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A coupled system of fractional differential equations on the half-line

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Abstract

In this paper, we consider a new fractional differential system on an unbounded domain

$$D^{\alpha}u(t)+\varphi\bigl(t,v(t),D^{\gamma_1}v(t)\bigr)=0,\quad t\in[0,+\infty),\alpha\in(2,3],$$
$$D^{\beta}v(t)+\psi\bigl(t,u(t),D^{\gamma_2}u(t)\bigr)=0,\quad t\in[0,+\infty),\beta\in(2,3],$$

subject to the conditions

$$I^{3-\alpha}u(t)|_{t=0}=0,\qquad D^{\alpha-2}u(t)|_{t=0}=\int_0^h g_1(s)u(s)\,ds,\qquad D^{\alpha-1}u(+\infty)=Mu(\xi)+a,$$
$$I^{3-\beta}v(t)|_{t=0}=0,\qquad D^{\beta-2}v(t)|_{t=0}=\int_0^h g_2(s)v(s)\,ds,\qquad D^{\beta-1}v(+\infty)=Nv(\eta)+b.$$

The nonlinear terms φ and ψ are dependent on the fractional derivative of lower order $\gamma_i\in(0,1)$, $i=1,2$, which creates additional complexity to verify the existence of solutions. Moreover, a proper choice of Banach space allows the solutions to be defined on the half-line. From some standard fixed point theorems, sufficient conditions for the existence and uniqueness of solutions to boundary value problems are developed. Finally, the main result is applied to an illustrative example.

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Keywords: Fractional differential equations; Existence and uniqueness; Unbounded domain; Standard fixed point theorems; Boundary value problem

1 Introduction

Fractional calculus has recently evolved as an excellent tool for mathematical modeling owing to its widespread applications in the fields of engineering, physics, electrodynamics of complex medium, photoelasticity, etc; one can see [1–12] and the references cited therein. Meanwhile, relevant theory of fractional differential and integral equations has been established, and the research on fractional differential equations for boundary value problems is in a stage of rapid development.

Based on some kinds of analytical techniques, boundary value problems involving fractional differential equations attracted a considerable attention; see [13–33] and the references therein. It not only has promotional value and practical significance in medical image processing, seismic analysis, and large-scale climate research, but also has important research potential on numerical analysis.

Recently, the study of coupled systems involving fractional differential equations appeared in the literature [4, 9, 10, 17, 18, 32, 33]. Much of the work has been considered on finite intervals; however, a study of boundary value problems on unbounded domain is well under way. Wang, Ahmad, and Zhang [34] studied a coupled system of fractional differential equations with m-point fractional boundary conditions

$$\begin{cases} D^p u(t) + f(t, v(t)) = 0, & p \in (2,3), \\ D^q v(t) + g(t, u(t)) = 0, & q \in (2,3), \\ u(0) = u'(0) = 0, & D^{p-1}u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \\ v(0) = v'(0) = 0, & D^{q-1}v(+\infty) = \sum_{i=1}^{m-2} \gamma_i v(\xi_i), \end{cases}$$

where $t \in J = [0, +\infty)$, $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $\beta_i, \gamma_i > 0$ such that $0 < \sum_{i=1}^{m-2} \beta_i u(\xi_i) < \Gamma(p)$ and $0 < \sum_{i=1}^{m-2} \gamma_i v(\xi_i) < \Gamma(q)$. D^p, D^q denote the standard Riemann–Liouville fractional derivatives. By virtue of standard fixed point theorems, the authors discussed the existence and uniqueness of solutions.

In [35], the authors investigated a class of fractional differential equations on an infinite interval

$$D_{0^+}^{\alpha} u(t) + f\bigl(t, u(t), D_{0^+}^{\alpha-1} u(t)\bigr) = 0, \quad t \in (0, +\infty),$$

with integral boundary conditions

$$\begin{cases} u(0) = 0, \\ D_{0^+}^{\alpha-1} u(\infty) = \int_0^\tau g_1(s) u(s)\, ds + a, \\ D_{0^+}^{\alpha-2} u(0) = \int_0^\tau g_2(s) u(s)\, ds + b, \end{cases}$$

where $2 < \alpha \leq 3$, $f : \mathbb{R}^+ \times (\mathbb{R}^+)^2 \to \mathbb{R}^+$, $f(t, u, v) \not\equiv 0$, and f satisfies L^1-Carathéodory conditions. Existence results for positive solutions to the boundary value problem were obtained in three cases by using Krasnoselskii's fixed point theorem.

To our knowledge, some remarkable results on the existence and multiplicity of solutions for fractional differential equations have been discussed widely on finite intervals [14–33]. Instead, it is relatively rare for work to be done related to existence results on infinite intervals [34–47].

In [31], the authors discussed the existence and uniqueness of positive solutions for the fractional differential equation

$$\begin{cases} D^\alpha u(t) = f(t, u(t), D^p u(t)), \\ I^{3-\alpha} u(0) = D^{\alpha-2} u(0) = u(1) = 0, \end{cases}$$

where $0 < p < 1$, $2 < \alpha < 3$, $t \in (0,1)$, D^α is the standard Riemann–Liouville fractional derivative of order α. By applying a nonlinear alternative of Leray–Schauder type and the Banach contraction theorem, the existence and uniqueness of solutions were obtained.

Motivated by the above papers, we are devoted to establishing some results on the existence and uniqueness of solutions for a new coupled system of nonlinear fractional differential equations

$$\begin{cases} D^{\alpha}u(t) = -\varphi(t,v(t),D^{\gamma_1}v(t)), \quad \gamma_1 \in (0,1), \\ D^{\beta}v(t) = -\psi(t,u(t),D^{\gamma_2}u(t)), \quad \gamma_2 \in (0,1), \\ I^{3-\alpha}u(0) = 0, \qquad D^{\alpha-2}u(0) = \int_0^h g_1(s)u(s)\,ds, \qquad D^{\alpha-1}u(+\infty) = Mu(\xi) + a, \\ I^{3-\beta}v(0) = 0, \qquad D^{\beta-2}v(0) = \int_0^h g_2(s)v(s)\,ds, \qquad D^{\beta-1}v(+\infty) = Nv(\eta) + b, \end{cases} \tag{1.1}$$

where $2 < \alpha,\beta \leq 3$, $0 < \gamma_i < 1$, $i = 1,2$, $t \in J = [0,+\infty)$, M, N are real numbers with $0 < M\xi^{\alpha-1} < \Gamma(\alpha)$, $0 < N\eta^{\beta-1} < \Gamma(\beta)$, $\xi,\eta,h > 0$, parameters $a,b \in \mathbb{R}^+$, $g_1,g_2 \in L^1[0,h]$ are nonnegative functions, $\varphi,\psi \in C(J \times \mathbb{R} \times \mathbb{R},\mathbb{R})$ and D^α, D^β denote the fractional derivatives of Riemann–Liouville type of order α and β. Our conclusion is a natural expansion of the previous results in [31].

In this paper, the aim is to deal with the new coupled system of fractional differential equations on infinite intervals. Sufficient conditions for the existence and uniqueness of unbounded solutions for system (1.1) are obtained base upon Schauder's fixed point theorem and the Banach contraction theorem. Unlike previous works, the main difficulty of this paper is that we have to construct an appropriate Banach space, because the functions φ, ψ contain the fractional derivatives.

2 Preliminaries and auxiliary results

For the convenience of the readers, we recall some useful definitions and lemmas.

Definition 2.1 ([1]) The fractional integral of Riemann–Liouville type of order $\alpha > 0$ of a function f is defined as

$$\bigl(I^\alpha f\bigr)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds, \quad \alpha > 0,$$

provided the integral exists.

Definition 2.2 ([1]) The fractional derivative of Riemann–Liouville type of order $\alpha > 0$ of a function f is given by

$$\bigl(D^\alpha f\bigr)(t) = \bigl(D^{\lceil\alpha\rceil}I^{\lceil\alpha\rceil-\alpha}f\bigr)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

For further analysis, let

$$\sigma_1 = \frac{1}{\Gamma(\alpha) - M\xi^{\alpha-1}}, \qquad \sigma_2 = \frac{1}{\Gamma(\beta) - N\eta^{\beta-1}},$$

$$\omega_1 = \int_0^h g_1(t)t^{\alpha-1}\,dt, \qquad \omega_2 = \int_0^h g_2(t)t^{\beta-1}\,dt,$$

$$\delta_1 = \int_0^h \bigl(1+t^{\alpha-1}\bigr)g_1(t)\,dt, \qquad \delta_2 = \int_0^h \bigl(1+t^{\beta-1}\bigr)g_2(t)\,dt,$$

$$T_1(t) = \frac{\sigma_1 M\xi^{\alpha-2}t^{\alpha-1} + t^{\alpha-2}}{\Gamma(\alpha-1)}, \qquad T_2(t) = \frac{\sigma_2 N\eta^{\beta-2}t^{\beta-1} + t^{\beta-2}}{\Gamma(\beta-1)},$$
$$l_1 = \frac{1+\sigma_1 M\xi^{\alpha-2}}{(1-\mu_1)\Gamma(\alpha-1)}, \qquad l_2 = \frac{1+\sigma_2 N\eta^{\beta-2}}{(1-\mu_2)\Gamma(\beta-1)}.$$

In this paper, we always assume that $g_i : [0,+\infty) \to [0,+\infty)$ are continuous, and $\mu_i = \int_0^h g_i(t)T_i(t)\,dt < 1$, $i = 1,2$.

Lemma 2.1 *Assume that* $f \in L^1(J)$ *with* $0 < M\xi^{\alpha-1} < \Gamma(\alpha)$, $\alpha \in (2,3]$. *Then the fractional differential equation*

$$D^\alpha u(t) + f(t) = 0, \quad t \in [0,+\infty),$$

with

$$\begin{cases} I^{3-\alpha}u(0) = 0, \\ D^{\alpha-2}u(0) = \int_0^h g_1(s)u(s)\,ds, \\ D^{\alpha-1}u(+\infty) = Mu(\xi) + a, \end{cases}$$

has the solution

$$u(t) = a\sigma_1 t^{\alpha-1} + \frac{a\sigma_1\omega_1 T_1(t)}{1-\mu_1} + \int_0^{+\infty} H(t,s)f(s)\,ds,$$

where

$$H(t,s) = H_1(t,s) + \frac{T_1(t)}{1-\mu_1}\int_0^h H_1(\tau,s)g_1(\tau)\,d\tau, \tag{2.1}$$
$$H_1(t,s) = \sigma_1 M t^{\alpha-1}G_1(\xi,s) + G_1(t,s), \tag{2.2}$$

and

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t < +\infty, \\ t^{\alpha-1}, & 0 \le t \le s < +\infty. \end{cases} \tag{2.3}$$

Proof First, we can reduce the above problem to an equivalent integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - I^\alpha f(t) \tag{2.4}$$

for some $c_i \in \mathbb{R}$, $i = 1,2,3$. By the condition $I^{3-\alpha}u(t)|_{t=0} = 0$, we have

$$\begin{aligned} I^{3-\alpha}u(t) &= c_1 I^{3-\alpha}t^{\alpha-1} + c_2 I^{3-\alpha}t^{\alpha-2} + c_3 I^{3-\alpha}t^{\alpha-3} - I^{3-\alpha}I^\alpha f(t) \\ &= c_1\frac{\Gamma(\alpha)}{\Gamma(3)}t^2 + c_2\frac{\Gamma(\alpha-1)}{\Gamma(2)}t + c_3\frac{\Gamma(\alpha-2)}{\Gamma(1)} - I^3 f(t), \end{aligned}$$

since $I^3 f(t) \to 0$ as $t \to 0$, we must set $c_3 = 0$. On application of $D^{\alpha-2}u(0) = \int_0^h g_1(s)u(s)\,ds$ and $D^{\alpha-1}u(+\infty) = Mu(\xi) + a$, we have

$$\begin{aligned} D^{\alpha-2}u(t) &= c_1 D^{\alpha-2}t^{\alpha-1} + c_2 D^{\alpha-2}t^{\alpha-2} - D^{\alpha-2}I^{\alpha}f(t) \\ &= c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) - \int_0^t (t-s)f(s)\,ds, \\ D^{\alpha-1}u(t) &= c_1 D^{\alpha-1}t^{\alpha-1} + c_2 D^{\alpha-1}t^{\alpha-2} - D^{\alpha-1}I^{\alpha}f(t) \\ &= c_1\Gamma(\alpha) - \int_0^t f(s)\,ds, \end{aligned}$$

that is,

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(\alpha) - M\xi^{\alpha-1}}\biggl(a + \int_0^{+\infty} f(s)\,ds + \frac{M\xi^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^h g_1(s)u(s)\,ds \\ &\quad - \frac{M}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}f(s)\,ds\biggr), \\ c_2 &= \frac{1}{\Gamma(\alpha-1)}\int_0^h g_1(s)u(s)\,ds. \end{aligned}$$

This implies

$$\begin{aligned} u(t) &= \sigma_1 t^{\alpha-1}\biggl(a + \int_0^{+\infty} f(s)\,ds + \frac{M\xi^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^h g_1(s)u(s)\,ds \\ &\quad - \frac{M}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}f(s)\,ds\biggr) \\ &\quad + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^h g_1(s)u(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds \\ &= a\sigma_1 t^{\alpha-1} + \frac{\sigma_1 M\xi^{\alpha-2}t^{\alpha-1} + t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^h g_1(s)u(s)\,ds - \frac{\sigma_1 M t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}f(s)\,ds \\ &\quad + \biggl(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\sigma_1 M\xi^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)}\biggr)\int_0^{+\infty} f(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds \\ &= a\sigma_1 t^{\alpha-1} + \frac{\sigma_1 M\xi^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty} f(s)\,ds - \frac{\sigma_1 M t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}f(s)\,ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty} f(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds + T_1(t)\int_0^h g_1(s)u(s)\,ds \\ &= a\sigma_1 t^{\alpha-1} + \sigma_1 M t^{\alpha-1}\int_0^{+\infty} G_1(\xi,s)f(s)\,ds + \int_0^{+\infty} G_1(t,s)f(s)\,ds \\ &\quad + T_1(t)\int_0^h g_1(s)u(s)\,ds. \end{aligned}$$

Multiplying both sides of the above equality by $g_1(t)$ and integrating from 0 to h, then

$$\begin{aligned} \int_0^h g_1(t)u(t)\,dt &= a\sigma_1\int_0^h g_1(t)t^{\alpha-1}\,dt + M\sigma_1\int_0^h g_1(t)t^{\alpha-1}\,dt\int_0^{+\infty} G_1(\xi,s)f(s)\,ds \\ &\quad + \int_0^h g_1(t)\int_0^{+\infty} G_1(t,s)f(s)\,ds\,dt + \int_0^h g_1(t)T_1(t)\,dt\int_0^h g_1(s)u(s)\,ds. \end{aligned}$$

Next we have

$$\int_0^h g_1(t)u(t)\,dt = \frac{a\sigma_1}{1-\mu_1}\int_0^h g_1(t)t^{\alpha-1}\,dt + \frac{M\sigma_1}{1-\mu_1}\int_0^h g_1(t)t^{\alpha-1}\,dt\int_0^{+\infty} G_1(\xi,s)f(s)\,ds$$
$$+ \frac{1}{1-\mu_1}\int_0^h g_1(t)\int_0^{+\infty} G_1(t,s)f(s)\,ds\,dt.$$

Finally, we can obtain

$$\begin{aligned} u(t) &= a\sigma_1 t^{\alpha-1} + \sigma_1 M t^{\alpha-1}\int_0^{+\infty} G_1(\xi,s)f(s)\,ds + \int_0^{+\infty} G_1(t,s)f(s)\,ds \\ &\quad + \frac{a\sigma_1 T_1(t)}{1-\mu_1}\int_0^h g_1(\tau)\tau^{\alpha-1}\,d\tau + \frac{T_1(t)}{1-\mu_1}\int_0^h g_1(\tau)\int_0^{+\infty} G_1(\tau,s)f(s)\,ds\,d\tau \\ &\quad + \frac{M\sigma_1 T_1(t)}{1-\mu_1}\int_0^h g_1(\tau)\tau^{\alpha-1}\,d\tau\int_0^{+\infty} G_1(\xi,s)f(s)\,ds \\ &= a\sigma_1 t^{\alpha-1} + \frac{a\sigma_1 T_1(t)}{1-\mu_1}\int_0^h g_1(\tau)\tau^{\alpha-1}\,d\tau + \int_0^{+\infty} H_1(t,s)f(s)\,ds \\ &\quad + \frac{T_1(t)}{1-\mu_1}\int_0^h g_1(\tau)\int_0^{+\infty} H_1(\tau,s)f(s)\,ds\,d\tau \\ &= a\sigma_1 t^{\alpha-1} + \frac{a\sigma_1\omega_1 T_1(t)}{1-\mu_1} + \int_0^{+\infty} H(t,s)f(s)\,ds. \end{aligned}$$

This completes the proof of the lemma. □

We can easily get the following result.

Lemma 2.2 *The function $G_1(t,s)$ defined by* (2.3) *satisfies*:

(i) *G_1 is continuous and $G_1(t,s) \geq 0$, $0 \leq t,s < +\infty$;*

(ii) *$G_1(t,s)$ is increasing in t, $0 \leq t,s < +\infty$.*

Remark 2.1 For $0 \leq t,s < +\infty$, we can easily obtain

$$\frac{G_1(t,s)}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}, \qquad \frac{G_1(\xi,s)t^{\alpha-1}}{1+t^{\alpha-1}} \leq \frac{\xi^{\alpha-1}}{\Gamma(\alpha)}, \quad \xi > 0.$$

Lemma 2.3 *The function $H(t,s)$ satisfies the following inequality*:

$$\frac{H(t,s)}{1+t^{\alpha-1}} \leq \sigma_1 + \frac{\delta_1(\sigma_1+\sigma_1^2 M\xi^{\alpha-2})}{(1-\mu_1)\Gamma(\alpha-1)} = \sigma_1(1+\delta_1 l_1), \quad \forall t,s \in [0,+\infty).$$

Proof From Remark 2.1, we have

$$\frac{H_1(t,s)}{1+t^{\alpha-1}} = \frac{\sigma_1 M t^{\alpha-1}G_1(\xi,s)}{1+t^{\alpha-1}} + \frac{G_1(t,s)}{1+t^{\alpha-1}} \leq \frac{1+\sigma_1 M\xi^{\alpha-1}}{\Gamma(\alpha)} = \sigma_1,$$

thus, from (2.1), we get

$$\frac{H(t,s)}{1+t^{\alpha-1}} = \frac{H_1(t,s)}{1+t^{\alpha-1}} + \frac{T_1(t)}{(1-\mu_1)(1+t^{\alpha-1})}\int_0^h H_1(\tau,s)g_1(\tau)\,d\tau$$

$$\begin{aligned}
&\leq \sigma_1 + \frac{(1+\sigma_1 M\xi^{\alpha-2})\sigma_1}{(1-\mu_1)\Gamma(\alpha-1)}\int_0^h \bigl(1+\tau^{\alpha-1}\bigr)g_1(\tau)\,d\tau \\
&= \sigma_1 + \frac{\delta_1(\sigma_1+\sigma_1^2 M\xi^{\alpha-2})}{(1-\mu_1)\Gamma(\alpha-1)} = \sigma_1(1+\delta_1 l_1).
\end{aligned}$$

The proof is completed. □

The general solution of

$$\begin{cases} D^\beta v(t) + g(t) = 0, & \beta\in(2,3], t\in[0,+\infty), \\ I^{3-\beta}v(t)|_{t=0} = 0, \quad D^{\beta-2}v(t)|_{t=0} = \int_0^h g_2(s)v(s)\,ds, & D^{\beta-1}v(+\infty) = Nv(\eta)+b \end{cases}$$

can be written by

$$v(t) = b\sigma_2 t^{\beta-1} + \frac{b\sigma_2\omega_2 T_2(t)}{1-\mu_2} + \int_0^{+\infty} K(t,s)g(s)\,ds,$$

where

$$\begin{aligned}
&K(t,s) = K_1(t,s) + \frac{T_2(t)}{1-\mu_2}\int_0^h K_1(\tau,s)g_2(\tau)\,d\tau, \\
&K_1(t,s) = \sigma_2 N t^{\beta-1}G_2(\xi,s) + G_2(t,s),
\end{aligned}$$

and $G_2(t,s)$ can be obtained from $G_1(t,s)$ by replacing α with β.

Hence, system (1.1) is equivalent to the following integral system:

$$\begin{cases} u(t) = a\sigma_1 t^{\alpha-1} + \frac{a\sigma_1\omega_1 T_1(t)}{1-\mu_1} + \int_0^{+\infty} H(t,s)\varphi(s,v(s),D^{\gamma_1}v(s))\,ds, \\ v(t) = b\sigma_2 t^{\beta-1} + \frac{b\sigma_2\omega_2 T_2(t)}{1-\mu_2} + \int_0^{+\infty} K(t,s)\psi(s,u(s),D^{\gamma_2}u(s))\,ds. \end{cases}$$

Define two spaces

$$\begin{aligned}
&X = \left\{u\in C(J), D^{\gamma_1}u\in C(J) \;\middle|\; \sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}} < +\infty, \sup_{t\in J}\frac{|D^{\gamma_1}u(t)|}{1+t^{\alpha-1-\gamma_1}} < +\infty\right\}, \\
&Y = \left\{v\in C(J), D^{\gamma_2}v\in C(J) \;\middle|\; \sup_{t\in J}\frac{|v(t)|}{1+t^{\beta-1}} < +\infty, \sup_{t\in J}\frac{|D^{\gamma_2}v(t)|}{1+t^{\beta-1-\gamma_2}} < +\infty\right\},
\end{aligned}$$

equipped with the norms

$$\|u\|_X = \sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}} + \sup_{t\in J}\frac{|D^{\gamma_1}u(t)|}{1+t^{\alpha-1-\gamma_1}}, \qquad \|v\|_Y = \sup_{t\in J}\frac{|v(t)|}{1+t^{\beta-1}} + \sup_{t\in J}\frac{|D^{\gamma_2}v(t)|}{1+t^{\beta-1-\gamma_2}},$$

where $0<\gamma_i<1$, $i=1,2$. $C(J)$ denotes the space of all continuous functions defined on $[0,+\infty)$.

Lemma 2.4 $(X,\|\cdot\|_X)$ *is a Banach space.*

Proof Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(X, \|\cdot\|_X)$; then $\forall\varepsilon > 0$, $\exists N(\varepsilon) > 0$ such that

$$\left|\frac{u_n(t)}{1+t^{\alpha-1}} - \frac{u_m(t)}{1+t^{\alpha-1}}\right| + \left|\frac{D^{\gamma_1}u_n(t)}{1+t^{\alpha-1-\gamma_1}} - \frac{D^{\gamma_1}u_m(t)}{1+t^{\alpha-1-\gamma_1}}\right| < \varepsilon,$$

for any $t \in J$ and $n, m > N(\varepsilon)$. We have $\lim_{n\to+\infty} \frac{u_n(t)}{1+t^{\alpha-1}} = \frac{u(t)}{1+t^{\alpha-1}}$, $u(t) \in C(J)$. Then, for $\frac{\Lambda_0}{2} = \sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}} > 0$, there exists $N > 0$ such that $|\frac{u_n(t)}{1+t^{\alpha-1}} - \frac{u(t)}{1+t^{\alpha-1}}| < \frac{\Lambda_0}{2}$, $n > N$. Further, set $\Lambda_i = \sup_{t\in J}\frac{|u_i(t)|}{1+t^{\alpha-1}}$, $i = 1, 2, \ldots, N$, and $\Lambda = \max\{\Lambda_i, i = 0, 1, 2, \ldots, N\}$. Then $\frac{|u_n(t)|}{1+t^{\alpha-1}} \le \Lambda$. Clearly, $\{\frac{u_n(t)}{1+t^{\alpha-1}}\}_{n=1}^{\infty}$ and $\{\frac{D^{\gamma_1}u_n(t)}{1+t^{\alpha-1-\gamma_1}}\}_{n=1}^{\infty}$ are Cauchy sequences in the space $C(J)$. Therefore, $\{\frac{D^{\gamma_1}u_n(t)}{1+t^{\alpha-1-\gamma_1}}\}_{n=1}^{\infty}$ converges uniformly to some $v \in C(J)$ and $\sup_{t\in J}|v(t)| < +\infty$. We need to prove that $v = \frac{D^{\gamma_1}u(t)}{1+t^{\alpha-1-\gamma_1}}$. For any $t \in J$, we have

$$\begin{aligned}
&\int_0^t (t-s)^{-\gamma_1-1}\left(1+s^{\alpha-1}\right)\frac{u_n(s)}{1+s^{\alpha-1}}\,ds \\
&\quad\le \Lambda\int_0^t (t-s)^{-\gamma_1-1}\left(1+s^{\alpha-1}\right)ds \\
&\quad= \Lambda t^{-\gamma_1}\int_0^1 (1-\tau)^{-\gamma_1-1}\,d\tau + \Lambda t^{\alpha-1-\gamma_1}\int_0^1 \tau^{\alpha-1}(1-\tau)^{-\gamma_1-1}\,d\tau \\
&\quad= \Lambda t^{-\gamma_1}B(1,-\gamma_1) + \Lambda t^{\alpha-1-\gamma_1}B(\alpha,-\gamma_1) \\
&\quad= \Lambda t^{-\gamma_1}\frac{\Gamma(-\gamma_1)}{\Gamma(1-\gamma_1)} + \Lambda t^{\alpha-1-\gamma_1}\frac{\Gamma(\alpha)\Gamma(-\gamma_1)}{\Gamma(\alpha-\gamma_1)}.
\end{aligned}$$

Furthermore, by Lebesgue's dominated convergence theorem, and considering the uniform convergence of $\{\frac{D^{\gamma_1}u_n(t)}{1+t^{\alpha-1-\gamma_1}}\}_{n=1}^{\infty}$, one has

$$\begin{aligned}
v(t) &= \lim_{n\to+\infty}\frac{D^{\gamma_1}u_n(t)}{1+t^{\alpha-1-\gamma_1}} \\
&= \lim_{n\to+\infty}\frac{1}{(1+t^{\alpha-1-\gamma_1})\Gamma(1-\gamma_1)}\cdot\frac{d}{dt}\int_0^t (t-s)^{-\gamma_1}u_n(s)\,ds \\
&= \frac{1}{(1+t^{\alpha-1-\gamma_1})\Gamma(1-\gamma_1)}\int_0^t (t-s)^{-\gamma_1-1}\left(1+s^{\alpha-1}\right)\frac{u(s)}{1+s^{\alpha-1}}\,ds \\
&= \frac{D^{\gamma_1}u(t)}{1+t^{\alpha-1-\gamma_1}}.
\end{aligned}$$

Thus

$$\lim_{n\to+\infty}\frac{u_n(t)}{1+t^{\alpha-1}} + \frac{D^{\gamma_1}u_n(t)}{1+t^{\alpha-1-\gamma_1}} = \frac{u(t)}{1+t^{\alpha-1}} + \frac{D^{\gamma_1}u(t)}{1+t^{\alpha-1-\gamma_1}}.$$

Therefore, we conclude that $(X, \|\cdot\|_X)$ is a Banach space. □

To prove the existence-uniqueness of solutions for system (1.1), we state the following compactness criterion.

Lemma 2.5 ([33]) *Let $U \subseteq Y$ be a bounded set; then U is relatively compact in Y if:*

(i) *for any* $u \in U$, $\frac{u(t)}{1+t^{\alpha-1}}$ *and* $D^{\alpha-1}u(t)$ *are equicontinuous on any compact interval of* J;
(ii) *for any* $\varepsilon > 0$, *there exists a constant* $T = T(\varepsilon) > 0$ *such that*

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon, \qquad \left|D^{\alpha-1}u(t_1) - D^{\alpha-1}u(t_2)\right| < \varepsilon,$$

for any $t_1, t_2 \geq T$ *and* $u \in U$.

Remark 2.2 According to Lemmas 2.4 and 2.5, it is clear that Z is relatively compact in X if the following conditions hold:
(i) for any $v \in Z$, $\frac{v(t)}{1+t^{\alpha-1}}$ and $\frac{D^{\gamma_1}v(t)}{1+t^{\alpha-1-\gamma_1}}$ are equicontinuous on any compact interval of J;
(ii) for any $\varepsilon > 0$, there exists a constant $L = L(\varepsilon) > 0$ such that

$$\left|\frac{v(t_1)}{1+t_1^{\alpha-1}} - \frac{v(t_2)}{1+t_2^{\alpha-1}}\right| + \left|\frac{D^{\gamma_1}v(t_1)}{1+t_1^{\alpha-1-\gamma_1}} - \frac{D^{\gamma_1}v(t_2)}{1+t_2^{\alpha-1-\gamma_1}}\right| < \varepsilon$$

for any $t_1, t_2 \geq L$ and $v \in Z$.

Lemma 2.6 (Schauder's fixed point theorem) *Let C be a nonempty, closed, bounded, and convex subset of a Banach space X. Suppose that $T : C \to C$ is a continuous and compact mapping. Then T has at least one fixed point in C.*

3 Main results

In our considerations, we work in the space $Q = \{(u, v) \mid u \in X, v \in Y\}$ endowed with the norm

$$\|(u,v)\|_Q = \max\{\|u\|_X, \|v\|_Y\}, \quad (u,v) \in Q.$$

By Lemma 2.4, Q is a Banach space. Let $T : Q \to Q$ be the operator defined as

$$T(u,v)(t) = \bigl(T_1 v(t), T_2 u(t)\bigr),$$

where

$$T_1 v(t) = a\sigma_1 t^{\alpha-1} + \frac{a\sigma_1\omega_1 T_1(t)}{1-\mu_1} + \int_0^{+\infty} H(t,s)\varphi\bigl(s, v(s), D^{\gamma_1}v(s)\bigr)\,ds,$$
$$T_2 u(t) = b\sigma_2 t^{\beta-1} + \frac{b\sigma_2\omega_2 T_2(t)}{1-\mu_2} + \int_0^{+\infty} K(t,s)\psi\bigl(s, u(s), D^{\gamma_2}u(s)\bigr)\,ds.$$

Notice that system (1.1) has a solution if and only if the operator T has a fixed point. For the forthcoming analysis, denote

$$L_1 = \sigma_1(1+\delta_1 l_1), \qquad L_2 = \sigma_2(1+\delta_2 l_2),$$
$$\zeta_1 = \frac{1+\sigma_1\Gamma(\alpha) + 2l_1\delta_1 + \sigma_1 M\xi^{\alpha-1}(1+2\omega_1 l_1)}{\Gamma(\alpha-\gamma_1)},$$
$$\zeta_2 = \frac{1+\sigma_2\Gamma(\beta) + 2l_2\delta_2 + \sigma_2 N\eta^{\beta-1}(1+2\omega_2 l_2)}{\Gamma(\beta-\gamma_2)},$$
$$\theta(s) = \max\{1+s^{\alpha-1}, 1+s^{\alpha-1-\gamma_1}, 1+s^{\beta-1}, 1+s^{\beta-1-\gamma_2}\}, \quad s \in [0,+\infty).$$

We need the following assumptions:

(H_1) There exist nonnegative functions $c_i(t), d_i(t) \in L^1(J) \cap C(J)$, $i = 1, 2, 3$, such that

$$\begin{aligned} &\left|\varphi(t,u,v)\right| \le c_1(t) + c_2(t)|u| + c_3(t)|v|, \quad t \in [0,+\infty), \\ &\int_0^{+\infty} c_1(t)\,dt < +\infty, \qquad \int_0^{+\infty} \bigl(c_2(t) + c_3(t)\bigr)\theta(t)\,dt < \max\left\{\frac{1}{2L_1}, \frac{1}{2\zeta_1}\right\}; \end{aligned}$$

and

$$\begin{aligned} &\left|\psi(t,u,v)\right| \le d_1(t) + d_2(t)|u| + d_3(t)|v|, \quad t \in [0,+\infty), \\ &\int_0^{+\infty} d_1(t)\,dt < +\infty, \qquad \int_0^{+\infty} \bigl(d_2(t) + d_3(t)\bigr)\theta(t)\,dt < \max\left\{\frac{1}{2L_2}, \frac{1}{2\zeta_2}\right\}. \end{aligned}$$

(H_2) For any $u, v, x, y \in \mathbb{R}$, there exist $\lambda_i(t) \in L^1(J) \cap C(J)$ with $\lambda_i(t) > 0$, $i = 1, 2$, such that

$$\begin{aligned} &\left|\varphi(t,u,v) - \varphi(t,x,y)\right| \le \lambda_1(t)\bigl(|u-x| + |v-y|\bigr), \quad t \in [0,+\infty), \\ &\left|\psi(t,u,v) - \psi(t,x,y)\right| \le \lambda_2(t)\bigl(|u-x| + |v-y|\bigr), \quad t \in [0,+\infty). \end{aligned}$$

This section is devoted to some existence and uniqueness results of system (1.1). In order to do this, define

$$B_R = \bigl\{(u,v) \in Q \mid \bigl\|(u,v)\bigr\|_Q \le R\bigr\},$$

where

$$\begin{aligned} R > \Biggl\{ &\frac{a\sigma_1(1+\omega_1 l_1) + L_1\int_0^{+\infty} c_1(s)\,ds}{\frac{1}{2} - L_1\int_0^{+\infty}(c_2(s)+c_3(s))\theta(s)\,ds}, \frac{\frac{2a\sigma_1\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma_1)}(1+\omega_1 l_1) + \zeta_1\int_0^{+\infty} c_1(s)\,ds}{\frac{1}{2} - \zeta_1\int_0^{+\infty}(c_2(s)+c_3(s))\theta(s)\,ds}, \\ &\frac{b\sigma_2(1+\omega_2 l_2) + L_2\int_0^{+\infty} d_1(s)\,ds}{\frac{1}{2} - L_2\int_0^{+\infty}(d_2(s)+d_3(s))\theta(s)\,ds}, \frac{\frac{2b\sigma_2\Gamma(\beta-1)}{\Gamma(\beta-\gamma_2)}(1+\omega_2 l_2) + \zeta_2\int_0^{+\infty} d_1(s)\,ds}{\frac{1}{2} - \zeta_2\int_0^{+\infty}(d_2(s)+d_3(s))\theta(s)\,ds}\Biggr\}. \end{aligned}$$

We observe that B_R is a bounded closed ball in the Banach space Q.

Lemma 3.1 *If* (H_1) *is satisfied, then* $T: B_R \to B_R$.

Proof First, for any $(u,v) \in B_R$, we know that

$$\|T_1 v\|_X = \sup_{t\in J}\frac{|T_1 v(t)|}{1+t^{\alpha-1}} + \sup_{t\in J}\frac{|D^{\gamma_1} T_1 v(t)|}{1+t^{\alpha-1-\gamma_1}}, \tag{3.1}$$

and from condition (H_1), we have

$$\begin{aligned} \frac{|T_1 v(t)|}{1+t^{\alpha-1}} &= \left|\frac{a\sigma_1 t^{\alpha-1}}{1+t^{\alpha-1}} + \frac{T_1(t)}{1+t^{\alpha-1}}\cdot\frac{a\sigma_1\omega_1}{1-\mu_1} + \int_0^{+\infty}\frac{H(t,s)}{1+t^{\alpha-1}}\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\,ds\right| \\ &\le a\sigma_1 + \frac{a\sigma_1\omega_1(1+\sigma_1 M\xi^{\alpha-2})}{(1-\mu_1)\Gamma(\alpha-1)} + \int_0^{+\infty}\sigma_1(1+\delta_1 l_1)\bigl|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\bigr|\,ds \\ &\le a\sigma_1(1+\omega_1 l_1) + L_1\int_0^{+\infty}\bigl(c_1(s) + c_2(s)\bigl|v(s)\bigr| + c_3(s)\bigl|D^{\gamma_1}v(s)\bigr|\bigr)\,ds \end{aligned}$$

$$\begin{aligned}
&\leq a\sigma_1(1+\omega_1 l_1) + L_1 \int_0^{+\infty} c_1(s)\,ds + L_1 \int_0^{+\infty} \bigl(c_2(s)+c_3(s)\bigr)\theta(s)\,ds \bigl\|(\nu,\nu)\bigr\|_Q \\
&< \frac{a\sigma_1(1+\omega_1 l_1) + L_1 \int_0^{+\infty} c_1(s)\,ds}{1-2L_1 \int_0^{+\infty}(c_2(s)+c_3(s))\theta(s)\,ds} < \frac{R}{2}.
\end{aligned}$$

In view of Lemma 2.1, one has

$$\begin{aligned}
D^{\gamma_1}u(t) &= D^{\gamma_1}\biggl(\sigma_1 t^{\alpha-1}\biggl(a + \int_0^{+\infty} f(s)\,ds + \frac{M\xi^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^h g_1(s)u(s)\,ds - MI^{\alpha} f(\xi)\biggr) \\
&\quad + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^h g_1(s)u(s)\,ds - I^{\alpha} f(t)\biggr) \\
&= \biggl(a + \int_0^{+\infty} f(s)\,ds + \frac{M\xi^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^h g_1(s)u(s)\,ds - MI^{\alpha} f(\xi)\biggr) \cdot \frac{\sigma_1 t^{\alpha-1-\gamma_1}\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} \\
&\quad + \frac{t^{\alpha-2-\gamma_1}}{\Gamma(\alpha-1-\gamma_1)} \int_0^h g_1(s)u(s)\,ds - I^{\alpha-\gamma_1} f(t),
\end{aligned}$$

and thus, we can easily show that

$$\begin{aligned}
\biggl|\int_0^h g_1(t)T_1\nu(t)\,dt\biggr| &= \biggl|\frac{a\sigma_1}{1-\mu_1} \int_0^h g_1(t)t^{\alpha-1}\,dt \\
&\quad + \frac{M\sigma_1}{1-\mu_1} \int_0^h g_1(t)t^{\alpha-1}\,dt \int_0^{+\infty} G_1(\xi,s)\varphi\bigl(s,\nu(s),D^{\gamma_1}\nu(s)\bigr)\,ds \\
&\quad + \frac{1}{1-\mu_1} \int_0^h g_1(t) \int_0^{+\infty} G_1(t,s)\varphi\bigl(s,\nu(s),D^{\gamma_1}\nu(s)\bigr)\,ds\,dt\biggr| \\
&\leq \frac{a\sigma_1\omega_1}{1-\mu_1} + \frac{\sigma_1\omega_1 M\xi^{\alpha-1}}{(1-\mu_1)\Gamma(\alpha)} \int_0^{+\infty} \bigl|\varphi\bigl(s,\nu(s),D^{\gamma_1}\nu(s)\bigr)\bigr|\,ds \\
&\quad + \frac{1}{(1-\mu_1)\Gamma(\alpha)} \int_0^h g_1(t)\bigl(1+t^{\alpha-1}\bigr)\,dt \int_0^{+\infty} \bigl|\varphi\bigl(s,\nu(s),D^{\gamma_1}\nu(s)\bigr)\bigr|\,ds \\
&\leq \frac{a\sigma_1\omega_1}{1-\mu_1} + \frac{\sigma_1\omega_1 M\xi^{\alpha-1}+\delta_1}{(1-\mu_1)\Gamma(\alpha)} \int_0^{+\infty} c_1(s)\,ds \\
&\quad + \frac{\sigma_1\omega_1 M\xi^{\alpha-1}+\delta_1}{(1-\mu_1)\Gamma(\alpha)} \int_0^{+\infty} \bigl(c_2(s)+c_3(s)\bigr)\theta(s)\,ds \bigl\|(\nu,\nu)\bigr\|_Q.
\end{aligned}$$

Further,

$$\begin{aligned}
\frac{|D^{\gamma_1}T_1\nu(t)|}{1+t^{\alpha-1-\gamma_1}} &= \biggl|\frac{\sigma_1 t^{\alpha-1-\gamma_1}\Gamma(\alpha)}{(1+t^{\alpha-1-\gamma_1})\Gamma(\alpha-\gamma_1)}\biggl(a + \int_0^{+\infty} \varphi\bigl(s,\nu(s),D^{\gamma_1}\nu(s)\bigr)\,ds \\
&\quad + \frac{M\xi^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^h g_1(s)T_1\nu(s)\,ds \\
&\quad - \frac{M}{\Gamma(\alpha)} \int_0^{\xi} (\xi-s)^{\alpha-1}\varphi\bigl(s,\nu(s),D^{\gamma_1}\nu(s)\bigr)\,ds\biggr) \\
&\quad + \frac{t^{\alpha-2-\gamma_1}}{(1+t^{\alpha-1-\gamma_1})} \int_0^h g_1(s)\frac{T_1\nu(s)}{\Gamma(\alpha-1-\gamma_1)}\,ds \\
&\quad - \frac{1}{\Gamma(\alpha-\gamma_1)} \int_0^t \frac{(t-s)^{\alpha-1-\gamma_1}}{1+t^{\alpha-1-\gamma_1}}\varphi\bigl(s,\nu(s),D^{\gamma_1}\nu(s)\bigr)\,ds\biggr|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{a\sigma_1\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} + \frac{\sigma_1\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\int_0^{+\infty}\bigl|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\bigr|\,ds \\
&\quad + \frac{\sigma_1 M\xi^{\alpha-1}}{\Gamma(\alpha-\gamma_1)}\int_0^{\xi}\bigl|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\bigr|\,ds \\
&\quad + \frac{\sigma_1 M\xi^{\alpha-2}(\alpha-1)+\alpha-1-\gamma_1}{\Gamma(\alpha-\gamma_1)}\left|\int_0^h g_1(s)T_1v(s)\,ds\right| \\
&\quad + \frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^t\left(1-\frac{s}{t}\right)^{\alpha-1-\gamma_1}\bigl|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\bigr|\,ds \\
&\leq \frac{1+\sigma_1\Gamma(\alpha)+2l_1\delta_1+\sigma_1 M\xi^{\alpha-1}(1+2\omega_1 l_1)}{\Gamma(\alpha-\gamma_1)}\int_0^{+\infty}c_1(s)\,ds \\
&\quad + \frac{1+\sigma_1\Gamma(\alpha)+2l_1\delta_1+\sigma_1 M\xi^{\alpha-1}(1+2\omega_1 l_1)}{\Gamma(\alpha-\gamma_1)} \\
&\quad \times \int_0^{+\infty}\bigl(c_2(s)+c_3(s)\bigr)\theta(s)\,ds\bigl\|(v,v)\bigr\|_Q \\
&\quad + a\sigma_1\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} + \frac{2\omega_1 l_1\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma_1)}\right) \\
&< \frac{\frac{2a\sigma_1\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma_1)}(1+\omega_1 l_1)+\zeta_1\int_0^{+\infty}c_1(s)\,ds}{1-2\zeta_1\int_0^{+\infty}(c_2(s)+c_3(s))\theta(s)\,ds} < \frac{R}{2},
\end{aligned}
$$

which implies that

$$
\|T_1v\|_X = \sup_{t\in J}\frac{|T_1v(t)|}{1+t^{\alpha-1}} + \sup_{t\in J}\frac{|D^{\gamma_1}T_1v(t)|}{1+t^{\alpha-1-\gamma_1}} < \frac{R}{2}+\frac{R}{2} = R. \tag{3.2}
$$

Similarly, we can obtain

$$
\begin{aligned}
\|T_2u\|_Y &= \sup_{t\in J}\frac{|T_2u(t)|}{1+t^{\beta-1}} + \sup_{t\in J}\frac{|D^{\gamma_2}T_2u(t)|}{1+t^{\beta-1-\gamma_2}} \\
&\leq \frac{b\sigma_2(1+\omega_2 l_2)+L_2\int_0^{+\infty}d_1(s)\,ds}{\frac{1}{2}-L_2\int_0^{+\infty}(d_2(s)+d_3(s))\theta(s)\,ds} + \frac{\frac{2b\sigma_2\Gamma(\beta-1)}{\Gamma(\beta-\gamma_2)}(1+\omega_2 l_2)+\zeta_2\int_0^{+\infty}d_1(s)\,ds}{\frac{1}{2}-\zeta_2\int_0^{+\infty}(d_2(s)+d_3(s))\theta(s)\,ds} \\
&< \frac{R}{2}+\frac{R}{2} = R.
\end{aligned} \tag{3.3}
$$

It shows that $\|T(u,v)\|_Q \leq R$, and T_1, T_2 are continuous on J. Thus $T: B_R \to B_R$ is well defined. □

Theorem 3.1 *If* (H_1) *holds, then system* (1.1) *has at least one solution.*

Proof First, the operator $T: B_R \to B_R$ is continuous owing to the continuity of φ and ψ. We are going to show that T is a completely continuous operator. By Lemma 3.1, T is bounded. We need to show that T is relatively compact by means of Remark 2.2. This part consists of two steps as follows.

Step 1 We show that T is equicontinuous on any compact interval of J.

Let ω be a bounded subset of B_R, $J_1 \subseteq [0,+\infty)$ be a compact interval. Then, for any $t_1, t_2 \in J_1$ with $t_1 < t_2$, $v \in \omega$, we have

$$\begin{aligned}
&\left|\frac{T_1 v(t_2)}{1+t_2^{\alpha-1}} - \frac{T_1 v(t_1)}{1+t_1^{\alpha-1}}\right| \\
&= \left|\frac{a\sigma_1 t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{a\sigma_1 t_1^{\alpha-1}}{1+t_1^{\alpha-1}} + \int_0^{+\infty}\left(\frac{H(t_2,s)}{1+t_2^{\alpha-1}} - \frac{H(t_1,s)}{1+t_1^{\alpha-1}}\right)\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\,ds\right. \\
&\quad \left. + \left(\frac{T_1(t_2)}{(1-\mu_1)(1+t_2^{\alpha-1})} - \frac{T_1(t_1)}{(1-\mu_1)(1+t_1^{\alpha-1})}\right)\cdot a\sigma_1\omega_1\right| \\
&\le \frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|a\sigma_1}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})} + \left|\frac{\sigma_1 M t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{\sigma_1 M t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right| \int_0^{+\infty} G_1(\xi,s)\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds \\
&\quad + \left|\frac{T_1(t_2)}{(1-\mu_1)(1+t_2^{\alpha-1})} - \frac{T_1(t_1)}{(1-\mu_1)(1+t_1^{\alpha-1})}\right| \\
&\quad \times \int_0^{+\infty}\int_0^h H_1(\tau,s)g_1(\tau)\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,d\tau\,ds \\
&\quad + \int_0^{+\infty}\frac{G_1(t_2,s)-G_1(t_1,s)}{1+t_2^{\alpha-1}}\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds \\
&\quad + \int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_2^{\alpha-1}} - \frac{G_1(t_1,s)}{1+t_1^{\alpha-1}}\right|\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds \\
&\le \frac{\sigma_1 M\xi^{\alpha-2}|t_2^{\alpha-1}-t_1^{\alpha-1}| + |t_2^{\alpha-2}-t_1^{\alpha-2}| + (t_1t_2)^{\alpha-2}|t_1-t_2|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})(1-\mu_1)\Gamma(\alpha-1)} \\
&\quad \times \left(a\sigma_1\omega_1 + \sigma_1\delta_1\int_0^{+\infty}\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds\right) \\
&\quad + \frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|a\sigma_1}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})} + \frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|\sigma_1 M\xi^{\alpha-1}}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})\Gamma(\alpha)}\int_0^{+\infty}\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds \\
&\quad + \int_0^{t_1}\left|\frac{(t_2^{\alpha-1}-t_1^{\alpha-1}) + (t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}}{\Gamma(\alpha)(1+t_2^{\alpha-1})}\right|\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds \\
&\quad + \int_{t_1}^{t_2}\left|\frac{(t_2^{\alpha-1}-t_1^{\alpha-1}) - (t_2-s)^{\alpha-1}}{\Gamma(\alpha)(1+t_2^{\alpha-1})}\right|\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds \\
&\quad + \int_{t_2}^{+\infty}\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma(\alpha)(1+t_2^{\alpha-1})}\right|\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds \\
&\quad + \int_0^{+\infty}\left|\frac{t_1^{\alpha-1}-t_2^{\alpha-1}}{\Gamma(\alpha)(1+t_2^{\alpha-1})}\right|\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds.
\end{aligned}$$

Then we have $|\frac{T_1 v(t_2)}{1+t_2^{\alpha-1}} - \frac{T_1 v(t_1)}{1+t_1^{\alpha-1}}| \to 0$ as $t_1 \to t_2$. Further, we know that

$$\begin{aligned}
&\left|\frac{D^{\gamma_1}T_1 v(t_2)}{1+t_2^{\alpha-1-\gamma_1}} - \frac{D^{\gamma_1}T_1 v(t_1)}{1+t_1^{\alpha-1-\gamma_1}}\right| \\
&= \left|\frac{t_1^{\alpha-1-\gamma_1} - t_2^{\alpha-1-\gamma_1}}{(1+t_1^{\alpha-1-\gamma_1})(1+t_2^{\alpha-1-\gamma_1})}\left(\frac{a\sigma_1\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} + \frac{\sigma_1\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\int_0^{+\infty}\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\,ds\right.\right. \\
&\quad - \frac{\sigma_1 M}{\Gamma(\alpha-\gamma_1)}\int_0^{\xi}(\xi-s)^{\alpha-1}\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma_1 M\xi^{\alpha-2}(\alpha-1)}{\Gamma(\alpha-\gamma_1)}\int_0^h g_1(s)T_1v(s)\,ds\Big)\\
&+\left(\frac{t_2^{\alpha-1-\gamma_1}}{1+t_2^{\alpha-1-\gamma_1}}-\frac{t_1^{\alpha-1-\gamma_1}}{1+t_1^{\alpha-1-\gamma_1}}\right)\int_0^h g_1(s)\frac{T_1v(s)}{\Gamma(\alpha-1-\gamma_1)}\,ds\\
&+\frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1-\gamma_1}}{1+t_1^{\alpha-1-\gamma_1}}\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\,ds\\
&-\frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1-\gamma_1}}{1+t_2^{\alpha-1-\gamma_1}}\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\,ds\bigg|\\
\le{}&\frac{|t_2^{\alpha-1-\gamma_1}-t_1^{\alpha-1-\gamma_1}|}{(1+t_1^{\alpha-1-\gamma_1})(1+t_2^{\alpha-1-\gamma_1})}\left(\frac{a\sigma_1\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}+\frac{\sigma_1\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\int_0^{+\infty}\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds\right.\\
&\left.+\frac{\sigma_1 M\xi^{\alpha-1}}{\Gamma(\alpha-\gamma_1)}\int_0^{\xi}\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds+\frac{2\sigma_1 M\xi^{\alpha-2}}{\Gamma(\alpha-\gamma_1)}\int_0^h g_1(s)T_1v(s)\,ds\right)\\
&+\frac{|t_2^{\alpha-1-\gamma_1}-t_1^{\alpha-1-\gamma_1}|}{(1+t_1^{\alpha-1-\gamma_1})(1+t_2^{\alpha-1-\gamma_1})\Gamma(\alpha-1-\gamma_1)}\int_0^h g_1(s)T_1v(s)\,ds\\
&+\frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^{t_1}\left|\frac{(t_1-s)^{\alpha-1-\gamma_1}}{1+t_1^{\alpha-1-\gamma_1}}-\frac{(t_2-s)^{\alpha-1-\gamma_1}}{1+t_2^{\alpha-1-\gamma_1}}\right|\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds\\
&+\frac{1}{\Gamma(\alpha-\gamma_1)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1-\gamma_1}}{1+t_2^{\alpha-1-\gamma_1}}\big|\varphi\big(s,v(s),D^{\gamma_1}v(s)\big)\big|\,ds,
\end{aligned}
$$

so $|\frac{D^{\gamma_1}T_1v(t_2)}{1+t_2^{\alpha-1-\gamma_1}}-\frac{D^{\gamma_1}T_1v(t_1)}{1+t_1^{\alpha-1-\gamma_1}}|\to 0$ as $t_1\to t_2$. Moreover, notice that $\varphi(t,v(t),D^{\gamma_1}v(t))$ is bounded on J_1. For any $v\in\omega$, $\frac{T_1v(t)}{1+t^{\alpha-1}}$ and $\frac{D^{\gamma_1}T_1v(t)}{1+t^{\alpha-1-\gamma_1}}$ are equicontinuous on J_1, that is, T_1 is equicontinuous. Similarly, we know that T_2 is also equicontinuous. Thus T is equicontinuous on J_1.

Step 2 We show that T is equiconvergent at ∞.

Since $\lim_{t\to+\infty}\frac{t^{\lambda-1}}{1+t^{\lambda-1}}=1$, for any $\varepsilon>0$, there exists a constant $\mu_1>0$, for each $t>\mu_1$, one has $|\frac{t^{\lambda-1}}{1+t^{\lambda-1}}-1|<\frac{\varepsilon}{2}$. Thus, for each $t_1,t_2>\mu_1$, we have

$$\left|\frac{t_2^{\lambda-1}}{1+t_2^{\lambda-1}}-\frac{t_1^{\lambda-1}}{1+t_1^{\lambda-1}}\right|\le\left|\frac{t_2^{\lambda-1}}{1+t_2^{\lambda-1}}-1\right|+\left|\frac{t_1^{\lambda-1}}{1+t_1^{\lambda-1}}-1\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Further, there exists $\varsigma\ge s$ such that $\lim_{t\to+\infty}\frac{(t-\varsigma)^{\lambda-1}}{1+t^{\lambda-1}}=1$. Then, for any $\varepsilon>0$, there exists $\mu_2>\varsigma>0$ such that, for each $t_1,t_2>\mu_2$, we have

$$
\begin{aligned}
\left|\frac{(t_2-s)^{\lambda-1}}{1+t_2^{\lambda-1}}-\frac{(t_1-s)^{\lambda-1}}{1+t_1^{\lambda-1}}\right|&\le\left|\frac{(t_2-\varsigma)^{\lambda-1}}{1+t_2^{\lambda-1}}-1\right|+\left|\frac{(t_1-\varsigma)^{\lambda-1}}{1+t_1^{\lambda-1}}-1\right|\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

Therefore, for any $\varepsilon>0$, choose $\mu\ge\max\{\mu_1,\mu_2\}$; then, for each $t_1,t_2>\mu$, one has

$$
\begin{aligned}
&\left|\frac{T_1v(t_2)}{1+t_2^{\alpha-1}}-\frac{T_1v(t_1)}{1+t_1^{\alpha-1}}\right|\\
&\quad\le\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|a\sigma_1+\left(\frac{\sigma_1 M\xi^{\alpha-2}}{\Gamma(\alpha-1)(1-\mu_1)}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha-1)(1-\mu_1)}\left|\frac{t_2^{\alpha-2}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-2}}{1+t_1^{\alpha-1}}\right|\biggr) \\
&\times\left(a\sigma_1\omega_1+\int_0^{+\infty}\int_0^h H_1(\tau,s)g_1(\tau)\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|d\tau\,ds\right) \\
&+\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\cdot\frac{\sigma_1 M\xi^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left(\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\right) \\
&\times\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{+\infty}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
\le{}&\varepsilon a\sigma_1+\frac{\varepsilon(1+\sigma_1 M\xi^{\alpha-2})}{\Gamma(\alpha-1)(1-\mu_1)}\left(a\sigma_1\omega_1+\sigma_1\delta_1\int_0^{+\infty}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds\right) \\
&+\frac{\varepsilon\sigma_1 M\xi^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds+\frac{2\varepsilon}{\Gamma(\alpha)}\int_0^{t_1}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{\varepsilon}{\Gamma(\alpha)}\int_{t_1}^{+\infty}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
\le{}&\varepsilon a\sigma_1(1+\omega_1 l_1)+\varepsilon\left(L_1+\frac{2}{\Gamma(\alpha)}\right)\int_0^{+\infty}\bigl(c_1(s)+c_2(s)\left|v(s)\right|+c_3(s)\left|D^{\gamma_1}v(s)\right|\bigr)\,ds.
\end{aligned}
$$

In addition, we can obtain

$$
\begin{aligned}
&\left|\frac{D^{\gamma_1}T_1v(t_2)}{1+t_2^{\alpha-1-\gamma_1}}-\frac{D^{\gamma_1}T_1v(t_1)}{1+t_1^{\alpha-1-\gamma_1}}\right| \\
\le{}&\left|\frac{t_2^{\alpha-1-\gamma_1}}{1+t_2^{\alpha-1-\gamma_1}}-\frac{t_1^{\alpha-1-\gamma_1}}{1+t_1^{\alpha-1-\gamma_1}}\right|\biggl(\frac{a\sigma_1\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}+\frac{\sigma_1\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\int_0^{+\infty}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{M\sigma_1}{\Gamma(\alpha-\gamma_1)}\int_0^{\xi}(\xi-s)^{\alpha-1}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{\sigma_1 M\xi^{\alpha-2}(\alpha-1)+\alpha-1-\gamma_1}{\Gamma(\alpha-\gamma_1)}\int_0^h g_1(s)T_1v(s)\,ds\biggr) \\
&+\frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^{t_1}\left|\frac{(t_1-s)^{\alpha-1-\gamma_1}}{1+t_1^{\alpha-1-\gamma_1}}-\frac{(t_2-s)^{\alpha-1-\gamma_1}}{1+t_2^{\alpha-1-\gamma_1}}\right|\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{1}{\Gamma(\alpha-\gamma_1)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1-\gamma_1}}{1+t_2^{\alpha-1-\gamma_1}}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
\le{}&\varepsilon a\sigma_1\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}+\frac{2\omega_1 l_1\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma_1)}\right)+\frac{\varepsilon\sigma_1\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\int_0^{+\infty}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{\varepsilon\sigma_1 M\xi^{\alpha-1}}{\Gamma(\alpha-\gamma_1)}\int_0^{\xi}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{2\varepsilon l_1(\sigma_1\omega_1 M\xi^{\alpha-1}+\delta_1)}{\Gamma(\alpha-\gamma_1)}\int_0^{+\infty}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{\varepsilon}{\Gamma(\alpha-\gamma_1)}\int_0^{t_1}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds \\
&+\frac{1}{\Gamma(\alpha-\gamma_1)}\int_{t_1}^{t_2}\left|\varphi\bigl(s,v(s),D^{\gamma_1}v(s)\bigr)\right|ds
\end{aligned}
$$

$$\leq \frac{2\varepsilon a\sigma_1\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma_1)}(1+\omega_1 l_1) + \left(\varepsilon\zeta_1 + \frac{1}{\Gamma(\alpha-\gamma_1)}\right)\int_0^{+\infty}\left(c_1(s)+c_2(s)\left|v(s)\right|+c_3(s)\left|D^{\gamma_1}v(s)\right|\right)ds.$$

Thus we have

$$\left|\frac{T_1v(t_2)}{1+t_2^{\alpha-1}} - \frac{T_1v(t_1)}{1+t_1^{\alpha-1}}\right| + \left|\frac{D^{\gamma_1}T_1v(t_2)}{1+t_2^{\alpha-1-\gamma_1}} - \frac{D^{\gamma_1}T_1v(t_1)}{1+t_1^{\alpha-1-\gamma_1}}\right| \leq \left(\varepsilon(L_1+\zeta_1)+\frac{1}{\Gamma(\alpha-\gamma_1)}+\frac{2\varepsilon}{\Gamma(\alpha)}\right)\int_0^{+\infty}\left(c_1(s)+c_2(s)\left|v(s)\right|+c_3(s)\left|D^{\gamma_1}v(s)\right|\right)ds + \varepsilon a\sigma_1(1+\omega_1 l_1)\left(1+\frac{2\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma_1)}\right).$$

Then, for all $\varepsilon > 0$, there exists $\mu > 0$ such that, for $t_1, t_2 > \mu$, $T_1 : \omega \to \omega$ is equiconvergent at infinity. Using the same argument, $T_2 : \omega \to \omega$ is also equiconvergent at infinity. Thus $T : \omega \to \omega$ is equiconvergent at infinity. By means of Remark 2.2, we know $T : B_R \to B_R$ is completely continuous.

According to Schauder's fixed point theorem, we conclude that T has at least one fixed point, that is, system (1.1) has at least one solution in B_R. □

Corollary 3.1 *Assume that*

(H_3) *there exist nonnegative functions* $a(t), b(t), a_i(t) \in L^1(J) \cap C(J)$, $i = 1, 2$, *such that*

$$\left|\varphi(t,u,v)\right| \leq a(t) + a_1(t)\left(|u|^{p_1}+|v|^{p_2}\right), \quad 0 < p_i < 1, i = 1, 2, t \in J,$$
$$\int_0^{+\infty} a(t)\,dt < +\infty, \qquad \int_0^{+\infty} a_1(t)\theta(t)\,dt < \max\left\{\frac{1}{4L_1}, \frac{1}{4\zeta_1}\right\};$$

and

$$\left|\psi(t,u,v)\right| \leq b(t) + a_2(t)\left(|u|^{q_1}+|v|^{q_2}\right), \quad 0 < q_i < 1, i = 1, 2, t \in J,$$
$$\int_0^{+\infty} b(t)\,dt < +\infty, \qquad \int_0^{+\infty} a_2(t)\theta(t)\,dt < \max\left\{\frac{1}{4L_2}, \frac{1}{4\zeta_2}\right\}.$$

Here, $a_i, b_i, i = 1, 2$, *are nonnegative constants, then system* (1.1) *has at least one solution.*

Proof In this case, let $p* = \max\{p_1, p_2\}$, $q* = \max\{q_1, q_2\}$, we take

$$R > \left\{\left(\frac{a\sigma_1(1+\omega_1 l_1)+L_1\int_0^{+\infty}a(s)\,ds}{\frac{1}{2}-2L_1\int_0^{+\infty}a_1(s)\theta(s)\,ds}\right)^{1/p*}, \left(\frac{\frac{2a\sigma_1\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma_1)}(1+\omega_1 l_1)+\zeta_1\int_0^{+\infty}a(s)\,ds}{\frac{1}{2}-2\zeta_1\int_0^{+\infty}a_1(s)\theta(s)\,ds}\right)^{1/p*}, \left(\frac{b\sigma_2(1+\omega_2 l_2)+L_2\int_0^{+\infty}b(s)\,ds}{\frac{1}{2}-2L_2\int_0^{+\infty}a_2(s)\theta(s)\,ds}\right)^{1/q*}, \left(\frac{\frac{2b\sigma_2\Gamma(\beta-1)}{\Gamma(\beta-\gamma_2)}(1+\omega_2 l_2)+\zeta_2\int_0^{+\infty}b(s)\,ds}{\frac{1}{2}-2\zeta_2\int_0^{+\infty}a_2(s)\theta(s)\,ds}\right)^{1/q*}\right\}.$$

The rest of the proof is similar to Theorem 3.1, so we omit the details. □

Remark 3.1 For the sake of simplicity, if $a(t) = b(t) = 0$ in condition (H_3), that is,

$$\begin{aligned} &\left|\varphi(t,u,v)\right| \le a_1(t)\bigl(|u|^{p_1} + |v|^{p_2}\bigr), \quad p_i > 1, i = 1,2, t \in J, \\ &\int_0^{+\infty} a_1(t)\theta(t)\,dt < \max\left\{\frac{1}{4L_1}, \frac{1}{4\zeta_1}\right\}; \end{aligned}$$

and

$$\begin{aligned} &\left|\psi(t,u,v)\right| \le a_2(t)\bigl(|u|^{q_1} + |v|^{q_2}\bigr), \quad q_i > 1, i = 1,2, t \in J, \\ &\int_0^{+\infty} a_2(t)\theta(t)\,dt < \max\left\{\frac{1}{4L_2}, \frac{1}{4\zeta_2}\right\}. \end{aligned}$$

Due to the different values of R, the conclusion of Theorem 3.1 is also true for the nonstrict inequalities $p_i, q_i > 1$. It should be replaced by a weak form which can be derived easily from (3.2) and (3.3).

When $h = 0$, the boundary conditions of system (1.1) are changed to the form:

$$\begin{cases} I^{3-\alpha}u(0) = D^{\alpha-2}u(0) = 0, & D^{\alpha-1}u(+\infty) = Mu(\xi) + a, \\ I^{3-\beta}v(0) = D^{\beta-2}v(0) = 0, & D^{\beta-1}v(+\infty) = Nv(\eta) + b. \end{cases} \tag{3.4}$$

Similar to Theorem 3.1, we can obtain the following result.

Theorem 3.2 *Assume that*

(H_1') *there exist nonnegative functions* $c_i(t), d_i(t) \in L^1(J)$, $i = 1,2,3$, *such that*

$$\begin{aligned} &\left|\varphi(t,u,v)\right| \le c_1(t) + c_2(t)|u| + c_3(t)|v|, \quad t \in [0,+\infty), \\ &\int_0^{+\infty} c_1(t)\,dt < +\infty, \qquad \int_0^{+\infty} \bigl(c_2(t) + c_3(t)\bigr)\theta(t)\,dt < \max\left\{\frac{1}{2L_1'}, \frac{1}{2\zeta_1'}\right\}, \end{aligned}$$

where

$$L_1' = \sigma_1, \qquad \zeta_1' = \frac{1 + \sigma_1(1 + M\xi^{\alpha-1})}{\Gamma(\alpha - \gamma_1)},$$

and

$$\begin{aligned} &\left|\psi(t,u,v)\right| \le d_1(t) + d_2(t)|u| + d_3(t)|v|, \quad t \in [0,+\infty), \\ &\int_0^{+\infty} d_1(t)\,dt < +\infty, \qquad \int_0^{+\infty} \bigl(d_2(t) + d_3(t)\bigr)\theta(t)\,dt < \max\left\{\frac{1}{2L_2'}, \frac{1}{2\zeta_2'}\right\}, \end{aligned}$$

where

$$L_2' = \sigma_2, \qquad \zeta_2' = \frac{1 + \sigma_2(1 + N\eta^{\beta-1})}{\Gamma(\beta - \gamma_2)}.$$

Then system (1.1) *with boundary condition* (3.4) *has at least one solution.*

Theorem 3.3 *Assume that* (H_1), (H_2) *hold, then system* (1.1) *has a unique solution if*

$$m_1 + m_2 < 1, \qquad n_1 + n_2 < 1,$$

where

$$m_1 = L_1 \int_0^{+\infty} \lambda_1(s)\theta(s)\,ds, \qquad n_1 = L_2 \int_0^{+\infty} \lambda_2(s)\theta(s)\,ds,$$
$$m_2 = \zeta_1 \int_0^{+\infty} \lambda_1(s)\theta(s)\,ds, \qquad n_2 = \zeta_2 \int_0^{+\infty} \lambda_2(s)\theta(s)\,ds.$$

Proof Let $u_i(t), v_i(t) \in C^1(J)$, $i = 1, 2$; then we have

$$\begin{aligned}
&\left|\frac{T_1 v_2(t)}{1+t^{\alpha-1}} - \frac{T_1 v_1(t)}{1+t^{\alpha-1}}\right| \\
&\quad\le \int_0^{+\infty} \frac{H(t,s)}{1+t^{\alpha-1}} \left|\varphi\big(s, v_2(s), D^{\gamma_1} v_2(s)\big) - \varphi\big(s, v_1(s), D^{\gamma_1} v_1(s)\big)\right| ds \\
&\quad\le L_1 \int_0^{+\infty} \lambda_1(s)\theta(s)\,ds \|v_2 - v_1\|_X = m_1 \|v_2 - v_1\|_X
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{D^{\gamma_1} T_1 v_2(t)}{1+t^{\alpha-1-\gamma_1}} - \frac{D^{\gamma_1} T_1 v_1(t)}{1+t^{\alpha-1-\gamma_1}}\right| \\
&\quad\le \frac{\sigma_1 \Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} \int_0^{+\infty} \left|\varphi\big(s, v_2(s), D^{\gamma_1} v_2(s)\big) - \varphi\big(s, v_1(s), D^{\gamma_1} v_1(s)\big)\right| ds \\
&\qquad + \frac{M\sigma_1 \xi^{\alpha-1}}{\Gamma(\alpha-\gamma_1)} \int_0^{\xi} \left|\varphi\big(s, v_2(s), D^{\gamma_1} v_2(s)\big) - \varphi\big(s, v_1(s), D^{\gamma_1} v_1(s)\big)\right| ds \\
&\qquad + \frac{1}{\Gamma(\alpha-\gamma_1)} \int_0^{t} \left|\varphi\big(s, v_2(s), D^{\gamma_1} v_2(s)\big) - \varphi\big(s, v_1(s), D^{\gamma_1} v_1(s)\big)\right| ds \\
&\qquad + \frac{2(1+\sigma_1 M \xi^{\alpha-2})}{\Gamma(\alpha-\gamma_1)} \int_0^{h} g_1(s)\big(T_1 v_2(s) - T_1 v_1(s)\big)\,ds \\
&\quad\le \zeta_1 \int_0^{+\infty} \lambda_1(s)\theta(s)\,ds \|v_2 - v_1\|_X = m_2 \|v_2 - v_1\|_X.
\end{aligned}$$

We can see that

$$\|T_1 v_2 - T_1 v_1\|_X \le (m_1 + m_2)\|v_2 - v_1\|_X.$$

Analogously, it can be proved that

$$\begin{aligned}
\left|\frac{T_2 u_2(t)}{1+t^{\beta-1}} - \frac{T_2 u_1(t)}{1+t^{\beta-1}}\right| &\le \sigma_2(1+\delta_2 l_2) \int_0^{+\infty} \lambda_2(s)\theta(s)\,ds \|u_2 - u_1\|_Y \\
&= n_1 \|u_2 - u_1\|_Y
\end{aligned}$$

and

$$\left|\frac{D^{\gamma_2} T_2 u_2(t)}{1+t^{\beta-1-\gamma_2}} - \frac{D^{\gamma_2} T_2 u_1(t)}{1+t^{\beta-1-\gamma_2}}\right|$$

$$
\begin{aligned}
&\le \left(\frac{\sigma_1(\Gamma(\alpha)+M\xi^{\alpha-1})+1}{\Gamma(\alpha-\gamma_1)}+\frac{(\sigma_1\omega_1 M\xi^{\alpha-1}+\delta_1)}{(1-\mu_1)\Gamma(\alpha)}\cdot\frac{2(1+\sigma_1 M\xi^{\alpha-2})}{\Gamma(\alpha-\gamma_1)}\right)\\
&\quad\times\int_0^{+\infty}\lambda_2(s)\theta(s)\,ds\|u_2-u_1\|_Y\\
&= n_2\|u_2-u_1\|_Y.
\end{aligned}
$$

Thus we know that

$$\|T_2u_2-T_2u_1\|_Y\le(n_1+n_2)\|u_2-u_1\|_Y.$$

In conclusion, we have

$$\bigl\|T(u_2,v_2)-T(u_1,v_1)\bigr\|_Q\le\max\{m_1+m_2,n_1+n_2\}\bigl\|(u_2,v_2)-(u_1,v_1)\bigr\|_Q.$$

Obviously, T is a contraction. By means of the Banach contraction theorem, T has a unique fixed point which is the unique solution of system (1.1). □

Corollary 3.2 *Assume that* (H_2), (H_3) *hold, then system* (1.1) *has a unique solution if* $m_1+m_2<1$, $n_1+n_2<1$.

Corollary 3.3 *On the basis of Remark* 3.1, *if condition* (H_2) *holds, then system* (1.1) *has a unique solution if* $m_1+m_2<1$, $n_1+n_2<1$. *In short, if* φ, ψ *are bounded and continuous on* $J\times R\times R$, *then there exists a solution for system* (1.1).

Remark 3.2 If $\varphi,\psi\in C(J\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, $\varphi(t,u,v),\psi(t,u,v)\not\equiv 0$, under condition (H_1) or (H_3), then system (1.1) has at least one positive solution. Further, the positive solution is unique if (H_1), (H_2) or (H_2), (H_3) are satisfied with $m_1+m_2<1$, $n_1+n_2<1$.

4 An example

Example 4.1 Consider the system

$$
\begin{cases}
D^{\frac52}u(t)+\frac{\sin t}{\sqrt{2+t^2}}+\frac{\sqrt{|v(t)D^{\frac12}v(t)|}}{72e^t(1+t^{\frac32})}=0, \quad t\in[0,+\infty),\\
D^{\frac52}v(t)+(2+\cos t)e^{-t^2}+\frac{\sin|u(t)|}{48e^t(1+t^{\frac32})}+\frac{\ln(1+|D^{\frac12}u(t)|)}{192e^{\sqrt t}(1+t^{\frac32})}=0,\\
I^{\frac12}u(0)=0, \quad D^{\frac12}u(0)=(\frac{3\pi^{\frac32}}{16}-\frac{\pi}{4})\int_0^1 s^4u(s)\,ds, \quad D^{\frac32}u(+\infty)=u(1)+2,\\
I^{\frac12}v(0)=0, \quad D^{\frac12}v(0)=(\frac{\pi^{\frac32}}{8}-\frac{\pi}{6})\int_0^1 s^2v(s)\,ds, \quad D^{\frac32}v(+\infty)=v(1)+2,
\end{cases}
\tag{4.1}
$$

where $\alpha=\beta=\frac52$, $\gamma_1=\gamma_2=\frac12$, $h=1$, $M,N,\xi,\eta=1$, $a=b=2$, $g_1(t)=(\frac{3\pi^{\frac32}}{16}-\frac{\pi}{4})t^4$, $g_2(t)=(\frac{\pi^{\frac32}}{8}-\frac{\pi}{6})t^2$, and

$$
\begin{aligned}
\varphi(t,u,v)&=\frac{\sin t}{\sqrt{2+t^2}}+\frac{\sqrt{|uv|}}{72e^t(1+t^{\frac32})},\\
\psi(t,u,v)&=(2+\cos t)e^{-t^2}+\frac{\sin|u|}{48e^t(1+t^{\frac32})}+\frac{\ln(1+|v|)}{192e^{\sqrt t}(1+t^{\frac32})}.
\end{aligned}
$$

Choose

$$c_1(t)=\frac{1}{\sqrt{2+t^2}},\qquad c_2(t)=c_3(t)=\frac{1}{144e^t(1+t^{\frac{3}{2}})},$$
$$d_1(t)=3e^{-t^2},\qquad d_2(t)=\frac{1}{48e^t(1+t^{\frac{3}{2}})},\qquad d_3(t)=\frac{1}{192e^{\sqrt{t}}(1+t^{\frac{3}{2}})}.$$

Obviously, $|\varphi(t,u,v)|\le c_1(t)+c_2(t)|u|+c_3(t)|v|$, $|\psi(t,u,v)|\le d_1(t)+d_2(t)|u|+d_3(t)|v|$, and by simple computations, we find that $0<M\xi^{\alpha-1}, N\eta^{\beta-1}<\Gamma(\frac{5}{2})\approx 1.329$, $\sigma_1=\sigma_2=\frac{4}{3\sqrt{\pi}-4}$, $T_1(t)=T_2(t)=\frac{8}{3\pi-4\sqrt{\pi}}t^{\frac{3}{2}}+\frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}$, $\mu_1=\int_0^1 g_1(t)T_1(t)\,dt=\frac{\sqrt{\pi}}{14}+\frac{3\pi-4\sqrt{\pi}}{24}<1$, $\mu_2=\int_0^1 g_2(t)T_2(t)\,dt=\frac{\pi}{8}-\frac{\sqrt{\pi}}{12}<1$, $\delta_1=\frac{9\pi^{\frac{3}{2}}-12\pi}{140}$, $\delta_2=\frac{7\pi^{\frac{3}{2}}}{96}-\frac{7\pi}{72}$, $\omega_1=\frac{3\pi^{\frac{3}{2}}-4\pi}{112}$, $\omega_2=\frac{\pi^{\frac{3}{2}}}{32}-\frac{\pi}{24}$, $l_1=\frac{1008}{(3\sqrt{\pi}-4)(168+16\sqrt{\pi}-21\pi)}$ and $l_2=\frac{144}{(3\sqrt{\pi}-4)(24+2\sqrt{\pi}-3\pi)}$. Further, we can obtain

$$\int_0^{+\infty} c_1(t)\,dt=\frac{\pi}{4}<+\infty,$$
$$\begin{aligned}\int_0^{+\infty}\bigl(c_2(t)+c_3(t)\bigr)\theta(t)\,dt&=\frac{1}{72}\int_0^1\frac{1+t}{e^t(1+t^{\frac{3}{2}})}\,dt+\frac{1}{72}\int_1^{+\infty}e^{-t}\,dt\\&\le\frac{1}{72}\left(2+\frac{1}{e}-\frac{4\ln 2}{3}\right)<\max\left\{\frac{1}{2L_1},\frac{1}{2\zeta_1}\right\}.\end{aligned}$$

Here,

$$L_1=\sigma_1(1+\delta_1 l_1)=\frac{4}{3\sqrt{\pi}-4}+\frac{3024\pi}{35(3\sqrt{\pi}-4)(168+16\sqrt{\pi}-21\pi)}\approx 4.617,$$
$$\zeta_1=\frac{6\sqrt{\pi}}{3\sqrt{\pi}-4}+\frac{4536\pi^{\frac{3}{2}}-1008\pi}{35(3\sqrt{\pi}-4)(168+16\sqrt{\pi}-21\pi)}\approx 11.747,$$
$$\int_0^{+\infty} d_1(t)\,dt=\frac{3\sqrt{\pi}}{2}<+\infty,$$
$$\begin{aligned}\int_0^{+\infty}\bigl(d_2(t)+d_3(t)\bigr)\theta(t)\,dt&=\int_0^1\left(\frac{1+t}{48e^t(1+t^{\frac{3}{2}})}+\frac{1+t}{192e^{\sqrt{t}}(1+t^{\frac{3}{2}})}\right)dt\\&\quad+\int_1^{+\infty}\left(\frac{1}{48e^t}+\frac{1}{192e^{\sqrt{t}}}\right)dt\\&\le\frac{5}{96}-\frac{5\ln 2}{144}+\frac{1}{32e}<\max\left\{\frac{1}{2L_2},\frac{1}{2\zeta_2}\right\},\end{aligned}$$
$$L_2=\sigma_2(1+\delta_2 l_2)=\frac{4}{3\sqrt{\pi}-4}+\frac{14\pi}{(3\sqrt{\pi}-4)(24+2\sqrt{\pi}-3\pi)}\approx 4.879,$$

and

$$\zeta_2=\frac{6\sqrt{\pi}}{3\sqrt{\pi}-4}+\frac{21\pi^{\frac{3}{2}}-7\pi}{(3\sqrt{\pi}-4)(24+2\sqrt{\pi}-3\pi)}\approx 12.050.$$

Then the conditions of Theorem 3.1 are satisfied, so system (4.1) has at least one solution.

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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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