Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Boundary value problem for a class of fractional integro-differential coupled systems with Hadamard fractional calculus and impulses

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Abstract

This paper considers the boundary value problem for a class of fractional integro-differential coupled systems with Hadamard fractional calculus and impulses. Some sufficient conditions of the existence and uniqueness are obtained by means of the Banach contraction principle and Leray–Schauder alternative. We also give some interesting examples to illustrate the effectiveness of our main results.

MSC: 34B10; 34B15; 34B37

Keywords: Boundary value problem; Hadamard fractional differential coupled system; Impulses; Leray–Schauder alternative theorem; Existence and uniqueness

1 Introduction

In 1695, L'Hôpital asked what was $\frac{d^n y}{dx^n}$ if $n = \frac{1}{2}$ in his letter to Leibniz. This year is generally regarded as the birthday of fractional calculus. Hereafter, Leibniz, J. Bernoulli, Euler, Lagrange, Laplace, Lacroix, Fourier, Abel, Cantor, De Morgen, Ya Sonin, Riemann, Liouville, Caputo, et al. have made important contributions to the definition of fractional calculus. In 1830s, Riemann and Liouville defined the integral and derivative which is now called Riemann–Liouville (R-L) fractional calculus by the Cauchy integral formula. Subsequently, many famous and important fractional integrals and derivatives have been proposed, for example, Grünwald–Letnikov fractional derivative, Caputo fractional derivative, Weyl fractional calculus, Hadamard fractional calculus, and so on. As for the history of fractional calculus, the readers can refer to the literature [1, 2].

The fractional-order calculus as a good tool is used to establish the mathematical model describing many actual phenomena and processes. For example, the fractional differential equations can describe the diffusion processes (see [3, 4]), the mechanical properties of materials (see [5–8]), the signal processing (see [9]), the image processing (see [10]), the behavior of viscoelastic and visco-plastic materials under external influences (see [11, 12]), the pharmacokinetics (see [13–15]), the bioengineering (see [16, 17]), the control theory (see [18, 19]), and so on. In addition there are some applications of fractional calculus within various fields of mathematics itself, e.g., in the analytical investigation of various types of special functions (see [20]). Therefore, the fractional differential equation has been widely focused and studied in depth. There have been some monographs and textbooks for the readers to learn use fractional calculus theories and methods (see [2, 21–25]). In the last few decades, there have been many papers dealing with fractional differential equation involving Riemann–Liouville and Caputo fractional derivatives (see [26–43]). In fact, the Hadamard fractional derivative is one of the most famous fractional calculi which was put forward by Hadamard in 1892. This type of fractional derivative differs from other types of derivatives. Its main feature is that the integral kernel contains a logarithmic function of arbitrary exponent in definition. Recently, there have been several papers dealing with Hadamard fractional differential equation (see [44–67]). However, these papers rarely considered the Hadamard fractional differential coupled equations. Therefore, it is interesting and challenging to study the Hadamard nonlinear fractional differential coupled system with impulses. So, in this paper we mainly study the following impulsive fractional differential coupled system with Hadamard fractional calculus:

$$\begin{cases} {}_{\mathrm{RLH}}D^{\alpha}_{t_k}[u(t) - {}_HJ^{\alpha}_{t_k}e(t,u(t),v(t))] = g(t,u(t),v(t)), & t \in J = [a,T], t \neq t_k, \\ {}_{\mathrm{RLH}}D^{\beta}_{t_k}[v(t) - {}_HJ^{\beta}_{t_k}f(t,u(t),v(t))] = h(t,u(t),v(t)), & t \in J = [a,T], t \neq t_k, \\ {}_{\mathrm{RLH}}D^{\alpha-1}_{t_k}u(t_k^+) - {}_{\mathrm{RLH}}D^{\alpha-1}_{t_k}u(t_k^-) = I_k(u(t_k)), & k = 1,2,\ldots,m, \\ {}_{\mathrm{RLH}}D^{\beta-1}_{t_k}v(t_k^+) - {}_{\mathrm{RLH}}D^{\beta-1}_{t_k}v(t_k^-) = J_k(v(t_k)), & k = 1,2,\ldots,m, \\ c \cdot {}_{\mathrm{RLH}}D^{\alpha-1}_{a}u(a) = u(T), \qquad d \cdot {}_{\mathrm{RLH}}D^{\beta-1}_{a}v(a) = v(T), \end{cases} \tag{1.1}$$

where $a > 0$, $1 < \alpha,\beta < 2$, $c,d \in \mathbb{R}$, $I_k, J_k \in C(\mathbb{R},\mathbb{R})$. ${}_{\mathrm{RLH}}D^{\alpha}_{t_k}$, ${}_{\mathrm{RLH}}D^{\beta}_{t_k}$ denote the left-sided Riemann–Liouville type Hadamard fractional derivatives of order α and β. ${}_HJ^{\alpha}_{t_k}$, ${}_HJ^{\beta}_{t_k}$ denote the left-sided Hadamard fractional integrals of order α and β. $e,f,g,h : J \times \mathbb{R}^2 \to \mathbb{R}$ are some given continuous functions and impulsive points, $\{t_k\}_{k=1}^m$ satisfies $a = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$. ${}_{\mathrm{RLH}}D^{\alpha-1}_{t_k}u(t_k^+)$, ${}_{\mathrm{RLH}}D^{\alpha-1}_{t_k}u(t_k^-)$, ${}_{\mathrm{RLH}}D^{\beta-1}_{t_k}v(t_k^+)$, and ${}_{\mathrm{RLH}}D^{\beta-1}_{t_k}v(t_k^-)$ represent the right and left limits and satisfy the left continuity at $t = t_k$, $k = 1,2,\ldots,m$.

In addition, some other inspiration for studying system (1.1) comes from the literature [48, 49]. In [48], the authors considered the existence and finite-time stability results of Hadamard type impulsive fractional differential equations as follows:

$$\begin{cases} {}_HD^{\alpha}_1 u(t) = f(t,u(t),\max_{\xi\in[\beta t,t]} u(\xi)), & \alpha \in (0,1), t \in [1,e]\backslash\Theta, \beta \in (0,1), \\ {}_HJ^{1-\alpha}_1 u(t_i^+) - {}_HJ^{1-\alpha}_1 u(t_i^-) = a_i u(t_i^-) + b_i, & a_i, b_i > 0, i = 1,2,\ldots,m, \\ {}_HJ^{1-\alpha}_1 u(1) = u_0, \quad u_0 > 0, \end{cases}$$

with initial condition $u(t) = \phi(t)$, $t \in [\beta,1]$, where ${}_HD^{\alpha}_1$ denotes the left-sided Riemann–Liouville type Hadamard fractional derivative of order α, ${}_HJ^{1-\alpha}_1$ denotes the left-sided Hadamard fractional integral of order $1-\alpha$, and $\Theta = \{t_1, t_2, \ldots, t_m\}$ satisfying $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$. $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $u(t_i^+) = \lim_{\epsilon\to 0^+} u(t_i + \epsilon)$, $u(t_i^-) = \lim_{\epsilon\to 0^-} u(t_i + \epsilon)$.

In [49], the author discussed the existence and uniqueness results of solutions for the Hadamard and Riemann–Liouville fractional neutral functional integro-differential equations with finite delay described by

$$\begin{cases} {}_HD^{\alpha}[u(t)-\sum_{i=1}^{m} I^{\beta_i}h_i(t,u_t)] = f(t,u_t), & t\in J=[1,T], \\ u(t)=\varphi(t), \quad t\in[1-r,1], r>0, \end{cases}$$

where ${}_HD^{\alpha}$ denotes the left-sided Riemann–Liouville type Hadamard fractional derivative of order α, $0<\alpha\leq 1$, I^{β_i} is the Riemann–Liouville fractional integral of order $\beta_i>0$, $i=1,2,\dots,m, f,h_i: J\times C([-r,0],\mathbb{R})\to\mathbb{R}$ are given continuous functions, $\varphi\in C([1-r,1],\mathbb{R})$ with $\varphi(1)=0$. For any function u defined on $[1-r,T]$ and any $t\in J$, $u_t(\theta)=u(t+\theta)$, $\theta\in[-r,0]$ denotes the element of $C([-r,0],\mathbb{R})$. The author derived the existence of solutions by the Leray–Schauder alternative and established the uniqueness of solutions by the Banach contraction principle.

The rest of this paper is organized as follows. In Sect. 2, we recall some useful preliminaries. In Sect. 3, we shall prove the existence and uniqueness of solutions for system (1.1). In Sect. 4, some examples are also provided to illustrate the effectiveness of our main results. Finally, the conclusion is given to simply recall our studied contents and obtained results in Sect. 5.

2 Preliminaries

In this section, we introduce some notations and definitions of Hadamard fractional calculus and present preliminary results needed in our proofs later.

Definition 2.1 ([22]) For $a\geq 0$, the left-sided Hadamard fractional integral of order $\alpha>0$ for a function $u:(a,\infty)\to\mathbb{R}$ is defined as

$${}_HJ_a^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}u(s)\frac{ds}{s},$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([22]) For $a\geq 0$, the left-sided Riemann–Liouville type Hadamard fractional derivative of order $\alpha>0$ for a function $u:(a,\infty)\to\mathbb{R}$ is defined by

$${}_{\mathrm{RLH}}D_a^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\ln\frac{t}{s}\right)^{n-\alpha+1}u(s)\frac{ds}{s}, \quad n-1<\alpha<n, n=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number $\alpha>0$, and $\Gamma(\cdot)$ is the gamma function.

Lemma 2.1 ([22]) *For $a>0$, assume that $u\in C(a,T)\cap L^1(a,T)$ with a left-sided Riemann–Liouville type Hadamard fractional derivative of order $\alpha>0$. Then*

$${}_HJ_a^{\alpha}\,{}_{\mathrm{RLH}}D_a^{\alpha}u(t)=u(t)+c_1\left(\ln\frac{t}{a}\right)^{\alpha-1}+c_2\left(\ln\frac{t}{a}\right)^{\alpha-2}+\cdots+c_n\left(\ln\frac{t}{a}\right)^{\alpha-n}$$

for some $c_i\in\mathbb{R}$, $i=1,2,\dots,n-1$, $n=[\alpha]+1$.

Lemma 2.2 ([44]) *Let $\alpha > 0$, $\beta > 0$, and $0 < a < \infty$. Then the following properties hold*:

$$\begin{aligned}
&{}_{\mathrm{RLH}}D_a^\alpha \left(\ln \frac{t}{a}\right)^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\ln \frac{x}{a}\right)^{\beta-\alpha-1}, \\
&{}_H J_a^\alpha \left(\ln \frac{t}{a}\right)^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\ln \frac{x}{a}\right)^{\beta+\alpha-1}, \\
&{}_{\mathrm{RLH}}D_a^\alpha {}_H J_a^\alpha u(t) = u(t).
\end{aligned}$$

Lemma 2.3 ([68]) *If E is a real Banach space and $F : E \to E$ is a contraction mapping, then F has a unique fixed point in E.*

Lemma 2.4 (Leray–Schauder alternative theorem [61]) *Let U be a normed linear space and $F : U \to U$ be a completely continuous operator (i.e., a map that restricted to any bounded set in U is compact). Let*

$$\varepsilon(F) = \{x \in U : x = kF(x), 0 < k < 1\},$$

then either the set $\varepsilon(F)$ is unbounded, or F has at least one fixed point.

For the convenient statements, we introduce the notation as follows: $t_0 = a$, $t_{m+1} = T$, $I_0(t) = J_0(t) \equiv 0$. Let $C[a,T]$ be the Banach space of all continuous functions from $[a,T] \to \mathbb{R}$ with the norm $\|\omega\| = \sup_{t\in[a,T]} |\omega(t)|$. For $1 < \gamma < 2$, we define

$$\begin{aligned}
\mathrm{PC}_\gamma[a,T] = \big\{&\omega : \omega(t) \in C(t_k,t_{k+1}], {}_{\mathrm{RLH}}D_{t_k}^{\gamma-1}\omega(t_k^-), {}_{\mathrm{RLH}}D_{t_k}^{\gamma-1}\omega(t_k^+) \text{ all exist} \\
&\text{and satisfy } {}_{\mathrm{RLH}}D_{t_k}^{\gamma-1}\omega(t_k^-) = {}_{\mathrm{RLH}}D_{t_k}^{\gamma-1}\omega(t_k), k = 0,1,2,\ldots,m\big\}.
\end{aligned}$$

Obviously, $\mathrm{PC}_\gamma[a,T]$ is a Banach space equipped with the norm $\|\omega\|_{\mathrm{PC}_\gamma} = \|\omega(t)\|_C$. The space $X = \mathrm{PC}_\alpha[a,T] \times \mathrm{PC}_\beta[a,T]$ equipped with the norm $\|(u,v)\| = \max\{\|u\|_{\mathrm{PC}_\alpha}, \|v\|_{\mathrm{PC}_\beta}\}$ is also a Banach space.

Definition 2.3 A pair of functions $(u(t),v(t)) \in X = \mathrm{PC}_\alpha[a,T] \times \mathrm{PC}_\beta[a,T]$ is called to be a solution of (1.1) if $(u(t),v(t))$ satisfy all the equations and boundary value conditions of system (1.1).

Lemma 2.5 *Assume that the functions $e, g \in C[a,T])$ and $I_k \in C(\mathbb{R},\mathbb{R})$. If $\delta \triangleq c\Gamma(\alpha) - (\ln \frac{T}{t_m})^{\alpha-1} \neq 0$, then, for given $v(t) \in \mathrm{PC}_\beta[a,T]$, a function $u(t) \in \mathrm{PC}_\alpha[a,T]$ is a solution of the impulsive Hadamard fractional differential equation*

$$\begin{cases}
{}_{\mathrm{RLH}}D_{t_k}^\alpha[u(t) - {}_H J_{t_k}^\alpha e(t,u(t),v(t))] = g(t,u(t),v(t)), \quad 1 < \alpha < 2, \\
{}_{\mathrm{RLH}}D_{t_k}^{\alpha-1}u(t_k^+) - {}_{\mathrm{RLH}}D_{t_k}^{\alpha-1}u(t_k^-) = I_k(u(t_k)), \quad k = 1,2,\ldots,m, \\
c \cdot {}_{\mathrm{RLH}}D_a^{\alpha-1}u(a) = u(T),
\end{cases} \tag{2.1}$$

if and only if $u(t) \in C[a,T] \cap \mathrm{PC}_\alpha[a,T]$ *is a solution of the integral equation*

$$u(t) = {}_HJ_{t_k}^{\alpha} g\big(t,u(t),v(t)\big) + {}_HJ_{t_k}^{\alpha} e\big(t,u(t),v(t)\big) + c^*\left(\ln\frac{t}{t_k}\right)^{\alpha-1} + \frac{\Lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\big[I_i\big(u(t_i)\big) + {}_HJ_{t_{i-1}}^{1} g\big(t_i,u(t_i),v(t_i)\big) + {}_HJ_{t_{i-1}}^{1} e\big(t_i,u(t_i),v(t_i)\big)\big]\left(\ln\frac{t}{t_k}\right)^{\alpha-1}, \quad t \in (t_k,t_{k+1}], \quad (2.2)$$

where $k = 0,1,2,\ldots,m$, $\Lambda = \begin{cases} 0, & t \in [a,t_1], \\ 1, & t \in (t_1,T], \end{cases}$ *and*

$$c^* = \frac{1}{\delta}\left({}_HJ_{t_m}^{\alpha} g\big(T,u(T),v(T)\big) + {}_HJ_{t_m}^{\alpha} e\big(T,u(T),v(T)\big) + \frac{(\ln\frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\big[I_i\big(u(t_i)\big) + {}_HJ_{t_{i-1}}^{1} g\big(t_i,u(t_i),v(t_i)\big) + {}_HJ_{t_{i-1}}^{1} e\big(t_i,u(t_i),v(t_i)\big)\big]\right).$$

Proof When $t \in [a,t_1] = [t_0,t_1]$, applying the Hadamard fractional integral operator on both sides of the first equation in (2.1), that is,

$${}_HJ_{t_0}^{\alpha}\,{}_{\mathrm{RLH}}D_{t_0}^{\alpha}\big[u(t) - {}_HJ_{t_0}^{\alpha} e\big(t,u(t),v(t)\big)\big] = {}_HJ_{t_0}^{\alpha} g\big(t,u(t),v(t)\big),$$

we have

$$u(t) = {}_HJ_{t_0}^{\alpha} g\big(t,u(t),v(t)\big) + {}_HJ_{t_0}^{\alpha} e\big(t,u(t),v(t)\big) + c_1\left(\ln\frac{t}{t_0}\right)^{\alpha-1} + d_1\left(\ln\frac{t}{t_0}\right)^{\alpha-2}, \quad (2.3)$$

where c_1 and d_1 are some constants. In the light of the existence of $u(a)$, we have $d_1 = 0$.

In view of Lemmas 2.1–2.2, we obtain

$$\begin{aligned} {}_{\mathrm{RLH}}D_{t_0}^{\alpha-1} u(t) &= {}_{\mathrm{RLH}}D_{t_0}^{\alpha-1}\,{}_HJ_{t_0}^{\alpha} g\big(t,u(t),v(t)\big) + {}_{\mathrm{RLH}}D_{t_0}^{\alpha-1}\,{}_HJ_{t_0}^{\alpha} e\big(t,u(t),v(t)\big) \\ &\quad + c_1\,{}_HD_{t_0}^{\alpha-1}\left(\ln\frac{t}{t_0}\right)^{\alpha-1} \\ &= {}_HJ_{t_0}^{1} g\big(t,u(t),v(t)\big) + {}_HJ_{t_0}^{1} e\big(t,u(t),v(t)\big) + c_1\Gamma(\alpha). \end{aligned} \quad (2.4)$$

(2.4) gives that

$${}_{\mathrm{RLH}}D_{t_0}^{\alpha-1} u(t_0) = c_1\Gamma(\alpha). \quad (2.5)$$

According to (2.3) and (2.4), we get

$${}_{\mathrm{RLH}}D_{t_0}^{\alpha-1} u\big(t_1^-\big) = {}_HJ_{t_0}^{1} g\big(t_1,u(t_1),v(t_1)\big) + {}_HJ_{t_0}^{1} e\big(t_1,u(t_1),v(t_1)\big) + c_1\Gamma(\alpha) \quad (2.6)$$

and

$$u(t) = {}_HJ_{t_0}^{\alpha} g\big(t,u(t),v(t)\big) + {}_HJ_{t_0}^{\alpha} e\big(t,u(t),v(t)\big) + c_1\left(\ln\frac{t}{t_0}\right)^{\alpha-1}, \quad t \in [t_0,t_1]. \quad (2.7)$$

When $t \in (t_1, t_2]$, there are similar to have

$$u(t) = {}_HJ_{t_1}^{\alpha} g\bigl(t, u(t), v(t)\bigr) + {}_HJ_{t_1}^{\alpha} e\bigl(t, u(t), v(t)\bigr) + c_2\left(\ln \frac{t}{t_1}\right)^{\alpha-1} + d_2\left(\ln \frac{t}{t_1}\right)^{\alpha-2}, \tag{2.8}$$

where c_2 and d_2 are some constants. In the light of the existence of ${}_{\mathrm{RLH}}D_{t_1}^{\alpha-1}u(t_1^+)$, we have $d_2 = 0$, and

$${}_{\mathrm{RLH}}D_{t_1}^{\alpha-1}u\bigl(t_1^+\bigr) = c_2\Gamma(\alpha). \tag{2.9}$$

It follows from (2.6), (2.9), and the second equation of (2.1) that

$$c_2 - c_1 = \frac{1}{\Gamma(\alpha)}\bigl[I_1\bigl(u(t_1)\bigr) + {}_HJ_{t_0}^{1} g\bigl(t_1, u(t_1), v(t_1)\bigr) + {}_HJ_{t_0}^{1} e\bigl(t_1, u(t_1), v(t_1)\bigr)\bigr] \tag{2.10}$$

and

$$u(t) = {}_HJ_{t_1}^{\alpha} g\bigl(t, u(t), v(t)\bigr) + {}_HJ_{t_1}^{\alpha} e\bigl(t, u(t), v(t)\bigr) + c_2\left(\ln \frac{t}{t_1}\right)^{\alpha-1}, \quad t \in (t_1, t_2]. \tag{2.11}$$

Repeating the above calculation process, for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we obtain

$$c_{k+1} - c_k = \frac{1}{\Gamma(\alpha)}\bigl[I_k\bigl(u(t_k)\bigr) + {}_HJ_{t_{k-1}}^{1} g\bigl(t_k, u(t_k), v(t_k)\bigr) + {}_HJ_{t_{k-1}}^{1} e\bigl(t_k, u(t_k), v(t_k)\bigr)\bigr] \tag{2.12}$$

and

$$u(t) = {}_HJ_{t_k}^{\alpha} g\bigl(t, u(t), v(t)\bigr) + {}_HJ_{t_k}^{\alpha} e\bigl(t, u(t), v(t)\bigr) + c_{k+1}\left(\ln \frac{t}{t_k}\right)^{\alpha-1}, \quad t \in (t_k, t_{k+1}]. \tag{2.13}$$

From (2.12) and (2.13), we have

$$\begin{aligned} c_{m+1} - c_1 = \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\bigl[I_k\bigl(u(t_k)\bigr) &+ {}_HJ_{t_{k-1}}^{1} g\bigl(t_k, u(t_k), v(t_k)\bigr) \\ &+ {}_HJ_{t_{k-1}}^{1} e\bigl(t_k, u(t_k), v(t_k)\bigr)\bigr] \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} u(T) = u(t_{m+1}) = {}&{}_HJ_{t_m}^{\alpha} g\bigl(T, u(T), v(T)\bigr) + {}_HJ_{t_m}^{\alpha} e\bigl(T, u(T), v(T)\bigr) \\ &+ c_{m+1}\left(\ln \frac{T}{t_m}\right)^{\alpha-1}. \end{aligned} \tag{2.15}$$

In the light of (2.5), (2.14), (2.15), and the third equation of (2.1), we have

$$\begin{aligned} c_1 = \frac{1}{\delta}\Biggl(&{}_HJ_{t_m}^{\alpha} g\bigl(T, u(T), v(T)\bigr) + {}_HJ_{t_m}^{\alpha} e\bigl(T, u(T), v(T)\bigr) + \frac{(\ln \frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\bigl[I_i\bigl(u(t_i)\bigr) \\ &+ {}_HJ_{t_{i-1}}^{1} g\bigl(t_i, u(t_i), v(t_i)\bigr) + {}_HJ_{t_{i-1}}^{1} e\bigl(t_i, u(t_i), v(t_i)\bigr)\bigr]\Biggr). \end{aligned} \tag{2.16}$$

Thus, for $k = 1, 2, \ldots, m$, we have

$$\begin{aligned} u(t) =& {}_HJ^{\alpha}_{t_k} g\bigl(t, u(t), v(t)\bigr) + {}_HJ^{\alpha}_{t_k} e\bigl(t, u(t), v(t)\bigr) + c_1 \left(\ln \frac{t}{t_k}\right)^{\alpha-1} \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \bigl[I_i\bigl(u(t_i)\bigr) + {}_HJ^{1}_{t_{i-1}} g\bigl(t_i, u(t_i), v(t_i)\bigr) \\ &+ {}_HJ^{1}_{t_{i-1}} e\bigl(t_i, u(t_i), v(t_i)\bigr)\bigr] \left(\ln \frac{t}{t_k}\right)^{\alpha-1}, \quad t \in (t_k, t_{k+1}]. \end{aligned} \tag{2.17}$$

Conversely, if $u(t)$ satisfies (2.2), it is easy to verify $u(t)$ satisfying (2.1). The proof is complete. □

Similarly, we obtain the following lemma.

Lemma 2.6 *Assume that the functions $f, h \in C[a, T]$ and $J_k \in C(\mathbb{R}, \mathbb{R})$. If $\rho \triangleq d\Gamma(\beta) - (\ln \frac{T}{t_m})^{\beta-1} \neq 0$, then, for given $u(t) \in \mathrm{PC}_\alpha[a, T]$, a function $v(t) \in \mathrm{PC}_\beta[a, T]$ is a solution of the impulsive Hadamard fractional differential equation*

$$\begin{cases} {}_{\mathrm{RLH}}D^{\beta}_{t_k}[v(t) - {}_HJ^{\beta}_{t_k} f(t, u(t), v(t))] = h(t, u(t), v(t)), \quad 1 < \beta < 2, \\ {}_{\mathrm{RLH}}D^{\beta-1}_{t_k} v(t_k^+) - {}_{\mathrm{RLH}}D^{\beta-1}_{t_k} v(t_k^-) = J_k(v(t_k)), \quad k = 1, 2, \ldots, m, \\ d \cdot {}_{\mathrm{RLH}}D^{\beta-1}_{a} v(a) = v(T) \end{cases} \tag{2.18}$$

if and only if $v(t) \in C[a, T] \cap \mathrm{PC}_\beta[a, T]$ is a solution of the integral equation

$$\begin{aligned} v(t) =& {}_HJ^{\beta}_{t_k} h\bigl(t, u(t), v(t)\bigr) + {}_HJ^{\beta}_{t_k} f\bigl(t, u(t), v(t)\bigr) + d^* \left(\ln \frac{t}{t_k}\right)^{\beta-1} \\ &+ \frac{\Lambda}{\Gamma(\beta)} \sum_{i=1}^{k} \bigl[J_i\bigl(v(t_i)\bigr) + {}_HJ^{1}_{t_{i-1}} h\bigl(t_i, u(t_i), v(t_i)\bigr) \\ &+ {}_HJ^{1}_{t_{i-1}} f\bigl(t_i, u(t_i), v(t_i)\bigr)\bigr] \left(\ln \frac{t}{t_k}\right)^{\beta-1}, \quad t \in (t_k, t_{k+1}], \end{aligned} \tag{2.19}$$

where $k = 0, 1, 2, \ldots, m$, $\Lambda = \begin{cases} 0, & t \in [a, t_1], \\ 1, & t \in (t_1, T], \end{cases}$ and

$$\begin{aligned} d^* =& \frac{1}{\rho} \Biggl({}_HJ^{\beta}_{t_m} h\bigl(T, u(T), v(T)\bigr) + {}_HJ^{\beta}_{t_m} f\bigl(T, u(T), v(T)\bigr) + \frac{(\ln \frac{T}{t_m})^{\beta-1}}{\Gamma(\beta)} \sum_{i=1}^{m} \bigl[J_i\bigl(v(t_i)\bigr) \\ &+ {}_HJ^{1}_{t_{i-1}} h\bigl(t_i, u(t_i), v(t_i)\bigr) + {}_HJ^{1}_{t_{i-1}} f\bigl(t_i, u(t_i), v(t_i)\bigr)\bigr] \Biggr). \end{aligned}$$

3 Main results

In this section, we shall employ Lemmas 2.3 and 2.4 to prove the existence of solutions to system (1.1). In the light of Lemmas 2.5 and 2.6, we define the operator $S : X = \mathrm{PC}_\alpha \times \mathrm{PC}_\beta \to X$ by

$$S(u, v)(t) = \bigl(S_1(u, v)(t), S_2(u, v)(t)\bigr)^T, \quad \forall (u, v) \in X, t \in [a, T], \tag{3.1}$$

where

$$\begin{aligned}S_1(u,v)(t) &= {}_HJ^{\alpha}_{t_k} g\bigl(t,u(t),v(t)\bigr) + {}_HJ^{\alpha}_{t_k} e\bigl(t,u(t),v(t)\bigr)\\ &\quad + c^*\left(\ln\frac{t}{t_k}\right)^{\alpha-1} + \frac{\Lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\bigl[I_i\bigl(u(t_i)\bigr) + {}_HJ^{1}_{t_{i-1}} g\bigl(t_i,u(t_i),v(t_i)\bigr)\\ &\quad + {}_HJ^{1}_{t_{i-1}} e\bigl(t_i,u(t_i),v(t_i)\bigr)\bigr]\left(\ln\frac{t}{t_k}\right)^{\alpha-1}, \quad t\in(t_k,t_{k+1}], 0\le k\le m,\end{aligned}$$

and

$$\begin{aligned}S_2(x,y)(t) &= {}_HJ^{\beta}_{t_k} h\bigl(t,u(t),v(t)\bigr) + {}_HJ^{\beta}_{t_k} f\bigl(t,u(t),v(t)\bigr)\\ &\quad + d^*\left(\ln\frac{t}{t_k}\right)^{\beta-1} + \frac{\Lambda}{\Gamma(\beta)}\sum_{i=1}^{k}\bigl[J_i\bigl(v(t_i)\bigr) + {}_HJ^{1}_{t_{i-1}} h\bigl(t_i,u(t_i),v(t_i)\bigr)\\ &\quad + {}_HJ^{1}_{t_{i-1}} f\bigl(t_i,u(t_i),v(t_i)\bigr)\bigr]\left(\ln\frac{t}{t_k}\right)^{\beta-1}, \quad t\in(t_k,t_{k+1}], 0\le k\le m.\end{aligned}$$

Solving system (1.1) is equivalent to finding the fixed point of the operator S defined by (3.1). Now we present and prove our main results.

Theorem 3.1 *If the following conditions* (H_1)–(H_3) *hold, then the Hadamard impulsive fractional differential coupled system* (1.1) *has a pair of unique solutions* $(u^*(t),v^*(t))\in \mathrm{PC}_\alpha\times\mathrm{PC}_\beta$.

(H_1) *Let* $e,f,g,h\in C[a,T]$, $I_k,J_k\in C(\mathbb{R},\mathbb{R})$, $k=1,2,\dots,m$. *For* $u_i,v_i\in\mathbb{R}$ $(i=1,2)$, *there exist some positive constants* $M_i,\overline{M}_i,N_i,\overline{N}_i (i=1,2)$, P_k, *and* Q_k $(k=1,2,\dots,m)$ *such that*

$$\begin{aligned}&\bigl|g(t,u_1,v_1)-e(t,u_2,v_2)\bigr|\le M_1|u_1-u_2|+M_2|v_1-v_2|,\\ &\bigl|e(t,u_1,v_1)-g(t,u_2,v_2)\bigr|\le N_1|u_1-u_2|+N_2|v_1-v_2|,\\ &\bigl|f(t,u_1,v_1)-f(t,u_2,v_2)\bigr|\le \overline{M}_1|u_1-u_2|+\overline{M}_2|v_1-v_2|,\\ &\bigl|h(t,u_1,v_1)-h(t,u_2,v_2)\bigr|\le \overline{N}_1|u_1-u_2|+\overline{N}_2|v_1-v_2|,\\ &\bigl|I_k(u_1)-I_k(v_1)\bigr|\le P_k|u_1-v_1|,\\ &\bigl|J_k(u_1)-J_k(v_1)\bigr|\le Q_k|u_1-v_1|, \quad k=1,2\dots,m.\end{aligned}$$

(H_2) $\delta\triangleq c\Gamma(\alpha)-(\ln\frac{T}{t_m})^{\alpha-1}>0$, $\rho\triangleq d\Gamma(\beta)-(\ln\frac{T}{t_m})^{\beta-1}>0$.

(H_3)

$$\begin{aligned}\kappa &\triangleq \frac{M_1+M_2+N_1+N_2}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha}\\ &\quad\times\left[1+\frac{1}{\delta}\left(\ln\frac{T}{a}\right)^{\alpha-1}+\frac{1}{\delta\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\right]\\ &\quad+\frac{\sum_{i=1}^{m}P_i}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha-1}\left[1+\left(\ln\frac{T}{a}\right)^{\alpha-1}\right]<1,\end{aligned}$$

$$\varrho \triangleq \frac{\overline{M}_1+\overline{M}_2+\overline{N}_1+\overline{N}_2}{\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta}$$
$$\times\left[1+\frac{1}{\rho}\left(\ln\frac{T}{a}\right)^{\beta-1}+\frac{1}{\rho\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta-1}+\frac{1}{\Gamma(\beta)}\right]$$
$$+\frac{\sum_{i=1}^{m}Q_i}{\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta-1}\left[1+\left(\ln\frac{T}{a}\right)^{\beta-1}\right]<1.$$

Proof Now, we apply the Banach contraction principle to prove that $S:X\to X$ defined by (3.1) has a unique fixed point. We shall show that S is a contraction. In fact, from (3.1) and conditions (H_1)–(H_2), for $t\in J=[a,T]$, $(u_1,v_1),(u_2,v_2)\in X$, we have

$$\begin{aligned}
&\bigl|S_1(u_1,v_1)(t)-S_1(u_2,v_2)(t)\bigr|\\
&=\Bigl|{}_HJ^{\alpha}_{t_k}\bigl[g\bigl(t,u_1(t),v_1(t)\bigr)-g\bigl(t,u_2(t),v_2(t)\bigr)\bigr]+{}_HJ^{\alpha}_{t_k}\bigl[e\bigl(t,u_1(t),v_1(t)\bigr)-e\bigl(t,u_2(t),v_2(t)\bigr)\bigr]\\
&+\frac{1}{\delta}\Bigl[{}_HJ^{\alpha}_{t_m}\bigl[g\bigl(T,u_1(T),v_1(T)\bigr)-g\bigl(T,u_2(T),v_2(T)\bigr)\bigr]+{}_HJ^{\alpha}_{t_m}\bigl[e\bigl(T,u_1(T),v_1(T)\bigr)\\
&-e\bigl(T,u_2(T),v_2(T)\bigr)\bigr]+\frac{(\ln\frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\bigl(\bigl[I_i\bigl(u_1(t_i)\bigr)-I_i\bigl(u_2(t_i)\bigr)\bigr]\\
&+{}_HJ^{1}_{t_{i-1}}\bigl[g\bigl(t_i,u_1(t_i),v_1(t_i)\bigr)-g\bigl(t_i,u_2(t_i),v_2(t_i)\bigr)\bigr]\\
&+{}_HJ^{1}_{t_{i-1}}\bigl[e\bigl(t_i,u_1(t_i),v_1(t_i)\bigr)-e\bigl(t_i,u_2(t_i),v_2(t_i)\bigr)\bigr]\bigr)\Bigr]\left(\ln\frac{t}{t_k}\right)^{\alpha-1}\\
&+\frac{\Lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\bigl(\bigl[I_i\bigl(u_1(t_i)\bigr)-I_i\bigl(u_2(t_i)\bigr)\bigr]+{}_HJ^{1}_{t_{i-1}}\bigl[g\bigl(t_i,u_1(t_i),v_2(t_i)\bigr)-g\bigl(t_i,u_1(t_i),v_2(t_i)\bigr)\bigr]\\
&+{}_HJ^{1}_{t_{i-1}}\bigl[e\bigl(t_i,u_1(t_i),v_1(t_i)\bigr)-e\bigl(t_i,u_2(t_i),v_2(t_i)\bigr)\bigr]\bigr)\left(\ln\frac{t}{t_k}\right)^{\alpha-1}\Bigr|\\
&\le{}_HJ^{\alpha}_{t_k}\bigl|g\bigl(t,u_1(t),v_1(t)\bigr)-g\bigl(t,u_2(t),v_2(t)\bigr)\bigr|+{}_HJ^{\alpha}_{t_k}\bigl|e\bigl(t,u_1(t),v_1(t)\bigr)-e\bigl(t,u_2(t),v_2(t)\bigr)\bigr|\\
&+\frac{1}{\delta}\Bigl[{}_HJ^{\alpha}_{t_m}\bigl|g\bigl(T,u_1(T),v_1(T)\bigr)-g\bigl(T,u_2(T),v_2(T)\bigr)\bigr|+{}_HJ^{\alpha}_{t_m}\bigl|e\bigl(T,u_1(T),v_1(T)\bigr)\\
&-e\bigl(T,u_2(T),v_2(T)\bigr)\bigr|+\frac{(\ln\frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\bigl(\bigl|I_i\bigl(u_1(t_i)\bigr)\\
&-I_i\bigl(u_2(t_i)\bigr)\bigr|+{}_HJ^{1}_{t_{i-1}}\bigl|g\bigl(t_i,u_1(t_i),v_1(t_i)\bigr)-g\bigl(t_i,u_2(t_i),v_2(t_i)\bigr)\bigr|\\
&+{}_HJ^{1}_{t_{i-1}}\bigl|e\bigl(t_i,u_1(t_i),v_1(t_i)\bigr)-e\bigl(t_i,u_2(t_i),v_2(t_i)\bigr)\bigr|\bigr)\Bigr]\left(\ln\frac{t}{t_k}\right)^{\alpha-1}\\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\bigl(\bigl|I_i\bigl(u_1(t_i)\bigr)-I_i\bigl(u_2(t_i)\bigr)\bigr|+{}_HJ^{1}_{t_{i-1}}\bigl|g\bigl(t_i,u_1(t_i),v_2(t_i)\bigr)-g\bigl(t_i,u_1(t_i),v_2(t_i)\bigr)\bigr|\\
&+{}_HJ^{1}_{t_{i-1}}\bigl|e\bigl(t_i,u_1(t_i),v_1(t_i)\bigr)-e\bigl(t_i,u_2(t_i),v_2(t_i)\bigr)\bigr|\bigr)\left(\ln\frac{t}{t_k}\right)^{\alpha-1}|\\
&\le{}_HJ^{\alpha}_{t_k}\bigl[M_1\bigl|u_1(t)-u_2(t)\bigr|+M_2\bigl|v_1(t)-v_2(t)\bigr|\bigr]
\end{aligned}$$

$$
\begin{aligned}
&+ {}_HJ^{\alpha}_{t_k}\big[N_1\big|u_1(t)-u_2(t)\big| + N_2\big|v_1(t)-v_2(t)\big|\big] \\
&+ \frac{1}{\delta}\Bigg[{}_HJ^{\alpha}_{t_m}\big[M_1\big|u_1(T)-u_2(T)\big| + M_2\big|v_1(T)-v_2(T)\big|\big] + {}_HJ^{\alpha}_{t_m}\big[N_1\big|u_1(T)-u_2(T)\big| \\
&+ N_2\big|v_1(T)-v_2(T)\big|\big] + \frac{(\ln\frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\big(P_i\big|u_1(t_i)-u_2(t_i)\big| \\
&+ {}_HJ^{1}_{t_{i-1}}\big[M_1\big|u_1(t_i)-u_2(t_i)\big| + M_2\big|v_1(t_i)-v_2(t_i)\big|\big] \\
&+ {}_HJ^{1}_{t_{i-1}}\big[N_1\big|u_1(t_i)-u_2(t_i)\big| + N_2\big|v_1(t_i)-v_2(t_i)\big|\big]\big)\Bigg]\left(\ln\frac{t}{t_k}\right)^{\alpha-1} \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\big(P_i\big|u_1(t_i)-u_2(t_i)\big| + {}_HJ^{1}_{t_{i-1}}\big[M_1\big|u_1(t_i)-u_2(t_i)\big| + M_2\big|v_1(t_i)-v_2(t_i)\big|\big] \\
&+ {}_HJ^{1}_{t_{i-1}}[N_1\big|u_1(t_i)-u_2(t_i)\big| + N_2\big[\big|v_1(t_i)-v_2(t_i)\big|\big]\big)\left(\ln\frac{t}{t_k}\right)^{\alpha-1} \\
\le{}& {}_HJ^{\alpha}_{t_k}\big[M_1\|u_1-u_2\|_{\mathrm{PC}_\alpha} + M_2\|v_1-v_2\|_{\mathrm{PC}_\beta}\big] \\
&+ {}_HJ^{\alpha}_{t_k}\big[N_1\|u_1-u_2\|_{\mathrm{PC}_\alpha} + N_2\|v_1-v_2\|_{\mathrm{PC}_\beta}\big] \\
&+ \frac{1}{\delta}\Bigg[{}_HJ^{\alpha}_{t_m}\big[M_1\|u_1-u_2\|_{\mathrm{PC}_\alpha} + M_2\|v_1-v_2\|_{\mathrm{PC}_\beta}\big] + {}_HJ^{\alpha}_{t_m}\big[N_1\|u_1-u_2\|_{\mathrm{PC}_\alpha} \\
&+ N_2\|v_1-v_2\|_{\mathrm{PC}_\beta}\big] + \frac{(\ln\frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\big(P_i\|u_1-u_2\|_{\mathrm{PC}_\alpha} + {}_HJ^{1}_{t_{i-1}}\big[M_1\|u_1-u_2\|_{\mathrm{PC}_\alpha} \\
&+ M_2\|v_1-v_2\|_{\mathrm{PC}_\beta}\big] + {}_HJ^{1}_{t_{i-1}}\big[N_1\|u_1-u_2\|_{\mathrm{PC}_\alpha} + N_2\|v_1-v_2\|_{\mathrm{PC}_\beta}\big]\big)\Bigg]\left(\ln\frac{t}{t_k}\right)^{\alpha-1} \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\big(P_i\|u_1-u_2\|_{\mathrm{PC}_\alpha} + {}_HJ^{1}_{t_{i-1}}\big[M_1\|u_1-u_2\|_{\mathrm{PC}_\alpha} + M_2\|v_1-v_2\|_{\mathrm{PC}_\beta}\big] \\
&+ {}_HJ^{1}_{t_{i-1}}[N_1\|u_1-u_2\|_{\mathrm{PC}_\alpha} + N_2\big[\|v_1-v_2\|_{\mathrm{PC}_\beta}\big]\big)\left(\ln\frac{t}{t_k}\right)^{\alpha-1} \\
\le{}& (M_1+M_2+N_1+N_2)\big\|(u_1-u_2,v_1-v_2)\big\|\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&+ \frac{1}{\delta}\Bigg[(M_1+M_2+N_1+N_2)\big\|(u_1-u_2,v_1-v_2)\big\|\frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&+ \frac{(\ln\frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\Big(P_i\big\|(u_1-u_2,v_1-v_2)\big\| \\
&+ (M_1+M_2+N_1+N_2)\big\|(u_1-u_2,v_1-v_2)\big\| \\
&\times\frac{1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\frac{ds}{s}\Big)\Bigg]\left(\ln\frac{t}{t_k}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\Big(P_i\big\|(u_1-u_2,v_1-v_2)\big\| \\
&+ (M_1+M_2+N_1+N_2)\big\|(u_1-u_2,v_1-v_2)\big\|\frac{1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\frac{ds}{s}\Big)\left(\ln\frac{t}{t_k}\right)^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{M_1+M_2+N_1+N_2}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha}\bigl\|(u_1-u_2,v_1-v_2)\bigr\| \\
&\quad+\frac{1}{\delta}\Biggl[\frac{M_1+M_2+N_1+N_2}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha}\bigl\|(u_1-u_2,v_1-v_2)\bigr\| \\
&\quad+\frac{(\ln\frac{T}{a})^{\alpha-1}}{\Gamma(\alpha)}\left(\sum_{i=1}^{m}P_i+\frac{M_1+M_2+N_1+N_2}{\Gamma(\alpha)}\ln\frac{T}{a}\right)\bigl\|(u_1-u_2,v_1-v_2)\bigr\|\Biggr]\left(\ln\frac{T}{a}\right)^{\alpha-1} \\
&\quad+\frac{1}{\Gamma(\alpha)}\left(\sum_{i=1}^{k}P_i+\frac{M_1+M_2+N_1+N_2}{\Gamma(\alpha)}\ln\frac{T}{a}\right)\left(\ln\frac{T}{a}\right)^{\alpha-1}\bigl\|(u_1-u_2,v_1-v_2)\bigr\| \\
&\le\Biggl\{\frac{M_1+M_2+N_1+N_2}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha}\left[1+\frac{1}{\delta}\left(\ln\frac{T}{a}\right)^{\alpha-1}+\frac{1}{\delta\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\right] \\
&\quad+\frac{\sum_{i=1}^{m}P_i}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha-1}\left[1+\left(\ln\frac{T}{a}\right)^{\alpha-1}\right]\Biggr\}\bigl\|(u_1-u_2,v_1-v_2)\bigr\| \\
&=\kappa\bigl\|(u_1-u_2,v_1-v_2)\bigr\|.
\end{aligned}
\tag{3.2}
$$

Similarly, we derive

$$
\begin{aligned}
&\bigl|S_2(u_1,v_1)(t)-S_2(u_2,v_2)(t)\bigr| \\
&\quad\le\Biggl\{\frac{\overline{M}_1+\overline{M}_2+\overline{N}_1+\overline{N}_2}{\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta}\left[1+\frac{1}{\rho}\left(\ln\frac{T}{a}\right)^{\beta-1}+\frac{1}{\rho\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta-1}+\frac{1}{\Gamma(\beta)}\right] \\
&\quad\quad+\frac{\sum_{i=1}^{m}Q_i}{\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta-1}\left[1+\left(\ln\frac{T}{a}\right)^{\beta-1}\right]\Biggr\}\bigl\|(u_1-u_2,v_1-v_2)\bigr\| \\
&\quad=\varrho\bigl\|(u_1-u_2,v_1-v_2)\bigr\|.
\end{aligned}
\tag{3.3}
$$

According to (H_3), (3.2), and (3.3), we get

$$
\begin{aligned}
\bigl\|S(u_1,v_1)-S(u_2,v_2)\bigr\| &= \max\bigl\{\bigl\|S_1(u_1,v_1)-S_1(u_2,v_2)\bigr\|_{\mathrm{PC}_\alpha},\bigl\|S_2(u_1,v_1)-S_2(u_2,v_2)\bigr\|_{\mathrm{PC}_\beta}\bigr\} \\
&\le \max\{\kappa,\varrho\}\bigl\|(u_1,v_1)\bigr\| < \bigl\|(u_1,v_1)\bigr\|.
\end{aligned}
\tag{3.4}
$$

Therefore, (3.4) means that $S: X\to X$ defined by (3.1) is a contraction. According to Lemma 2.3, S has a unique fixed point $(u^*(t),v^*(t))\in X$, which is a pair of unique solutions of system (1.1). The proof of Theorem 3.1 is completed. □

Theorem 3.2 *Let $e,f,g,h\in C[a,T])$, I_k, $J_k\in C(\mathbb{R},\mathbb{R})$, $k=1,2,\dots,m$. Assume that there exist some positive constants L_1, L_2, $\overline{L}_1$, $\overline{L}_2$, O_k, and $\overline{O}_k$ $(k=1,2,\dots,m)$ such that*

(H_4) *$|e(t,u,v)|\le L_1$, $|g(t,u,v)|\le L_2$, $|f(t,u,v)|\le \overline{L}_1$, $|h(t,u,v)|\le\overline{L}_2$, $|I_k(u)|\le O_k$ and $|J_k(v)|\le\overline{O}_k$ $(k=1,2,\dots,m)$ for all $t\in(a,T]$, $u,v\in\mathbb{R}$.*

If (H_2) and (H_4) hold, then the Hadamard impulsive fractional differential coupled system (1.1) has at least a pair of solutions $(u^*(t),v^*(t))$.

Proof Define the operator $S:X\to X$ as (3.1). In order to apply the Leray–Schauder alternative theorem, we need first to prove that S is completely continuous. Indeed, in view of the continuities of e, g, f, h, I_k, and J_k, it is easy to know that T is continuous.

Now we show that the operator S is uniformly bounded. Let $r > 0$, $B_r = \{(u,v) \in X, \|(u,v)\| \le r\}$ be any bounded subset of X. For all $(u,v) \in B_r$, $t \in [a,T]$, it follows from (H_4) that

$$\begin{aligned}
&\bigl|S_1(u,v)(t)\bigr| \\
&= \Biggl| {}_HJ_{t_k}^{\alpha} g\bigl(t,u(t),v(t)\bigr) + {}_HJ_{t_k}^{\alpha} e\bigl(t,u(t),v(t)\bigr) + \frac{1}{\delta}\Biggl({}_HJ_{t_m}^{\alpha} g\bigl(T,u(T),v(T)\bigr) \\
&\quad + {}_HJ_{t_m}^{\alpha} e\bigl(T,u(T),v(T)\bigr) + \frac{(\ln\frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\bigl[I_i\bigl(u(t_i)\bigr) + {}_HJ_{t_{i-1}}^{1} g\bigl(t_i,u(t_i),v(t_i)\bigr) \\
&\quad + {}_HJ_{t_{i-1}}^{1} e\bigl(t_i,u(t_i),v(t_i)\bigr)\bigr]\Biggr)\left(\ln\frac{t}{t_k}\right)^{\alpha-1} + \frac{\Lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\bigl[I_i\bigl(u(t_i)\bigr) \\
&\quad + {}_HJ_{t_{i-1}}^{1} g\bigl(t_i,u(t_i),v(t_i)\bigr) + {}_HJ_{t_{i-1}}^{1} e\bigl(t_i,u(t_i),v(t_i)\bigr)\bigr]\left(\ln\frac{t}{t_k}\right)^{\alpha-1}\Biggr| \\
&\le {}_HJ_{t_k}^{\alpha}\bigl|g\bigl(t,u(t),v(t)\bigr)\bigr| + {}_HJ_{t_k}^{\alpha}\bigl|e\bigl(t,u(t),v(t)\bigr)\bigr| + \frac{1}{\delta}\Biggl({}_HJ_{t_m}^{\alpha}\bigl|g\bigl(T,u(T),v(T)\bigr)\bigr| \\
&\quad + {}_HJ_{t_m}^{\alpha}\bigl|e\bigl(T,u(T),v(T)\bigr)\bigr| + \frac{(\ln\frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\bigl[\bigl|I_i\bigl(u(t_i)\bigr)\bigr| + {}_HJ_{t_{i-1}}^{1}\bigl|g\bigl(t_i,u(t_i),v(t_i)\bigr)\bigr| \\
&\quad + {}_HJ_{t_{i-1}}^{1}\bigl|e\bigl(t_i,u(t_i),v(t_i)\bigr)\bigr|\bigr]\Biggr)\left(\ln\frac{t}{t_k}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\bigl[\bigl|I_i\bigl(u(t_i)\bigr)\bigr| \\
&\quad + {}_HJ_{t_{i-1}}^{1}\bigl|g\bigl(t_i,u(t_i),v(t_i)\bigr)\bigr| + {}_HJ_{t_{i-1}}^{1}\bigl|e\bigl(t_i,u(t_i),v(t_i)\bigr)\bigr|\bigr]\left(\ln\frac{t}{t_k}\right)^{\alpha-1} \\
&\le \frac{L_1+L_2}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} + \frac{1}{\delta}\left(\frac{L_1+L_2}{\Gamma(\alpha)}\int_{t_m}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}\frac{ds}{s} + \frac{(\ln\frac{T}{t_m})^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\biggl[O_i\right. \\
&\quad \left. + \frac{L_1+L_2}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\frac{ds}{s}\biggr]\right)\left(\ln\frac{t}{t_k}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\left[O_i + \frac{L_1+L_2}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\frac{ds}{s}\right]\left(\ln\frac{t}{t_k}\right)^{\alpha-1} \\
&\le \frac{L_1+L_2}{\Gamma(\alpha)}\int_{a}^{T}\left(\ln\frac{T}{a}\right)^{\alpha-1}\frac{ds}{s} + \frac{1}{\delta}\left(\frac{L_1+L_2}{\Gamma(\alpha)}\int_{a}^{T}\left(\ln\frac{T}{a}\right)^{\alpha-1}\frac{ds}{s}\right. \\
&\quad \left. + \frac{(\ln\frac{T}{a})^{\alpha-1}}{\Gamma(\alpha)}\left[\sum_{i=1}^{m}O_i + \frac{L_1+L_2}{\Gamma(\alpha)}\int_{a}^{T}\frac{ds}{s}\right]\right)\left(\ln\frac{T}{a}\right)^{\alpha-1} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left[\sum_{i=1}^{m}O_i + \frac{L_1+L_2}{\Gamma(\alpha)}\int_{a}^{T}\frac{ds}{s}\right]\left(\ln\frac{T}{a}\right)^{\alpha-1} \\
&= \frac{L_1+L_2}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha}\left[1 + \frac{1}{\delta}\left(\ln\frac{T}{a}\right)^{\alpha-1} + \frac{1}{\delta\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\right] \\
&\quad + \frac{\sum_{i=1}^{m}P_i}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha-1}\left[1 + \left(\ln\frac{T}{a}\right)^{\alpha-1}\right] \triangleq A.
\end{aligned} \tag{3.5}$$

Similarly, we have

$$\begin{aligned}\left|S_2(u,v)(t)\right| \le{}& \frac{\overline{L}_1+\overline{L}_2}{\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta}\left[1+\frac{1}{\rho}\left(\ln\frac{T}{a}\right)^{\beta-1}+\frac{1}{\rho\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta-1}+\frac{1}{\Gamma(\beta)}\right]\\&+\frac{\sum_{i=1}^{m}\overline{O}_i}{\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta-1}\left[1+\left(\ln\frac{T}{a}\right)^{\beta-1}\right]\triangleq B.\end{aligned}\tag{3.6}$$

From (3.5) and (3.6), we know that S is uniformly bounded.

Next, we show that the operator S is equicontinuous. In fact, for any $\tau_1,\tau_2\in[a,T]$, $\tau_1<\tau_2$, $\tau_2-\tau_1$ is small enough such that $\tau_1,\tau_2\in[t_k,t_{k+1}]$, $k=0,1,2,\dots,m$, we have

$$\begin{aligned}&\left|S_1(u,v)(\tau_2)-S_1(u,v)(\tau_1)\right|\\
&=\Big|{}_HJ_{t_k}^{\alpha}g\big(\tau_2,u(\tau_2),v(\tau_2)\big)-{}_HJ_{t_k}^{\alpha}g\big(\tau_1,u(\tau_1),v(\tau_1)\big)\\
&\quad+{}_HJ_{t_k}^{\alpha}e\big(\tau_2,u(\tau_2),v(\tau_2)\big)-{}_HJ_{t_k}^{\alpha}e\big(\tau_1,u(\tau_1),v(\tau_1)\big)\\
&\quad+c^*\left[\left(\ln\frac{\tau_2}{t_k}\right)^{\alpha-1}-\left(\ln\frac{\tau_1}{t_k}\right)^{\alpha-1}\right]+\frac{\Lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\Big[I_i\big(u(t_i)\big)+{}_HJ_{t_{i-1}}^{1}g\big(t_i,u(t_i),v(t_i)\big)\\
&\quad+{}_HJ_{t_{i-1}}^{1}e\big(t_i,u(t_i),v(t_i)\big)\Big]\left[\left(\ln\frac{\tau_2}{t_k}\right)^{\alpha-1}-\left(\ln\frac{\tau_1}{t_k}\right)^{\alpha-1}\right]\Big|\\
&=\Big|\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}g\big(s,u(s),v(s)\big)\frac{ds}{s}-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}g\big(s,u(s),v(s)\big)\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}e\big(s,u(s),v(s)\big)\frac{ds}{s}-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}e\big(s,u(s),v(s)\big)\frac{ds}{s}\\
&\quad+c^*\left[\left(\ln\frac{\tau_2}{t_k}\right)^{\alpha-1}-\left(\ln\frac{\tau_1}{t_k}\right)^{\alpha-1}\right]+\frac{\Lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\Big[I_i\big(u(t_i)\big)+{}_HJ_{t_{i-1}}^{1}g\big(t_i,u(t_i),v(t_i)\big)\\
&\quad+{}_HJ_{t_{i-1}}^{1}e\big(t_i,u(t_i),v(t_i)\big)\Big]\left[\left(\ln\frac{\tau_2}{t_k}\right)^{\alpha-1}-\left(\ln\frac{\tau_1}{t_k}\right)^{\alpha-1}\right]\Big|\\
&\le\frac{1}{\Gamma(\alpha)}\left\{\int_{t_k}^{\tau_1}\left[\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}-\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}\right]\left|g\big(s,u(s),v(s)\big)\right|\frac{ds}{s}+\int_{\tau_1}^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\right.\\
&\quad\times\left|g\big(s,u(s),v(s)\big)\right|\frac{ds}{s}+\int_{t_k}^{\tau_1}\left[\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}-\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}\right]\left|e\big(s,u(s),v(s)\big)\right|\frac{ds}{s}\\
&\quad\left.+\int_{\tau_1}^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\left|e\big(s,u(s),v(s)\big)\right|\frac{ds}{s}\right\}+\left|c^*\right|\left[\left(\ln\frac{\tau_2}{t_k}\right)^{\alpha-1}-\left(\ln\frac{\tau_1}{t_k}\right)^{\alpha-1}\right]\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\Big[\left|I_i\big(u(t_i)\big)\right|+{}_HJ_{t_{i-1}}^{1}\left|g\big(t_i,u(t_i),v(t_i)\big)\right|+{}_HJ_{t_{i-1}}^{1}\left|e\big(t_i,u(t_i),v(t_i)\big)\right|\Big]\\
&\quad\times\left[\left(\ln\frac{\tau_2}{t_k}\right)^{\alpha-1}-\left(\ln\frac{\tau_1}{t_k}\right)^{\alpha-1}\right]\\
&\le\frac{1}{\Gamma(\alpha)}\left\{L_1\int_{t_k}^{\tau_1}\left[\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}-\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}\right]\frac{ds}{s}+L_1\int_{\tau_1}^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\frac{ds}{s}\right.\end{aligned}$$

$$+ L_2 \int_{t_k}^{\tau_1} \left[\left(\ln \frac{\tau_2}{s} \right)^{\alpha-1} - \left(\ln \frac{\tau_1}{s} \right)^{\alpha-1} \right] \frac{ds}{s} + L_2 \int_{\tau_1}^{\tau_2} \left(\ln \frac{\tau_2}{s} \right)^{\alpha-1} \frac{ds}{s} \Bigg\}$$
$$+ |c^*| \left[\left(\ln \frac{\tau_2}{t_k} \right)^{\alpha-1} - \left(\ln \frac{\tau_1}{t_k} \right)^{\alpha-1} \right] + \frac{1}{\Gamma(\alpha)} \left[\sum_{i=1}^{m} O_i + \frac{L_1}{\Gamma(\alpha)} \int_a^{t_m} \frac{ds}{s} \right.$$
$$\left. + \frac{L_2}{\Gamma(\alpha)} \int_a^{t_m} \frac{ds}{s} \right] \left[\left(\ln \frac{\tau_2}{t_k} \right)^{\alpha-1} - \left(\ln \frac{\tau_1}{t_k} \right)^{\alpha-1} \right] \to 0, \quad \text{as } \tau_1 \to \tau_2. \tag{3.7}$$

We similarly get

$$\left| S_2(u,v)(\tau_2) - S_2(u,v)(\tau_1) \right| \to 0, \quad \text{as } \tau_1 \to \tau_2. \tag{3.8}$$

(3.7) and (3.8) mean that S is equicontinuous. By the Ascoli–Arzelá theorem, we know that S is completely continuous.

Finally, we prove that the set $\varepsilon(S) = \{(u,v) \in X | (u,v) = \lambda S(u,v), 0 < \lambda < 1\}$ is bounded. Let $(u,v) \in \varepsilon(S)$, then $(u,v) = \lambda S(u,v)$, for any $t \in [a,T]$, we have

$$u(t) = \lambda S_1(u,v)(t), \qquad v(t) = \lambda S_2(u,v)(t).$$

For $t \in (t_k, t_{k+1}]$, $k = 0,1,2,\ldots,m$, it follows from (3.5) and (3.6) that

$$\left| u(t) \right| = \left| \lambda S_1(u,v)(t) \right| = \lambda \left| S_1(u,v)(t) \right| \leq \lambda A \tag{3.9}$$

and

$$\left| v(t) \right| = \left| \lambda S_2(u,v)(t) \right| = \lambda \left| S_2(u,v)(t) \right| \leq \lambda B. \tag{3.10}$$

(3.9) and (3.10) implicate that $\varepsilon(S)$ is bounded for any $t \in [a,T]$. In view of Lemma 2.4, the operator S defined by (3.1) has at least one fixed point. Hence, the Hadamard impulsive fractional differential coupled system (1.1) has at least a pair of solutions $(u^*(t), v^*(t))$. The proof is completed. □

4 Illustrative examples

Consider the nonlinear Hadamard fractional integro-differential coupled system with impulses as follows:

$$\begin{cases} {}_{\mathrm{RLH}}D_{t_k}^{\frac{3}{2}}[u(t) - {}_H J_{t_k}^{\frac{3}{2}} e(t,u(t),v(t))] = g(t,u(t),v(t)), & t \in J = [1,e], t \neq t_1 = \frac{4}{3}, \\ {}_{\mathrm{RLH}}D_{t_k}^{\frac{5}{4}}[v(t) - {}_H J_{t_k}^{\frac{5}{4}} f(t,u(t),v(t))] = h(t,u(t),v(t)), & t \in J = [1,e], t \neq t_1 = \frac{4}{3}, \\ {}_{\mathrm{RLH}}D_{t_1}^{\frac{1}{2}} u(t_1^+) - {}_{\mathrm{RLH}}D_{t_1}^{\frac{1}{2}} u(t_1^-) = I_1(u(t_1)), \\ {}_{\mathrm{RLH}}D_{t_1}^{\frac{1}{4}} v(t_1^+) - {}_{\mathrm{RLH}}D_{t_1}^{\frac{1}{4}} v(t_1^-) = J_1(v(t_1)), \\ 3 {}_H D_1^{\frac{1}{2}} u(1) = u(e), \qquad 4 {}_H D_1^{\frac{1}{4}} v(1) = v(e). \end{cases} \tag{4.1}$$

Case 1 Take $g(t,u,v) = \frac{(u^2+v^3)\cos 2t}{180}$, $e(t,u,v) = \frac{(u+v)\sin t}{50}$, $f(t,u,v) = \frac{e^{-t}(\sqrt[3]{u}+\sqrt[5]{v})}{150}$, $h(t,u,v) = \frac{\sin u \cos v \arcsin t}{20\pi}$, $I_1(u) = \frac{u^2}{10}$, $J_1(v) = \frac{u^4}{20}$. Obviously, $e,g,f,h \in C[1,e]$, $I_1, J_1 \in C(\mathbb{R},\mathbb{R})$. By the simple calculation, we have

$$\left| g(t,u_1,v_1) - g(t,u_2,v_2) \right| \leq \frac{1}{90} |u_1 - u_2| + \frac{1}{60} |v_1 - v_2|,$$

$$\begin{aligned}
&\big|e(t,u_1,v_1)-e(t,u_2,v_2)\big| \le \frac{1}{50}|u_1-u_2| + \frac{1}{50}|v_1-v_2|,\\
&\big|f(t,u_1,v_1)-f(t,u_2,v_2)\big| \le \frac{1}{50}|u_1-u_2| + \frac{1}{30}|v_1-v_2|,\\
&\big|h(t,u_1,v_1)-h(t,u_2,v_2)\big| \le \frac{1}{10}|u_1-u_2| + \frac{1}{10}|v_1-v_2|,\\
&\big|I_1(u_1)-e(u_2)\big| \le \frac{1}{5}|u_1-u_2|, \qquad \big|J_1(v_1)-J_1(v_2)\big| \le \frac{1}{5}|v_1-v_2|,
\end{aligned}$$

that is, $M_1 = \frac{1}{90}, M_2 = \frac{1}{60}, N_1 = N_2 = \frac{1}{50}, \overline{M}_1 = \frac{1}{50}, \overline{M}_2 = \frac{1}{30}, \overline{N}_1 = \overline{N}_1 = \frac{1}{10}, P_1 = Q_1 = \frac{1}{5}$. Thus, we obtain

$$\begin{aligned}
&\delta = c\Gamma(\alpha) - \left(\ln\frac{T}{t_1}\right)^{\alpha-1} \approx 1.8147 > 0, \qquad \rho = d\Gamma(\beta) - \left(\ln\frac{T}{t_1}\right)^{\beta-1} \approx 2.7069 > 0,\\
&\kappa = \frac{M_1+M_2+N_1+N_2}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha}\left[1+\frac{1}{\delta}\left(\ln\frac{T}{a}\right)^{\alpha-1} + \frac{1}{\delta\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\right]\\
&\quad + \frac{\sum_{i=1}^m P_i}{\Gamma(\alpha)}\left(\ln\frac{T}{a}\right)^{\alpha-1}\left[1+\left(\ln\frac{T}{a}\right)^{\alpha-1}\right] \approx 0.7038 < 1,\\
&\varrho = \frac{\overline{M}_1+\overline{M}_2+\overline{N}_1+\overline{N}_2}{\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta}\left[1+\frac{1}{\rho}\left(\ln\frac{T}{a}\right)^{\beta-1} + \frac{1}{\rho\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta-1} + \frac{1}{\Gamma(\beta)}\right]\\
&\quad + \frac{\sum_{i=1}^m Q_i}{\Gamma(\beta)}\left(\ln\frac{T}{a}\right)^{\beta-1}\left[1+\left(\ln\frac{T}{a}\right)^{\beta-1}\right] \approx 0.7379 < 1.
\end{aligned}$$

Therefore, conditions (H_1)–(H_3) of Theorem 3.1 hold. Then (4.1) has a pair of unique solutions $(u^*(t), v^*(t)) \in \mathrm{PC}_{\frac{3}{2}}[1,e] \times \mathrm{PC}_{\frac{5}{4}}[1,e]$.

Case 2 Take $e(t,u,v) = g(t,u,v) = f(t,u,v) = h(t,u,v) = \sin\sqrt[2]{3}t + e^{-(u+v)^2} + \arctan(tuv)$, $I_1(u) = \arccos u^2$, $J_1(v) = \frac{1}{1+v^2}$. Obviously, $e,g,f,h \in C[1,e]$, $I_1, J_1 \in C(\mathbb{R},\mathbb{R})$. $|e(t,u,v)| = |g(t,u,v)| = |f(t,u,v)| = |h(t,u,v)| = |\sin\sqrt{3}t + e^{-(u+v)^2} + \arctan(tuv)| \le \frac{\pi}{2} + 1 + \frac{\pi}{2} = \pi + 1$, $|I_1(u)| = |\arccos u^2| \le \pi$, $|J_1(v)| = |\frac{1}{1+v^2}| \le 1$. Thus, conditions (H_1) and (H_4) hold. According to Theorem 3.2, we know that (4.1) has at least a pair of solutions $(u^*(t), v^*(t))$.

5 Conclusions

In describing some phenomena and processes of many fields such as physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, capacitor theory, electrical circuits, biology, control theory, fitting of experimental data, and so on, the fractional differential equation is better and more accurate than the integer-order differential equations. Therefore, the study of fractional differential equations has attracted the eyes of many scholars. Good papers involving the dynamics of the fractional differential equation are emerging in large numbers. However, it was noticed that most of these works are based on Riemann–Liouville and Caputo fractional derivatives. In fact, another kind of fractional derivatives was introduced by Hadamard in 1892. It differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition contains a logarithmic function of arbitrary exponent. Relatively speaking, this fractional differential equation with Hadamard derivatives is still studied less than that of Riemann–Liouville and Caputo. So it is worth studying the Hadamard fractional differential equations. In this paper, we consider the boundary value problem for a class of fractional integro-differential coupled systems with Hadamard fractional calculus and impulses. By means of the Banach contraction principle and Leray–Schauder alternative theorem, some new sufficient criteria are established to guarantee the existence and uniqueness of solutions.

Acknowledgements
The authors thank the referees for a number of suggestions which have improved many aspects of this article.

Funding
This work was supported by the National Natural Sciences Foundation of Peoples Republic of China under Grant (Nos. 11161025, 11661047).

Availability of data and materials
Not applicable.

Competing interests
The authors declare to have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 15 January 2019 Accepted: 31 May 2019 Published online: 10 June 2019

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