leq \upsilon\big(h_0^s+\epsilon\big)^{\frac{1}{p-1}} \int_1^e G_3(e,s)\left(\int_1^e H_3(s,\tau)\big(\|u\|+\|v\|+\|w\|\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&\leq \sigma_3\big\|(u,v,w)\big\|_Y, \quad \text{for all } t\in[1,e].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\big\|Q(u,v,w)\big\|_Y &= \big\|\big(Q_1(u,v,w),Q_2(u,v,w),Q_3(u,v,w)\big)\big\|_Y \\
&= \big\|Q_1(u,v,w)\big\| + \big\|Q_2(u,v,w)\big\| + \big\|Q_3(u,v,w)\big\| \\
&\leq \sigma_1\big\|(u,v,w)\big\|_Y + \sigma_2\big\|(u,v,w)\big\|_Y + \sigma_3\big\|(u,v,w)\big\|_Y \\
&= (\sigma_1+\sigma_2+\sigma_3)\big\|(u,v,w)\big\|_Y \\
&= \big\|(u,v,w)\big\|_Y.
\end{aligned}
$$

Hence, $\|Q(u,v,w)\|_Y \leq \|(u,v,w)\|_Y$. Define the set

$$
\Omega_1 = \big\{(u,v,w)\in Y : \big\|(u,v,w)\big\|_Y < R_1\big\},
$$

then

$$
\big\|Q(u,v,w)\big\|_Y \leq \big\|(u,v,w)\big\|_Y, \quad \text{for } (u,v,w)\in P\cap\partial\Omega_1. \tag{10}
$$

On the other hand, by the definitions of f_∞^i, g_∞^i and h_∞^i, there exists $\overline{R}_2>0$ such that

$$
\begin{aligned}
f(t,u,v) &\geq \big(f_\infty^i-\epsilon\big)(u+v+w)^{p-1}, \quad t\in I, u,v,w\geq 0, u+v+w\geq \overline{R}_2, \\
g(t,u,v) &\geq \big(g_\infty^i-\epsilon\big)(u+v+w)^{p-1}, \quad t\in I, u,v,w\geq 0, u+v+w\geq \overline{R}_2, \\
h(t,u,v) &\geq \big(h_\infty^i-\epsilon\big)(u+v+w)^{p-1}, \quad t\in I, u,v,w\geq 0, u+v+w\geq \overline{R}_2,
\end{aligned}
$$

Let $R_2 = \max\{2R_1, \frac{\overline{R_2}}{m}\}$. Choose $(u,v,w)\in P$ with $\|(u,v,w)\|_Y = R_2$. Then

$$
\min_{t\in I}\big(u(t)+v(t)+w(t)\big) \geq m\big\|(u,v,w)\big\|_Y = mR_2 \geq \overline{R}_2.
$$

Also we have

$$\begin{aligned}
&Q_1(u,v,w)(t)\\
&\quad= \lambda \int_1^e G_1(t,s)\left(\int_1^e H_1(s,\tau)f\big(\tau,u(\tau),v(\tau),w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad\geq \lambda m \int_{t\in I} G_1(e,s)\left(\int_1^e H_1(s,\tau)\big(f_\infty^i-\epsilon\big)\big(u(\tau)+v(\tau)+w(\tau)\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad\geq \lambda m^2\big(f_\infty^i-\epsilon\big)^{\frac{1}{p-1}} \int_{t\in I} G_1(e,s)\left(\int_{t\in I}\delta(\tau)H_1(\tau,\tau)\big(\|(u,v,w)\|_Y\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad\geq \sigma_1\|(u,v,w)\|_Y,\\
&Q_2(u,v,w)(t)\\
&\quad= \lambda \int_1^e G_2(t,s)\left(\int_1^e H_2(s,\tau)g\big(\tau,u(\tau),v(\tau),w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad\geq \mu m \int_{t\in I} G_2(e,s)\left(\int_1^e H_2(s,\tau)\big(g_\infty^i-\epsilon\big)\big(u(\tau)+v(\tau)+w(\tau)\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad\geq \mu m^2\big(g_\infty^i-\epsilon\big)^{\frac{1}{p-1}} \int_{t\in I} G_2(e,s)\left(\int_{t\in I}\delta(\tau)H_2(\tau,\tau)\big(\|(u,v,w)\|_Y\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad\geq \sigma_2\|(u,v,w)\|_Y,\\
&Q_3(u,v,w)(t)\\
&\quad= \lambda \int_1^e G_3(t,s)\left(\int_1^e H_3(s,\tau)h\big(\tau,u(\tau),v(\tau),w(\tau)\big)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad\geq \upsilon m \int_{s\in I} G_3(e,s)\left(\int_1^e H_3(s,\tau)\big(h_\infty^i-\epsilon\big)\big(u(\tau)+v(\tau)+w(\tau)\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad\geq \upsilon m^2\big(h_\infty^i-\epsilon\big)^{\frac{1}{p-1}} \int_{t\in I} G_3(e,s)\left(\int_{\tau\in I}\delta(\tau)H_3(\tau,\tau)\big(\|(u,v,w)\|_Y\big)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad\geq \sigma_3\|(u,v,w)\|_Y.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Q(u,v,w)\|_Y &= \big\|\big(Q_1(u,v,w),Q_2(u,v,w),Q_3(u,v,w)\big)\big\|_Y\\
&= \|Q_1(u,v,w)\| + \|Q_2(u,v,w)\| + \|Q_3(u,v,w)\|\\
&\geq \sigma_1\|(u,v,w)\|_Y + \sigma_2\|(u,v,w)\|_Y + \sigma_3\|(u,v,w)\|_Y\\
&= (\sigma_1+\sigma_2+\sigma_3)\|(u,v,w)\|_Y\\
&= \|(u,v,w)\|_Y.
\end{aligned}$$

Hence, $\|Q(u,v,w)\|_Y \geq \|(u,v,w)\|_Y$. Define the set

$$\Omega_2 = \big\{(u,v,w)\in Y : \|(u,v,w)\|_Y < R_2\big\}$$

then

$$\left\|Q(u,v,w)\right\|_{Y} \geq \left\|(u,v,w)\right\|_{Y}, \quad \text{for } (u,v,w) \in P \cap \partial\Omega_2. \tag{11}$$

Therefore, by (10), (11) and Theorem 3.5, we conclude that Q has at least one fixed point $(u,v,w) \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $R_1 \leq \|(u,v,w)\|_Y \leq R_2$.

(b) Let $\lambda \in (L_1,\infty)$, $\mu \in (L_3,\infty)$, $\upsilon \in (L_5,\infty)$ and let $\epsilon > 0$ be a positive number such that $\epsilon < f_\infty^i$, $\epsilon < g_\infty^i$, $\epsilon < h_\infty^i$. We have

$$\begin{aligned}
&\sigma_1\left[m^2\left(f_\infty^i - \epsilon\right)^{\frac{1}{p-1}} \int_{t\in I} G_1(e,s)\left(\int_{t\in I} \delta(\tau)H_1(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \leq \lambda,\\
&\sigma_2\left[m^2\left(g_\infty^i - \epsilon\right)^{\frac{1}{p-1}} \int_{t\in I} G_2(e,s)\left(\int_{t\in I} \delta(\tau)H_2(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \leq \mu,\\
&\sigma_3\left[m^2\left(h_\infty^i - \epsilon\right)^{\frac{1}{p-1}} \int_{t\in I} G_3(e,s)\left(\int_{t\in I} \delta(\tau)H_3(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \leq \upsilon,\\
&\epsilon \leq \frac{\sigma_1}{\lambda}\left[\int_1^e G_1(e,s)\left(\int_1^e H_1(s,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1},\\
&\epsilon \leq \frac{\sigma_2}{\mu}\left[\int_1^e G_2(e,s)\left(\int_1^e H_2(s,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1},\\
&\epsilon \leq \frac{\sigma_3}{\upsilon}\left[\int_1^e G_3(e,s)\left(\int_1^e H_3(s,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1}.
\end{aligned}$$

By the definitions of $f_0^s = 0$, $g_0^s = 0$ and $h_0^s = 0$, there exists $R_1 > 0$ such that

$$\begin{aligned}
&f(t,u,v,w) \leq \epsilon^{p-1}(u+v+w)^{p-1}, \quad 0 \leq u+v+w \leq R_1,\\
&g(t,u,v,w) \leq \epsilon^{p-1}(u+v+w)^{p-1}, \quad 0 \leq u+v+w \leq R_1,\\
&h(t,u,v,w) \leq \epsilon^{p-1}(u+v+w)^{p-1}, \quad 0 \leq u+v+w \leq R_1.
\end{aligned}$$

Let $(u,v,w) \in P$ with $\|(u,v,w)\|_Y = R_1$ i.e. $\|u\| + \|v\| + \|w\| = R_1$. Then we have

$$\begin{aligned}
Q_1(u,v,w)(t) &= \lambda \int_1^e G_1(t,s)\left(\int_1^e H_1(s,\tau)f\left(\tau,u(\tau),v(\tau),w(\tau)\right)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\\
&\leq \lambda \int_1^e G_1(e,s)\left(\int_1^e H_1(s,\tau)\epsilon^{p-1}\left(u(\tau)+v(\tau)+w(\tau)\right)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\\
&\leq \lambda\epsilon \int_1^e G_1(e,s)\left(\int_1^e H_1(s,\tau)\left(\|u\|+\|v\|+\|w\|\right)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\\
&\leq \sigma_1\left\|(u,v,w)\right\|_Y, \quad \text{for all } t \in [1,e].
\end{aligned}$$

Hence, $\|Q_1(u,v,w)\| \leq \sigma_1\|(u,v,w)\|_Y$. In a similar manner, we conclude that

$$\left\|Q_2(u,v,w)\right\| \leq \sigma_2\left\|(u,v,w)\right\|_Y, \qquad \left\|Q_3(u,v,w)\right\| \leq \sigma_3\left\|(u,v,w)\right\|_Y.$$

Therefore,

$$\begin{aligned}\|Q(u,v,w)\|_Y &= \|(Q_1(u,v,w),Q_2(u,v,w),Q_3(u,v,w))\|_Y\\ &= \|Q_1(u,v,w)\| + \|Q_2(u,v,w)\| + \|Q_3(u,v,w)\|\\ &\le \sigma_1\|(u,v,w)\|_Y + \sigma_2\|(u,v,w)\|_Y + \sigma_3\|(u,v,w)\|_Y\\ &= (\sigma_1+\sigma_2+\sigma_3)\|(u,v,w)\|_Y\\ &= \|(u,v,w)\|_Y.\end{aligned}$$

Hence, $\|Q(u,v,w)\|_Y \le \|(u,v,w)\|_Y$. Define the set

$$\Omega_1 = \{(u,v,w)\in Y : \|(u,v,w)\|_Y < R_1\},$$

then

$$\|Q(u,v,w)\|_Y \le \|(u,v,w)\|_Y, \quad \text{for } (u,v,w)\in P\cap\partial\Omega_1. \tag{12}$$

By the definitions of $f^i_\infty, g^i_\infty, h^i_\infty \in (0,\infty)$, there exists $\overline{R}_2 > 0$ such that

$$\begin{aligned}f(t,u,v,w) &\ge (f^i_\infty - \epsilon)(u+v+w)^{p-1}, \quad u+v+w \ge \overline{R}_2,\\ g(t,u,v,w) &\ge (g^i_\infty - \epsilon)(u+v+w)^{p-1}, \quad u+v+w \ge \overline{R}_2,\\ h(t,u,v,w) &\ge (h^i_\infty - \epsilon)(u+v+w)^{p-1}, \quad u+v+w \ge \overline{R}_2.\end{aligned}$$

Define the set

$$\Omega_2 = \{(u,v,w)\in Y \mid \|(u,v,w)\|_Y < R_2\}$$

and proceeding in a similar manner of proof (a), we get

$$\|Q(u,v,w)\|_Y \ge \|(u,v,w)\|_Y, \quad \text{for } (u,v,w)\in P\cap\partial\Omega_2. \tag{13}$$

Therefore, by (12), (13) and Theorem 3.5, we conclude that Q has at least one fixed point $(u,v,w)\in P\cap(\overline{\Omega}_2\setminus\Omega_1)$ with $R_1 \le \|(u,v,w)\|_Y \le R_2$. Similarly, we can prove the remaining. □

Prior to our next result, we define the positive numbers M_1, M_2, M_3, M_4, M_5 and M_6 by

$$\begin{aligned}M_1 &= \rho_1\left[m^2(f^i_0-\epsilon)^{\frac{1}{p-1}}\int_{s\in I} G_1(e,s)\left(\int_{\tau\in I}\delta(\tau)H_1(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1},\\ M_2 &= \rho_1\left[(f^s_\infty+\epsilon)^{\frac{1}{p-1}}\int_1^e G_1(e,s)\left(\int_1^e H_1(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1},\\ M_3 &= \rho_2\left[m^2(g^i_0-\epsilon)^{\frac{1}{p-1}}\int_{s\in I} G_2(e,s)\left(\int_{\tau\in I}\delta(\tau)H_2(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1},\\ M_4 &= \rho_2\left[(g^s_\infty+\epsilon)^{\frac{1}{p-1}}\int_1^e G_2(e,s)\left(\int_1^e H_2(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right]^{-1},\end{aligned}$$

$$M_5 = \rho_3\left[m^2\left(h_0^i - \epsilon\right)^{\frac{1}{p-1}} \int_{s\in I} G_3(e,s)\left(\int_{\tau\in I} \delta(\tau)H_3(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1},$$

$$M_6 = \rho_3\left[\left(h_\infty^s + \epsilon\right)^{\frac{1}{p-1}} \int_1^e G_3(e,s)\left(\int_1^e H_3(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1},$$

where $\rho_1, \rho_2, \rho_3 > 0$ are three positive numbers with $\rho_1 + \rho_2 + \rho_3 = 1$.

Theorem 4.3 *Assume that the conditions* (A1)–(A3) *hold.*

(a) *If* $f_0^i, g_0^i, h_0^i f_\infty^s, g_\infty^s, h_\infty^s \in (0,\infty)$, $M_1 < M_2$, $M_3 < M_4$ *and* $M_5 < M_6$ *then for each* $\lambda \in (M_1, M_2)$, $\mu \in (M_3, M_4)$ *and* $\upsilon \in (M_5, M_6)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [1,e]$, *for problem* (1)–(2).

(b) *If* $f_\infty^s = g_\infty^s = h_\infty^s = 0$, $f_0^i, g_0^i, h_0^i \in (0,\infty)$, *then for each* $\lambda \in (M_1,\infty)$, $\mu \in (M_3,\infty)$ *and* $\upsilon \in (M_5,\infty)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [1,e]$, *for problem* (1)–(2).

(c) *If* $f_\infty^s, g_\infty^s, h_\infty^s \in (0,\infty)$, $f_0^i = g_0^i = h_0^i = \infty$ *then for each* $\lambda \in (0, M_2)$, $\mu \in (0, M_4)$ *and* $\upsilon \in (0, M_6)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [1,e]$, *for problem* (1)–(2).

(d) *If* $f_\infty^s = g_\infty^s = h_\infty^s = 0$, $f_0^i = g_0^i = h_0^i = \infty$, *then for each* $\lambda \in (0,\infty)$, $\mu \in (0,\infty)$ *and* $\upsilon \in (0,\infty)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [1,e]$, *for problem* (1)–(2).

Proof (a) For any $\lambda \in (M_1, M_2)$, $\mu \in (M_3, M_4)$, $\upsilon \in (M_5, M_6)$, there exists $0 < \epsilon < \min\{f_0^i, g_0^i, h_0^i\}$ such that

$$\rho_1\left[m^2\left(f_0^i - \epsilon\right)^{\frac{1}{p-1}} \int_{s\in I} G_1(e,s)\left(\int_{\tau\in I} \delta(\tau)H_1(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \le \lambda,$$

$$\rho_1\left[\left(f_\infty^s + \epsilon\right)^{\frac{1}{p-1}} \int_1^e G_1(e,s)\left(\int_1^e H_1(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \ge \lambda,$$

$$\rho_2\left[m^2\left(g_0^i - \epsilon\right)^{\frac{1}{p-1}} \int_{s\in I} G_2(e,s)\left(\int_{\tau\in I} \delta(\tau)H_2(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \le \mu,$$

$$\rho_2\left[\left(g_\infty^s + \epsilon\right)^{\frac{1}{p-1}} \int_1^e G_2(e,s)\left(\int_1^e H_2(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \ge \mu,$$

$$\rho_3\left[m^2\left(h_0^i - \epsilon\right)^{\frac{1}{p-1}} \int_{s\in I} G_3(e,s)\left(\int_{\tau\in I} \delta(\tau)H_3(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \le \upsilon,$$

$$\rho_3\left[\left(h_\infty^s + \epsilon\right)^{\frac{1}{p-1}} \int_1^e G_3(e,s)\left(\int_1^e H_3(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^{-1} \ge \upsilon.$$

By the definitions of $f_0^i, g_0^i \in (0,\infty)$ and $h_0^i \in (0,\infty)$, we deduce that there exists $R_3 > 0$ such that

$$f(t,u,v,w) \ge \left(f_0^i - \epsilon\right)(u+v+w)^{p-1}, \quad t \in I, u, v, w \ge 0, u+v+w \le R_3,$$

$$g(t,u,v,w) \ge \left(g_0^i - \epsilon\right)(u+v+w)^{p-1}, \quad t \in I, u, v, w \ge 0, u+v+w \le R_3,$$

$$h(t,u,v,w) \ge \left(h_0^i - \epsilon\right)(u+v+w)^{p-1}, \quad t \in I, u, v, w \ge 0, u+v+w \le R_3.$$

Let $(u,v,w)\in P$ with $\|(u,v,w)\|_Y = R_3$, that is, $\|u\|+\|v\|+\|w\| = R_3$. Because $u(t)+v(t)+w(t)\leq \|u\|+\|v\|+\|w\| = R_3$ for all $t\in[1.e]$, we have

$$\begin{aligned}
&Q_1(u,v,w)(t)\\
&\quad = \lambda \int_1^e G_1(t,s)\biggl(\int_1^e H_1(s,\tau)f\bigl(\tau,u(\tau),v(\tau),w(\tau)\bigr)\frac{d\tau}{\tau}\biggr)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad \geq \lambda m \int_{s\in I} G_1(e,s)\biggl(\int_{\tau\in I} \delta_1(\tau)H_1(\tau,\tau)f\bigl(\tau,u(\tau),v(\tau),w(\tau)\bigr)\frac{d\tau}{\tau}\biggr)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad \geq \lambda m \int_{s\in I} G_1(e,s)\biggl(\int_{\tau\in I} H_1(\tau,\tau)\bigl(f_0^i-\epsilon\bigr)\bigl(u(\tau)+v(\tau)+w(\tau)\bigr)^{p-1}\frac{d\tau}{\tau}\biggr)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\quad \geq \lambda m^2\bigl(f_0^i-\epsilon\bigr)^{\frac{1}{p-1}} \int_{s\in I} G_1(e,s)\biggl(\int_{\tau\in I} \delta(\tau)H_1(\tau,\tau)\frac{d\tau}{\tau}\biggr)^{\frac{1}{p-1}}\frac{ds}{s}\bigl(\|u\|+\|v\|+\|w\|\bigr)\\
&\quad \geq \rho_1\bigl\|(u,v,w)\bigr\|.
\end{aligned}$$

Hence, $\|Q_1(u,v,w)\|\geq \rho_1\|(u,v,w)\|$. In a similar manner, we conclude that $\|Q_2(u,v,w)\|\geq \rho_2\|(u,v,w)\|$, $\|Q_3(u,v,w)\|\geq \rho_3\|(u,v,w)\|$. Therefore,

$$\begin{aligned}
\bigl\|Q(u,v,w)\bigr\|_Y &= \bigl\|\bigl(Q_1(u,v,w),Q_2(u,v,w),Q_3(u,v,w)\bigr)\bigr\|_Y\\
&= \bigl\|Q_1(u,v,w)\bigr\| + \bigl\|Q_2(u,v,w)\bigr\| + \bigl\|Q_3(u,v,w)\bigr\|\\
&\geq \rho_1\bigl\|(u,v,w)\bigr\|_Y + \rho_2\bigl\|(u,v,w)\bigr\|_Y + \rho_3\bigl\|(u,v,w)\bigr\|_Y\\
&= (\rho_1+\rho_2+\rho_3)\bigl\|(u,v,w)\bigr\|_Y\\
&= \bigl\|(u,v,w)\bigr\|_Y.
\end{aligned}$$

Hence, $\|Q(u,v,w)\|_Y\geq \|(u,v,w)\|_Y$. Define the set

$$\Omega_3 = \bigl\{(u,v,w)\in Y : \bigl\|(u,v,w)\bigr\|_Y < R_3\bigr\},$$

then

$$\bigl\|Q(u,v,w)\bigr\|_Y \geq \bigl\|(u,v,w)\bigr\|_Y, \quad \text{for } (u,v,w)\in P\cap\partial\Omega_3. \tag{14}$$

On the other hand, we define $f^\star, g^\star.h^\star : [1,e]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ as follows:

$$\begin{aligned}
f^\star(t,x) &= \max_{0\leq u+v+w\leq x} f(t,u,v,w),\\
g^\star(t,x) &= \max_{0\leq u+v+w\leq x} g(t,u,v),\\
h^\star(t,x) &= \max_{0\leq u+v+w\leq x} h(t,u,v), \quad \text{for all } x\in\mathbb{R}^+.
\end{aligned}$$

Then

$$f(t,u,v,w)\leq f^\star(t,x), g(t,u,v,w)\leq g^\star(t,x), h(t,u,v,w)\leq h^\star(t,x), \quad u+v+w\leq x.$$

It follows that the functions $f^\star$, $g^\star$, $h^\star$ are nondecreasing and satisfy the conditions

$$\limsup_{x\to+\infty}\max_{t\in[1,e]}\frac{f^\star(t,x)}{x^{p-1}}\le f^s_\infty,$$
$$\limsup_{x\to+\infty}\max_{t\in[1,e]}\frac{g^\star(t,x)}{x^{p-1}}\le g^s_\infty,$$
$$\limsup_{x\to+\infty}\max_{t\in[1,e]}\frac{h^\star(t,x)}{x^{p-1}}\le h^s_\infty.$$

Next, by the definitions of $f^s_\infty, g^s_\infty, h^s_\infty \in (0,\infty)$, there exists $\overline{R}_4 > 0$ such that, for any $t \in [1,e]$, $x \ge \overline{R}_4$, we have

$$f^\star(t,x)\le\left(f^s_\infty+\epsilon\right)x^{p-1},\qquad g^\star(t,x)\le\left(g^s_\infty+\epsilon\right)x^{p-1},$$
$$h^\star(t,x)\le\left(h^s_\infty+\epsilon\right)x^{p-1},\quad x\ge\overline{R}_4.$$

Let $R_4 = \max\{2R_3, \overline{R}_4\}$ and $\Omega_4 = \{(u,v,w) \in Y : \|(u,v,w)\|_Y < R_4\}$. For any $(u,v,w) \in P \cap \partial\Omega_4$ and $t \in [1,e]$, we have

$$f\bigl(t,u(t),v(t),w(t)\bigr)\le f^\star\bigl(t,\bigl\|(u,v,w)\bigr\|_Y\bigr),\quad t\in[1,e],$$
$$g\bigl(t,u(t),v(t),w(t)\bigr)\le g^\star\bigl(t,\bigl\|(u,v,w)\bigr\|_Y\bigr),\quad t\in[1,e],$$
$$h\bigl(t,u(t),v(t),w(t)\bigr)\le h^\star\bigl(t,\bigl\|(u,v,w)\bigr\|_Y\bigr),\quad t\in[1,e],$$

then

$$\begin{aligned}
Q_1(u,v)(t)&\le\lambda\int_1^e G_1(e,s)\left(\int_1^e H_1(s,\tau)f^\star\bigl(\tau,\bigl\|(u,v,w)\bigr\|_Y\bigr)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\le\lambda\int_1^e G_1(e,s)\left(\int_1^e H_1(\tau,\tau)\bigl(f^s_\infty+\epsilon\bigr)\bigl(\bigl\|(u,v,w)\bigr\|_Y\bigr)^{p-1}\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&=\lambda\bigl(f^s_\infty+\epsilon\bigr)^{\frac{1}{p-1}}\int_1^e G_1(e,s)\left(\int_1^e H_1(\tau,\tau)\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\bigl\|(u,v,w)\bigr\|_Y\\
&\le\rho_1\bigl\|(u,v,w)\bigr\|_Y,
\end{aligned}$$

so $\|Q_1(u,v,w)\| \le \rho_1\|(u,v,w)\|_Y$, $(u,v,w) \in P \cap \partial\Omega_4$. In a similar manner, we deduce $\|Q_2(u,v,w)\| \le \rho_2\|(u,v,w)\|_Y$, $\|Q_3(u,v,w)\| \le \rho_3\|(u,v,w)\|_Y$. Therefore,

$$\begin{aligned}
\bigl\|Q(u,v,w)\bigr\|_Y&=\bigl\|\bigl(Q_1(u,v,w),Q_2(u,v,w),Q_3(u,v,w)\bigr)\bigr\|_Y\\
&=\bigl\|Q_1(u,v,w)\bigr\|+\bigl\|Q_2(u,v,w)\bigr\|+\bigl\|Q_3(u,v,w)\bigr\|\\
&\le\rho_1\bigl\|(u,v,w)\bigr\|_Y+\rho_2\bigl\|(u,v,w)\bigr\|_Y+\rho_3\bigl\|(u,v,w)\bigr\|_Y\\
&=(\rho_1+\rho_2+\rho_3)\bigl\|(u,v,w)\bigr\|_Y\\
&=\bigl\|(u,v,w)\bigr\|_Y.
\end{aligned}$$

Hence, $\|Q(u,v,w)\|_Y \le \|(u,v,w)\|_Y$. Define the set

$$\Omega_4=\bigl\{(u,v,w)\in Y:\bigl\|(u,v,w)\bigr\|_Y<R_4\bigr\},$$

then

$$\left\|Q(u,v,w)\right\|_Y \le \left\|(u,v,w)\right\|_Y, \quad \text{for } (u,v,w)\in P\cap\partial\Omega_4. \tag{15}$$

Therefore, by (14), (15) and Theorem 3.5, we conclude that Q has at least one fixed point $(u,v,w)\in P\cap(\overline{\Omega}_4\setminus\Omega_3)$ with $R_3\le\|(u,v,w)\|_Y\le R_4$. The proofs of the cases (b)–(d) are similar to that of (a) and we shall omit them. □

5 Example

Let us consider an example to illustrate the above result.

Here $n=m=l=3$, $\alpha_1=\alpha_2=\alpha_3=\frac{5}{2}$, $\beta_1=\beta_2=\beta_3=\frac{3}{2}$, $p_1=q_1=r_1=\frac{5}{4}$, $p_2=q_2=r_2=\frac{1}{2}$, $\mu_1=\frac{1}{2}$, $\lambda_1=\frac{1}{3}$, $\xi=\frac{3}{2}$. Let $p=2$, we consider the Hadamard fractional differential equations

$$\begin{cases} {}^HD^{3/2}_{1^+}(\phi_p({}^HD^{5/2}_{1^+}u(t)))=\lambda^{p-1}f(t,u(t),v(t),w(t)), & t\in(1,e),\\ {}^HD^{3/2}_{1^+}(\phi_p({}^HD^{5/2}_{1^+}v(t)))=\mu^{p-1}g(t,u(t),v(t),w(t)), & t\in(1,e),\\ {}^HD^{3/2}_{1^+}(\phi_p({}^HD^{5/2}_{1^+}w(t)))=\upsilon^{p-1}h(t,u(t),v(t),w(t)), & t\in(1,e),\end{cases} \tag{16}$$

with the three-point boundary conditions

$$\begin{cases} u(1)=u'(1)=0, \qquad (1/2)u^{5/4}(e)=(1/3)u^{5/4}(3/2),\\ \phi_p({}^HD^{5/2}_{1^+}u(1))=0=({}^HD^{1/2}_{1^+}(\phi_p({}^HD^{5/2}_{1^+}u(e)))),\\ v(1)=v'(1)=0, \qquad (1/2)v^{5/4}(e)=(1/3)v^{5/4}(3/2),\\ \phi_p({}^HD^{5/2}_{1^+}v(1))=0=({}^HD^{1/2}_{1^+}(\phi_p({}^HD^{5/2}_{1^+}v(e)))),\\ w(1)=w'(1)=0, \qquad (1/2)w^{5/4}(e)=(1/3)w^{5/4}(3/2),\\ \phi_p({}^HD^{5/2}_{1^+}w(1))=0=({}^HD^{1/2}_{1^+}(\phi_p({}^HD^{5/2}_{1^+}w(e)))),\end{cases} \tag{17}$$

where

$$\begin{aligned} f(t,u,v,w)&=\frac{(\ln t+1)[800(u+v+w)+1](u+v+w)(4+\sin v)}{u+v+w+1},\\ g(t,u,v,w)&=\frac{(\sqrt{\ln t+1})[400(u+v+w)+1](u+v+w)(3+\cos w)}{u+v+w+1},\\ h(t,u,v,w)&=\frac{(\ln t+2)[200(u+v+w)+1](u+v+w)(2+\sin u)}{u+v+w+1}.\end{aligned}$$

After simple calculations, we get

$$\begin{aligned} &f_0^s=8, \quad f_0^i=5, \quad f_\infty^s=8000, \quad f_\infty^i=3000,\\ &g_0^s=5.6569, \quad g_0^i=4.4722, \quad g_\infty^s=37085, \quad g_\infty^i=894.43,\\ &h_0^s=6, \quad h_0^i=4.5, \quad h_\infty^s=1800, \quad h_\infty^i=450, \quad m=0.125,\\ &L_1=0.05903556919\sigma_1, \quad L_2=0.1282828756\sigma_1, \quad L_3=0.08446316582\sigma_2,\\ &L_4=0.1195672885\sigma_2, \quad L_5=0.094532722\sigma_3, \quad L_6=0.1076892453\sigma_3,\end{aligned}$$

where $\sigma_1,\sigma_2,\sigma_3>0$ are three positive real numbers such that $\sigma_1+\sigma_2+\sigma_3=1$.

Employing Theorem 4.2 of (a), for each $\lambda \in (L_1, L_2)$, $\mu \in (L_3, L_4)$ and $\upsilon \in (L_5, L_6)$, there exists a positive solution $(u(t), v(t), w(t))$ of the Hadamard fractional differential equation (16)–(17).

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