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Existence and uniqueness of solutions for mixed fractional q-difference boundary value problems

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Abstract
In this paper, we investigate the existence and uniqueness of solutions for mixed fractional q-difference boundary value problems involving the Riemann–Liouville and the Caputo fractional derivative. By using the Guo–Krasnoselskii fixed point theorem and Banach contraction mapping principle as well as Schaefer's fixed point theorem, we obtain the main results. In addition, several examples are given to illustrate the main results.

MSC: 65L10; 26A33; 39A13; 34A12

Keywords: Boundary value problem; Fractional q-difference equations; Existence and uniqueness

1 Introduction

In recent years, the theory of fractional differential equations has become an important aspect of differential equations; see [1–8]. Fractional differential equations can describe many phenomena in various fields of science and engineering, such as physics, mechanics, chemistry, control engineering, etc. For an extensive collection of such results, we refer to the monographs by Kilbas et al. [1], Podlubny [2] and Samko et al. [9].

In 2011, Zhao and Sun et al. [10] studied the existence of positive solutions for the nonlinear fractional differential equation boundary value problem

$$\begin{aligned} &D_{0^+}^{\alpha} u(t) = \lambda f\bigl(u(t)\bigr), \quad 0 < t < 1, \\ &u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0, \end{aligned}$$

where $1 < \alpha \leq 2$ is a real number, $D_{0^+}^{\alpha}$ is the Caputo fractional derivative, $\lambda > 0$ and $f : [0, +\infty) \to (0, +\infty)$ is continuous, by using the properties of the Green function and Guo–Krasnoselskii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problems are established.

In 2011, Feng and Sun et al. [11] discussed the boundary value problem to the following system of fractional differential equations:

$$\begin{aligned} &D^{\alpha}_{0^+}u(t) + f\bigl(t, v(t)\bigr) = 0, \quad 0 < t < 1, \\ &D^{\beta}_{0^+}v(t) + f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = u'(0) = v(0) = v(1) = v'(0) = 0, \end{aligned}$$

where $2 < \alpha, \beta \leq 3$ and $f, g : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\lim_{t\to 0^+} f(t, \cdot) = +\infty$, $\lim_{t\to 0^+} g(t, \cdot) = +\infty$ and $D^{\alpha}_{0^+}$, $D^{\beta}_{0^+}$ are both Riemann–Liouville type, by using Green's function and its corresponding properties, the authors transform the derivative systems into equivalent integral systems. The existence is based on a nonlinear alternative of Leray–Schauder type and Krasnoseskii's fixed point theorem in cones.

For more information about boundary value problems of fractional differential equations; see [12–15].

More recently, an attempt has been made to develop a discrete fractional calculus, and some results (in various directions) are already available in the literature; see [16–21].

The q-difference calculus or quantum calculus is an old subject that was initially developed by Jackson [22]. It is rich in history and applications as the reader can confirm in the work by Ernst [23].

In 2014, Li, Han, and Sun et al. [24] investigated the following boundary value problems for fractional q-difference equations with nonlocal conditions:

$$\begin{aligned} &\bigl({}^cD^{\alpha}_q x\bigr)(t) + f\bigl(t, {}^cD^{\sigma}_q x(t)\bigr) = 0, \quad 0 < t < 1, \\ &x(0) = y(x), \quad \gamma(D_q x)(1) - \beta D^2_q x(1) = 0, \end{aligned}$$

where $0 < q < 1$, $1 < \alpha < 2$, $0 < \sigma < 1$, $\beta, \gamma \geq 0$, and $\frac{(1-t)^{(\alpha-2)}}{(1-t)^{(\alpha-3)}} \geq \frac{[\alpha]_q\beta}{\gamma}$, $f : C((0,1) \times \mathbb{R})$ and y is a continuous functional.

In 2015, Li and Han et al. [25] considered the following fractional q-difference equation:

$$\bigl(D^{\alpha}_q u\bigr)(t) + \lambda h(t) f\bigl(u(t)\bigr) = 0, \quad 0 < t < 1,$$

subjected to

$$u(0) = D_q u(0) = D_q u(1) = 0,$$

where $0 < q < 1$, $2 < \alpha < 3$, $f : C([0,\infty), (0,\infty))$, $h : C((0,1), (0,\infty))$, the author prove the existence of positive solutions for the above boundary value problems by utilizing a fixed point theorem in cone. Several existence results for positive solutions in terms of different values of the parameter λ are obtained. Motivated by the above papers, we study the following equations:

$$D^{\alpha}_q u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to the boundary conditions

$$u(0) = {}^cD^{\beta}_q u(0) = {}^cD^{\beta}_q u(1) = 0, \tag{1.2}$$

where $0 < \beta \leq 1$, $2 < \alpha \leq 2 + \beta$, D_q^α, ${}^cD_q^\beta$ are the Riemann–Liouville fractional q-derivative and Caputo fractional q-derivative of order α, β, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

As far as we know, most of the papers studied the fractional q-difference using a single-derivative (Riemann–Liouville or Caputo derivative), in this paper, we consider the existence and uniqueness of the solution to the fractional q-difference boundary value problem of mixed derivative (Riemann–Liouville and Caputo derivative), because of the Caputo derivative with respect to constant is zero, however, the Riemann–Liouville derivative with respect to constant is not zero, which adds to the difficulty of research, especially to the computational difficulties.

The paper is arranged as follows. In Sect. 2, we introduce some basic knowledge and a definition of q-fractional integral together with some basic lemmas, which are necessary for the main results. In Sect. 3, we derive some useful main results. In Sect. 4, we present several examples to illustrate our main results.

2 Basic definitions and preliminaries

In this section, we list some useful definitions and preliminaries, which are useful for the proof of the main results.

For $q \in (0,1)$, we define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^k$ with $k \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ is

$$(a-b)^0 = 1, \qquad (a-b)^{(k)} = \prod_{i=0}^{k-1}\bigl(a - bq^i\bigr), \quad k \in \mathbb{N}, a, b \in \mathbb{R}.$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(a-b)^{(\gamma)} = a^\gamma \prod_{i=0}^{\infty} \frac{a - bq^i}{a - bq^{i+r}}, \quad a \neq 0.$$

Note if $b = 0$, then $a^{(\gamma)} = a^\gamma$. We also use the notation $0^{(\gamma)} = 0$ for $\gamma \geq 0$.

The definition of the q-Gamma is

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R}\backslash\{0,-1,-2\cdots\},$$

then $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

For $\forall x, y > 0$, $B_q(x,y) = \int_0^1 t^{x-1}(1-qt)^{(y-1)}\,d_qt$, especially, $B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$. The q-integral of a function f defined on the interval $[0,b]$ is given by

$$(I_qf)(x) = \int_0^x f(t)\,d_qt = x(1-q)\sum_{n=0}^{\infty} f\bigl(xq^n\bigr)q^n, \quad x \in [0,b].$$

If $a \in [0,b]$ and f is defined on the interval $[0,b]$, its q-integral from a to b is defined by

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt.$$

Definition 2.1 ([26]) Let $\alpha \geq 0$ and f be a real function defined on a certain interval $[a,b]$. The Riemann–Liouville fractional q-integral of order α is defined by

$$\begin{aligned} &\bigl(I_q^0 f\bigr)(t) = f(t), \\ &\bigl(I_q^\alpha f\bigr)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s)\,d_qs, \quad \alpha > 0, t \in [a,b]. \end{aligned}$$

Definition 2.2 ([26]) The fractional q-derivative of the Riemann–Liouville type of order $\alpha \geq 0$ of a continuous and differential function f is given by

$$\begin{aligned} &\bigl(D_q^0 f\bigr)(t) = f(t), \quad t \in [a,b], \\ &\bigl(D_q^\alpha f\bigr)(t) = \bigl(D_q^l I_q^{l-\alpha} f\bigr)(t), \quad \alpha > 0, t \in [a,b], \end{aligned}$$

where l is the smallest integer greater than or equal to α.

Definition 2.3 ([26]) Let $\alpha \geq 0$, and the Caputo fractional q-derivatives of f be defined by

$$\bigl({}^cD_q^\alpha f\bigr)(x) = \bigl(I_q^{\lceil\alpha\rceil-\alpha} D_q^{\lceil\alpha\rceil} f\bigr)(x),$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal α.

If $f(x) = x^{\beta-1}$, $\beta \notin \mathbb{N}$. Then according to [27], we know that

$${}^cD_q^\alpha f(x) = \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)} x^{\beta-\alpha-1}.$$

Lemma 2.1 ([28]) *Let $\alpha, \beta \geq 0$ and $f : [a,b] \to \mathbb{R}$ be a continuous function defined on $[a,b]$ and its derivative exist. Then the following formulas hold*:

$$\begin{aligned} &D_q^\alpha \bigl(I_q^\alpha f\bigr)(t) = f(t), \\ &I_q^\alpha I_q^\beta f(t) = I_q^{\alpha+\beta} f(t). \end{aligned}$$

Lemma 2.2 ([28]) *Let $f : [a,b] \to \mathbb{R}$ be differentiable, p be a positive integer. Then the following equality holds*:

$$I_q^\alpha D_q^p f(t) = D_q^p I_q^\alpha f(t) - \sum_{k=0}^{p-1} \frac{t^{\alpha-p+k}}{\Gamma_q(\alpha-p+k+1)} \bigl(D_q^k f\bigr)(0), \quad t \in [a,b].$$

Theorem 2.1 ([29]) *Let X be a Banach space and let $P \subset X$ be a cone. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator such that*

(1) $\|Tu\| \geq \|u\|$ *for any* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for any* $u \in P \cap \partial\Omega_2$ *or*
(2) $\|Tu\| \leq \|u\|$ *for any* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for any* $u \in P \cap \partial\Omega_2$.

Then T *has a fixed point in* $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 2.3 ([30]) *Let* B *be a Banach space with* $C \subseteq B$ *closed and convex. Assume* U *is a relatively open subset of* C *with* $0 \in U$ *and* $T : \overline{U} \to C$ *is a continuous, compact map. Then either*

(1) T *has a fixed point in* $\overline{U}$; *or*
(2) *there exist* $U \in \partial U$ *and* $\lambda \in (0,1)$ *with* $U = \lambda TU$.

3 Main results

Consider the following fractional q-difference equation:

$$\begin{aligned} &D_q^\alpha u(t) + h(t) = 0, \quad 0 < t < 1, \\ &u(0) = {}^cD_q^\beta u(0) = {}^cD_q^\beta u(1) = 0, \end{aligned} \tag{3.1}$$

where h is a continuous function.

At first, we need an integral representation of the solution to problem (3.1).

Lemma 3.1 *Function* $u \in [0,1]$ *is a solution to* (3.1) *if and only if* u *is a solution to the integral equation*

$$u(t) = \int_0^1 G(t,qs)h(s)\,d_qs, \quad t \in [0,1],$$

in which $G(t,qs)$, *the Green function linked to* (3.1), *is given by*

$$G(t,qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1} - (t-qs)^{(\alpha-1)}, & 0 \leq qs \leq t \leq 1, \\ (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}, & 0 \leq t \leq qs \leq 1. \end{cases} \tag{3.2}$$

Proof At first, we prove the necessity, from (3.1), we can see that

$$\left(D_q^\alpha u\right)(t) = -h(t).$$

Integrating the two sides of the equation, we obtain

$$\left(I_q^\alpha D_q^\alpha u\right)(t) = -I_q^\alpha h(t).$$

Then according to Definition 2.2, we know

$$\left(I_q^\alpha D_q^3 I_q^{3-\alpha} u\right)(t) = -I_q^\alpha h(t).$$

Finally, by making use of Lemmas 2.1 and 2.2 with $p = 3$, we derive that

$$\left(D_q^3 I_q^\alpha I_q^{3-\alpha} u\right)(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} - I_q^\alpha h(t),$$

i.e. the general solution to (3.1) is

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s,$$

in which C_1, C_2, C_3 are real constants. According to the boundary value conditions $u(0) = 0$, we can know that $C_3 = 0$. Taking the β-order of Caputo q-derivative of $u(t)$, we get

$$\begin{aligned} {}^cD_q^\beta u(t) = {}& -\frac{1}{\Gamma_q(\alpha-\beta)} \int_0^t (t-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)} C_1 t^{\alpha-\beta-1} \\ & + \frac{\Gamma_q(\alpha-1)}{\Gamma_q(\alpha-\beta-1)} C_2 t^{\alpha-\beta-2}. \end{aligned}$$

By the boundary condition ${}^cD_q^\beta u(0) = 0$, we get

$$C_2 = 0.$$

Then the boundary condition (1.2) yields

$$-\frac{1}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)} C_1 = 0,$$

so

$$C_1 = \frac{\int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s}{\Gamma_q(\alpha)}.$$

Therefore,

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_t^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \\ &\quad - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t \bigl[(1-qs)^{(\alpha-\beta-1)} t^{\alpha-1} - (t-qs)^{(\alpha-1)}\bigr] h(s)\, d_q s \\ &\quad + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_t^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \\ &= \int_0^1 G(t,qs) h(s)\, d_q s. \end{aligned}$$

Next, we will prove the sufficiency.

Due to

$$u(t) = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s,$$

then according to Definitions 2.1 and 2.2 as well as the definition of the q-Gamma function, we can get $D_q^\alpha t^{\alpha-1} = 0$, hereafter, we derive that $D_q^\alpha u(t) = -h(t)$, hence $D_q^\alpha u(t) + h(t) = 0$. On the basis of the expression of $u(t)$, we know $u(0) = 0$. Taking the q-derivative of Caputo on both sides of this equation, we get

$$^cD_q^\beta u(t) = \frac{t^{\alpha-\beta-1}}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_qs - \frac{1}{\Gamma_q(\alpha-\beta)} \int_0^t (t-qs)^{(\alpha-\beta-1)} h(s)\, d_qs,$$

in the light of the expression of $^cD_q^\beta u(t)$, we can obtain

$$\begin{aligned} &^cD_q^\beta u(0) = 0, \\ &^cD_q^\beta u(1) = \frac{1}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_qs - \frac{1}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_qs \\ &\qquad\quad = 0. \end{aligned}$$

Thus we complete the proof. □

The properties of the Green function play an important role in this paper.

Lemma 3.2 *The Green function G defined by* (3.2) *satisfies the following requirements*:
(1) $G(t,qs) \geq 0$, $0 \leq t, qs \leq 1$;
(2) $\beta qs(1-qs)^{(\alpha-1)} t^{\alpha-1} \leq \Gamma_q(\alpha) G(t,qs) \leq [\alpha-1]_q (1-qs)^{(\alpha-\beta-1)} t^{\alpha-2} qs$, $\forall t, qs \in [0,1]$.

Proof (1) Let

$$\begin{aligned} G_1(t,qs) &= \frac{1}{\Gamma_q(\alpha)} \bigl[(1-qs)^{(\alpha-\beta-1)} t^{\alpha-1} - (t-qs)^{(\alpha-1)}\bigr], \quad 0 \leq qs \leq t \leq 1, \\ G_2(t,qs) &= \frac{1}{\Gamma_q(\alpha)} (1-qs)^{(\alpha-\beta-1)} t^{\alpha-1}, \quad 0 \leq t \leq qs \leq 1. \end{aligned}$$

When $0 \leq t \leq qs \leq 1$, it is clear that $G_2(t,qs) > 0$.
When $0 \leq qs \leq t \leq 1$,

$$\begin{aligned} G_1(t,qs) &= \frac{1}{\Gamma_q(\alpha)} \bigl[(1-qs)^{(\alpha-\beta-1)} t^{\alpha-1} - (t-qs)^{(\alpha-1)}\bigr] \\ &= \frac{1}{\Gamma_q(\alpha)} \biggl[(1-qs)^{(\alpha-\beta-1)} t^{\alpha-1} - t^{\alpha-1}\biggl(1-q\frac{s}{t}\biggr)^{(\alpha-1)}\biggr] \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \biggl[(1-qs)^{(\alpha-\beta-1)} - \biggl(1-q\frac{s}{t}\biggr)^{(\alpha-1)}\biggr]. \end{aligned}$$

We use $(1-qs)^{(\alpha-\beta-1)} \geq (1-q\frac{s}{t})^{(\alpha-\beta-1)} \geq (1-q\frac{s}{t})^{(\alpha-1)}$, i.e., $G_1(t,qs) > 0$. All in all, we obtain $G(t,qs) \geq 0$.

(2) When $0 \leq qs \leq t \leq 1$, we have

$$\int_{t-qs}^t x^{\alpha-2}\, d_qx = \int_0^t x^{\alpha-2}\, d_qx - \int_0^{t-qs} x^{\alpha-2}\, d_qx$$

$$
\begin{aligned}
&= (1-q)\sum_{n=0}^{\infty} tq^n \left(tq^n\right)^{\alpha-2} - (1-q)\sum_{m=0}^{\infty}(t-qs)q^m\left[(t-qs)q^m\right]^{\alpha-2} \\
&= (1-q)t^{\alpha-1}\sum_{n=0}^{\infty} q^{n(\alpha-1)} - (1-q)(t-qs)^{\alpha-1}\sum_{m=0}^{\infty} q^{m(\alpha-1)} \\
&= (1-q)t^{\alpha-1}\frac{1}{1-q^{\alpha-1}} - (1-q)(t-qs)^{\alpha-1}\frac{1}{1-q^{\alpha-1}} \\
&= \frac{1-q}{1-q^{\alpha-1}}\left[t^{\alpha-1} - (t-qs)^{\alpha-1}\right],
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\int_{t-qs}^{t} x^{\alpha-2}\, d_q x &= \frac{1-q}{1-q^{\alpha-1}}\left[t^{\alpha-1} - (t-qs)^{\alpha-1}\right] = \frac{1}{[\alpha-1]_q}\left[t^{\alpha-1} - (t-qs)^{\alpha-1}\right], \\
\Gamma_q(\alpha)G(t,qs) &= (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1} - (t-qs)^{(\alpha-1)} \\
&\le (1-qs)^{(\alpha-\beta-1)}\left[t^{\alpha-1} - (t-qs)^{(\alpha-1)}\right] \\
&= (1-qs)^{(\alpha-\beta-1)}[\alpha-1]_q \int_{t-qs}^{t} x^{\alpha-2}\, d_q x.
\end{aligned}
$$

Then, according to the monotonicity of $x^{\alpha-2}$ and the property of the q-integral, we have

$$
\Gamma_q(\alpha)G(t,qs) \le [\alpha-1]_q(1-qs)^{(\alpha-\beta-1)}t^{\alpha-2}qs.
$$

On the other hand, $\frac{1}{(1-qs)^{\beta}} \ge 1 + \beta qs$.

In fact, $g(x)$ is decreasing with respect to x, i.e. $g'(x) < 0$, $g(x) \le g(0) = 1$, so $\frac{1}{(1-qs)^{\beta}} \ge 1 + \beta qs$ holds. Hence,

$$
\begin{aligned}
\Gamma_q(\alpha)G(t,qs) &= (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1} - (t-qs)^{(\alpha-1)} \\
&\ge (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1} - (t-qst)^{(\alpha-1)} \\
&= (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1} - t^{\alpha-1}(1-qs)^{(\alpha-1)} \\
&= t^{\alpha-1}(1-qs)^{(\alpha-1)}\left[(1-qs)^{(-\beta)} - 1\right] \\
&\ge \beta qs(1-qs)^{(\alpha-1)}t^{\alpha-1},
\end{aligned}
$$

when $qs = 1$, we have $t = 1$, it is obvious that $\Gamma_q(\alpha)G(t,qs) \ge \beta qs(1-qs)^{(\alpha-1)}t^{\alpha-1}$. When $0 \le t \le qs < 1$,

$$
\begin{aligned}
\Gamma_q(\alpha)G(t,qs) &= (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1} \\
&\le (1-qs)^{(\alpha-\beta-1)}t^{\alpha-2}qs \\
&\le [\alpha-1]_q(1-qs)^{(\alpha-\beta-1)}t^{\alpha-2}qs, \\
\Gamma_q(\alpha)G(t,qs) &= (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1} \\
&= (1-qs)^{(-\beta)}(1-qs)^{(\alpha-1)}t^{\alpha-1} \\
&\ge (1+\beta qs)(1-qs)^{(\alpha-1)}t^{\alpha-1} \\
&\ge \beta qs(1-qs)^{(\alpha-1)}t^{\alpha-1},
\end{aligned}
$$

when $qs = 1$, it is obvious that $\Gamma_q(\alpha)G(t,qs) \geq \beta qs(1-qs)^{(\alpha-1)}t^{\alpha-1}$. Thus we complete the proof. □

Let $B = C([0,1],\mathbb{R})$. We use the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|x\| = \sup_{t\in[0,1]} \bigl|x(t)\bigr|.$$

Define the cone $P \subset B = C[0,1]$ by $P = \{x \in B | x(t) \geq 0, \forall t \in [0,1]\}$.

Denote $C_1 = \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-\beta-1)}\,d_qs$, $C_2 = \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}\,d_qs$, $A = M(C_1 + C_2)$.

Theorem 3.1 *Let $0 < \beta \leq 1$, $2 < \alpha \leq 2 + \beta$, and $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. Suppose there exist two positive constants $r_1 > r_2 > 0$ such that $f(t,u) \leq N_1 r_1$, $(t,u) \in [0,1] \times [0,r_1]$, $f(t,u) \geq N_2 r_2$, $(t,u) \in [0,1] \times [0,r_2]$, where $N_1 = \frac{\Gamma_q(\alpha)}{[\alpha-1]_q B_q(2,\alpha-\beta)}$, $N_2 = \frac{\Gamma_q(\alpha)2^{\alpha-1}}{\beta B_q(2,\alpha)}$. Then the boundary value problem (1.1)–(1.2) has at least one solution with $r_2 \leq \|u\| \leq r_1$.*

Proof By making use of Lemma 3.1, we know that $u \in C[0,1]$ is a solution of (1.1)–(1.2) if and only if u is a solution of the following integral equation:

$$u(t) = \int_0^1 G(t,qs)f\bigl(s,u(s)\bigr)\,d_qs.$$

Let $T : P \to B$ be the operator defined by

$$Tu(t) = \int_0^1 G(t,qs)f\bigl(s,u(s)\bigr)\,d_qs.$$

It is easy to see that the operator T is continuous in view of the continuity of G and f. In order to apply the Arzela–Ascoli theorem, we need to prove that $T : P \to P$ is a completely continuous.

In fact, suppose $\Omega \subset P$ is bounded, i.e., there exists a positive $N > 0$ such that $\|u\| \leq N$ for all $u \in \Omega$. Let $M = \max_{\|u\|\leq N} |f(u)| + 1$. Then, for $u \in \Omega$, $t \in [0,1]$, we have

$$\begin{aligned}
\bigl|(Tu)(t)\bigr| &= \biggl|\int_0^1 \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad - \int_0^t \frac{1}{\Gamma_q(\alpha)}(t-qs)^{(\alpha-1)}f\bigl(s,u(s)\bigr)\,d_qs\biggr| \\
&\leq \int_0^1 \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs \\
&\quad + \int_0^t \frac{1}{\Gamma_q(\alpha)}(t-qs)^{(\alpha-1)}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs \\
&< M(C_1 + C_2) = A.
\end{aligned}$$

Therefore, $T(\Omega)$ is uniform bounded.

On the other hand, for any given $\varepsilon > 0$, setting $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{2M(C_1+C_2)}\}$, for each $0 \le t_1 \le t_2 \le 1$ and $|t_2 - t_1| < \delta$, $u \in \Omega$, we have $|Tu(t_2) - Tu(t_1)| < \varepsilon$, that is to say, $T(\Omega)$ is equicontinuous. In fact,

$$\begin{aligned}
\bigl|Tu(t_2) - Tu(t_1)\bigr| &= \biggl|\int_0^1 G(t_2, qs)f\bigl(s,u(s)\bigr)\,d_qs - \int_0^1 G(t_1, qs)f\bigl(s,u(s)\bigr)\,d_qs\biggr| \\
&= \biggl|\int_0^1 \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-\beta-1)}t_2^{\alpha-1}f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad - \int_0^{t_2} \frac{1}{\Gamma_q(\alpha)}(t_2-qs)^{(\alpha-1)}f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad - \int_0^1 \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-\beta-1)}t_1^{\alpha-1}f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad + \int_0^{t_1} \frac{1}{\Gamma_q(\alpha)}(t_1-qs)^{(\alpha-1)}f\bigl(s,u(s)\bigr)\,d_qs\biggr| \\
&= \biggl|\int_0^1 \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-\beta-1)}f\bigl(s,u(s)\bigr)\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)\,d_qs\biggr| \\
&\quad + \biggl|\int_0^{t_1} \frac{1}{\Gamma_q(\alpha)}f\bigl(s,u(s)\bigr)\bigl[(t_1-qs)^{(\alpha-1)} - (t_2-qs)^{(\alpha-1)}\bigr]\,d_qs\biggr| \\
&\quad + \biggl|\int_{t_1}^{t_2} \frac{1}{\Gamma_q(\alpha)}(t_2-qs)^{(\alpha-1)}f\bigl(s,u(s)\bigr)\,d_qs\biggr| \\
&\le \biggl|\int_0^1 \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-\beta-1)}f\bigl(s,u(s)\bigr)\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)\,d_qs\biggr| \\
&\quad + \biggl|\int_0^{t_1} \frac{1}{\Gamma_q(\alpha)}f\bigl(s,u(s)\bigr)(1-qs)^{(\alpha-1)}\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)\,d_qs\biggr| \\
&\quad + \biggl|\int_{t_1}^{t_2} \frac{1}{\Gamma_q(\alpha)}(t_2-qs)^{(\alpha-1)}f\bigl(s,u(s)\bigr)\,d_qs\biggr|.
\end{aligned}$$

Now we discuss $t_2^{\alpha-1} - t_1^{\alpha-1}$:

(1) $0 \le t_1 < \delta$, $\delta < t_2 < 2\delta$, $t_2^{\alpha-1} - t_1^{\alpha-1} \le t_2^{\alpha-1} \le 2\delta$,

(2) $0 < t_1 < t_2 < \delta$, $t_2^{\alpha-1} - t_1^{\alpha-1} < t_2^{\alpha-2} < \delta^{\alpha-2} < (\alpha-2)\delta < \delta < 2\delta$,

(3) $\delta \le t_1 < t_2 < 1$, $t_2^{\alpha-1} - t_1^{\alpha-1} \le [\alpha-1](t_2 - t_1) < 2\delta$.

Thus, we have $|Tu(t_2) - Tu(t_1)| < 2M\delta(C_1 + C_2) < \varepsilon (t_1 \to t_2)$, by making use of the Arzela–Ascoli theorem, $T(\Omega)$ is a relatively compact in P, and the operator $T : P \to P$ is completely continuous.

Next we define $\Omega_i = \{u \in P : \|u\| \le r_i, i = 1, 2\}$.

For $t \in [0,1]$ and $u \in P \cap \partial\Omega_1$, we know that

$$\begin{aligned}
Tu(t) &= \max_{t\in[0,1]} \int_0^1 G(t, qs)f\bigl(s,u(s)\bigr)\,d_qs \\
&\le \frac{N_1 r_1}{\Gamma_q(\alpha)} \int_0^1 [\alpha-1]_q(1-qs)^{(\alpha-\beta-1)}t^{\alpha-2}qs\,d_qs \\
&\le \frac{N_1 r_1[\alpha-1]_q}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-\beta-1)}qs\,d_qs
\end{aligned}$$

$$= \frac{[\alpha-1]_q N_1 r_1}{\Gamma_q(\alpha)} B_q(2, \alpha-\beta)$$
$$= \|u\|,$$

we have $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$.

For $t \in [\frac{1}{2}, 1]$ and $u \in P \cap \partial\Omega_2$, we derive that

$$\begin{aligned}
(Tu)(t) &\geq \int_0^1 \min_{t\in[\frac{1}{2},1]} G(t,qs) f\bigl(s,u(s)\bigr)\, d_qs \\
&\geq \frac{1}{\Gamma_q(\alpha)} N_2 r_2 \int_0^1 \beta qs(1-qs)^{(\alpha-1)} t^{\alpha-1}\, d_qs \\
&\geq \frac{1}{\Gamma_q(\alpha)} N_2 r_2 \int_0^1 \beta qs(1-qs)^{(\alpha-1)} \left(\frac{1}{2}\right)^{\alpha-1} d_qs \\
&= \frac{1}{\Gamma_q(\alpha)} \frac{N_2 r_2}{2^{\alpha-1}} \int_0^1 \beta qs(1-qs)^{(\alpha-1)}\, d_qs \\
&= \frac{\beta}{\Gamma_q(\alpha)} \frac{N_2 r_2}{2^{\alpha-1}} B_q(2,\alpha) \\
&= \|u\|,
\end{aligned}$$

we have $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. By Theorem 2.1, we see that the operator T has a fixed point in $u \in P \cap (\overline{\Omega_2} \backslash \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$. Thus we complete the proof. □

Theorem 3.2 *Assume that* $0 < \beta \leq 1$, $2 < \alpha \leq 2+\beta$, $0 < q < 1$, *and the following condition holds*:

(H_1): *there exists a constant* $L > 0$, *such that* $|f(t,u) - f(t,v)| \leq L|u-v|$, $\forall t \in [0,1]$, $u, v \in \mathbb{R}$. *If*

$$0 < L < \frac{\Gamma_q(\alpha)}{B_q(2,\alpha-\beta)[\alpha-1]_q},$$

then the boundary value problem (1.1)–(1.2) *has a unique solution in* $[0,1]$.

Proof According to (H_1) and Lemma 3.2, we have

$$\begin{aligned}
\bigl|Tu(t) - Tv(t)\bigr| &= \left| \int_0^1 G(t,qs) f\bigl(s,u(s)\bigr)\, d_qs - \int_0^1 G(t,qs) f\bigl(s,v(s)\bigr)\, d_qs \right| \\
&\leq \int_0^1 G(t,qs) \bigl| f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr|\, d_qs \\
&\leq L \int_0^1 G(t,qs)|u-v|\, d_qs \\
&\leq L \int_0^1 \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} (1-qs)^{(\alpha-\beta-1)} t^{\alpha-2} qs|u-v|\, d_qs \\
&\leq L \int_0^1 \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} (1-qs)^{(\alpha-\beta-1)} qs|u-v|\, d_qs \\
&= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} L|u-v| B_q(2,\alpha-\beta) = K|u-v|,
\end{aligned}$$

from the condition $0 < L < \frac{\Gamma_q(\alpha)}{B_q(2,\alpha-\beta)[\alpha-1]_q}$, we can draw the conclusion that $0 < K < 1$. By the Banach contraction mapping principle, we see that T has a unique fixed point in E, that is to say, the boundary value problem (1.1)–(1.2) has a unique solution. Thus we complete the proof. □

Theorem 3.3 *Assume $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Suppose the following conditions are satisfied*:

(H_2): *there exists a continuous function $\varphi : (0,+\infty) \to \mathbb{R}$ with $f(t,z) \le \varphi(z)$ on $[0,1] \times (0,+\infty)$*;

(H_3): *there exists $r > 0$ so that $\|\varphi\| \le \frac{r}{C_1+C_2}$*.

Then the boundary value problem (1.1)–(1.2) *has a solution.*

Proof We will prove the result by using Schaefer's fixed point theorem. First of all, we define the closed set $K \subset B$, $K = \{x(t) \in C[0,1] | x(t) \ge 0\}$. Let $U = \{x | x \in K, \|x\| < r\}$, $\overline{U} = \{x | x \in K, \|x\| \le r\}$. We will prove that $T : \overline{U} \to K$ is continuous. Since f is continuous, choosing $\{u_n\}$ to be a sequence such that $u_n \to u$ ($n \to \infty$) in $\overline{U}$, for any $t \in [0,1]$, then we have

$$\begin{aligned} \bigl|(Tu_n)(t) - (Tu)(t)\bigr| &\le \int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)} t^{\alpha-1}}{\Gamma_q(\alpha)} \bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u(s)\bigr)\bigr|\, d_q s \\ &\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u(s)\bigr)\bigr|\, d_q s \\ &\le \int_0^1 \frac{(1-qs)^{(\alpha-\beta-1)} t^{\alpha-1}}{\Gamma_q(\alpha)} \max_{s\in[0,1]} \bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u(s)\bigr)\bigr|\, d_q s \\ &\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \max_{s\in[0,1]} \bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u(s)\bigr)\bigr|\, d_q s. \end{aligned}$$

Since f is continuous, we obtain $\|T(u_n) - T(u)\| \to 0$ as $n \to \infty$.

Next we will prove that $T : \overline{U} \to K$ is a compact map, for any $0 < \eta \le r$. Let $E = \{x \in K | \|x\| \le \eta\}$. According to Theorem 3.1, we see that $T(E)$ is relatively compact in K, and the operator $T : \overline{U} \to K$ is completely continuous.

At last, we suppose $U \in \partial U$ is a solution of $u(t) = \lambda Tu(t)$, for $\lambda \in (0,1)$, we know that $T : \overline{U} \to K$ is completely continuous. According to (H_2) and (H_3), we know

$$\begin{aligned} u(t) = \lambda Tu(t) &= \lambda \biggl[\int_0^1 \frac{1}{\Gamma_q(\alpha)} (1-qs)^{(\alpha-\beta-1)} t^{\alpha-1} f\bigl(s,u(s)\bigr)\, d_q s \\ &\quad - \int_0^t \frac{1}{\Gamma_q(\alpha)} (t-qs)^{(\alpha-1)} f\bigl(s,u(s)\bigr)\, d_q s \biggr] \\ &< \int_0^1 \frac{1}{\Gamma_q(\alpha)} (1-qs)^{(\alpha-\beta-1)} f\bigl(s,u(s)\bigr)\, d_q s + \int_0^t \frac{1}{\Gamma_q(\alpha)} (t-qs)^{(\alpha-1)} f\bigl(s,u(s)\bigr)\, d_q s \\ &\le \int_0^1 \frac{1}{\Gamma_q(\alpha)} (1-qs)^{(\alpha-\beta-1)} f\bigl(s,u(s)\bigr)\, d_q s + \int_0^1 \frac{1}{\Gamma_q(\alpha)} (1-qs)^{(\alpha-1)} f\bigl(s,u(s)\bigr)\, d_q s \\ &= (C_1 + C_2)\|\varphi\| \le r, \end{aligned}$$

where r is satisfied with (H_3), that is to say, there is no $U \in \partial U$ such that $u = \lambda Tu$ for $\lambda \in (0,1)$, as a consequence of Lemma 2.3, T has a fixed point $t \in \overline{U}$, which is a solution of the boundary value problem (1.1)–(1.2). Thus we complete the proof. □

4 Examples

Example 4.1 Consider the following fractional q-difference boundary value problem:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{9}{4}} u(t) + (1+2t^2)\arctan(\frac{1}{3}(u+1)) = 0, \quad 0 < t < 1, \\ u(0) = {}^cD_{\frac{1}{2}}^{\frac{1}{4}} u(0) = {}^cD_{\frac{1}{2}}^{\frac{1}{4}} u(1) = 0, \end{cases} \tag{4.1}$$

where $\alpha = \frac{9}{4}$, $\beta = \frac{1}{4}$, $q = \frac{1}{2}$, $f(t,u(t)) = (1+2t^2)\arctan(\frac{1}{3}(u+1))$, $\forall (t,u) \in [0,1] \times \mathbb{R}$,

$$\begin{aligned} \bigl|f(t,u(t) - f(t,v(t))\bigr| &= \left|\bigl(1+2t^2\bigr)\arctan\left(\frac{1}{3}(u+1)\right) - \bigl(1+2t^2\bigr)\arctan\left(\frac{1}{3}(v+1)\right)\right| \\ &\leq \frac{1}{3}\bigl(1+2t^2\bigr)|u-v| \\ &< |u-v|, \quad t \in [0,1], u,v \in \mathbb{R}. \end{aligned}$$

We use

$$\begin{aligned} &B_q(2,\alpha-\beta) = B_{\frac{1}{2}}(2,2) = \int_0^1 t\left(1-\frac{1}{2}t\right) d_{\frac{1}{2}} t, \\ &\int_0^1 t\left(1-\frac{1}{2}t\right) d_{\frac{1}{2}} t = 1 \cdot \left(1-\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{1}{2^n}\left(1-\frac{1}{2^{n+1}}\right) \cdot \frac{1}{2^n} = \frac{3}{7}. \end{aligned}$$

Set $L = 2$, we have $0 < L < \frac{\Gamma_{\frac{1}{2}}(\frac{9}{4})}{B_q(2,\alpha-\beta)[\alpha-1]_q}$, according to Theorem 3.2, we know (4.1) has a unique solution.

Example 4.2 Consider the following fractional q-difference boundary value problem:

$$\begin{cases} {}^cD_q^{\frac{5}{2}} u(t) + \frac{\sin t\Gamma_q(\frac{5}{2})}{\int_0^1 (1-qs)\,d_qs + \int_0^1 (1-qs)^{(\frac{3}{2})}\,d_qs} = 0, \\ u(0) = {}^cD_q^{\beta} u(0) = {}^cD_q^{\beta} u(1) = 0, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} &f\bigl(t,x(t)\bigr) = \frac{\sin t\Gamma_q(\frac{5}{2})}{\int_0^1 (1-qs)\,d_qs + \int_0^1 (1-qs)^{(\frac{3}{2})}\,d_qs} = 0, \\ &\alpha = \frac{5}{2}, \qquad \beta = \frac{1}{2}, \qquad C_1 = \int_0^1 (1-qs)\,d_qs, \qquad C_2 = \int_0^1 (1-qs)^{(\frac{3}{2})}\,d_qs, \\ &\varphi(z) = \frac{\Gamma_q(\frac{5}{2})}{\int_0^1 (1-qs)\,d_qs + \int_0^1 (1-qs)^{(\frac{3}{2})}\,d_qs} = \frac{1}{C_1 + C_2}, \end{aligned}$$

we choose $r = 1$, according to Theorem 3.3, we can derive that $f(t,z) \leq \varphi(z)$, $\|\varphi\| \leq \frac{1}{C_1+C_2}$, the boundary value problem (4.2) has a solution.

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