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A singular fractional Kelvin–Voigt model involving a nonlinear operator and their convergence properties

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Abstract

In this paper, we focus on a generalized singular fractional order Kelvin–Voigt model with a nonlinear operator. By using analytic techniques, the uniqueness of solution and an iterative scheme converging to the unique solution are established, which are very helpful to govern the process of the Kelvin–Voigt model. At the same time, the corresponding eigenvalue problem is studied and the property of solution for the eigenvalue problem is established. Some examples are given to illuminate the main results.

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1 Introduction

Many physical phenomena in natural sciences and engineering often exhibit some singular behavior; for example, in linear elastic fracture mechanics, the stress near the crack tip possesses a singularity of $r^{-0.5}$ [1], where r is the distance measured from the crack tip. Recently, Fisk [2] found that quantum fluctuations at absolute zero may push a system into a different phase or state, and the singular phenomenon happens near the quantum critical points in certain materials. In recent years, because of the importance of the singular behavior in critical point, the study of the singular problems has attracted much attention, for details, see [3–18] and the references cited therein.

On the other hand, relaxation processes deviating from the classical exponential (Debye) behavior are often encountered in the dynamics of complex materials [19]. In many cases a stretched exponential (Kohlrausch–Williams–Watts) decay is often exhibited by experimentally observed relaxation functions [19]

$$\Phi(t)e^{-(\frac{t}{\tau})^{\alpha}}, \quad 1<\alpha<1, \tag{1.1}$$

or a scaling decay

$$\Phi(t)\left(\frac{t}{\tau}\right)^{-\beta}, \quad 1<\beta<1. \tag{1.2}$$

The above processes (1.1) and (1.2) show that an appropriate tool to describe phenomenologically hybrid dynamical features is fractional calculus, which is incorporated into standard constitutive equations in a variety of works, mainly in the field of viscoelasticity. Defining that $\sigma(t)$ is the stress and $\epsilon(t)$ is the strain, Schiessel et al. [19] considered a system whose stress decays after a shear jump in an algebraic manner and obtained a standard fractional order viscoelasticity Kelvin–Voigt model

$$\sigma(t) = E\tau^{\alpha}\frac{d^{\alpha}}{dt^{\alpha}}\epsilon(t) + E\tau^{\beta}\frac{d^{\beta}}{dt^{\beta}}\epsilon(t), \tag{1.3}$$

where $\alpha > \beta > 0$, E is a constant, and $\frac{d^{\alpha}}{dt^{\alpha}}$ is the Riemann–Liouville derivative $\mathscr{D}_t{}^{\alpha}$ with an order of α. Thus the fractional order Kelvin–Voigt model (1.3) can be generalized by the following mathematical model:

$$\frac{d^{\alpha}}{dt^{\alpha}}\epsilon(t) = f\left(t, \frac{d^{\beta}}{dt^{\beta}}\epsilon(t)\right),$$

with $\alpha > \beta > 0$.

In this paper, we focus on the following generalized singular Kelvin–Voigt model:

$$\mathfrak{B}\big(\mathscr{D}_t{}^{\alpha}\epsilon(t)\big)\mathscr{D}_t{}^{\alpha}\epsilon(t) = f\big(t, -\epsilon(t), -\mathscr{D}_t{}^{\gamma}\epsilon(t)\big), \quad t \in (0,1), \tag{1.4}$$

subject to nonlocal boundary condition

$$\mathscr{D}_t{}^{\gamma}\epsilon(0) = 0, \qquad \mathscr{D}_t{}^{\gamma}\epsilon(1) = \int_0^1 \mathscr{D}_t{}^{\gamma}\epsilon(s)\,d\chi(s), \tag{1.5}$$

where $\mathscr{D}_t{}^{\alpha}$, $\mathscr{D}_t{}^{\gamma}$ are the standard Riemann–Liouville derivatives with the order of $0 < \gamma < 1 < \alpha \leq 2$, $\alpha - \gamma > 1$, $\int_0^1 \mathscr{D}_t{}^{\gamma}\epsilon(s)\,d\chi(s)$ is denoted by a Riemann–Stieltjes integral, χ is a function of bounded variation, and $d\chi$ can be a signed measure, $\mathfrak{B} \in \mathcal{Y}$ is a nonlinear operator with an individual property

$$\begin{aligned}\mathcal{Y} = \big\{&\mathfrak{B} \in C^2\big([0,+\infty),[0,+\infty)\big) : \text{there exists a constant } \sigma > 0 \\ &\text{such that, for any } 0 < c < 1, \mathfrak{B}(cs) \leq c^{\sigma}\mathfrak{B}(s)\big\}.\end{aligned}$$

In particular, in the generalized Kelvin–Voigt model (1.4)–(1.5), we allow that the nonlinearity $f(t,u,v)$ has singularity at both $u = 0$ and (or) $v = 0$.

In the past decades, a large number of numerical and analytical results have been obtained for various differential equations with physical background [20–77]. Recently, some new type functions and inequalities such as noninstantaneous impulsive inequalities [78], Gronwall–Bellman–Bihari inequalities [79], Mittag-Leffler functions [80], generalized Gauss hypergeometric functions [81], and asymptotical-analytic technique [82] have been developed to improve and perfect fractional calculus and its application. In particular, Saoudi and Agarwal et al. [83] employed the method of Nehari manifold combined with the fibering maps to establish the existence of solutions to the boundary value problem for the nonlinear fractional differential equations with Riemann–Liouville fractional derivative. This work shows that the critical point theory and variational methods are also very effective tools in determining the existence of solutions for fractional order differential equations.

However, up to now, few results have been reported for the generalized Kelvin–Voigt model (1.4)–(1.5) when f has singularity on the strain. This present paper aims to study the singular case for the generalized fractional order Kelvin–Voigt model (1.4)–(1.5). Notice that model (1.4)–(1.5) involves a nonlinear operator, which implies that model (1.4)–(1.5) includes many interesting and important models as special cases such as Marwell model, Zener model, Poynting–Thoinson model. If $\mathfrak{B}(x) = E\tau^{\alpha}, f = \sigma(t) - E\tau^{\beta}\frac{d^{\beta}\epsilon(t)}{dt^{\beta}}$, then model (1.4)–(1.5) reduces to the standard fractional order viscoelasticity Kelvin–Voigt model (1.3). If $\mathfrak{B}(x) = |x|^{p-2}$, $p \geq 2$, model (1.4)–(1.5) becomes the form

$$\begin{cases} \varphi_p(\mathscr{D}_{\boldsymbol{t}}{}^{\alpha}\epsilon(t)) = f(t, -\epsilon(t), -\mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon(t)), & t \in (0,1), \\ \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon(0) = 0, \qquad \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon(1) = \int_0^1 \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon(s)\,d\chi(s), \end{cases}$$

which is a p-Poisson equation [10, 84]. Thus model (1.4)–(1.5) is more generalized than viscoelasticity Kelvin–Voigt model (1.3). To the best of our knowledge, no results have been reported on the existence and uniqueness of solutions for model (1.4)–(1.5) when f can be singular at the points of the strain vanishing.

2 Preliminaries and lemmas

Before we give a detailed description of preliminaries and lemmas, we first establish some properties of an inverse operator for the operator $s\mathfrak{B}(s)$.

Proposition 2.1 *If $\mathfrak{B} \in \mathcal{Y}$, let $\mathfrak{L}(s) = s\mathfrak{B}(s)$, then $\mathfrak{L}$ has a nonnegative increasing inverse mapping $\mathfrak{L}^{-1}(s)$, and for any $0 < c < 1$,*

$$\mathfrak{L}^{-1}(cs) \geq c^{\frac{1}{1+\sigma}}\mathfrak{L}^{-1}(s). \tag{2.1}$$

Proof Firstly, we prove that $\mathfrak{B}$ is an increasing operator if $\mathfrak{B} \in \mathcal{Y}$. In fact, for any $\mathfrak{B} \in \mathcal{Y}$ and $s, t \in [0, +\infty)$, without loss of generality, let $0 \leq s < t$. If $s = 0$, obviously $\mathfrak{B}(s) \leq \mathfrak{B}(t)$ holds. If $s \neq 0$, let $c_0 = s/t$, then $0 < c_0 < 1$. It follows from the property of $\mathfrak{B}$ that

$$\mathfrak{B}(s) = \mathfrak{B}(c_0 t) \leq c_0^{\sigma}\mathfrak{B}(t) < \mathfrak{B}(t),$$

which implies that $\mathfrak{B}$ is an increasing operator. Thus we have $\mathfrak{L}'(s) = (s\mathfrak{B}(s))' > 0$ for any $s > 0$, i.e, $\mathfrak{L}$ is a bijection on $(0, \infty)$ and has a nonnegative increasing inverse mapping $\mathfrak{L}^{-1}(s)$.

On the other hand, for any $0 < c < 1$, let $b = c^{\frac{1}{1+\sigma}}$, then $0 < b < 1$. Thus we have

$$\mathfrak{L}(bx) = bx\mathfrak{B}(bx) \leq b^{1+\sigma}x\mathfrak{B}(x) = b^{1+\sigma}\mathfrak{L}(x) \quad \text{for } x > 0.$$

Consequently, let $s = \mathfrak{L}(x)$, then

$$b\mathfrak{L}^{-1}(s) = bx \leq \mathfrak{L}^{-1}\bigl(b^{1+\sigma}\mathfrak{L}(x)\bigr) = \mathfrak{L}^{-1}(cs),$$

that is,

$$c^{\frac{1}{1+\sigma}}\mathfrak{L}^{-1}(s)\le\mathfrak{L}^{-1}(cs).$$

The proof is completed. □

Remark 2.1 Clearly, if $r\ge 1$, we have

$$\mathfrak{L}^{-1}(rs)\le r^{\frac{1}{1+\sigma}}\mathfrak{L}^{-1}(s). \tag{2.2}$$

Remark 2.2 The operator set $\mathcal{Y}$ includes a large class of operators, and the standard type of operators is $\mathfrak{B}(s)=\sum_{i=1}^{n}s^{\alpha_i}$, $\alpha_i>0$. In fact, take $\sigma=\min\{\alpha_1,\dots,\alpha_n\}>0$, then for any $0<c<1$, one has

$$\mathfrak{B}(cs)\le c^{\sigma}\mathfrak{B}(s).$$

Now based on Proposition 2.1, we transform model (1.4)–(1.5) to a convenient form

$$\begin{cases}\mathscr{D}_{\boldsymbol{t}}{}^{\alpha}\epsilon(t)=\mathfrak{L}^{-1}(f(t,-\epsilon(t),-\mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon(t))), & t\in(0,1),\\ \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon(0)=0, \qquad \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon(1)=\int_0^1\mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon(s)\,d\chi(s), & \end{cases} \tag{2.3}$$

and then, with the help of a simple transformation $y=-\epsilon$, (2.3) can be rewritten as follows:

$$\begin{cases}-\mathscr{D}_{\boldsymbol{t}}{}^{\alpha}y(t)=\mathfrak{L}^{-1}(f(t,y(t),\mathscr{D}_{\boldsymbol{t}}{}^{\gamma}y(t))), & t\in(0,1),\\ \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}y(0)=0, \qquad \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}y(1)=\int_0^1\mathscr{D}_{\boldsymbol{t}}{}^{\gamma}y(s)\,d\chi(s). & \end{cases} \tag{2.4}$$

Next we recall the theory of Riemann–Liouville fractional calculus, which will be used in the rest of this paper.

Definition 2.1 ([85]) The Riemann–Liouville fractional integral of order $\alpha>0$ of a function $x:(0,+\infty)\to\mathbb{R}$ is given by

$$I^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)\,ds$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2 ([85]) The Riemann–Liouville fractional derivative of order $\alpha>0$ of a continuous function $x:(0,+\infty)\to\mathbb{R}$ is given by

$$\mathscr{D}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t(t-s)^{n-\alpha-1}x(s)\,ds,$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwisely defined on $(0,+\infty)$.

Proposition 2.2 ([85])

(1) *If* $x, y : (0, +\infty) \to \mathbb{R}$ *with order* $\alpha > 0$, *then*

$$\mathscr{D}_t^{\alpha}\big(u(t) + v(t)\big) = \mathscr{D}_t^{\alpha} u(t) + \mathscr{D}_t^{\alpha} v(t).$$

(2) *If* $u \in L^1(0,1)$, $\nu > \gamma > 0$, *then*

$$I^{\nu} I^{\gamma} x(t) = I^{\nu+\gamma} u(t), \qquad \mathscr{D}_t^{\gamma} I^{\nu} u(t) = I^{\nu-\gamma} u(t), \qquad \mathscr{D}_t^{\gamma} I^{\gamma} u(t) = u(t). \tag{2.5}$$

(3) *If* $\alpha > 0$, $\gamma > 0$, *then*

$$\mathscr{D}_t^{\alpha} t^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} t^{\gamma-\alpha-1}.$$

(4) *Let* $\alpha > 0$, *and* $f(x)$ *is integrable, then*

$$I^{\alpha} \mathscr{D}_t^{\alpha} f(x) = f(x) + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \cdots + c_n x^{\alpha-n}, \tag{2.6}$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, *n is the smallest integer greater than or equal to* α.

Lemma 2.1 *Let* $y(t) = I^{\gamma}\varphi(t)$, $\varphi(t) \in C[0,1]$, *then model* (2.4) *is equivalent to the following integro-differential equation*:

$$\begin{cases} -\mathscr{D}_t^{\alpha-\gamma}\varphi(t) = \mathfrak{L}^{-1}(f(t, I^{\gamma}\varphi(t), \varphi(t))), \\ \varphi(0) = 0, \qquad \varphi(1) = \int_0^1 \varphi(s)\,d\chi(s). \end{cases} \tag{2.7}$$

Proof In fact, let $y(t) = I^{\gamma}\varphi(t)$, $\varphi(t) \in C[0,1]$, then from (2.5) one has

$$\mathscr{D}_t^{\gamma} y(t) = \mathscr{D}_t^{\gamma} I^{\gamma}\varphi(t) = \varphi(t). \tag{2.8}$$

On the other hand, notice $1 < \alpha \le 2$ and $1 < \alpha - \gamma < 2$, so by Definitions 2.1, 2.2 and (2.5), we also have

$$\begin{aligned} \mathscr{D}_t^{\alpha} y(t) &= \frac{d^2}{dt^2}\big(I^{2-\alpha} y(t)\big) = \frac{d^2}{dt^2}\big(I^{2-\alpha} I^{\gamma}\varphi(t)\big) = \frac{d^2}{dt^3}\big(I^{2-\alpha+\gamma}\varphi(t)\big) \\ &= \mathscr{D}_t^{\alpha-\gamma}\varphi(t). \end{aligned} \tag{2.9}$$

By (2.8), we have $\mathscr{D}_t^{\gamma} y(0) = \varphi(0) = 0$, $\varphi(1) = \int_0^1 \varphi(s)\,d\chi(s)$. And then it follows from (2.8) and (2.9) that

$$-\mathscr{D}_t^{\alpha-\gamma}\varphi(t) = \mathfrak{L}^{-1}\big(f\big(t, I^{\gamma}\varphi(t), \varphi(t)\big)\big).$$

Thus, model (2.4) is transformed into the integro-differential equation (2.7).

Conversely, if $\varphi \in C([0,1], [0,+\infty))$ is a solution for the integro-differential equation (2.7). Then letting $y(t) = I^{\gamma}\varphi(t)$ and using (2.8)and (2.9), we get

$$-\mathscr{D}_t^{\alpha} y(t) = -\mathscr{D}_t^{\alpha-\gamma}\varphi(t) = \mathfrak{L}^{-1}\big(f\big(t, I^{\gamma}\varphi(t), \varphi(t)\big)\big) = \mathfrak{L}^{-1}\big(f\big(t, y(t), \mathscr{D}_t^{\gamma} y(t)\big)\big), \quad 0 < t < 1,$$

and $\mathscr{D}_t{}^{\gamma}y(0) = \varphi(0) = 0$, $\mathscr{D}_t{}^{\gamma}y(1) = \int_0^1 \mathscr{D}_t{}^{\gamma}y(s)\,d\chi(s)$. Consequently, the integro-differential equation (2.7) is transformed into model (2.4). □

Thus in order to establish the existence and uniqueness of solution of model (1.4)–(1.5), we only need to focus on the integro-differential equation (2.7). We have the following lemma.

Lemma 2.2 ([86]) *Assume* $1 < \alpha - \gamma < 2$, *for a given function* $h \in L^1(0,1)$, *the boundary value problem*

$$\begin{cases} -\mathscr{D}_t{}^{\alpha-\gamma}\varphi(t) = h(t), & 0 < t < 1, \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{2.10}$$

has the unique solution

$$\varphi(t) = \int_0^1 G(t,s)h(s)\,ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha-\gamma)} \begin{cases} [t(1-s)]^{\alpha-\gamma-1}, & 0 \le t \le s \le 1, \\ [t(1-s)]^{\alpha-\gamma-1} - (t-s)^{\alpha-\gamma-1}, & 0 \le s \le t \le 1. \end{cases}$$

On the other hand, it follows from (2.6) that the unique solution of the following problem

$$\begin{cases} -\mathscr{D}_t{}^{\alpha-\gamma}\varphi(t) = 0, & 0 < t < 1, \\ \varphi(0) = 0, \qquad \varphi(1) = 1, \end{cases}$$

is $t^{\alpha-\gamma-1}$. Let

$$\mathcal{C} = \int_0^1 t^{\alpha-\gamma-1}\,d\chi(t), \qquad \mathcal{G}(s) = \int_0^1 G(t,s)\,d\chi(t).$$

According to the strategy of [87], the Green function of the integro-differential equation (2.7) is

$$H(t,s) = \frac{t^{\alpha-\gamma-1}}{1-\mathcal{C}}\mathcal{G}(s) + G(t,s). \tag{2.11}$$

Lemma 2.3 ([87]) *Assume* $0 \le \mathcal{C} < 1$ *and* $\mathcal{G}(s) \ge 0$ *for* $s \in [0,1]$, *then the functions* $G(t,s)$ *and* $H(t,s)$ *have the following properties*:

(1) $G(t,s) > 0$, $H(t,s) > 0$ *for* $t,s \in (0,1)$.

(2) *There exist two positive constants* a, b *such that*

$$at^{\alpha-\gamma-1}\mathcal{G}(s) \le H(t,s) \le bt^{\alpha-\gamma-1}, \quad t,s \in [0,1].$$

Our main tool is the fixed point theorem of mixed monotone operator. For convenience of the reader, here we first recall some definitions, notations, and known results; for details, see [88].

Let $(E, \|\cdot\|)$ be a real Banach space and P be a cone of E. Define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \subset P$. The cone P is called solid cone if its interior $\mathring{P}$ is nonempty and P is called normal if there exists a constant $M > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq M\|y\|$. The least positive number satisfying the above is called the normal constant of P.

Given $e \in P$ with $\|e\| \leq 1$, $e \neq \theta$. Define a subset of P as follows:

$$P_e = \{\varphi \in P : \text{there exist } \lambda > 0 \text{ and } \mu > 0 \text{ such that } \lambda e \leq \varphi \leq \mu e\}.$$

Obviously $P_e \subset P$, and if $e \in \mathring{P}$, then $P_e = \mathring{P}$.

Definition 2.3 Let P be a normal cone of a Banach space E. $A : P_e \times P_e \to P_e$ is called mixed monotone operator if $A(u, v)$ is nondecreasing in u and nonincreasing in v, i.e.,

$$u_1 \leq u_2, u_1, u_2 \in P \quad \text{implies} \quad A(u_1, v) \leq A(u_2, v)$$

for any $v \in P_e$, and

$$v_1 \leq v_2, v_1, v_2 \in P_e \quad \text{implies} \quad A(u, v_1) \geq A(u, v_2)$$

for any $u \in P_e$. The element $w^* \in P_e$ is called a fixed point of A if $A(w^*, w^*) = w^*$.

Lemma 2.4 ([89]) *Assume that $A : P_e \times P_e \to P_e$ is a mixed monotone operator. If there exists a constant $0 \leq \kappa < 1$ such that*

$$A\left(cx, \frac{1}{c}y\right) \geq c^{\kappa} A(x, y), \quad x, y \in P_e, 0 < c < 1. \tag{2.12}$$

Then the operator A has a unique fixed point $w^* \in P_e$. Moreover, for any initial value $(x_0, y_0) \in P_e \times P_e$, by constructing successively the sequences $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, we have $\|x_n - w^*\| \to 0$, $\|y_n - w^*\| \to 0$ as $n \to +\infty$, and

$$\left\|x_n - w^*\right\| = o\left(1 - r^{\kappa^n}\right), \qquad \left\|y_n - w^*\right\| = o\left(1 - r^{\kappa^n}\right),$$

where $0 < r < 1$, r is a constant from (x_0, y_0).

Lemma 2.5 ([88]) *Assume that $A : P_e \times P_e \to P_e$ is a mixed monotone operator and there exists $0 < \kappa < 1$ such that (2.12) holds. If w_λ^* is a unique solution of the equation*

$$A(x, x) = \lambda x, \quad \lambda > 0,$$

in P_e, then $\|w_\lambda^* - w_{\lambda_0}^*\| \to 0$, $\lambda \to \lambda_0$. If $0 < \kappa < \frac{1}{2}$, then $0 < \lambda_1 < \lambda_2$ implies that $w_{\lambda_1}^* \geq w_{\lambda_2}^*$, $w_{\lambda_1}^* \neq w_{\lambda_2}^*$, and

$$\lim_{\lambda \to 0^+} \left\|w_\lambda^*\right\| = +\infty, \qquad \lim_{\lambda \to +\infty} \left\|w_\lambda^*\right\| = 0.$$

3 Main results

To ensure the nonnegativity of Green function of model (2.7) and the development of our work, the following conditions are necessary:

(A0) χ is a function of bounded variation satisfying $\mathcal{G}(s) \geq 0$ for $s \in [0,1]$ and $0 \leq \mathcal{C} < 1$.

(A1) There exist two continuous functions $g : [0,1] \times [0,+\infty)^2 \to [0,+\infty)$, $h : [0,1] \times (0,+\infty)^2 \to [0,+\infty)$ with $g(t,1,1) > 0$, $h(t,1,1) > 0$ such that

$$f(t,x,y) = g(t,x,y) + h(t,x,y),$$

and for all $t \in [0,1]$, $g(t,x,y)$ is nondecreasing and $h(t,x,y)$ is nonincreasing in $x,y > 0$, respectively.

(A2) There exists a constant $0 < \kappa < 1 + \sigma$ such that, for all $x,y > 0$, $t \in [0,1]$ and for any $c \in (0,1)$,

$$g(t,cx,cy) \geq c^{\kappa} g(t,x,y), \qquad h\big(t,c^{-1}x,c^{-1}y\big) \geq c^{\kappa} h(t,x,y).$$

Remark 3.1 It follows from (A2) that for $r \geq 1$ and for all $x,y > 0$, $t \in [0,1]$

$$g(t,rx,ry) \leq r^{\kappa} g(t,x,y), \qquad h\big(t,r^{-1}x,r^{-1}y\big) \leq r^{\kappa} h(t,x,y).$$

Remark 3.2 Condition (A2) implies that h can be allowed to be singular on $x = y = 0$, and the order of singularity can be larger than 1, for example, $h(t,x,y) = x^{-\rho} + y^{-\theta}$, $\rho,\theta \in (1,1+\sigma)$.

Now define our work space $E = C[0,1]$ with the norm $\|\varphi\| := \max_{t\in[0,1]} |\varphi(t)|$ and a cone $P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0,1]\}$. Clearly, P is a normal cone of E with normal constant 1.

Denote

$$m_1 = \min_{t\in[0,1]} g(t,1,1), \qquad m_2 = \min_{t\in[0,1]} h(t,1,1),$$
$$M_1 = \max_{t\in[0,1]} g(t,1,1), \qquad M_2 = \max_{t\in[0,1]} h(t,1,1),$$

and

$$e(t) = t^{\alpha-\gamma-1}, \quad t \in [0,1].$$

Take a subset of P

$$P_e = \left\{\varphi \in P : \frac{1}{\eta} e(t) \leq \varphi(t) \leq \eta e(t), t \in [0,1]\right\}, \tag{3.1}$$

where

$$\eta > \max\left\{\left[\frac{b(1+\sigma)}{1+\sigma-\kappa(\alpha-1)}\mathfrak{L}^{-1}\big(\rho^{\kappa} M_1 + \varrho^{-\kappa} M_2\big)\right]^{\frac{1+\sigma}{1+\sigma-\kappa}}, 1, \rho^{-1},\right.$$
$$\left. 2\varrho, \left[a\mathfrak{L}^{-1}\big(\rho^{-\kappa} m_2\big)\int_0^1 \mathcal{G}(s)\,ds\right]^{-\frac{1+\sigma}{1+\sigma-\kappa}}\right\},$$

and

$$\rho = \max\left\{\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}, 1\right\}, \qquad \varrho = \min\left\{\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}, 1\right\}.$$

Then P_e is nonempty since $e(t) \in P_e$.

Now we state our main result as follows.

Theorem 3.1 *Assume that* (A0)–(A2) *hold. Then the singular Kelvin–Voigt model* (1.4)–(1.5) *has a unique solution* ϵ^*, *and there exist two constants* $0 < \nu < \mu$ *such that*

$$-\mu t^{\alpha-1} \le \epsilon^*(t) \le -\nu t^{\alpha-1}. \tag{3.2}$$

Moreover, for any initial $u_0, v_0 \in P_e$, *where* P_e *is defined by* (3.1), *construct successively two sequences*

$$u_n = \int_0^1 H(t,s)\mathfrak{L}^{-1}\bigl(g\bigl(s, I^\gamma u_{n-1}(s), u_{n-1}(s)\bigr) + h\bigl(s, I^\gamma v_{n-1}(s), v_{n-1}(s)\bigr)\bigr)\,ds,$$
$$n = 1, 2, \ldots, \tag{3.3}$$
$$v_n = \int_0^1 H(t,s)\mathfrak{L}^{-1}\bigl(g\bigl(s, I^\gamma v_{n-1}(s), v_{n-1}(s)\bigr) + h\bigl(s, I^\gamma u_{n-1}(s), u_{n-1}(s)\bigr)\bigr)\,ds,$$
$$n = 1, 2, \ldots, \tag{3.4}$$

then $u_n(t)$, $v_n(t)$ *converge uniformly to* $-\mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon^*(t)$ *on* $[0,1]$ *as* $n \to \infty$, *i.e.*,

$$\bigl\|u_n + \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon^*\bigr\| \to 0, \qquad \bigl\|v_n + \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon^*\bigr\| \to 0, \quad n \to \infty. \tag{3.5}$$

Furthermore, there exists a constant $0 < r < 1$ *such that*

$$\bigl\|u_n + \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon^*\bigr\| = o\bigl(1 - r^{\kappa^n}\bigr), \qquad \bigl\|v_n + \mathscr{D}_{\boldsymbol{t}}{}^{\gamma}\epsilon^*\bigr\| = o\bigl(1 - r^{\kappa^n}\bigr), \tag{3.6}$$

where r *depends on the initial value* (u_0, v_0).

Proof To obtain the uniqueness of positive solution for problem (1.4)–(1.5), we define an operator $A : P_e \times P_e \to P$ by

$$A(u,v)(t) = \int_0^1 H(t,s)\mathfrak{L}^{-1}\bigl(g\bigl(s, I^\gamma u(s), u(s)\bigr) + h\bigl(s, I^\gamma v(s), v(s)\bigr)\bigr)\,ds. \tag{3.7}$$

Firstly we show that $A : P_e \times P_e \to P$ is well defined. In fact, from the definition of Riemann–Liouville fractional integral, we have

$$I^\gamma e(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}s^{\alpha-\gamma-1}\,ds = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}t^{\alpha-1} \le \rho t^{\alpha-1}, \quad t \in [0,1]. \tag{3.8}$$

On the other hand, for any $u, v \in P_e$, we have

$$\frac{1}{\eta}e(t) \le u(t) \le \eta e(t), \frac{1}{\eta}e(t) \le v(t) \le \eta e(t), \quad t \in [0,1]. \tag{3.9}$$

Thus it follows from (3.8)–(3.9), (A1)–(A2), and $\eta\rho > 1$ that

$$g\big(t, I^{\gamma}u(t), u(t)\big) \le g\big(t, \eta\rho t^{\alpha-1}, \eta t^{\alpha-\gamma-1}\big) \le g(t, \eta\rho, \eta\rho) \le \rho^{\kappa}\eta^{\kappa}M_1, \tag{3.10}$$

and

$$\begin{aligned} h\big(t, I^{\gamma}v(t), v(t)\big) &\le h\left(t, \frac{\Gamma(\alpha-\gamma)}{\eta\Gamma(\alpha)}t^{\alpha-1}, \frac{1}{\eta}t^{\alpha-\gamma-1}\right) \\ &\le h\left(t, \frac{\varrho}{\eta}t^{\alpha-1}, \frac{\varrho}{\eta}t^{\alpha-\gamma-1}\right) \le \left(\frac{\varrho}{\eta}t^{\alpha-1}\right)^{-\kappa} h(t,1,1) \\ &\le \varrho^{-\kappa}\eta^{\kappa}t^{-\kappa(\alpha-1)}M_2. \end{aligned} \tag{3.11}$$

And then (3.10)–(3.11) and (2.2) yield

$$\begin{aligned} &\mathfrak{L}^{-1}\big(s, g\big(I^{\gamma}u(s), u(s)\big) + h\big(s, I^{\gamma}v(s), v(s)\big)\big) \\ &\quad \le \mathfrak{L}^{-1}\big(\rho^{\kappa}\eta^{\kappa}M_1 + \varrho^{-\kappa}\eta^{\kappa}s^{-\kappa(\alpha-1)}M_2\big) \\ &\quad \le s^{-\frac{\kappa(\alpha-1)}{1+\sigma}}\eta^{\frac{\kappa}{1+\sigma}}\mathfrak{L}^{-1}\big(\rho^{\kappa}M_1 + \varrho^{-\kappa}M_2\big). \end{aligned} \tag{3.12}$$

Notice that $0 < \kappa < 1 + \sigma$, we have

$$\frac{\kappa(\alpha-1)}{1+\sigma} < 1. \tag{3.13}$$

Thus, by using Lemma 2.3 and combining (3.12) and (3.13), one gets

$$\begin{aligned} A(u,v)(t) &\le \frac{b(1+\sigma)}{1+\sigma-\kappa(\alpha-1)}\eta^{\frac{\kappa}{1+\sigma}}\mathfrak{L}^{-1}\big(\rho^{\kappa}M_1 + \varrho^{-\kappa}M_2\big)t^{\alpha-\gamma-1} \le \eta t^{\alpha-\gamma-1} \\ &< +\infty, \quad t \in [0,1], \end{aligned}$$

which implies that $A : P_e \times P_e \to P$ is well defined and

$$A(u,v)(t) \le \eta t^{\alpha-\gamma-1}, \quad t \in [0,1]. \tag{3.14}$$

On the other hand, notice that $\eta\rho > 1$, it follows from (3.8) (3.9) and (A1)–(A2) that

$$\begin{aligned} h\big(t, I^{\gamma}v(t), v(t)\big) &\ge h\left(t, \frac{\eta\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}t^{\alpha-1}, \eta t^{\alpha-\gamma-1}\right) \\ &\ge h\big(t, \eta\rho t^{\alpha-1}, \eta\rho t^{\alpha-\gamma-1}\big) \\ &\ge h(t, \eta\rho, \eta\rho) \ge \eta^{-\kappa}\rho^{-\kappa}h(t,1,1) \ge \eta^{-\kappa}\rho^{-\kappa}m_2, \end{aligned}$$

which implies

$$
\begin{aligned}
A(u,v)(t) &= \int_0^1 H(t,s)\mathfrak{L}^{-1}\big(g\big(s,I^{\gamma}u(s),u(s)\big)+h\big(s,I^{\gamma}v(s),v(s)\big)\big) \\
&\geq \int_0^1 H(t,s)\mathfrak{L}^{-1}\big(h\big(s,I^{\gamma}v(s),v(s)\big)\big) \\
&\geq a\mathfrak{L}^{-1}\big(\eta^{-\kappa}\rho^{-\kappa}h(s,1,1)\big)\int_0^1 \mathcal{G}(s)dst^{\alpha-\gamma-1} \\
&\geq a\eta^{-\frac{\kappa}{1+\sigma}}\mathfrak{L}^{-1}\big(\rho^{-\kappa}m_2\big)\int_0^1 \mathcal{G}(s)dst^{\alpha-\gamma-1} \\
&\geq \frac{1}{\eta}t^{\alpha-\gamma-1}, \quad t\in[0,1].
\end{aligned}
\tag{3.15}
$$

Hence, (3.14) and (3.15) guarantee that $A: P_e \times P_e \to P_e$.

Next, we prove that $A: P_e\times P_e \to P_e$ is a mixed monotone operator. In fact, for any $u_1, u_2 \in P_e$ and $u_1 \leq u_2$, from the monotonicity of I^{γ}, $\mathfrak{L}^{-1}$, and g, we have

$$
\begin{aligned}
A(u_1,v)(t) &= \int_0^1 H(t,s)\mathfrak{L}^{-1}\big(g\big(s,I^{\gamma}u_1(s),u_1(s)\big)+h\big(s,I^{\gamma}v(s),v(s)\big)\big)\,ds \\
&\leq \int_0^1 H(t,s)\mathfrak{L}^{-1}\big(g\big(s,I^{\gamma}u_2(s),u_2(s)\big)+h\big(s,I^{\gamma}v(s),v(s)\big)\big)\,ds \\
&= A(u_2,v)(t),
\end{aligned}
\tag{3.16}
$$

which implies that

$$
A(u_1,v)(t) \leq A(u_2,v)(t), \quad v\in P_e, \tag{3.17}
$$

that is, $A(u,v)$ is nondecreasing in u for any $v \in P_e$. Similar to (3.16), if $v_1 \geq v_2$, $v_1, v_2 \in P_e$, the following formula is also valid:

$$
A(u,v_1)(t) \leq A(u,v_2)(t), \quad u\in P_e. \tag{3.18}
$$

So it follows from (3.17) and (3.18) that $A: P_e\times P_e \to P_e$ is a mixed monotone operator.

Finally, we prove that the operator A satisfies condition (2.12). For any $u, v \in P_e$ and $0 < c < 1$, it follows from (A2) that

$$
\begin{aligned}
A\left(cu,\frac{1}{c}v\right)(t) &= \int_0^1 H(t,s)\mathfrak{L}^{-1}\big(g\big(s,cI^{\gamma}u(s),cu(s)\big)+h\big(s,c^{-1}I^{\gamma}v(s),c^{-1}v(s)\big)\big)\,ds \\
&\geq \int_0^1 H(t,s)\mathfrak{L}^{-1}\big(c^{\kappa}g\big(s,I^{\gamma}u(s),u(s)\big)+c^{\kappa}h\big(s,I^{\gamma}v(s),v(s)\big)\big)\,ds \\
&\geq c^{\frac{\kappa}{1+\sigma}}\int_0^1 H(t,s)\mathfrak{L}^{-1}\big(g\big(s,I^{\gamma}u(s),u(s)\big)+h\big(s,I^{\gamma}v(s),v(s)\big)\big)\,ds \\
&= c^{\frac{\kappa}{1+\sigma}}A(u,v)(t), \quad t\in[0,1].
\end{aligned}
\tag{3.19}
$$

Since $0 < \kappa < 1 + \sigma$, we have $0 < \frac{\kappa}{1+\sigma} < 1$. It follows from (3.19) that (2.12) holds, thus Lemma 2.4 assures that the operator A has a unique fixed point $\varphi^* \in P_e$. Moreover, for any initial value $(u_0, v_0) \in P_e \times P_e$, construct successively the sequences:

$$u_n = \int_0^1 H(t,s)\mathfrak{L}^{-1}\big(g\big(s, I^\gamma u_{n-1}(s), u_{n-1}(s)\big) + h\big(s, I^\gamma v_{n-1}(s), v_{n-1}(s)\big)\big)\,ds, \quad n = 1, 2, \ldots,$$

$$v_n = \int_0^1 H(t,s)\mathfrak{L}^{-1}\big(g\big(s, I^\gamma v_{n-1}(s), v_{n-1}(s)\big) + h\big(s, I^\gamma u_{n-1}(s), u_{n-1}(s)\big)\big)\,ds, \quad n = 1, 2, \ldots.$$

Then $u_n(t)$, $v_n(t)$ converge uniformly to $\varphi^*(t)$ on $[0,1]$ as $n \to \infty$, i.e., $\|u_n - \varphi^*\| \to 0$, $\|v_n - \varphi^*\| \to 0$ as $n \to \infty$, and there exists a constant $0 < r < 1$ which depends on (x_0, y_0) such that

$$\|u_n - \varphi^*\| = o\big(1 - r^{\kappa^n}\big), \qquad \|v_n - \varphi^*\| = o\big(1 - r^{\kappa^n}\big).$$

In the end, by Lemma 2.1, the abstract Kelvin–Voigt model (1.4)–(1.5) has a unique solution $\epsilon^* = -I^\gamma \varphi^*(t)$. Since $\varphi^* \in P_e$, we have

$$-\mu t^{\alpha-1} = -\frac{\eta\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}t^{\alpha-1} \le \epsilon^*(t) = -I^\gamma \varphi^*(t) \le -\frac{\Gamma(\alpha-\gamma)}{\eta\Gamma(\alpha)}t^{\alpha-1} = -\nu t^{\alpha-1}$$

and $\|u_n - \varphi^*\| = \|u_n + \mathscr{D}_t{}^\gamma \epsilon^*\| \to 0$, $\|v_n - \varphi^*\| = \|v_n + \mathscr{D}_t{}^\gamma \epsilon^*\| \to 0$ as $n \to \infty$. Moreover, there exists a constant $0 < r < 1$ (which depends on the initial value (u_0, v_0)) such that

$$\|u_n + \mathscr{D}_t{}^\gamma \epsilon^*\| = o\big(1 - r^{\kappa^n}\big), \qquad \|v_n + \mathscr{D}_t{}^\gamma \epsilon^*\| = o\big(1 - r^{\kappa^n}\big).$$

Thus (3.2)–(3.6) hold and the proof of Theorem 3.1 is completed. □

Next we consider the following eigenvalue problem of model (1.4)–(1.5):

$$\begin{cases} \mathfrak{B}(\frac{1}{\lambda}\mathscr{D}_t{}^\alpha x(t))\mathscr{D}_t{}^\alpha x(t) = \lambda f(t, -x(t), -\mathscr{D}_t{}^\gamma x(t)), & t \in (0,1), \\ \mathscr{D}_t{}^\gamma x(0) = 0, \qquad \mathscr{D}_t{}^\gamma x(1) = \int_0^1 \mathscr{D}_t{}^\gamma x(s)\,d\chi(s). \end{cases} \tag{3.20}$$

According to Theorem 3.1 and Lemma 2.1, define an operator $A_\lambda : P_e \times P_e \to P$

$$A_\lambda(u,v)(t) = \lambda \int_0^1 H(t,s)\mathfrak{L}^{-1}\big(g\big(s, I^\gamma u(s), u(s)\big) + h\big(s, I^\gamma v(s), v(s)\big)\big)\,ds, \tag{3.21}$$

we have the following property of solution.

Theorem 3.2 *Assume that* (A0)–(A2) *hold. Then the eigenvalue problem* (3.20) *has a unique solution w_λ^*. Moreover, $0 < \lambda_1 < \lambda_2$ implies that $w_{\lambda_1}^* \le w_{\lambda_2}^*$, $w_{\lambda_1}^* \ne w_{\lambda_2}^*$. If $\kappa \in (0, \frac{1+\sigma}{2})$, then*

$$\lim_{\lambda \to 0^+} \|\mathscr{D}_t{}^\gamma \epsilon_\lambda^*\| = +\infty, \qquad \lim_{\lambda \to +\infty} \|\mathscr{D}_t{}^\gamma \epsilon_\lambda^*\| = 0. \tag{3.22}$$

Proof It follows from Theorem 3.1 that the operator A_λ (3.21) has a unique fixed point $\varphi_\lambda^* \in P_e$, which implies that the eigenvalue problem (3.20) has a unique solution $w_\lambda^* = -I^\gamma \varphi_\lambda^*$.

By Lemma 2.5, we have $0 < \lambda_1 < \lambda_2$ implies that $\varphi^*_{\lambda_1} \le \varphi^*_{\lambda_2}$, $\varphi^*_{\lambda_1} \ne \varphi^*_{\lambda_2}$, that is, $w^*_{\lambda_1} \ge w^*_{\lambda_2}$, $w^*_{\lambda_1} \ne w^*_{\lambda_2}$, and if $\kappa \in (0, \frac{1+\sigma}{2})$, then

$$\lim_{\lambda\to 0^+} \left\| \mathscr{D}_t{}^{\gamma} \epsilon^*_\lambda \right\| = +\infty, \qquad \lim_{\lambda\to +\infty} \left\| \mathscr{D}_t{}^{\gamma} \epsilon^*_\lambda \right\| = 0,$$

that is (3.22), so this completes the proof of Theorem 3.2. □

4 Numerical examples

Now we give some examples to illustrate our results.

Example 4.1 Let $\mathfrak{B}(x) = x^2$, consider the following abstract singular Kelvin–Voigt model:

$$\begin{cases} \mathfrak{B}(\mathscr{D}_t{}^{\frac{3}{2}}\epsilon(t))\mathscr{D}_t{}^{\frac{3}{2}}\epsilon(t) = \frac{2+t^2}{1+t^2}(\epsilon^{\frac{2}{3}}(t) + [-\mathscr{D}_t{}^{\frac{1}{4}}\epsilon(t)]^2 + \epsilon^{-\frac{4}{3}}(t) + [-\mathscr{D}_t{}^{\frac{1}{4}}\epsilon(t)]^{-\frac{2}{3}}), \\ \mathscr{D}_t{}^{\frac{1}{4}}\epsilon(0) = 0, \qquad \mathscr{D}_t{}^{\frac{1}{4}}\epsilon(1) = \int_0^1 \mathscr{D}_t{}^{\frac{1}{4}}\epsilon(s)\,d\chi(s), \end{cases} \tag{4.1}$$

where χ is a bounded variation function satisfying

$$\chi(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}), \\ 2, & t \in [\frac{1}{2}, \frac{3}{4}), \\ 1, & t \in [\frac{3}{4}, 1]. \end{cases}$$

Thus by simple computation, the problem (4.1) reduces to the following singular multi-point boundary value problem:

$$\begin{cases} \mathfrak{B}(\mathscr{D}_t{}^{\frac{3}{2}}\epsilon(t))\mathscr{D}_t{}^{\frac{3}{2}}\epsilon(t) = \frac{2+t^2}{1+t^2}(\epsilon^{\frac{2}{3}}(t) + [-\mathscr{D}_t{}^{\frac{1}{4}}\epsilon(t)]^2 + \epsilon^{-\frac{4}{3}}(t) + [-\mathscr{D}_t{}^{\frac{1}{4}}\epsilon(t)]^{-\frac{2}{3}}), \\ \mathscr{D}_t{}^{\frac{1}{4}}\epsilon(0) = 0, \qquad \mathscr{D}_t{}^{\frac{1}{4}}\epsilon(1) = 2\mathscr{D}_t{}^{\frac{1}{4}}(\frac{1}{2}) - \mathscr{D}_t{}^{\frac{1}{4}}(\frac{3}{4}). \end{cases} \tag{4.2}$$

Corollary 4.1 *The abstract singular Kelvin–Voigt model* (4.1) *has a unique positive solution* ϵ^*, *and there exist two constants* $0 < \nu < \mu$ *such that*

$$-\mu t^{\frac{1}{2}} \le \epsilon^*(t) \le -\nu t^{\frac{1}{2}}. \tag{4.3}$$

Moreover, for any initial $u_0, v_0 \in P_e$, *construct successively two sequences*:

$$\begin{aligned} u_n ={}& \int_0^1 \left[4t^{\frac{1}{4}}\mathcal{G}(s) + G(t,s)\right]\left(\frac{2+s^2}{1+s^2}\right)^{\frac{1}{3}} \\ &\times \left(\left(I^{\frac{1}{4}}u_{n-1}(s)\right)^{\frac{2}{3}} + u^2_{n-1}(s) + \left(I^{\frac{1}{4}}v_{n-1}(s)\right)^{-\frac{4}{5}} + v^{\frac{2}{3}}_{n-1}(s)\right)^{\frac{1}{3}}\,ds, \quad n = 1, 2, \ldots, \\ v_n ={}& \int_0^1 \left[4t^{\frac{1}{4}}\mathcal{G}(s) + G(t,s)\right]\left(\frac{2+s^2}{1+s^2}\right)^{\frac{1}{3}} \\ &\times \left(\left(I^{\frac{1}{4}}v_{n-1}(s)\right)^{\frac{2}{3}} + v^2_{n-1}(s) + \left(I^{\frac{1}{4}}u_{n-1}(s)\right)^{-\frac{4}{5}} + u^{\frac{2}{3}}_{n-1}(s)\right)^{\frac{1}{3}}\,ds, \quad n = 1, 2, \ldots, \end{aligned}$$

which converge uniformly to $-\mathscr{D}_t{}^{\frac{1}{4}}\epsilon^*(t)$ *on* $[0,1]$ *as* $n \to \infty$, *and there exists a constant* $0 < r < 1$ *such that*

$$\left\| u_n + \mathscr{D}_t{}^{\frac{1}{4}}\epsilon^* \right\| = o\left(1 - r^{2^n}\right), \qquad \left\| v_n + \mathscr{D}_t{}^{\frac{1}{4}}\epsilon^* \right\| = o\left(1 - r^{2^n}\right).$$

Proof We only need to consider the equivalent equation (4.2). Firstly, comparing with the general model (1.4)–(1.5), one gets

$$\gamma = \frac{1}{4}, \qquad \alpha = \frac{3}{2}, \qquad \sigma = 2, \qquad f(t,u,v) = \frac{2+t^2}{1+t^2}\left(u^{\frac{2}{3}} + v^2 + u^{-\frac{4}{5}} + v^{-\frac{2}{3}}\right)$$

and

$$\mathcal{C} = \int_0^1 t^{\alpha-1}\,d\chi(t) = 2 \times \left(\frac{1}{2}\right)^{\frac{1}{4}} - \left(\frac{3}{4}\right)^{\frac{1}{4}} = 0.7512 < 1, \qquad \mathcal{G}(s) \geq 0.$$

Next let $g(t,u,v) = \frac{2+t^2}{1+t^2}(u^{\frac{2}{3}} + v^2)$, $h(t,u,v) = \frac{2+t^2}{1+t^2}(u^{-\frac{4}{5}} + v^{-\frac{2}{3}})$, then for any $u, v > 0$ and $0 < c < 1$, we have

$$\begin{aligned} g(t,cu,cv) &= \frac{2+t^2}{1+t^2}\left(c^{\frac{2}{3}}u^{\frac{2}{3}} + c^2v^2\right) \geq c^2 g(t,u,v), \\ h\left(t,c^{-1}u,c^{-1}v\right) &= \frac{2+t^2}{1+t^2}\left(\left(c^{-1}u\right)^{-\frac{4}{5}} + \left(c^{-1}v\right)^{-\frac{2}{3}}\right) \geq c^2 h(t,u,v). \end{aligned}$$

Take $\kappa = 2$, then $0 < \kappa < 1 + \sigma$, and $g : [0,1] \times [0,+\infty)^2 \to [0,+\infty)$, $h : [0,1] \times (0,+\infty)^2 \to [0,+\infty)$ are continuous, and for all $t \in [0,1]$, $g(t,u,v)$ is nondecreasing and $h(t,u,v)$ is nonincreasing in $u, v > 0$, respectively. Thus all the conditions of Theorem 3.1 hold, according to Theorem 3.1, the singular Kelvin–Voigt model (4.1) has a unique positive solution ϵ^*, and there exist two constants $0 < \nu < \mu$ such that (4.3) holds.

Moreover, for any initial $u_0, v_0 \in P_e$, construct successively two sequences:

$$\begin{aligned} u_n &= \int_0^1 \left[4t^{\frac{1}{4}}\mathcal{G}(s) + G(t,s)\right]\left(\frac{2+s^2}{1+s^2}\right)^{\frac{1}{3}} \\ &\quad \times \left(\left(I^{\frac{1}{4}}u_{n-1}(s)\right)^{\frac{2}{3}} + u_{n-1}^2(s) + \left(I^{\frac{1}{4}}v_{n-1}(s)\right)^{-\frac{4}{5}} + v_{n-1}^{\frac{2}{3}}(s)\right)^{\frac{1}{3}}\,ds, \quad n = 1,2,\ldots, \\ v_n &= \int_0^1 \left[4t^{\frac{1}{4}}\mathcal{G}(s) + G(t,s)\right]\left(\frac{2+s^2}{1+s^2}\right)^{\frac{1}{3}} \\ &\quad \times \left(\left(I^{\frac{1}{4}}v_{n-1}(s)\right)^{\frac{2}{3}} + v_{n-1}^2(s) + \left(I^{\frac{1}{4}}u_{n-1}(s)\right)^{-\frac{4}{5}} + u_{n-1}^{\frac{2}{3}}(s)\right)^{\frac{1}{3}}\,ds, \quad n = 1,2,\ldots, \end{aligned}$$

which converge uniformly to $-\mathscr{D}_t^{\frac{1}{4}}\epsilon^*(t)$ on $[0,1]$ as $n \to \infty$, and there exists a constant $0 < r < 1$ such that

$$\left\|u_n + \mathscr{D}_t^{\frac{1}{4}}\epsilon^*\right\| = o\left(1 - r^{2^n}\right), \qquad \left\|v_n + \mathscr{D}_t^{\frac{1}{4}}\epsilon^*\right\| = o\left(1 - r^{2^n}\right). \qquad \square$$

Example 4.2 Let $\mathfrak{B}(x) = x^{\frac{1}{2}}$, consider the following eigenvalue problem of singular Kelvin–Voigt model:

$$\begin{cases} \mathfrak{B}(\mathscr{D}_t^{\frac{5}{3}}\epsilon(t))\mathscr{D}_t^{\frac{5}{3}}\epsilon(t) = \lambda\left[\frac{2+\sin t}{1+e^t+\cos t} + 2t^2[-\mathscr{D}_t^{\frac{1}{3}}\epsilon(t)]^{\frac{2}{3}} + \epsilon^{-\frac{1}{2}}(t)\right], \\ \mathscr{D}_t^{\frac{1}{3}}\epsilon(0) = 0, \qquad \mathscr{D}_t^{\frac{1}{3}}\epsilon(1) = \int_0^1 \mathscr{D}_t^{\frac{1}{3}}\epsilon(s)\,d\chi(s), \end{cases} \tag{4.4}$$

where χ is a bounded variation function satisfying

$$\chi(t)=\begin{cases}0, & t\in[0,\frac{1}{3}),\\ \frac{5}{2}, & t\in[\frac{1}{3},\frac{2}{3}),\\ 2, & t\in[\frac{2}{3},1].\end{cases}$$

Thus by simple computation, the problem (4.4) reduces to the following singular multi-point boundary value problem:

$$\begin{cases}\mathfrak{B}(\mathscr{D}_t^{\frac{5}{3}}\epsilon(t))\mathscr{D}_t^{\frac{5}{3}}\epsilon(t)=\lambda[\frac{2+\sin t}{1+e^t+\cos t}+2t^2[-\mathscr{D}_t^{\frac{1}{3}}\epsilon(t)]^{\frac{2}{3}}+\epsilon^{-\frac{1}{2}}(t)],\\ \mathscr{D}_t^{\frac{1}{3}}\epsilon(0)=0, \qquad \mathscr{D}_t^{\frac{1}{3}}\epsilon(1)=\frac{5}{2}\mathscr{D}_t^{\frac{1}{3}}(\frac{1}{3})-\frac{1}{2}\mathscr{D}_t^{\frac{1}{3}}(\frac{2}{3}).\end{cases} \tag{4.5}$$

Corollary 4.2 *The abstract singular Kelvin–Voigt model* (4.4) *has a unique positive solution* ϵ^*, *and there exist two constants* $0<\nu<\mu$ *such that*

$$-\mu t^{\frac{2}{3}}\le\epsilon^*(t)\le-\nu t^{\frac{2}{3}}. \tag{4.6}$$

Moreover, for any initial $u_0,v_0\in P_e$, *construct successively two sequences*:

$$u_n=\lambda\int_0^1\left[\frac{t^{\frac{1}{3}}}{0.9815}\mathcal{G}(s)+G(t,s)\right]\left(\frac{2+\sin s}{1+e^s+\cos s}+2s^2u_{n-1}^{\frac{2}{3}}(s)+\left(I^{\frac{1}{3}}v_{n-1}(s)\right)^{-\frac{1}{2}}\right)^{\frac{2}{3}}ds,$$
$$n=1,2,\ldots,$$
$$v_n=\lambda\int_0^1\left[\frac{t^{\frac{1}{3}}}{0.9815}\mathcal{G}(s)+G(t,s)\right]\left(\frac{2+\sin s}{1+e^s+\cos s}+2s^2v_{n-1}^{\frac{2}{3}}(s)+\left(I^{\frac{1}{3}}u_{n-1}(s)\right)^{-\frac{1}{2}}\right)^{\frac{2}{3}}ds,$$
$$n=1,2,\ldots,$$

which converge uniformly to $-\mathscr{D}_t^{\frac{1}{4}}\epsilon^*(t)$ *on* $[0,1]$ *as* $n\to\infty$, *and there exists a constant* $0<r<1$ *such that*

$$\left\|u_n+\mathscr{D}_t^{\frac{1}{4}}\epsilon^*\right\|=o\left(1-r^{2^n}\right), \qquad \left\|v_n+\mathscr{D}_t^{\frac{1}{4}}\epsilon^*\right\|=o\left(1-r^{2^n}\right).$$

In addition, we also have

$$\lim_{\lambda\to0^+}\left\|\mathscr{D}_t^{\gamma}\epsilon_\lambda^*\right\|=+\infty, \qquad \lim_{\lambda\to+\infty}\left\|\mathscr{D}_t^{\gamma}\epsilon_\lambda^*\right\|=0.$$

Proof We consider equation (4.5). Let

$$\gamma=\frac{1}{3}, \qquad \alpha=\frac{5}{3}, \qquad \sigma=\frac{1}{2}, \qquad f(t,u,v)=\frac{2+\sin t}{1+e^t+\cos t}+2t^2u^{\frac{2}{3}}+v^{-\frac{1}{2}}$$

and

$$\mathcal{C}=\int_0^1t^{\alpha-1}\,d\chi(t)=\frac{5}{2}\times\left(\frac{1}{3}\right)^{\frac{2}{3}}-\frac{1}{2}\times\left(\frac{2}{3}\right)^{\frac{2}{3}}=0.0185<1, \qquad \mathcal{G}(s)\ge0.$$

Take $g(t,u,v) = \frac{2+\sin t}{2(1+e^t+\cos t)} + 2t^2u^{\frac{2}{3}}$, $h(t,u,v) = \frac{2+\sin t}{2(1+e^t+\cos t)} + v^{-\frac{1}{2}}$, then $g(t,1,1) > 0$, $h(t,1,1) > 0$, and for any $u,v > 0$ and $0 < c < 1$, we have

$$\begin{aligned} g(t,cu,cv) &= \frac{2+\sin t}{2(1+e^t+\cos t)} + 2t^2(cu)^{\frac{2}{3}} \geq c^{\frac{2}{3}}\left[\frac{2+\sin t}{2(1+e^t+\cos t)} + 2t^2u^{\frac{2}{3}}\right] \\ &= c^{\frac{2}{3}}g(t,u,v), \\ h\bigl(t,c^{-1}u,c^{-1}v\bigr) &= \frac{2+\sin t}{2(1+e^t+\cos t)} + (cv)^{-\frac{1}{2}} \geq c^{\frac{2}{3}}\left[\frac{2+\sin t}{2(1+e^t+\cos t)} + (cv)^{-\frac{1}{2}}\right] \\ &= c^{\frac{2}{3}}h(t,u,v). \end{aligned}$$

Take $\kappa = \frac{2}{3}$, then $0 < \kappa < 1 + \sigma = \frac{3}{2}$, and $g : [0,1] \times [0,+\infty)^2 \to [0,+\infty)$, $h : [0,1] \times (0,+\infty)^2 \to [0,+\infty)$ are continuous, and for all $t \in [0,1]$, $g(t,u,v)$ is nondecreasing and $h(t,u,v)$ is nonincreasing in $u,v > 0$, respectively. Thus all the conditions of Theorem 3.1 hold, according to Theorem 3.1, the singular Kelvin–Voigt model (4.4) has a unique positive solution ϵ^*, and there exist two constants $0 < \nu < \mu$ such that (4.6) holds.

Moreover, for any initial $u_0, v_0 \in P_e$, construct successively two sequences:

$$\begin{aligned} u_n &= \lambda \int_0^1 \left[\frac{t^{\frac{1}{3}}}{0.9815}\mathcal{G}(s) + G(t,s)\right]\left(\frac{2+\sin s}{1+e^s+\cos s} + 2s^2u_{n-1}^{\frac{2}{3}}(s) + \bigl(I^{\frac{1}{3}}v_{n-1}(s)\bigr)^{-\frac{1}{2}}\right)^{\frac{2}{3}}ds, \\ &\quad n = 1,2,\ldots, \\ v_n &= \lambda \int_0^1 \left[\frac{t^{\frac{1}{3}}}{0.9815}\mathcal{G}(s) + G(t,s)\right]\left(\frac{2+\sin s}{1+e^s+\cos s} + 2s^2v_{n-1}^{\frac{2}{3}}(s) + \bigl(I^{\frac{1}{3}}u_{n-1}(s)\bigr)^{-\frac{1}{2}}\right)^{\frac{2}{3}}ds, \\ &\quad n = 1,2,\ldots, \end{aligned}$$

which converge uniformly to $-\mathscr{D}_t^{\frac{1}{3}}\epsilon^*(t)$ on $[0,1]$ as $n \to \infty$, and there exists a constant $0 < r < 1$ such that

$$\bigl\|u_n + \mathscr{D}_t^{\frac{1}{3}}\epsilon^*\bigr\| = o\bigl(1-r^{2^n}\bigr), \qquad \bigl\|v_n + \mathscr{D}_t^{\frac{1}{3}}\epsilon^*\bigr\| = o\bigl(1-r^{2^n}\bigr).$$

In particular, if we take $\lambda = 3,5$, then $\epsilon_3^*(t) \leq \epsilon_5^*(t)$. Since $\kappa = \frac{2}{3} \in (0, \frac{1+\sigma}{2}) = (0, \frac{3}{4})$, we have

$$\lim_{\lambda\to0^+}\bigl\|\mathscr{D}_t^{\gamma}\epsilon_\lambda^*\bigr\| = +\infty, \qquad \lim_{\lambda\to+\infty}\bigl\|\mathscr{D}_t^{\gamma}\epsilon_\lambda^*\bigr\| = 0. \qquad \square$$

5 Conclusion

In this work, we introduce a new nonlinear operator to generalize a standard Kelvin–Voigt model. By using the fixed point theorem of the mixed monotone operator, we not only establish the uniqueness of solution of this model, but also give an iterative scheme converging to the unique solution of the model. Especially, a nonlinear function of the model may have stronger singularity at some points of the strain vanishing, which can describe the case of instantaneous fracture of relaxation processes.

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Authors' contributions
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