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A linear, stabilized, non-spatial iterative, partitioned time stepping method for the nonlinear Navier–Stokes/Navier–Stokes interaction model

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Abstract

In this paper, a linear, stabilized, non-spatial iterative, partitioned time stepping method is developed and studied for the nonlinear Navier–Stokes/Navier–Stokes interaction. A backward Euler scheme is utilized for the temporal discretization while a linear Oseen scheme for the trilinear term is used to affect the spatial discretization approximated by the equal order elements. Therefore, we only solve a linear Stokes problem without spatial iterative per time step for each individual domain. Then, the method exploits properties of the Navier–Stokes/Navier–Stokes system to establish the stability and convergence by rigorous analysis. Finally, numerical experiments are presented to show the performance of the proposed method.

MSC: 35Q10; 65N30; 76D05

Keywords: Partitioned time stepping methods; Fluid–fluid interface; Navier–Stokes equations; Convergence; Numerical experiments

1 Introduction

The Navier–Stokes equations are useful because they describe the physics of many realistic problems of academic and industrial interest. They may be used to model weather, ocean currents, water flow, and many other phenomena. Many important applications need an accurate solution of multi-domain, multi-physics coupling of one fluid with another (e.g., the Navier–Stokes with the Navier–Stokes problems) [3, 4, 31, 32]. The uncoupled methods for two fluids are coupled through their shared interface by a rigid-lid coupling condition, i.e., no penetration and a slip with a friction condition allowing a jump in the tangential velocities across the shared interface [12]. Physics-based uncoupled methods are different from the traditional ones in the sense that they focus on decomposing different physical domains by directly using the given physical interface conditions, which is the key idea of the method proposed in this paper. Moreover, these methods allow existing highly optimized codes for each subproblem to be used in parallel as black boxes at each time step to solve the coupled problem.

Efficient stabilized finite element methods have been widely used in scientific computation to achieve high accuracy for the Navier–Stokes equations approximated by the equal order elements in practice. While these methods have been shown to be very successful, the theory ensuring their convergence and advantages for a coupled problem is still under development. Recently, some results have been obtained for partitioned time stepping methods for the fluid–fluid interaction by using the finite element methods [12, 13, 25]. In this paper, we shall follow the state-of-the-art convergence theory by using the geometric averaging at three time levels of the slip velocity at the interface to compute a friction coefficient and further establish stability and convergence of the presented method for the coupled fluid–fluid model. We stress that the extension of the general convergence theory to the partitioned time stepping method for the Navier–Stokes/Navier–Stokes interaction is derived from that in [12]. Here, in order to ensure the balance between the spatial and temporal computing allocation, an unconditional stable backward Euler scheme is utilized for the temporal discretization while the linear Oseen scheme is applied for the trilinear term with a non-spatial iterative correction per time step. The method presented results in a better coefficient matrix of the form $(a_{ij})_{N\times N} = \nu(\nabla\phi_i, \nabla\phi_j) + ((\mathbb{C}\cdot\nabla)\phi_i, \phi_j)$, improving the model presented with small viscosity [16, 17]. However, the difficulty for the numerical analysis arises from the trilinear term and the whole system for the presented discrete finite element scheme.

The rest of paper is organized as follows. In Sect. 2, we introduce the fluid–fluid model using two Navier–Stokes problems. In Sect. 3, the stability of the Navier–Stokes/Navier–Stokes interaction model is analyzed. In Sect. 4, the convergence of the presented method is analyzed. Finally, we present several numerical examples to illustrate the features of the proposed method in Sect. 5.

2 Preliminary

A coupled Navier–Stokes/Navier–Stokes problem is stated as follows:

$$u_{i,t} - \nu_i\Delta u_i + u_i\cdot\nabla u_i + \nabla p_i = f_i \quad \text{in } \Omega_i, \tag{1}$$

$$-\nu_i n_i\cdot\nabla u_i\cdot\tau = k|u_i - u_j|(u_i - u_j)\cdot\tau \quad \text{on } I, \tag{2}$$

$$u_i\cdot n_i = 0 \quad \text{on } I, \tag{3}$$

$$\nabla\cdot u_i = 0 \quad \text{in } \Omega_i, \tag{4}$$

$$u_i(x,0) = u_i^0(x) \quad \text{in } \Omega_i, \tag{5}$$

$$u_i = 0 \quad \text{on } \Gamma_i = \partial\Omega_i \setminus I. \tag{6}$$

Here, $i, j = 1, 2$, $i \neq j$. Let the domain $\Omega = \Omega_1 \cup \Omega_2$ consist of two subdomains Ω_1 and Ω_2 of R^d, $d = 2, 3$, with the outward unit normal vectors n_1 and n_2, respectively, coupled across the interface I. The viscosity $\nu_i > 0$, the body force $f_i : [0, T] \to H^1(\Omega_i)$, and the parameter $k \in R$ are given, $i = 1, 2$. Also, $u_i : \Omega_i \times [0, T] \to R^d$ and $p_i : \Omega_i \times [0, T] \to R$ represent velocity and pressure on the subdomains Ω_i, respectively, $i = 1, 2$.

For the mathematical problem (1)–(6), the following Hilbert spaces are introduced [1]:

$$X_i = \{v_i \in [H^1(\Omega_i)]^d : v_i = 0 \text{ on } \Gamma_i \text{ and } v_i\cdot n_i = 0 \text{ on } I\},$$

$$M_i = \left\{q_i \in L^2(\Omega_i) : \int_{\Omega_i} q_i\,dx = 0\right\}.$$

Multiplying (1) by $v_i \in X_i$ and (4) by $q_i \in M_i$, integrating and applying the divergence theorem, the above coupled problem is equivalent to finding $(u_i, p_i) \in (X_i, M_i)$ such that

$$\begin{aligned} &(u_{i,t}, v_i) + a(u_i, v_i) - d(v_i, p_i) + b(u_i, u_i, v_i) + \kappa \int_I \big|[\mathbf{u}]\big|[\mathbf{u}] v_i \, ds = (f_i, v_i), \\ &d(u_i, q_i) = 0, \quad \forall (v_i, q_i) \in (X_i, M_i), \end{aligned} \tag{7}$$

where $[\cdot]$ denotes the jump of the indicated quantity across the interface I: $[\mathbf{u}] = u_i - u_j$ and

$$(u_{i,t}, v_i) = \int_{\Omega_i} \frac{\partial u_i}{\partial t} v_i \, dx, \quad i = 1, 2.$$

The continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are defined on $X_i \times X_i$ and $X_i \times M_i$, respectively, by

$$\begin{aligned} &a(u_i, v_i) = \nu_i (\nabla u_i, \nabla v_i), \quad u_i, v_i \in X_i, \\ &d(v_i, q_i) = -(v_i, \nabla q_i) = (\operatorname{div} v_i, q_i), \quad v_i \in X_i, q_i \in M_i. \end{aligned}$$

These bilinear terms satisfy the following continuity and *inf–sup* properties:

$$\begin{aligned} &\big|a(u_i, v_i)\big| \le \nu \|\nabla u_i\|_0 \|\nabla v_i\|_0, \\ &\big|d(v_i, p_i)\big| \le C \|\nabla v_i\|_0 \|p_i\|_0, \\ &\sup_{v_i \in X_i} \frac{|d(v_i, q_i)|}{\|\nabla v_i\|_0} \ge \beta \|q_i\|_0 \qquad \forall q_i \in M_i, \beta > 0, \end{aligned} \tag{8}$$

where the positive constants C and β only depend on Ω. Similarly, by using the divergence theorem, (3) and (6), the trilinear term $b(\cdot,\cdot,\cdot)$ can be defined as follows [36]:

$$\begin{aligned} b(u_i, v_i, w_i) &= \frac{1}{2}(u_i \cdot \nabla v_i, w_i) + \frac{1}{2}\big((\operatorname{div} u_i) v_i, w_i\big) \\ &= \frac{1}{2}(u_i \cdot \nabla v_i, w_i) - \frac{1}{2}(u_i \cdot \nabla w_i, v_i), \quad u_i, v_i, w_i \in X_i. \end{aligned} \tag{9}$$

Obviously, the trilinear term $b(\cdot,\cdot,\cdot)$ satisfies the following skew-symmetry property [36]:

$$b(u_i, v_i, w_i) = -b(u_i, w_i, v_i).$$

A realistic model would contain many more complex terms. Here, we mainly focus on an algorithmic issue so we assume that the solution of (1)–(6) to be approximated is a strong solution. Moreover, the energetic stability of the monolithic problem is valid:

$$\sum_{i=1}^{2} \left(\frac{1}{2} \frac{d}{dt} \|u_i\|_0^2 + \nu_i \|\nabla u_i\|_0^2 \right) + \kappa \int_I \big|[\mathbf{u}]\big|^3 \, ds = \sum_{i=1}^{2} (f_i, v_i). \tag{10}$$

3 Stabilizations for Galerkin approximations

Given a respective shape-regular and conforming triangulation $\mathcal{T}_{h_i}$ of Ω_i, the finite element method is to solve (7) in two pairs of finite dimensional spaces $(X_i^h, M_i^h) \subset (X_i, M_i)$ [9, 11, 15, 36].

Stabilization of the Stokes' problem using local pressure projections dates back to the papers of Silvester [33, 34], Brecker and Braack [2], Brezzi and Fortin [6], Brezzi and Pitkiranta [7], Dohrmann and Bochev [14], Connors [23] and [8]. They provide a wide theoretical framework for these methods. The aim of the section is to give an elementary application in the spirit of these papers for a class of pressure projection method with equal order distribution for both velocity and pressure, which are computationally convenient and efficient in a parallel and multigrid context. Then, the unstable velocity–pressure pairs of the equal-order finite elements are defined as follows [22, 26–29, 38]:

$$X_i^h = \{v_h \in X : v_h|_K \in [R_r(K)]^d, \forall K \in K_h\},$$
$$M_i^h = \{q_h \in M : q_h|_K \in R_r(K), \forall K \in K_h\}, \quad r = 1, 2.$$

In order to analyze the stabilzation of the Galerkin approximations for the Navier–Stokes/Navier–Stokes interaction, we assume that π_i^h denotes the interpolation operator from the richer space M_h into the smaller space $\bar{M}_h \subset M_h$ such that $X_h \times \bar{M}_h$ satisfies the inf–sup condition and $\operatorname{div} X_h \subset M_h$.

Lemma 3.1 *It holds that*

$$\sup_{v_i^h \in X_i^h} \frac{d(v_i^h, q_i^h)}{\|\nabla v_i^h\|_0} + G^{1/2}(q_i^h, q_i^h) \geq \beta_0 \|q_i^h\|_0, \quad q_i^h \in M_i^h,$$

where the positive constant β_0 only depends on Ω and the stabilized term $G(\cdot,\cdot)$ is defined as follows:

$$G(q_i^h, q_i^h) \sim \begin{cases} \|q_i^h - \pi_i q_i^h\|_0, & r = 1, \\ \|h\nabla(q_i^h - \pi_i q_i^h)\|_0, & r = 2. \end{cases} \tag{11}$$

Proof For a bounded Lipschitz connected domain Ω and for any $p_i^h \in L^2(\Omega)$, there exist a positive constant $C_0 > 0$ and $v_i \in [H^1(\Omega_i)]^d$ satisfying

$$\operatorname{div} v_i = p_i^h$$

such that

$$\|v_i\|_1 \leq C_0 \|p_i^h\|_0$$

and

$$\|p_i^h\|_0^2 = d(v_i, p_i^h).$$

Then, there exists a linear operator $\tilde{\pi}_i^h : [H^1(\Omega_i)]^d \to X_h$ such that the orthogonality relation holds [5, 10]:

$$(v_i - \tilde{\pi}_i^h v_i, q_h) = 0, \quad \forall q_h \in M_h, \tag{12}$$

$$\|\nabla \tilde{\pi}_i^h v_i\|_0 \le C\|v_i\|_1 \le C_1 \|p_i^h\|_0, \tag{13}$$

where C_1 only depends on Ω. Noting that $\operatorname{div} \tilde{\pi}_h v \in M_h$ and thus $d(\tilde{\pi}_h v, p_i^h - \pi_h p_i^h) = 0$, and using the definition of $\tilde{\pi}_h$, we obtain

$$\begin{aligned}
\|p_i^h\|_0^2 &= d(v_i, p_i^h) \\
&= d(v_i, p_i^h - \pi_i^h p_i^h) + d(v_i, \pi_i^h p_i^h) \\
&= d(v_i - \tilde{\pi}_h v_i, p_i^h - \pi_i^h p_i^h) + d(\tilde{\pi}_h v_i, \pi_i^h p_i^h) \\
&= d(v_i - \tilde{\pi}_h v_i, p_i^h - \pi_i^h p_i^h) + d(\tilde{\pi}_i^h v_i, p_i^h),
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
d(v_i - \tilde{\pi}_i^h v_i, p_i^h - \pi_i^h p_i^h) &\le C\|p_i^h - \pi_i^h p_i^h\|_0 \|v_i\|_1 \\
&\le G^{1/2}(p_i^h, p_i^h)\|p_i^h\|_0, \quad r = 1,
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
d(v_i - \tilde{\pi}_i^h v_i, p_i^h - \pi_i^h p_i^h) &= -(\nabla(p_i^h - \pi_i^h p_i^h), v_i - \tilde{\pi}_i^h v_i) \\
&\le Ch\|\nabla(p_i^h - \pi_i^h p_i^h)\|_0 \|v_i\|_1 \\
&\le G^{1/2}(p_i^h, p_i^h)\|p_i^h\|_0, \quad r = 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{v_i^h \in X_i^h} \frac{d(v_i^h, q_i^h)}{\|\nabla v_i^h\|_0} + G^{1/2}(p_i^h, p_i^h) &\ge \frac{d(\tilde{\pi}_i v_i, q_i^h)}{\|\nabla \tilde{\pi}_i v_i\|_0} + G^{1/2}(p_i^h, p_i^h) \\
&\ge \beta_0 \|p_i^h\|_0^2.
\end{aligned}$$

For more details, the result related to the well-posedness of the Navier–Stokes/Navier–Stokes interaction can be found in [12, 13, 18, 20, 30, 37]. □

4 Stability

In this section, we are now in a position to state a discrete finite element scheme. We let $(u_i^n, p_i^n) =: (u_{i,h}^n, p_{i,h}^n)$, $i = 1, 2$, denote a discrete approximation to $u_i(t^n)$, where the discrete time t^n is calculated from the uniform time step size $\tau = T/N$ by $t^n = n\tau$, $n = 0, 1, \ldots, N$.

From the point of view of implementation, the method presented consists of several subroutines for solving the nonlinear fluid–fluid interaction. First, the first guess u_i^0 can be defined by (5). Then, we solve the Stokes equations approximated by the lower order finite element pairs to obtain the second initial datum u_i^1. Furthermore, the following solution

u_i^{n+1}, $n = 0, 1, 2, \ldots$, can be obtained by the following equations (16) and (17). For the numerical treatment of the time derivative term, we use the fully discrete backward Euler approximation. As for the partitioned scheme, we apply the Oseen scheme with a non-spatial iterative correction to simplify the trilinear term per time step and further obtain a better stiffness matrix. Especially, recalling the standard geometric averaging of the jump in [12, 13], we replace the term $u_j^{n+1}|u_i^{n+1} - u_j^{n+1}|$ by $|u_i^n - u_j^n|u_i^{n+1}$ and $u_j^n|u_i^n - u_j^n|^{1/2}|u_i^{n-1} - u_j^{n-1}|^{1/2}$ in order to decouple the fluid–fluid interaction, which is also a key idea to obtain the unconditionally stable partitioning.

The linear, stabilized, non-spatial iterative, partitioned time stepping method is defined as follows:

Step I. *Find* $(u_i^1, p_i^1) \in X_i^h \times M_i^h$ *satisfying the following* **Stokes** *equations:*

$$a(u_i^1, v_i) - d(v_i, p_i^1) + d(u_i^1, q_i) + G(p_i^1, q_i) = (f, v_i) \quad \forall (v_i, q_i) \in X_i^h \times M_i^h.$$

Moreover, set the iterative step $m = 1, 2, \ldots$, the error of two successive solutions

$$e_i^m = \sqrt{\left(u_i^m - u_i^{m-1}\right)^2 + \left(p_i^m - p_i^{m-1}\right)^2} < \varepsilon$$

with a sufficiently small iterative tolerance $\varepsilon > 0$.

Step II. *Solve the Navier–Stokes/Navier–Stokes interaction: Given* $\tau > 0, f_i \in [H^{-1}(\Omega_i)]^d$ $(i = 1, 2)$, *find* $(u_1^{n+1}, p_1^{n+1}) \in X_1^h \times M_1^h$ *such that*

$$\begin{aligned}
&\frac{(u_1^{n+1} - u_1^n, v_1)}{\tau} + a\left(u_1^{n+1}, v_1\right) - d\left(v_1, p_1^{n+1}\right) + d\left(u_1^{n+1}, q_1\right) + G\left(p_1^{n+1}, q_1\right) \\
&\quad + b_1\left(u_1^n, u_1^{n+1}, v_1\right) + k\int_I \left|\left[\mathbf{u}^n\right]\right| u_1^{n+1} v_1 \, ds - k\int_I \left|\left[\mathbf{u}^n\right]\right|^{1/2}\left|\left[\mathbf{u}^{n-1}\right]\right|^{1/2} u_2^n \cdot v_1 \, ds \\
&= \left(f_1\left(t^{n+1}\right), v_1\right), \quad \forall (v_1, q_1) \in X_1^h \times M_1^h,
\end{aligned} \tag{16}$$

and $(u_2^{n+1}, p_2^{n+1}) \in X_2^h \times M_2^h$ *such that*

$$\begin{aligned}
&\frac{(u_2^{n+1} - u_2^n, v_2)}{\tau} + a\left(u_2^{n+1}, v_2\right) - d\left(v_2, p_2^{n+1}\right) + d\left(u_2^{n+1}, q_2\right) + G\left(p_2^{n+1}, q_2\right) \\
&\quad + b_2\left(u_2^n, u_2^{n+1}, v_2\right) + k\int_I \left|\left[\mathbf{u}^n\right]\right| u_2^{n+1} v_2 \, ds - k\int_I \left|\left[\mathbf{u}^n\right]\right|^{1/2}\left|\left[\mathbf{u}^{n-1}\right]\right|^{1/2} u_1^n \cdot v_2 \, ds \\
&= \left(f_2\left(t^{n+1}\right), v_2\right) \quad \forall (v_2, q_2) \in X_2^h \times M_2^h.
\end{aligned} \tag{17}$$

This, of course, dictates that the overall structure of the linear, stablized, non-spatial iterative, partition time step method will be much the same as for a standard finite element method, as described in [12]. The key point of the presented method is to use a linear, non-spatial iterative, partitioned time stepping method for the nonlinear Navier–Stokes/Navier–Stokes interaction model.

Routine:

$[u_h^0, p_h^0]$ = **Stokes**$(\mathcal{T}_h, f)$;

for $i = 0, 1, 2, \ldots, T/dt$;

while $(e_m > \varepsilon)$ **do**

$(u_h^m, p_h^m) \rightarrow (u_h^{m-1}, p_h^{m-1})$;

$[u_h^m, p_h^m] = \mathbf{LNS}(\mathcal{T}_h, u_h^{m-1}, f)$;
end while
end for

In this section, we aim to establish a result concerning the unconditional stability of the scheme (16)–(17).

Lemma 4.1 *Let u_i^n, $i = 1, 2$, $n = 1, 2, \ldots, m$, be the solutions of equations (16) and (17). Then we have the following energy inequality*:

$$\begin{aligned}
&\|\mathbf{u}^{m+1}\|_0^2 + \sum_{n=1}^{m}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_0^2 + \tau\sum_{n=1}^{m}\left(\nu_1\|\nabla u_1^{n+1}\|_0^2 + \nu_2\|\nabla u_2^{n+1}\|_0^2\right) \\
&\quad + \kappa\tau\int_I\left|[\mathbf{u}^n]\right|\left(\left|u_1^{n+1}\right|^2 + \left|u_2^{n+1}\right|^2\right)ds \\
&\le \|\mathbf{u}^1\|_0^2 + \kappa\tau\int_I\left|[\mathbf{u}^0]\right|\left(\left|u_1^1\right|^2 + \left|u_2^1\right|^2\right)ds \\
&\quad + \sum_{n=1}^{m}\left(\frac{\tau}{\nu_1}\|f_1(t^{n+1})\|_0^2 + \frac{\tau}{\nu_2}\|f_2(t^{n+1})\|_{-1}^2\right),
\end{aligned} \tag{18}$$

where $u_i^n = (u_1^n, u_2^n)$ with the norm $\|\mathbf{u}^n\|_0 = (\sum_{i=1}^2 \|u_i^n\|_0^2)^{1/2}$.

Proof Noting that

$$b\left(u_i^n, u_i^{n+1}, u_i^{n+1}\right) = 0,$$

we start by testing (16) and (17) with $(v_i, q_i) = 2(\tau u_i^{n+1}, p_i^{n+1})$, respectively, to obtain

$$\begin{aligned}
&2\left(u_i^{n+1} - u_i^n, u_i^{n+1}\right) + 2\nu_i\tau\|\nabla u_i^{n+1}\|_0^2 + 2G\left(p_i^{n+1}, p_i^{n+1}\right) \\
&\quad + 2\kappa\tau\left(\int_I\left|u_i^{n+1}\right|^2\left|[\mathbf{u}^n]\right|ds - \int_I\left|[\mathbf{u}^n]\right|^{1/2}\left|[\mathbf{u}^{n-1}]\right|^{1/2}u_{i+1}^n\cdot u_i^{n+1}\,ds\right) \\
&= 2\tau\left(f_i\left(t^{n+1}\right), u_i^{n+1}\right) \quad \forall (v_i, q_i) \in X_i^h \times M_i^h, i = 1, 2,
\end{aligned} \tag{19}$$

where $u_3 = u_1$ when the index is out of bounds. Using the identity

$$2(a - b, a) = a^2 + (a - b)^2 - b^2, \tag{20}$$

and combing with (19) with $i = 1, 2$, we obtain the following result:

$$\begin{aligned}
&\|\mathbf{u}^{n+1}\|_0^2 - \|\mathbf{u}^n\|_0^2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_0^2 + 2\nu_1\tau\|\nabla u_1^{n+1}\|_0^2 + 2\nu_2\tau\|\nabla u_2^{n+1}\|_0^2 \\
&\quad + 2\kappa\tau\sum_{i=1}^{2}\left(\int_I\left|u_i^{n+1}\right|^2\left|[\mathbf{u}^n]\right|ds - \int_I\left|[\mathbf{u}^n]\right|^{1/2}\left|[\mathbf{u}^{n-1}]\right|^{1/2}u_{i+1}^n\cdot u_i^{n+1}\,ds\right) \\
&\quad + 2G\left(p_i^{n+1}, p_i^{n+1}\right) \\
&= 2\tau\sum_{i=1}^{2}\left(f_i\left(t^{n+1}\right), u_i^{n+1}\right),
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&\kappa\left(\int_I |u_i^{n+1}|^2 |[\mathbf{u}^n]|\,ds - \int_I |[\mathbf{u}^n]|^{1/2} |[\mathbf{u}^{n-1}]|^{1/2} u_{i+1}^n \cdot u_i^{n+1}\,ds\right) \\
&\quad = \frac{\kappa}{2}\int_I |u_i^{n+1}|^2 |[\mathbf{u}^n]|\,ds - \frac{\kappa}{2}\int_I |u_{i+1}^n|^2 |[\mathbf{u}^{n-1}]|\,ds \\
&\qquad + \frac{\kappa}{2}\int_I \bigl|u_i^{n+1}|[\mathbf{u}^n]|^{1/2} - u_{i+1}^n |[\mathbf{u}^{n-1}]|^{1/2}\bigr|^2\,ds, \\
&2\tau\left|\sum_{i=1}^2 (f_i(t^{n+1}), u_i^{n+1})\right| \\
&\quad \le 2\tau\gamma \sum_{i=1}^2 \|f_i(t^{n+1})\|_0 \|\nabla u_i^{n+1}\|_0 \\
&\quad \le \sum_{i=1}^2 \left(\tau\nu_i \|\nabla u_i^{n+1}\|_0^2 + \frac{\tau\gamma^2}{\nu_i}\|f_i(t^{n+1})\|_0^2\right),
\end{aligned} \tag{22}$$

where the positive constant is derived from the Poincaré inequality. Then, substituting these into (21), we infer that

$$\begin{aligned}
&\|\mathbf{u}^{n+1}\|_0^2 - \|\mathbf{u}^n\|_0^2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_0^2 + \nu_1\tau\|\nabla u_1^{n+1}\|_0^2 + \nu_2\tau\|\nabla u_2^{n+1}\|_0^2 \\
&\qquad + \kappa\tau\int_I \bigl\{|u_1^{n+1}|^2 + |u_2^{n+1}|^2\bigr\}|[\mathbf{u}^n]|\,ds - \kappa\tau\int_I \bigl\{|u_1^n|^2 + |u_2^n|^2\bigr\}|[\mathbf{u}^{n-1}]|\,ds \\
&\qquad + \kappa\tau\sum_{i=1}^2 \bigl|u_i^{n+1}|[\mathbf{u}^n]|^{1/2} - u_{i+1}^n|[\mathbf{u}^{n-1}]|^{1/2}\bigr|^2\,ds \\
&\quad \le \tau\sum_{i=1}^2 \frac{\|f_i(t^{n+1})\|_0^2}{\nu_i}.
\end{aligned} \tag{23}$$

Summing over $n = 1, 2, \ldots, m$ yields the desired result. □

5 Convergence

In this section, we consider the convergence of the presented method for the Navier–Stokes/Navier–Stokes interaction. First, we provide the discrete Gronwall's inequality [21, 24], which will be useful in the subsequent analysis.

Let a^{n+1}, b^{n+1}, c^{n+1}, d^{n+1} and D^{n+1}, $n = 0, 1, 2, \ldots, m$, be five nonnegative sequences satisfying

$$a^{m+1} + \sum_{n=0}^m b^{n+1} + \tau\sum_{n=0}^m c^{n+1} \le C_1 + C_2\tau\sum_{n=0}^m D^{n+1}a^{n+1} + C_3\tau\sum_{n=0}^m d^{n+1}. \tag{24}$$

Then we have the following result:

$$a^{m+1} + \sum_{n=0}^m b^{n+1} + \tau\sum_{n=0}^m c^{n+1} \le \exp\left(C_2\tau\sum_{n=0}^m \sigma^{n+1}\right)\left(C_1 + C_3\tau\sum_{n=0}^m d^{n+1}\right), \tag{25}$$

where

$$\sigma^{n+1} = \frac{D^{n+1}}{1-\tau D^{n+1}}.$$

In order to analyze convergence of the partitioned time stepping methods for the fluid–fluid interaction, we introduce the following Stokes projection by finding $(R_h(v_i, q_i), Q_h(v_i, q_i)) \in X_i^h \times M_i^h$ such that

$$\begin{aligned} &a\big(v_i - R_h(v_i,q_i), v_h\big) - d\big(v_h, q_i - Q_h(v_i,q_i)\big) + d\big(v_i - R_h(v_i,q_i), q_h\big) = 0, \\ &\quad (v_h, q_h) \in X_i^h \times M_i^h, \end{aligned} \tag{26}$$

which is well-defined and satisfies the following optimal approximation property:

$$\begin{aligned} &\big\| v_i - R_h(v_i,q_i)\big\|_0 + h\big(\big\|\nabla\big(v_i - R_h(v_i,q_i)\big)\big\|_0 + \big\|q_i - Q_h(v_i,q_i)\big\|_0\big) \\ &\quad \le Ch^2\big(\|v_i\|_2 + \|q_i\|_1\big), \quad i = 1,2. \end{aligned} \tag{27}$$

Theorem 5.1 *Assume that the initial data u_i^0 and u_i^1 satisfy the following estimate*:

$$\big\|\nabla\big(u_i(t^0) - u_i^0\big)\big\|_0 + \big\|\nabla\big(u_i(t^1) - u_i^1\big)\big\|_0 \le Ch, \quad i = 1,2. \tag{28}$$

Moreover, the time step τ satisfies the relation $\tau < 1/D^{n+1}$ with D^{n+1} defined by (43) below. Let (u_i, p_i) and (u_i^{n+1}, p_i^{n+1}) be the solutions of (1)–(6) and (16)–(17), respectively, with $u_i \in L^2([0,T], H^2(\Omega_i)\cap X_i)$, $p_i \in L^2([0,T], H^1(\Omega_i)\cap M_i)$, $u_{i,t} \in L^2([0,T], X_i)$ and $u_{i,tt} \in L^2([0,T], L^2(\Omega_i))$. Then it holds that

$$\begin{aligned} &\tau \sum_{n=0}^{m} \big(\nu_1 \big\|\nabla\big(u_1(t^{n+1}) - u_1^{n+1}\big)\big\|_0^2 + \nu_2 \big\|\nabla\big(u_2(t^{n+1}) - u_2^{n+1}\big)\big\|_0^2\big) \\ &\quad \le C\big(\tau^2 + h^{2r}\big), \quad r = 1,2, \end{aligned} \tag{29}$$

where C denotes a positive constant depending on the data $(\nu_i, \Omega_i, u_i, p_i, f_i)$, $i = 1,2$, which may stand for different values at different occurrences.

Proof Here, we analyze convergence on each subdomain independently. For convenience, we set $(e_i^j, \eta_i^j) = (R_h u_i(t^j) - u_i^j, Q_h p_i(t^j) - p_i^j)$ and $E_i^j = u_i(t^j) - R_h u_i(t^j)$. First, using the Stokes projection, we subtract (16) or (17) from (7) with $(v_i, q_i) = (e_i^{n+1}, \eta_i^{n+1}) \in X_i^h \times M_i^h$ to obtain

$$\begin{aligned} &\left(\frac{\partial u_i(t^{n+1})}{\partial t} - \frac{(u_i^{n+1} - u_i^n)}{\tau}, e_i^{n+1}\right) + a\big(e_i^{n+1}, e_i^{n+1}\big) + 2G\big(\eta_i^{n+1}, \eta_i^{n+1}\big) \\ &\qquad + b\big(u_i(t^{n+1}), u_i(t^{n+1}), e_i^{n+1}\big) - b\big(u_i^n, u_i^{n+1}, e_i^{n+1}\big) \\ &\qquad + \kappa\left(\int_I u_i(t^{n+1})\big|\big[\mathbf{u}(t^{n+1})\big]\big| \cdot e_i^{n+1}\,ds - \int_I u_i^{n+1}\big|\big[\mathbf{u}^n\big]\big| \cdot e_i^{n+1}\,ds\right) \\ &\qquad + \kappa\left(\int_I u_j^n\big|\big[\mathbf{u}^n\big]\big|^{1/2}\big|\big[\mathbf{u}^{n-1}\big]\big|^{1/2} \cdot e_i^{n+1}\,ds - \int_I u_j(t^{n+1})\big|\big[\mathbf{u}(t^{n+1})\big]\big| \cdot e_i^{n+1}\,ds\right) \\ &\quad = 2G\big(p_i(t^{j+1}), \eta_i\big), \end{aligned} \tag{30}$$

where $i = 1, j = 2$ or $i = 2, j = 1$. We analyze each term in the above equality. Note that

$$\begin{aligned}
&\left(\frac{\partial u_i(t^{n+1})}{\partial t} - \frac{(u_i^{n+1} - u_i^n)}{\tau}, e_i^{n+1}\right) \\
&\quad = \frac{1}{\tau}\left(\left(u_i(t^{n+1}) - u_i^{n+1}\right) - \left(u_i(t^n) - u_i^n\right), e_i^{n+1}\right) - \frac{1}{\tau}\left(u_i(t^{n+1}) - u_i(t^n), e_i^{n+1}\right) \\
&\qquad + \left(\frac{\partial u_i(t^{n+1})}{\partial t}, e_i^{n+1}\right) \\
&\quad = \frac{1}{\tau}\left(E_i^{n+1} - E_i^n, e_i^{n+1}\right) + \frac{1}{\tau}\left(e_i^{n+1} - e_i^n, e_i^{n+1}\right) - \left(RHS_i^{n+1}, e_i^{n+1}\right),
\end{aligned}$$

where $(RHS_i^{n+1}, v) = (\frac{u_i(t^{n+1}) - u_i(t^n)}{\tau} - \frac{\partial u_i(t^{n+1})}{\partial t}, v_i)$, $i = 1, 2$. Also, we see that

$$\left|\left(E_i^{n+1} - E_i^n, e_i^{n+1}\right)\right| \le \frac{1}{4\varepsilon_1 \nu_i}\left\|E_i^{n+1} - E_i^n\right\|_{-1}^2 + \varepsilon_1 \nu_i \left\|\nabla e_i^{n+1}\right\|_0^2, \tag{31}$$

$$\left|\left(RHS_i, e_i^{n+1}\right)\right| \le \frac{1}{4\varepsilon_2 \nu_i}\left\|RHS_i^{n+1}\right\|_{-1}^2 + \varepsilon_2 \nu_i \left\|\nabla e_i^{n+1}\right\|_0^2, \tag{32}$$

where $\varepsilon_i > 0$, $i = 1, 2$. For the trilinear terms, it is easy to see that

$$\begin{aligned}
&b\left(u_i(t^{n+1}), u_i(t^{n+1}), e_i^{n+1}\right) - b\left(u_i^n, u_i^{n+1}, e_i^{n+1}\right) \\
&\quad = b\left(u_i(t^{n+1}) - u_i(t^n), u_i(t^{n+1}), e_i^{n+1}\right) + b\left(E_i^n, u_i(t^{n+1}), e_i^{n+1}\right) \\
&\qquad + b\left(e_i^n, u_i(t^{n+1}), e_i^{n+1}\right) - b\left(u_i^n, u_i(t^{n+1}) - u_i^{n+1}, e_i^{n+1}\right) \\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{33}$$

To estimate these trilinear terms, using a classical result in [36], we see that

$$\begin{aligned}
|I_1| &\le C\left\|\nabla\left(u_i(t^{n+1}) - u_i(t^n)\right)\right\|_0 \left\|\nabla u_i(t^{n+1})\right\|_0 \left\|\nabla e_i^{n+1}\right\|_0 \\
&\le \varepsilon_3 \nu_i \left\|\nabla e_i^{n+1}\right\|_0^2 + \frac{C}{4\varepsilon_3 \nu_i}\left\|\nabla u_i(t^{n+1})\right\|_0^2 \left\|\nabla\left(u_i(t^{n+1}) - u_i(t^n)\right)\right\|_0^2.
\end{aligned}$$

Applying the Young inequality and the skew-symmetry of the trilinear term yields that

$$\begin{aligned}
|I_2 + I_4| &= \left|b\left(E_i^n, u_i(t^{n+1}), e_i^{n+1}\right) - b\left(u_i^n, E_i^{n+1}, e_i^{n+1}\right)\right| \\
&\le \left(\left\|\nabla u_i^n\right\|_0 + \left\|\nabla u_i(t^{n+1})\right\|_0\right)\left(\left\|\nabla E_i^n\right\|_0 + \left\|\nabla E_i^{n+1}\right\|_0\right)\left\|\nabla e_i^{n+1}\right\|_0 \\
&\le \varepsilon_4 \nu_i \left\|\nabla e_i^{n+1}\right\|_0^2 + \frac{c}{4\varepsilon_4 \nu_i}\left(\left\|\nabla u_i^n\right\|_0^2 + \left\|\nabla u_i(t^{n+1})\right\|_0^2\right)\left(\left\|E_i^n\right\|_0^2 + \left\|E_i^{n+1}\right\|_0^2\right).
\end{aligned}$$

For the third term I_3, using the following inequality [20]:

$$\|\phi\|_{L^4} \le C\|\phi\|_0^{\frac{1}{2} - \frac{1}{4}(d-2)} \|\nabla\phi\|_0^{\frac{1}{2} + \frac{1}{4}(d-2)} \quad \forall \phi \in X_i, d = 2, 3, \tag{34}$$

and applying the Young inequality and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
|I_3| &\le C\|e_i^n\|_0^{1/2}\|\nabla e_i^n\|_0^{1/2}\|\nabla u_i(t^{n+1})\|_0\|\nabla e_i^{n+1}\|_0 \\
&\le \varepsilon_5\nu_i\|\nabla e_i^{n+1}\|_0^2 + C\|e_i^n\|_0\|\nabla e_i^n\|_0\|\nabla u_i(t^{n+1})\|_0^2 \\
&\le \varepsilon_5\nu_i\|\nabla e_i^{n+1}\|_0^2 + \varepsilon_6\nu_i\|\nabla e_i^n\|_0^2 + C\|e_i^n\|_0^2\|\nabla u_i(t^{n+1})\|_0^4
\end{aligned}
$$

for $d = 2$, and

$$
\begin{aligned}
|I_3| &\le C\|e_i^n\|_0^{1/4}\|\nabla e_i^n\|_0^{3/4}\|\nabla u_i(t^{n+1})\|_0\|\nabla e_i^{n+1}\|_0 \\
&\le \varepsilon_5\nu_i\|\nabla e_i^{n+1}\|_0^2 + C\|e_i^n\|_0^{1/2}\|\nabla e_i^n\|_0^{3/2}\|\nabla u_i(t^{n+1})\|_0^2 \\
&\le \varepsilon_5\nu_i\|\nabla e_i^{n+1}\|_0^2 + \varepsilon_6\nu_i\|\nabla e_i^n\|_0^2 + C\|e_i^n\|_0^2\|\nabla u_i(t^{n+1})\|_0^8
\end{aligned}
$$

for $d = 3$.

Setting

$$
\begin{aligned}
&\overline{|[\mathbf{u}(t^{n+1})]|} = \frac{1}{2}\big(|[\mathbf{u}(t^{n+1})]| + |[\mathbf{u}(t^n)]|\big), \\
&\overline{|\mathbf{u}^n|} = \frac{1}{2}\big(|\mathbf{u}^n| + |\mathbf{u}^{n-1}|\big),
\end{aligned}
$$

and using the same approach as in [12], we get

$$
\begin{aligned}
I_5 &= \int_I u_i(t^{n+1})|[\mathbf{u}(t^{n+1})]|\cdot e_i^{n+1}\,ds - \int_I u_i^{n+1}|[\mathbf{u}^n]|\cdot e_i^{n+1}\,ds \\
&= \int_I u_i(t^{n+1})\big(|[\mathbf{u}(t^{n+1})]| - |[\mathbf{u}(t^n)]|\big)\cdot e_i^{n+1}\,ds \\
&\quad + \int_I u_i(t^{n+1})\big(|[\mathbf{u}(t^n)]| - |[P_h\mathbf{u}(t^n)]|\big)\cdot e_i^{n+1}\,ds \\
&\quad + \int_I u_i(t^{n+1})\big(|[P_h\mathbf{u}(t^n)]| - |[\mathbf{u}^n]|\big)\cdot e_i^{n+1}\,ds \\
&\quad + \int_I E_i^{n+1}|[\mathbf{u}^n]|\cdot e_i^{n+1}\,ds + \int_I |[\mathbf{u}^n]||e_i^{n+1}|^2\,ds \\
&\le \int_I u_i(t^{n+1})\big(|[\mathbf{u}(t^{n+1})]| - |[\mathbf{u}(t^n)]|\big)\cdot e_i^{n+1}\,ds + \int_I u_i(t^{n+1})|[E_i^n]|\cdot e_i^{n+1}\,ds \\
&\quad + \int_I u_i(t^{n+1})|[e_i^n]|\cdot e_i^{n+1}\,ds \\
&\quad + \int_I E_i^{n+1}|[\mathbf{u}^n]|\cdot e_i^{n+1}\,ds + \int_I |[\mathbf{u}^n]||e_i^{n+1}|^2\,ds.
\end{aligned}
\tag{35}
$$

Noting that

$$
\begin{aligned}
&\big|\overline{|[\mathbf{u}(t^n)]|} - |[\mathbf{u}(t^{n+1})]|\big| \\
&\quad = \frac{1}{2}\big(|[\mathbf{u}(t^n)]| - |[\mathbf{u}(t^{n+1})]|\big) + \frac{1}{2}\big(|[\mathbf{u}(t^{n-1})]| - |[\mathbf{u}(t^{n+1})]|\big) \\
&\quad \le \frac{1}{2}\big(|[\mathbf{u}(t^n) - \mathbf{u}(t^{n+1})]| + |[\mathbf{u}(t^{n-1}) - \mathbf{u}(t^{n+1})]|\big),
\end{aligned}
\tag{36}
$$

and

$$\begin{aligned}
&\big|[\mathbf{u}^n]\big|^{1/2}\big|[\mathbf{u}^{n-1}]\big|^{1/2} - \overline{\big|[\mathbf{u}^n]\big|} \\
&\quad = \frac{1}{2}\big(\big|[\mathbf{u}^n]\big|^{1/2} - \big|[\mathbf{u}^{n-1}]\big|^{1/2}\big)^2 \\
&\quad \le \frac{1}{2}\big|\big|[\mathbf{u}^n]\big| - \big|[\mathbf{u}^{n-1}]\big|\big| \\
&\quad \le \frac{1}{2}\big|[\mathbf{u}^n - \mathbf{u}^{n-1}]\big| \\
&\quad \le \frac{1}{2}\big|[\mathbf{u}^n - \mathbf{u}(t^n) + \mathbf{u}(t^{n-1}) - \mathbf{u}^{n-1}]\big| + \frac{1}{2}\big|[\mathbf{u}(t^n) - \mathbf{u}(t^{n-1})]\big| \\
&\quad \le \big|\overline{[\mathbf{e}^n]}\big| + \big|\overline{[\mathbf{E}^n]}\big| + \big|\overline{[\mathbf{e}^{n-1}]} + \overline{[\mathbf{E}^{n-1}]}\big| + \frac{1}{2}\big|[\mathbf{u}(t^n) - \mathbf{u}(t^{n-1})]\big|, \qquad (37)
\end{aligned}$$

the following error bound holds:

$$\begin{aligned}
I_6 &= \int_I u_j^n\big|[\mathbf{u}^n]\big|^{1/2}\big|[\mathbf{u}^{n-1}]\big|^{1/2} \cdot e_i^{n+1}\,ds - \int_I u_j(t^{n+1})\big|[\mathbf{u}(t^{n+1})]\big| \cdot e_i^{n+1}\,ds \\
&\le \int_I u_j^n\big(\big|[\mathbf{u}^n]\big|^{1/2}\big|[\mathbf{u}^{n-1}]\big|^{1/2} - \overline{|\mathbf{u}^n|}\big) \cdot e_i^{n+1}\,ds + \int_I u_j^n\big(\overline{|\mathbf{u}^n|} - P_h\overline{|\mathbf{u}^n|}\big) \cdot e_i^{n+1}\,ds \\
&\quad + \int_I u_j^n\big(P_h\overline{|\mathbf{u}^n|} - \big|[\mathbf{u}(t^n)]\big|\big) \cdot e_i^{n+1}\,ds \\
&\quad + \int_I u_j^n\big(\overline{\big|[\mathbf{u}(t^{n+1})]\big|} - \big|[\mathbf{u}(t^{n+1})]\big|\big) \cdot e_i^{n+1}\,ds \\
&\quad - \int_I \big(E_j^n + e_j^n\big)\big|[\mathbf{u}(t^{n+1})]\big| \cdot e_i^{n+1}\,ds \\
&\quad + \int_I \big(u_j(t^n) - u_j(t^{n+1})\big)\big|[\mathbf{u}(t^{n+1})]\big| \cdot e_i^{n+1}\,ds \\
&\le \int_I u_j^n\big|[\mathbf{u}(t^n) - \mathbf{u}(t^{n-1})]\big| \cdot e_i^{n+1}\,ds + \int_I u_j^n\overline{\big|[\mathbf{E}^n]\big|} \cdot e_i^{n+1}\,ds + \int_I u_j^n\big|\overline{[\mathbf{e}^n]}\big| \cdot e_i^{n+1}\,ds \\
&\quad + \int_I u_j^n\big(\big|[\mathbf{u}(t^n) - \mathbf{u}(t^{n+1})]\big| + \big|[\mathbf{u}(t^{n-1}) - \mathbf{u}(t^{n+1})]\big|\big) \cdot e_i^{n+1}\,ds \\
&\quad + \int_I \big(u_j(t^n) - u_j(t^{n+1})\big)\big|[\mathbf{u}(t^{n+1})]\big| \cdot e_i^{n+1}\,ds \\
&\quad - \int_I \big(E_j^n + e_j^n\big)\big|[\mathbf{u}(t^{n+1})]\big| \cdot e_i^{n+1}\,ds. \qquad (38)
\end{aligned}$$

Applying the same approach as in [12], we obtain

$$\begin{aligned}
&|I_5 + I_6| \\
&\quad \le \varepsilon_7\nu_i\big\|\nabla e_i^{n+1}\big\|_0^2 + C\nu_i^{-3}\big(1 + \big\|u_2^n\big\|_I^4 + \big\|[\mathbf{u}(t^{n+1})]\big\|_I^4\big)\big\|e_i^{n+1}\big\|_0^2 \\
&\qquad + \varepsilon_8\sum_{i=1}^{2}\nu_i\big(\big\|\nabla e_i^n\big\|_0^2 + \big\|\nabla e_i^{n-1}\big\|_0^2\big) + CL^{n+1}P^{n+1} \\
&\qquad + CM^{n+1}\sum_{i=1}^{2}\nu_i^{-3}\big(\big\|e_i^n\big\|_0^2 + \big\|e_i^{n-1}\big\|_0^2\big), \qquad (39)
\end{aligned}$$

where L^{n+1}, M^{n+1} and P^{n+1} can be defined by the following bound terms on interface I as follows:

$$L^{n+1} = \sum_{i=1}^{2}\big(\|u_i(t^{n+1})\|_I^2 + \|u_i^{n+1}\|_I^2 + \|u_i^n\|_I^2\big),$$

$$M^{n+1} = \sum_{i=1}^{2}\big(\|u_i(t^{n+1})\|_I^4 + \|u_i^n\|_I^4\big),$$

$$\begin{aligned} P^{n+1} = \sum_{i=1}^{2}\big\{&\big(\|u_i(t^{n+1}) - u_i(t^n)\|_I^2 + \|u_i(t^{n+1}) - u_i(t^{n-1})\|_I^2\big) \\ &+ \big(\|E_i^{n-1}\|_I^2 + \|E_i^n\|_I^2 + \|E_i^{n+1}\|_I^2\big)\big\}. \end{aligned}$$

Obviously, $\|\cdot\|_I$ is bounded by the corresponding L^2-norm. In addition, we can infer that the estimate of P^{n+1} has the order of $O(\tau^2 + h^{2r})$, $r = 1, 2$.

Choosing $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_7 = 1/4$, $\varepsilon_6 = 1/8$, and $\varepsilon_8 = 1/16$, and combining all these inequalities with (30) yields that

$$\begin{aligned} &\frac{1}{2\tau}\big(\|e_i^{n+1}\|_0^2 - \|e_i^n\|_0^2 + \|e_i^{n+1} - e_i^n\|_0^2\big) + \frac{3\nu_i}{4}\|\nabla e_i^{n+1}\|_0^2 \\ &\le C\left(\frac{1}{\tau}\|E_i^{n+1} - E_i^n\|_{-1}^2 + \|RHS_i^{n+1}\|_{-1}^2 + \|p_i - \Pi_h p_i\|_0^2\right) \\ &\quad + C\|\nabla u_i(t^{n+1})\|_0^2\|\nabla(u_i(t^{n+1}) - u_i(t^n))\|_0^2 \\ &\quad + \big(\|\nabla u_i^n\|_0^2 + \|\nabla u_i(t^{n+1})\|_0^2\big)\big(\|E_i^n\|_0^2 + \|E_i^{n+1}\|_0^2\big) + C\|e_i^n\|_0^2\|\nabla u_i(t^{n+1})\|_0^{2^d} \\ &\quad + C\nu_i^{-3}\big(1 + \|u_2^n\|_I^4 + \|[\mathbf{u}(t^{n+1})]\|_I^4\big)\|e_i^{n+1}\|_0^2 + CL^{n+1}P^{n+1} \\ &\quad + \sum_{i=1}^{2}\frac{\nu_i}{16}\big(\|\nabla e_i^n\|_0^2 + \|\nabla e_i^{n-1}\|_0^2\big) + \frac{\nu_i}{8}\|\nabla e_i^n\|_0^2 \\ &\quad + CM^{n+1}\sum_{i=1}^{2}\nu_i^{-3}\big(\|e_i^n\|_0^2 + \|e_i^{n-1}\|_0^2\big). \end{aligned} \tag{40}$$

Summing over $i = 1, 2$ for the above inequality and moving the term on the 5th line on the right-hand side of (40) to its left-hand side, t the term $\frac{\nu_i}{8}\|\nabla e_i^n\|_0^2$ can be exactly absorbed. Thus we find that

$$\begin{aligned} &\frac{1}{2\tau}\big(\|\mathbf{e}^{n+1}\|_0^2 - \|\mathbf{e}^n\|_0^2 + \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_0^2\big) + \sum_{i=1}^{2}\frac{\nu_i}{2}\|\nabla e_i^{n+1}\|_0^2 \\ &\quad + \sum_{i=1}^{2}\frac{\nu_i}{4}\big(\|\nabla e_i^{n+1}\|_0^2 - \|\nabla e_i^n\|_0^2\big) + \sum_{i=1}^{2}\frac{\nu_i}{16}\big(\|\nabla e_i^n\|_0^2 - \|\nabla e_i^{n-1}\|_0^2\big) \\ &\le C\sum_{i=1}^{2}\left(\frac{1}{\tau}\|E_i^{n+1} - E_i^n\|_{-1}^2 + \|RHS_i^{n+1}\|_{-1}^2 + \|p_i - \Pi_h p_i\|_0^2\right) \\ &\quad + C\sum_{i=1}^{2}\|\nabla u_i(t^{n+1})\|_0^2\|\nabla(u_i(t^{n+1}) - u_i(t^n))\|_0^2 \end{aligned}$$

$$+ C\sum_{i=1}^{2}\big(\|\nabla u_i^n\|_0^2 + \|\nabla u_i(t^{n+1})\|_0^2\big)\big(\|E_i^n\|_0^2 + \|E_i^{n+1}\|_0^2\big) + C\sum_{i=1}^{2}L^{n+1}P^{n+1}$$
$$+ C\sum_{i=1}^{2}\nu_i^{-3}\big(1 + \|u_2^n\|_I^4 + \|[\mathbf{u}(t^{n+1})]\|_I^4\big)\|e_i^{n+1}\|_0^2 + C\sum_{i=1}^{2}\|e_i^n\|_0^2\|\nabla u_i(t^{n+1})\|_0^{2^d}$$
$$+ C\sum_{i=1}^{2}\nu_i^{-3}M^{n+1}\big(\|e_i^n\|_0^2 + \|e_i^{n-1}\|_0^2\big).$$

Noting the bounds of $\|\nabla u_i(t^{n+1})\|_0$, $\sum_{n=0}^{m-1}\|\nabla u_i^{n+1}\|_0$, and $\|\mathbf{u}_{tt}\|_0$ yields

$$\begin{aligned}
&\|E_i^{n+1} - E_i^n\|_{-1}^2 + \tau\|RHS_i^{n+1}\|_{-1}^2 \\
&\quad\le C(\tau^2 + h^2)\big(\|\mathbf{u}_{tt}\|^2_{L^2([t_{n-1},t_{n+1}],L^2(\Omega_i))} \\
&\qquad + \|\mathbf{u}\|_{L^2([t_{n-1},t_{n+1}],H^2(\Omega_i))} + \|p\|_{L^2([t_{n-1},t_{n+1}],H^1(\Omega_i))}\big), \\
&L^{n+1}P^{n+1} \\
&\quad\le C(\tau^2 + h^{2r})\big(\|\mathbf{u}_t\|^2_{L^2([t_{n-1},t_{n+1}],L^2(\Omega_i))} + \|\mathbf{u}\|^2_{L^2([t_{n-1},t_{n+1}],H^2(\Omega_i))}\big).
\end{aligned}$$

Summing over $n = 1, 2, \ldots, m-1$, multiplying by 2τ, using the classical estimates, and rewritting the last term of the right-hand side of the above inequality as

$$\begin{aligned}
&\sum_{n=1}^{m-1}\left(\sum_{i=1}^{2}\nu_i^{-3}M^{n+1}\big(\|e_i^n\|_0^2 + \|e_i^{n-1}\|_0^2\big)\right) \\
&\quad\le C\sum_{i=1}^{2}\nu_i^{-3}(M^2 + M^3)\big(\|e_i^1\|_0^2 + \|e_i^0\|_0^2\big) + C\sum_{n=1}^{m-1}\nu_i^{-3}(M^{n+1} + M^{n+2})\|\mathbf{e}^n\|_0^2,
\end{aligned} \tag{41}$$

we obtain

$$\begin{aligned}
&\|\mathbf{e}^m\|_0^2 + \sum_{n=1}^{m-1}\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_0^2 + \tau\sum_{n=1}^{m-1}\sum_{i=1}^{2}\frac{\nu_i}{2}\|\nabla e_i^{n+1}\|_0^2 \\
&\qquad + \tau\sum_{i=1}^{2}\frac{\nu_i}{16}\big(\|\nabla e_i^{m-1}\|_0^2 + 4\|\nabla e_i^m\|_0^2\big) \\
&\quad\le \tau\sum_{i=1}^{2}\frac{\nu_i}{16}\big(\|\nabla e_i^0\|_0^2 + 4\|\nabla e_i^1\|_0^2\big) \\
&\qquad + \big(1 + C\tau\nu_i^{-3}(M^2 + M^3)\big)\big(\|\mathbf{e}^0\|_0^2 + \|\mathbf{e}^1\|_0^2\big) \\
&\qquad + C(\tau^2 + h^{2r}) + C\tau\sum_{n=1}^{m-1}D^{n+1}\|\mathbf{e}^{n+1}\|_0^2,
\end{aligned} \tag{42}$$

where D^{n+1} is defined by

$$\begin{aligned}
D^{n+1} &= \big(\nu_i^{-3}\big(1 + M^n + M^{n+1} + \|u_2^n\|_I^4 + \|[\mathbf{u}(t^{n+1})]\|_I^4\big) \\
&\qquad + \|\nabla\mathbf{u}(t^{n+1})\|_0^{2^d}\big), \quad d = 2, 3,
\end{aligned} \tag{43}$$

and C is dependent of the data (Ω_i, ν_i, f_i). Setting

$$a^{j+1} = \|e^{j+1}\|_0^2, \qquad b^{j+1} = \|e^{j+1} - e^j\|_0^2, \qquad c^{j+1} = \sum_{i=1}^{2} \|\nabla e_i^{j+1}\|_0^2,$$

and using Gronwall's inequality in Lemma 4.1, (28) and (42) yields the desired result. □

6 Numerical results

In this section, we assess numerical performance of the stabilized methods for the presented model. It will be checked by a known analytical solution problem. The main goal of the experiment is to verify convergence rates of the scheme (16)–(17). Here, we denote errors by

$$\mathrm{Err}(u_i) = \left(\tau \sum_{n=0}^{m} \|\nabla\left(u_i(t^{n+1}) - u_i^{n+1}\right)\|_{0,\Omega_i}^2\right)^{1/2},$$

$$\mathrm{Err}(p_i) = \left(\tau \sum_{n=0}^{m} \|p_i(t^{n+1}) - p_i^{n+1}\|_{0,\Omega_i}^2\right)^{1/2},$$

where $i = 1, 2$. All numerical computations are implemented by open source software Freefem [19].

Example 1 The computations of the experiment are carried out in the domains $\Omega_1 = (0,1) \times (0,1)$ and $\Omega_2 = (0,1) \times (-1,0)$. The prescribed exact solutions are given [13, 37] by

$$\begin{aligned}
p_1(t,x,y) &= p_2(t,x,y) = \exp(-t)\cos(\pi x)\sin(\pi y),\\
u_{1,1}(t,x,y) &= -\alpha x^2 \exp(-t)(x-1)^2(y-1),\\
u_{1,2}(t,x,y) &= \alpha xy\exp(-t)\left(6x + y - 3xy + 2x^2y - 4x^2 - 2\right),\\
u_{2,1}(t,x,y) &= -\alpha x\exp(-t)(x-1)\left(y^2x(x-1)\left(\frac{\mu_1}{\mu_2} + 1\right)\right.\\
&\quad \left. - \frac{\mu_1^{1/2}y^2\exp(t/2)}{(\alpha\kappa)^{1/2}} - x(x-1) + \frac{\mu_1^{1/2}\exp(t/2)}{(\alpha\kappa)^{1/2}} + \frac{\mu_1 xy(x-1)}{\mu_2}\right),\\
u_{2,2}(t,x,y) &= -\frac{\alpha y\exp(-t)(2x-1)}{3\mu_2(\alpha\kappa)^{1/2}}\left(6\mu_2x^2(\alpha\kappa)^{1/2} - 6\mu_2 x(\alpha\kappa)^{1/2} - 3\mu_1^{1/2}\mu_2\exp(t/2)\right.\\
&\quad - 2\mu_1x^2y^2(\alpha\kappa)^{1/2} - 2\mu_2x^2y^2(\alpha\kappa)^{1/2} + 3\mu_1xy(\alpha\kappa)^{1/2} + 2\mu_1xy^2(\alpha\kappa)^{1/2}\\
&\quad \left. - 3\mu_1x^2y(\alpha\kappa)^{1/2} + 2\mu_2xy^2(\alpha\kappa)^{1/2} + \mu_1^{1/2}\mu_2y^2\exp(t/2)\right),
\end{aligned}$$

with an arbitrary positive constant α. Here, (u_i, p_i), $i = 1, 2$ are the solutions of the original problem (1)–(6) and the right-hand sides $f = (f_1, f_2)$ can be obtained by (1). Moreover, $u_2 = (u_{2,1}, u_{2,2})$ satisfies the three interface conditions in [13].

Firstly, in the first example, we choose the same parameter values $\mu_1 = 0.5$, $\mu_2 = 0.05$, $\alpha = 1$ and $\kappa = 100$ as in [13]. The Euler scheme is used for the time discretization at $T = 1$,

Table 1 Errors for stabilized P_1–P_1 pair with $\tau = h$

$1/h$	Err(u_1)	Rate	Err(u_2)	Rate	Err(p_1)	Rate	Err(p_2)	Rate
8	7.2271E–2	–	2.9717E–1	–	1.4967E–2	–	1.1617E–2	–
32	1.7025E–2	1.043	7.0463E–2	1.038	1.8549E–3	1.506	1.9019E–3	1.305
64	8.4129E–3	1.017	3.4862E–2	1.015	6.8180E–4	1.444	7.2067E–4	1.400

Table 2 Errors for stabilized P_2–P_2 pair with $\tau = h^2$

$1/h$	Err(u_1)	Rate	Err(u_2)	Rate	Err(p_1)	Rate	Err(p_2)	Rate
4	2.6926E–2	–	1.7999E–1	–	1.7270E–2	–	1.6852E–2	–
8	5.7926E–3	2.215	3.4207E–2	2.396	4.2766E–3	2.014	4.2613E–3	1.984
16	1.3445E–3	2.107	6.9341E–3	2.303	1.0623E–3	2.009	1.1131E–3	1.914

Table 3 Errors for the different small viscosities based on stabilized P_1–P_1 pair

	$\mu_1 = 1.0E{-}4$ $\mu_2 = 1.0E{-}4$	$\mu_1 = 1.0E{-}4$ $\mu_2 = 1.0E{-}5$	$\mu_1 = 1.0E{-}5$ $\mu_2 = 1.0E{-}5$
Err(u_1)	1.3843E–2	1.3838E–2	1.3864E–2
Err(u_2)	2.0461E–2	5.3043E–2	2.0475E–2
Err(p_1)	1.9068E–3	1.9070E–3	1.9063E–3
Err(p_2)	9.7819E–4	2.6972E–3	9.7897E–4

Table 4 Errors for the different small viscosities based on stabilized P_2–P_2 pair

	$\mu_1 = 1.0E-4$ $\mu_2 = 1.0E-4$	$\mu_1 = 1.0E-4$ $\mu_2 = 1.0E-5$	$\mu_1 = 1.0E-5$ $\mu_2 = 1.0E-5$
Err(u_1)	4.9393E–2	4.9283E–2	5.4135E–2
Err(u_2)	4.9196E–2	5.3924E–2	5.3904E–2
Err(p_1)	3.0189E–4	3.0190E–4	3.0190E–4
Err(p_2)	3.1022E–4	7.7659E–4	3.1042E–4

with the time step $\tau = h$. Three values of space size $h = 1/8, 1/32, 1/64$ are chosen. We display the convergence orders and errors of the presented method in Tables 1–2 by P_r–P_r, $r = 1, 2$. From Tables 1–2, it can be easily seen that the method completely agree with the expected results in theory.

Secondly, we test the presented method with small viscosities. Here, we choose $\alpha = 1$, $\kappa = 100$, $h = 1/20$ and the time step $\tau = 0.005$. Then, we list the numerical errors with different small viscosities at $T = 0.1$ in Tables 3–4. Obviously, the presented method can deal with these problems involving small viscosities.

Example 2 In this example, we test the presented method for a submarine mountain problem. This problem describes the fluid, which flows in a domain including the submarine mountain. In this case, the subdomain Ω_2 is nonconvex. As is known, the viscosity of the fluid at submarine location is bigger than that at surface location. So we take $\mu_1 = 0.001$ and $\mu_2 = 0.01$ in this example.

Set $\Omega_1 = [0, 1] \times [0, 0.1]$ and $\Omega_2 = \{(x, y) : \frac{7}{40}(1 - (2x - 1)\sin(7x - 3.5)) \le y \le 0\}$. The initial conditions are chosen as follows:

$$p_1(0, x, y) = p_2(0, x, y) = \cos(\pi x)\sin(\pi y),$$
$$u_{1,1}(0, x, y) = x^2(1 - x)^2(0.1 - y),$$

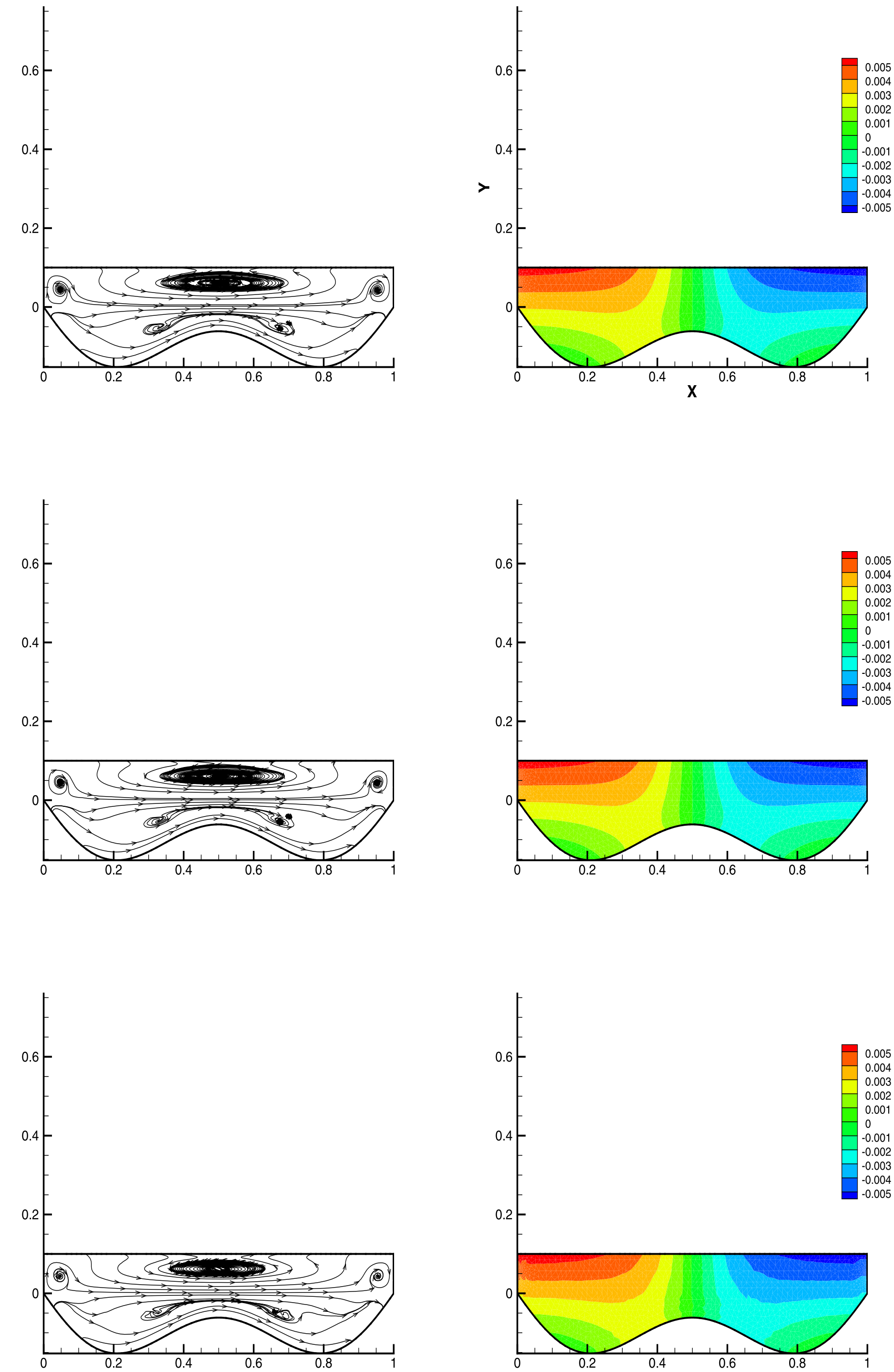

Figure 1 The numerical streamlines and isobars: the stabilized P_2–P_2 pair (the first line), the P_2–P_1 pair (the second line) and the stabilized P_1–P_1 pair (the third line)

$$u_{1,2}(0,x,y) = xy\bigl(-0.2 + y + 0.6x - 3xy - 0.4x^2 + 2x^2y\bigr),$$
$$u_{2,1}(0,x,y) = x^2(1-x)^2(0.1+y),$$
$$u_{2,2}(0,x,y) = xy\bigl(-0.2 - y + 0.6x + 3xy - 0.4x^2 - 2x^2y\bigr).$$

We apply the presented method to get the numerical solution with $h = 1/70$ and $\tau = 1/40$. In Fig. 1, we present profiles for the numerical velocity and pressure with different methods at the final time $T = 5$ and $\kappa = 100$. From this figure, we can see that the stabilized methods are stable and the unphysical oscillations do not appear, and the numerical results of these stabilized methods completely agreement with those obtained by the classical P_2–P_1 pair [35]. Besides, we can find that the presence of the submarine mountain affects the fluid.

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