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Nonlinear boundary value problems of a class of elliptic equations involving critical variable exponents

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Abstract
In this paper, we first obtain the existence of solutions for a class of elliptic equations involving critical variable exponents and nonlinear boundary values by the mountain pass theorem and concentration compactness principle. Then, under suitable assumptions, we obtain a sequence of solutions with positive energies going towards infinity by Fountain Theorem.

Keywords: Variable exponent Sobolev space; Weak solution; Mountain pass theorem; Fountain Theorem; Variational method

1 Introduction

In the studies of electrorheological fluids, nonlinear elasticity, and image restoration in practical applications, the classical Lebesgue and Sobolev spaces are inapplicable; see [1–3]. Such problems are inhomogeneous and nonlinear with variable exponential growth conditions. So we need to study the problems based on the theory of variable exponent Lebesgue and Sobolev spaces.

Since Kováčik and Rákosník first studied the $L^{p(x)}$ spaces and $W^{k,p(x)}$ spaces in [4], a lot of research has been done concerning these kinds of variable exponent spaces. The existence of solutions for $p(x)$-Laplacian Dirichlet problems on bounded domains has been widely discussed. For example, in [5] and [6], some results as regards the existence of solutions under some conditions are obtained.

The nonlinear elliptic boundary value problems appear when we study the conformal deformations on Riemannian manifolds with boundary. The study of nonlinear elliptic boundary value problems with p-Laplacian has become an interesting topic in recent years. Many results have been obtained on this kind of problems; see [7–9]. In the fractional Laplacian setting, the existence of solutions for the problem has been obtained; see [10–14].

But at present there are few papers on the study of nonlinear elliptic boundary value problems with $p(x)$-Laplacian. So this topic is worth further discussing.

In this paper, we consider the problem

$$\begin{cases} \operatorname{div}(a(x), \nabla u) + |u|^{p(x)-2}u = f(x,u) + h(x)|u|^{p^*(x)-2}u, & x \in \Omega, \\ a(x, \nabla u) \cdot \nu(x) = b(x,u), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p(x)$ is Lipschitz continuous and satisfies $1 < p_1 \le p(x) \le p_2 < N$, $p^*(x) = \frac{Np(x)}{N-p(x)}$. We assume that $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and we have the continuous derivative with respect to η of a function $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$. Suppose that a and A satisfy the following hypotheses:

(A1) For $\forall x \in \overline{\Omega}$, the equality $A(x, 0) = 0$ holds.

(A2) There exists a positive constant c_0 such that

$$\left|a(x,\eta)\right| \le c_0\left(1 + |\eta|^{p(x)-1}\right)$$

for all $x \in \overline{\Omega}$ and $\eta \in \mathbb{R}^N$.

(A3) For all $x \in \overline{\Omega}$ and $\eta_1, \eta_2 \in \mathbb{R}^N$, the following inequality holds:

$$0 \le \left[a(x,\eta_1) - a(x,\eta_2)\right] \cdot (\eta_1 - \eta_2),$$

where equality holds if and only if $\eta_1 = \eta_2$.

(A4) For all $x \in \overline{\Omega}$ and $\eta \in \mathbb{R}^N$, the inequalities

$$|\eta|^{p(x)} \le a(x,\eta) \cdot \eta \le p(x)A(x,\eta)$$

hold true.

(A5) For all $x \in \overline{\Omega}$ and $\eta \in \mathbb{R}^N$, the equality $A(x, -\eta) = A(x, \eta)$ holds true.

The above type of assumptions can be found in other papers too; for example, see [15, 16]. But in [16], the authors establish the existence of a solution for an elliptic problem with Dirichlet boundary conditions, and in [15], the authors consider the subcritical case. In the present paper, the problem involves not only the critical Sobolev exponents, but also the nonlinear boundary conditions. Because of the critical exponents, the compactness of the embedding fails, so to recover the loss of the compactness, we use the concentration compactness principle in [17].

Throughout this paper, we assume that the following conditions hold:

(F1) $f \in C(\overline{\Omega} \times \mathbb{R})$, $f(x,0) \equiv 0$ and $|f(x,t)| \le C_1(1 + |t|^{\alpha_1(x)-1})$, $\alpha_1 \in C(\overline{\Omega})$ with $p(x) \ll \alpha_1(x) \ll p^*(x)$, and $F(x,t) > 0$ in $\Omega_0 \times \mathbb{R}$ for some nonempty open set $\Omega_0 \subset \Omega$, where C_1 is a positive constant.

($\widetilde{\text{F}}1$) $f \in C(\overline{\Omega} \times \mathbb{R})$, $|f(x,t)| \le C_1(1 + |t|^{\alpha_1(x)-1})$, $\alpha_1 \in C(\overline{\Omega})$ with $1 \le \alpha_1(x) \ll p(x)$ and $F(x,t) > 0$ in $\Omega_0 \times \mathbb{R}$ for some nonempty open set $\Omega_0 \subset \Omega$.

(F2) $f(x,t) = -f(x,-t)$ for any $(x,t) \in \overline{\Omega} \times \mathbb{R}$.

(F3) For any $(x,t) \in \overline{\Omega} \times \mathbb{R}$, there exists a function $\mu_1(x) \in C^1(x)$ such that $\mu_1(x) \gg p(x)$ and $0 \le \mu_1(x)F(x,t) \le f(x,t)t$, where $F(x,t) = \int_0^t f(x,s)\,ds$.

(F4) $f(x,t) = o(|t|^{p(x)-1})$ hold uniformly for any $x \in \overline{\Omega}$, as $t \to 0$.

(B1) $b \in C(\overline{\Omega} \times \mathbb{R})$, $b(x,0) \equiv 0$ and $|b(x,t)| \le C_2|t|^{\alpha_2(x)-1}$, $\alpha_2 \in C(\overline{\Omega})$ with $p(x) \ll \alpha_2(x) \ll p_*(x)$, and $B(x,t) > 0$ in $\partial\Omega \times \mathbb{R}$, where C_2 is a positive constant and $p_*(x) = \frac{(N-1)p(x)}{N-p(x)}$.

($\widetilde{\text{B}}1$) $b \in C(\overline{\Omega} \times \mathbb{R})$, $|b(x,t)| \le C_2(1 + |t|^{\alpha_2(x)-1})$, $\alpha_2 \in C(\overline{\Omega})$ with $p(x) \ll \alpha_2(x) \ll p_*(x)$ and $B(x,t) > 0$ in $\partial\Omega \times \mathbb{R}$.

(B2) $b(x,t) = -b(x,-t)$ for any $(x,t) \in \overline{\Omega} \times \mathbb{R}$.

(B3) For any $(x,t) \in \partial\Omega \times \mathbb{R}$, there exists a function $\mu_2(x) \in C^1(x)$ such that $\mu_2(x) \gg p(x)$ and $0 \le \mu_2(x)B(x,t) \le b(x,t)t$, where $B(x,t) = \int_0^t b(x,s)\,ds$.

(H1) For any $x \in \Omega$, there exists $h_1 > 0$ such that $h(x) \ge h_1$ and $h(x) \in L^\infty(\Omega)$.

2 Preliminaries

We first recall some facts on spaces $L^{p(x)}$ and $W^{k,p(x)}$. For details see [4, 18, 19].

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p:\Omega\to[1,\infty]$, we denote

$$\rho_{p(x)}(u)=\int_{\Omega\setminus\Omega_\infty}|u|^{p(x)}\,dx+\sup_{x\in\Omega_\infty}\bigl|u(x)\bigr|,$$

where $\Omega_\infty=\{x\in\Omega:p(x)=\infty\}$.

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\rho_{p(x)}(tu)<\infty$, for some $t>0$. $L^{p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{L^{p(x)}}=\inf\biggl\{\lambda>0:\rho_{p(x)}\biggl(\frac{u}{\lambda}\biggr)\le 1\biggr\}.$$

For any $p\in\mathbf{P}(\Omega)$, we define the conjugate function $p'(x)$ as

$$p'(x)=\begin{cases}\infty, & x\in\Omega_1=\{x\in\Omega:p(x)=1\},\\ 1, & x\in\Omega_\infty,\\ \frac{p(x)}{p(x)-1}, & x\in\Omega\setminus(\Omega_1\cup\Omega_\infty).\end{cases}$$

Theorem 2.1 *Let $p\in\mathbf{P}(\Omega)$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$,*

$$\int_\Omega|uv|\,dx\le 2\|u\|_{L^{p(x)}}\|v\|_{L^{p'(x)}}.$$

For any $p\in\mathbf{P}(\Omega)$, we denote

$$p_1=\inf_{x\in\Omega}p(x),\qquad p_2=\sup_{x\in\Omega}p(x),$$

and we denote by $p(x)\ll q(x)$ the fact that $\inf_{x\in\Omega}(q(x)-p(x))>0$.

Theorem 2.2 *Let $p\in\mathbf{P}(\Omega)$ with $p_2<\infty$. For any $u\in L^{p(x)}(\Omega)$, we have*

(1) *if $\|u\|_{L^{p(x)}}\ge 1$, then $\|u\|_{L^{p(x)}}^{p_1}\le\int_\Omega|u|^{p(x)}\,dx\le\|u\|_{L^{p(x)}}^{p_2}$;*

(2) *if $\|u\|_{L^{p(x)}}<1$, then $\|u\|_{L^{p(x)}}^{p_2}\le\int_\Omega|u|^{p(x)}\,dx\le\|u\|_{L^{p(x)}}^{p_1}$.*

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u\in L^{p(x)}(\Omega)$ such that $|\nabla u|\in L^{p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{W^{1,p(x)}}=\|u\|_{L^{p(x)}}+\|\nabla u\|_{L^{p(x)}}.$$

For $u\in W^{1,p(x)}(\Omega)$, if we define

$$|||u|||=\inf\biggl\{t>0:\int_\Omega\frac{|u|^{p(x)}+|\nabla u|^{p(x)}}{t^{p(x)}}\,dx\le 1\biggr\},$$

then $|||\cdot|||$ and $\|\cdot\|_{W^{1,p(x)}}$ are equivalent norms on $W^{1,p(x)}(\Omega)$. In fact, we have

$$\frac{1}{2}\|u\|_{W^{1,p(x)}}\le|||u|||\le 2\|u\|_{W^{1,p(x)}}.$$

Theorem 2.3 *For any $u \in W^{1,p(x)}(\Omega)$, we have*

(1) *if $|||u||| \geq 1$, then $|||u|||^{p_1} \leq \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx \leq |||u|||^{p_2}$;*

(2) *if $|||u||| < 1$, then $|||u|||^{p_2} \leq \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx \leq |||u|||^{p_1}$.*

Theorem 2.4 *Let Ω be a bounded domain with the cone property. If $p \in C(\bar{\Omega})$ satisfying $1 < p_1 \leq p(x) \leq p_2 < N$ and q is a measurable function defined on Ω with*

$$p(x) \leq q(x) \ll p^*(x), \quad \text{a.e. } x \in \Omega,$$

then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

Theorem 2.5 *Let Ω be a domain with the cone property. If p is Lipschitz continuous and satisfies $1 < p_1 \leq p(x) \leq p_2 < N$, q is a measurable function defined on Ω with*

$$p(x) \leq q(x) \leq p^*(x), \quad \text{a.e. } x \in \Omega,$$

then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous.

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Suppose that $p \in C(\bar{\Omega})$ and $1 < p_1 \leq p(x) \leq p_2 < N$. If $q \in C(\partial\Omega)$ satisfies the condition*

$$1 \leq q(x) < p_*(x), \quad \forall x \in \partial\Omega,$$

then the boundary trace embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\partial\Omega)$ is compact.

In the proof of the main results, we will use the following principle of concentration compactness in $W^{1,p(x)}(\Omega)$, established in [17].

Theorem 2.7 *Assume that p is Lipschitz continuous on $\bar{\Omega}$ and satisfies $1 < p_1 \leq p(x) \leq p_2 < N$, and Ω is a bounded domain in $\mathbb{R}^N$. Let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ with $\|\nabla u_n\|_{L^{p(x)}} \leq 1$ such that*

$$\begin{aligned} &u_n \to u \quad \text{weakly in } W^{1,p(x)}(\Omega), \\ &|\nabla u_n|^{p(x)} \to \mu \quad \text{weak-}* \text{ in } M(\bar{\Omega}), \\ &|u_n|^{p^*(x)} \to \nu \quad \text{weak-}* \text{ in } M(\bar{\Omega}), \end{aligned}$$

as $n \to \infty$. Denote

$$C_* = \sup\left\{ \int_\Omega |u|^{p^*(x)}\,dx : \|\nabla u_n\|_{L^{p(x)}} \leq 1, u \in W^{1,p(x)}(\Omega) \right\}.$$

Then the limit measures are of the form

$$\begin{aligned} &\mu = |\nabla u|^{p(x)} + \sum_{j\in J} \mu_j \delta_{x_j} + \widetilde{\mu}, \qquad \mu(\bar{\Omega}) \leq 1, \\ &\nu = |u|^{p^*(x)} + \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \nu(\bar{\Omega}) \leq C_*, \end{aligned}$$

where J is a countable set, $\{\mu_j\}, \{\nu_j\} \subset [0,\infty)$, $\{x_j\} \subset \bar{\Omega}$, $\widetilde{\mu} \in M(\Omega)$ is a non-atomic nonnegative measure. The atoms and the regular part satisfy the generalized Sobolev inequality

$$\begin{aligned} &\nu(\bar{\Omega}) \le C_* \max\bigl\{\mu(\bar{\Omega})^{p_2^*/p_1}, \mu(\bar{\Omega})^{p_1^*/p_2}\bigr\}, \\ &\nu_j \le C_* \max\bigl\{\mu_j^{p_2^*/p_1}, \mu_j^{p_1^*/p_2}\bigr\}, \quad \forall j \in J, \end{aligned} \tag{2.1}$$

where $p_1^* = \inf_{x\in\Omega} p^*(x)$, $p_2^* = \sup_{x\in\Omega} p^*(x)$.

3 Existence of solutions for the problems

Set

$$\begin{aligned} &\Lambda(u) = \int_\Omega A(x,\nabla u)\,dx, \\ &K(u) = \int_\Omega F(x,u)\,dx, \\ &L(u) = \int_{\partial\Omega} B(x,u)\,dx, \\ &I(u) = \Lambda(u) + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx - \int_\Omega \frac{h(x)}{p^*(x)}|u|^{p^*(x)}\,dx - K(u) - L(u). \end{aligned}$$

We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of $p(x)$-Laplacian problem (1.1), if, for any $v \in W^{1,p(x)}(\Omega)$,

$$\begin{aligned} \langle I'(u), v\rangle = {}& \int_\Omega a(x,\nabla u)\nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx - \int_\Omega h(x)|u|^{p^*(x)-2}uv\,dx \\ & - \int_\Omega f(x,u)v\,dx - \int_{\partial\Omega} b(x,u)v\,dS = 0. \end{aligned}$$

So next we need only to consider the existence of nontrivial critical points of $I(u)$.

Lemma 3.1 ([16], Lemma 1) *The functional Λ is well-defined on $W^{1,p(x)}(\Omega)$, and for all $u, v \in W^{1,p(x)}$,*

$$\langle \Lambda'(u), v\rangle = \int_\Omega a(x,\nabla u)\nabla v\,dx.$$

Lemma 3.2 ([5], Lemma 2.9) *Suppose that f satisfies* (F1) *or* ($\widetilde{\mathrm{F}}$1). *Then $K(u)$ is weakly continuous.*

Lemma 3.3 ([5], Theorem 2.10) *Suppose that f satisfies* (F1) *or* ($\widetilde{\mathrm{F}}$1). *Then $K(u)$ is differentiable on $W^{1,p(x)}$, and, for all $u, v \in W^{1,p(x)}$,*

$$\langle K'(u), v\rangle = \int_\Omega f(x,u)v\,dx.$$

In the same way, the function L leads to a conclusion similar to Lemma 3.2 and Lemma 3.3.

Lemma 3.4 ([20], Theorem 4.1) *The mapping a is an operator of type S_+, that is, if $u_n \to u$ weakly in $W^{1,p(x)}(\Omega)$ and*

$$\lim_{n\to\infty} \sup \int_\Omega a(x,\nabla u)\cdot(\nabla u_n - \nabla u)\,dx \le 0, \tag{3.1}$$

then $u_n \to u$ strongly in $W^{1,p(x)}(\Omega)$.

Theorem 3.1 *Assume hypotheses* (F1), (F3), (F4), (B1), (B3) *and* (H1) *are fulfilled. Then there exists $M > 0$ such that, whenever $h(x) \le M$, the problem has a nontrivial solution.*

Proof (1) There exists $r > 0$ such that $\inf\{I(u) : |||u||| = r, u \in W^{1,p(x)}(\Omega)\} > c$.

From (F1), (F4) and (B1) we have

$$\begin{aligned}
&\left|F(x,u)\right| \le \varepsilon|u|^{p(x)} + C(\varepsilon)|u|^{p^*(x)},\\
&\left|B(x,u)\right| \le C_2\frac{1}{\alpha_2(x)}|u|^{\alpha_2(x)}.
\end{aligned}$$

Next, from (A4),

$$\begin{aligned}
I(u) ={}& \int_\Omega A(x,\nabla u)\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx + \int_\Omega \frac{h(x)}{p^*(x)}|u|^{p^*(x)}\,dx\\
&- \int_\Omega F(x,u)\,dx - \int_{\partial\Omega} B(x,u)\,dx\\
\ge{}& \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|u|^{p(x)} - \frac{h(x)}{p^*(x)}|u|^{p^*(x)}\,dx\\
&- \int_\Omega \left(\varepsilon|u|^{p(x)} + C(\varepsilon)|u|^{p^*(x)}\right)dx - \int_{\partial\Omega} \frac{C_2}{\alpha_2(x)}|u|^{\alpha_2(x)}\,dS\\
\ge{}& \frac{1}{p_2}\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{1}{p_2}\int_\Omega |u|^{p(x)}\,dx - \frac{h_1}{p_1^*}\int_\Omega |u|^{p^*(x)}\,dx\\
&- \int_\Omega \varepsilon|u|^{p(x)} + C(\varepsilon)|u|^{p^*(x)}\,dx - \int_{\partial\Omega} \frac{C_2}{\alpha_2(x)}|u|^{\alpha_2(x)}\,dS.
\end{aligned}$$

Let $\varepsilon < \frac{1}{2p_2}$, we get

$$I(u) \ge \int_\Omega (\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{2p_2} - C|u|^{p^*(x)}\,dx - \int_{\partial\Omega} \frac{C_2}{\alpha_2(x)}|u|^{\alpha_2(x)}\,dS. \tag{3.2}$$

As $\alpha_2(x)$, $p(x)$ are continuous on $\overline{\Omega}$, there exists $\delta_1 > 0$ such that $|\alpha_2(x) - \alpha_2(y)| < \varepsilon$ and $|p(x)-p(y)| < \varepsilon$ for any $\varepsilon \in (0,1)$ whenever $|x-y| < \delta_1$. Take $x \in \overline{\Omega}$, for any $y \in B_{\delta_1(x)}(x) \cap \overline{\Omega}$, we have

$$p(y) < p(x) + \varepsilon$$

and

$$\alpha_2(y) > \alpha_2(x) - \varepsilon.$$

As $p(x) \ll \alpha_2(x)$, take $\varepsilon = \frac{1}{4}\inf_{x\in\overline{\Omega}}(\alpha_2(x) - p(x))$, we have

$$\alpha_2(x) - \varepsilon - \bigl(p(x) + \varepsilon\bigr) \geq \frac{1}{2}\inf_{x\in\overline{\Omega}}\bigl(\alpha_2(x) - p(x)\bigr) > 0,$$

then

$$p(y) < p(x) + \varepsilon < \alpha_2(x) - \varepsilon < \alpha_2(y),$$

and further

$$p_x^- = \sup_{y\in\overline{B_{\delta_1}(x)}} p(y) < \alpha_{2x}^- = \inf_{y\in\overline{B_{\delta_1}(x)}}\bigl(\alpha_2(x)\bigr).$$

In the same manner, we get

$$\alpha_{2x}^+ = \sup_{y\in\overline{B_{\delta_1}(x)}} \alpha_2(x) < p_x^{*-} = \inf_{y\in\overline{B_{\delta_1}(x)}} p^*(x).$$

$\{B_{\delta_x}(x), x \in \overline{\Omega}\}$ is an open covering of $\overline{\Omega}$. Since $\overline{\Omega}$ is compact, we can pick a finite sub-covering $\{B_{\delta_i}(x_i)\}_{i=1}^k$ for $\overline{\Omega}$ from the covering $\{B_{\delta_x}(x), x \in \overline{\Omega}\}$ such that $\bigcup_{i=1}^k B_{\delta_i}(x_i) \supset \overline{\Omega}$. Denote $\delta_l = \min\{\delta_i, i = 1, 2, \dots, k\}$, we can use all the hypercubes whose length of the side is $\frac{\delta_l}{2}$ to divide the entire space $\mathbb{R}^N$, then $\bigcup_{i=1}^k B_{\delta_i}(x_i) \cap \Omega$ is divided by finite open regions $\{\Omega_i\}_{i=1}^m$ which mutually have no common points, and $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega_i}$. Then

$$p_i^- = \inf_{x\in\overline{\Omega_i}} < p_i^+ = \sup_{x\in\overline{\Omega_i}} p(x) < \alpha_{2i}^- = \inf_{x\in\overline{\Omega_i}} \alpha_2(x) < p_i^{*-} = \inf_{x\in\overline{\Omega_i}} p^*(x). \tag{3.3}$$

By Theorems 2.5 and 2.6, we know that there exist $c_4, c_5 > 1$ such that

$$\|u\|_{L^{p^*}(\Omega_i)} \leq c_4 |||u|||_{\Omega_i}, \qquad \|u\|_{L^{\alpha_2}(\partial\Omega_i)} \leq c_5 |||u|||_{\Omega_i},$$

where $i = 1, 2, \dots, m$.

Take $|||u||| \leq [\max(c_4, c_5)]^{-1}$, then $|||u|||_{\Omega_i} < [\max(c_4, c_5)]^{-1}$ and

$$\|u\|_{L^{p^*}(\Omega_i)} < 1, \qquad \|u\|_{L^{\alpha_2}(\partial\Omega_i)} < 1,$$

then we have

$$\begin{aligned}
&\int_\Omega \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{2p_2} - C|u|^{p^*(x)}\right) dx - \int_{\partial\Omega} \frac{C_2}{\alpha_2(x)}|u|^{\alpha_2(x)}\, dS \\
&\quad = \sum_{i=1}^m \int_{\Omega_i} (\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{2p_2} - C|u|^{p^*(x)}\, dx - \sum_{i=1}^m \int_{\partial\Omega_i} \frac{C_2}{\alpha_2(x)}|u|^{\alpha_2(x)}\, dS \\
&\quad \geq \sum_{i=1}^m \left(\frac{1}{2p_2}|||u|||_{\Omega_i}^{p_i^+} - C|||u|||_{\Omega_i}^{p_i^{*-}} - C|||u|||_{\Omega_i}^{\alpha_{2i}^-}\right).
\end{aligned}$$

Let

$$g(t) = \frac{1}{2p_2} t^{p_i^+} - Ct^{p_i^{*-}} - Ct^{\alpha_{2i}^-}. \tag{3.4}$$

By (3.3), there exists $0 < t_i < 1$ such that $g(t)$ is positive and increasing for any $t \in (0, t_i]$.

Take $t_k = \min\{t_i, i = 1, 2, \ldots, m\}$. Since $|||u||| \le \sum_{i=1}^m |||u|||_{\Omega_i}$, when $|||u||| = r < t_k$, there exists j such that $\frac{r}{m} \le |||u|||_{\Omega_j} \le r < t_j$, then

$$\begin{aligned} I(u) &\ge \int_\Omega \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{2p_2} - C|u|^{p^*(x)} \right) dx - \int_{\partial\Omega} \frac{C_2}{\alpha_2(x)} |u|^{\alpha_2(x)}\, dS \\ &\ge \frac{1}{2p_2} |||u|||_{\Omega_j}^{p_j^+} - C|||u|||_{\Omega_j}^{p_j^{*-}} - C|||u|||_{\Omega_j}^{\alpha_{2j}^-} \\ &\ge \left(\frac{r}{m}\right)^{p_j^+} \left[\frac{1}{2p_2} - C\left(\frac{r}{m}\right)^{p_j^{*-}-p_j^+} - C\left(\frac{r}{m}\right)^{\alpha_{2j}^- - p_j^+} \right] \\ &\ge \left(\frac{r}{m}\right)^{p_j^+} \left[\frac{1}{2p_2} - C\left(\frac{r}{m}\right)^{\alpha_{2j}^- - p_j^+} - C\left(\frac{r}{m}\right)^{\alpha_{2j}^- - p_j^+} \right]. \end{aligned}$$

Take

$$r = \min\left\{ m\left(\frac{1}{4Cp_2}\right)^{\frac{1}{\alpha_{2j}^- - p_j^+}}, t_k \right\},$$

we have $I(u) \ge c$, where $c = \frac{1}{4p_2}(\frac{r}{m})^{p_j^+}$.

(2) There exists $e \in W^{1,p(x)}(\Omega)$ such that $|||e||| > r$, then we have $I(e) < 0$.

From (F1) and (F3), we have

$$F(x,u) \ge C|u|^{\mu_1(x)},$$

for any $(x,t) \in \Omega_0 \times \mathbb{R}$.

Next from (A1) and (A2), for any $x \in \overline{\Omega}$,

$$A(x, \nabla u) = \int_0^1 a(x, t\nabla u)\, dt \le c_0 \left(|\nabla u| + \frac{1}{p(x)} |\nabla u|^{p(x)} \right)$$

and

$$\begin{aligned} I(u) \le{}& \int_\Omega c_0 |\nabla u|\, dx + \int_\Omega \frac{c_0}{p(x)} |\nabla u|^{p(x)} + \frac{1}{p(x)} |u|^{p(x)}\, dx - \int_\Omega \frac{h(x)}{p^*(x)} |u|^{p^*(x)} \\ &- \int_\Omega C|u|^{\mu_1(x)}\, dx - \int_{\partial\Omega} B(x,u)\, dx. \end{aligned}$$

Pick $x_0 \in \Omega_0$. As μ_1, p is continuous on $\overline{\Omega}$, there exists $0 < 2R < 1$ such that

$$p_{2x_0} = \sup_{x \in B_{2R}(x_0)} p(x) < \mu_{1x_0}^- = \inf_{x \in B_{2R}(x_0)} \mu_1(x) \le \mu_{1x_0}^+ = \sup_{x \in B_{2R}(x_0)} \tag{3.5}$$

for $B_{2R}(x_0) \subset \Omega_0$. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ such that $\phi \equiv 1$ for any $x \in B_{2R}(x_0)$, $0 \le \phi \le 1$ and $|\nabla\phi| \le \frac{1}{R}$. Then, for $s > 1$,

$$\begin{aligned} I(s\phi) &\le \int_\Omega c_0 s|\nabla\phi|\, dx + \int_\Omega \frac{c_0}{p(x)} |\nabla s\phi|^{p(x)} + \frac{1}{p(x)} |s\phi|^{p(x)}\, dx - \int_\Omega C|s\phi|^{\mu_1(x)}\, dx \\ &\le \frac{c_0}{R} \int_{B_{2R}(x_0)} s\, dx + \int_{B_{2R}(x_0)} \frac{c_0}{p_1 R^{p_{2x_0}}} s^{p(x)} + \frac{1}{p_1} s^{p(x)}\, dx \end{aligned}$$

$$
\begin{aligned}
&-\int_{B_{2R}(x_0)} Cs\mu_{1x_0}(x)|\phi|^{\mu_1(x)}\,dx \\
&\leq \frac{c_0}{R}\int_{B_{2R}(x_0)} s\,dx + \int_{B_{2R}(x_0)}\left(\frac{c_0}{p_1R^{p_{2x_0}}}+\frac{1}{p_1}\right)s^{p(x)}\,dx \\
&-\int_{B_{2R}(x_0)} Cs\mu_{1x_0}^{-}|\phi|^{\mu_1(x)}\,dx \\
&\leq \int_{B_{2R}(x_0)} s^{p(x)}(\frac{c_0}{R}s^{1-p(x)}+\frac{c_0}{p_1R^{p_{2x_0}}}+\frac{1}{p_1}-\overline{C}s^{\mu_{1x_0}^{-}-p(x)}\,dx,
\end{aligned}
$$

where $\overline{C}=\frac{C\int_{B_{2R}(x_0)}|\phi|^{\mu_1(x)}\,dx}{|B_{2R}(x_0)|}$.

As $\phi\equiv 1$ for any $x\in B_{2R}(x_0)$, $\int_{B_{2R}(x_0)}|\phi|^{\mu_1(x)}\,dx>0$, thus $\overline{C}>0$.

As $p(x)>1$, if s is sufficiently large, then $s^{1-p(x)}<1$. Thus

$$
\begin{aligned}
I(s\phi) &\leq \int_{B_{2R}(x_0)} s^{p(x)}(\frac{c_0}{R}+\frac{c_0}{p_1R^{p_1}}+\frac{1}{p_1}-\overline{C}s^{\mu_1^{-}-p(x)}\,dx \\
&= \int_{B_{2R}(x_0)} s^{p(x)}\left(C-s^{\mu_{1x_0}^{-}-p(x)}\right)dx.
\end{aligned}
$$

Because $\mu_{1x_0}^{-}-p_{2x_0}>0$, when s is sufficiently large, we have $\|\|s\phi\|\|>r$ and $I(s\phi)<0$.

(3) The functional I satisfies the (PS) condition (i.e. any sequence $\{u_n\}\subset W^{1,p(x)}(\Omega)$ with $I(u_n)\leq c$ and $I'(u_n)\to 0$ as $i\to\infty$ in $W^{-1,p'(x)}$ possesses a convergent subsequence).

(i) First, we show that the (PS) sequence $\{u_n\}\subset W^{1,p(x)}$ is bounded.

Note that $p(x)$ is Lipschitz continuous, then there exists a Lipschitz continuous function $\nu(x)$ such that $p(x)\ll\nu(x)\leq p^*(x)$ and

$$
\nu_1=\inf_{x\in\Omega}\nu(x)\leq\sup_{x\in\Omega}\nu(x)=\nu_2. \tag{3.6}
$$

Take

$$
\nu(x)=p(x)+\min\left\{\inf_{x\in\Omega}\bigl(\mu_1(x)-p(x)\bigr),\inf_{x\in\Omega}\bigl(\mu_2(x)-p(x)\bigr),\inf_{x\in\Omega}\bigl(p^*(x)-p(x)\bigr)\right\},
$$

we obtain

$$
\begin{aligned}
&I(u_n)-\left\langle I'(u_n),\frac{u_n}{\nu(x)}\right\rangle \\
&\geq \int_\Omega(\frac{1}{p(x)}a(x,\nabla u_n)\nabla u_n-\frac{1}{\nu(x)}a(x,\nabla u_n)\nabla u_n+a(x,\nabla u_n)\frac{u_n}{\nu(x)^2}\nabla\nu(x) \\
&+\int_\Omega\left(\frac{1}{p(x)}-\frac{1}{\nu(x)}\right)|u_n|^{p(x)}\,dx-\int_\Omega\left(\frac{1}{p^*(x)}-\frac{1}{\nu(x)}\right)h(x)|u_n|^{p^*(x)}\,dx \\
&-\int_\Omega\left(F(x,u_n)-\frac{1}{\nu(x)}f(x,u_n)u_n\right)dx-\int_{\partial\Omega}\left(B(x,u_n)-\frac{1}{\nu(x)}b(x,u_n)u_n\right)dS \\
&\geq \int_\Omega\left(\frac{1}{p(x)}-\frac{1}{\nu(x)}\right)\left(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\right)dx+\int_\Omega a(x,\nabla u_n)\frac{u_n}{\nu(x)^2}\nabla\nu(x)\,dx \\
&+\int_\Omega\left(\frac{1}{\nu(x)}-\frac{1}{p^*(x)}\right)h(x)|u_n|^{p^*(x)}\,dx
\end{aligned}
$$

$$\geq \frac{l_1}{\nu_2 p_2}\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx + \frac{l_2 h_1}{\nu_2 p_2^*}\int_\Omega |u_n|^{p^*(x)}\,dx$$
$$-\frac{c_0 M}{\nu_1^2}\int_\Omega |u_n|\left(1+|\nabla u_n|^{p(x)-1}\right)dx,$$

where $l_1 = \inf_{x\in\Omega}\{\nu(x) - p(x)\}$, $l_2 = \inf_{x\in\Omega} p^*(x) - \nu(x)|$, $M = \sup_{x\in\Omega}|\nabla\nu(x)|$.

By the Young inequality, we have

$$\begin{aligned}\int_\Omega |u_n|\,dx &\leq \int_\Omega \varepsilon_1\frac{1}{p(x)}|u_n|^{p(x)} + \frac{p(x)-1}{p(x)}\varepsilon_1^{\frac{1}{1-p(x)}}\,dx\\ &\leq \frac{\varepsilon_1}{p_1}\int_\Omega |u_n|^{p(x)} + C(\varepsilon_1).\end{aligned}\tag{3.7}$$

Take $\varepsilon_1 = \min\{1, \frac{\nu_1^2 p_1 l_1}{2c_0 M\nu_2 p_2}\}$ such that $\frac{c_0 M\varepsilon_1}{\nu_1^2 p_1} \leq \frac{l_1}{2\nu_2 p_2}$.

By the Young inequality, we have

$$\int_\Omega |\nabla u_n|^{p(x)-1}|u_n|\,dx \leq \varepsilon_2\int_\Omega |\nabla u_n|^{p(x)}\,dx + \frac{\varepsilon_2^{1-p_2}}{p_1}\int_\Omega |u_n|^{p(x)}\,dx.$$

Take $\varepsilon_2 = \min\{1, \frac{\nu_1^2 l_1}{2c_0 M\nu_2 p_2}\}$ such that $\frac{c_0 M\varepsilon_2}{\nu_1^2} \leq \frac{l_1}{2\nu_2 p_2}$.

By the Young inequality again, we have

$$\begin{aligned}\int_\Omega |u_n|^{p(x)}\,dx &\leq \varepsilon_3\frac{p(x)}{p^*(x)}|u_n|^{p^*(x)} + \frac{p^*(x)-p(x)}{p^*(x)}\varepsilon_3^{\frac{p(x)}{p(x)-p^*(x)}}\,dx\\ &\leq \frac{p_2\varepsilon_3}{p_1^*}\int_\Omega |u_n|^{p^*(x)}\,dx + C(\varepsilon_3).\end{aligned}$$

Take $\varepsilon_3 = \min\{1, \frac{h_1 l_2\nu_1^2 p_1 p_1^*}{2c_0 M\nu_2 p_2^* p_2\varepsilon_2^{1-p_2}}\}$ such that $\frac{c_0 M p_2\varepsilon_2^{1-p_2}}{\nu_1^2 p_1 p_1^*} \leq \frac{l_1}{2\nu_2 p_2}$. Then

$$\begin{aligned}&I(u_n) - \left\langle I'(u_n), \frac{u_n}{\nu(x)}\right\rangle\\ &\quad\geq \frac{l_1}{2\nu_2 p_2}\int_\Omega\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx + \frac{l_2 h_1}{2\nu_2 p_2^*}\int_\Omega |u_n|^{p^*(x)}\,dx - C\\ &\quad\geq \frac{l_1}{2\nu_2 p_2}\int_\Omega\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx - C.\end{aligned}$$

As

$$\int_\Omega\left|\frac{u_n\nu_1}{\nu(x)\|u_n\|_{L^{p(x)}}}\right|^{p(x)}dx \leq \int_\Omega\left|\frac{u_n}{\|u_n\|_{L^{p(x)}}}\right|^{p(x)}dx \leq 1,$$

we have $\|\frac{u_n}{\nu}\|_{L^{p(x)}} \leq \frac{\|u_n\|_{L^{p(x)}}}{\nu_1}$. Since

$$\left\|u_n\nabla\frac{1}{\nu(x)}\right\|_{L^{p(x)}} = \left\|\frac{u_n\nabla\nu(x)}{\nu^2(x)}\right\|_{L^{p(x)}} \leq \frac{M}{\nu_1^2}\|u_n\|_{L^{p(x)}},$$

we have

$$\left\|\nabla\frac{u_n}{\nu(x)}\right\|_{L^{p(x)}} = \left\|u_n\nabla\frac{1}{\nu(x)}+\frac{\nabla u_n}{\nu(x)}\right\|_{L^{p(x)}} \le \frac{M}{\nu_1^2}\|u_n\|_{L^{p(x)}}+\frac{M+\nu_1}{\nu_1^2}\|u_n\|_{W^{1,p(x)}},$$

so

$$\left\|\frac{u_n}{\nu(x)}\right\|_{W^{1,p(x)}} \le \left\|\frac{u_n}{\nu}\right\|_{L^{p(x)}}+\left\|\nabla\frac{u_n}{\nu(x)}\right\|_{L^{p(x)}} \le \frac{\|u_n\|_{L^{p(x)}}}{\nu_1}+\frac{M+\nu_1}{\nu_1^2}\|u_n\|_{W^{1,p(x)}} \le C\|u_n\|_{W^{1,p(x)}},$$

where C is constant. Moreover, $\frac{\|u_n\|_{W^{1,p(x)}}}{2} \le |||u||| \le 2\|u_n\|_{W^{1,p(x)}}$, we have

$$\left|\left|\left|\frac{u_n}{\nu(x)}\right|\right|\right| \le 2\left\|\frac{u_n}{\nu}\right\|_{W^{1,p(x)}} \le \frac{4C}{|||u|||},$$

when n is sufficiently large, we obtain

$$C + C|||u_n||| \ge \frac{l_1}{2\nu_2 p_2}\int_\Omega \left(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\right)dx \ge \frac{l_1}{2\nu_2 p_2}|||u_n|||^{p_1}.$$

By the Young inequality, we have

$$|||u_n||| \le \frac{\varepsilon}{p_1}|||u_n|||^{p_1}+C(\varepsilon).$$

Take $\varepsilon = \frac{l_1 p_1}{4\nu_2 p_2 C}$ such that

$$C + \frac{l_1}{4\nu_2 p_2}|||u_n|||^{p_1} \ge \frac{l_1}{2\nu_2 p_2}|||u_n|||^{p_1},$$

then $\{u_n\} \subset W^{1,p(x)}(\Omega)$ is bounded.

(ii) Next, we show that the (PS) sequence $\{u_n\} \subset W^{1,p(x)}(\Omega)$ possesses a convergent subsequence.

We know that $\{u_n\}$ is bounded. As $W^{1,p(x)}(\Omega)$ is reflexive, passing to a subsequence (still denoted by $\{u_n\}$), we may assume that there exists $u \in W^{1,p(x)}(\Omega)$ such that $u_n \to u$ weakly in $W^{1,p(x)}$ and $u_n \to u$ a.e. on Ω.

From the definition of (PS) sequence, we obtain $\lim_{n\to\infty}\langle I'(u_n), u_n - u\rangle = 0$, i.e.

$$\begin{aligned}\lim_{n\to\infty}\biggl[&\int_\Omega a(x,\nabla u_n)(\nabla u_n-\nabla u)\,dx+\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,dx\\ &-\int_\Omega f(x,u_n)(u_n-u)\,dx-\int_\Omega h(x)|u_n|^{p^*(x)-2}(u_n-u)\,dx\\ &-\int_{\partial\Omega} b(x,u_n)(u_n-u)\,dx\biggr]=0.\end{aligned}$$

As $p(x) < p^*(x)$, the embedding $W^{1,p(x)} \to L^{p(x)}(\Omega)$ is compact, so $u_n \to u$ strongly in $L^{p(x)}(\Omega)$. Hence when $n \to \infty$,

$$\left| \int_\Omega |u_n|^{p(x)-2} u_n (u_n - u)\, dx \right| \le \left\| |u_n|^{p(x)-1} \right\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)} \|u_n - u\|_{L^{p(x)}} \to 0.$$

From Theorems 2.1 and 2.4,

$$\left| \int_\Omega f(x,u_n)(u_n - u)\, dx \right| \le \left\| f(x,u_n) \right\|_{L^{\frac{\alpha_1(x)}{\alpha_1(x)-1}}(\Omega)} \|u_n - u\|_{L^{\alpha_1(x)}} \to 0.$$

From Theorems 2.1 and 2.6,

$$\left| \int_\Omega b(x,u_n)(u_n - u)\, dx \right| \le 2 \left\| b(x,u_n) \right\|_{L^{\frac{\alpha_2(x)}{\alpha_2(x)-1}}(\Omega)} \|u_n - u\|_{L^{\alpha_2(x)}} \to 0.$$

If we could verify that $u_n \to u$ strongly in $L^{p^*(x)}(\Omega)$, we can obtain

$$\int_\Omega h(x)|u_n|^{p^*(x)-2}(u_n - u)\, dx| \le 2 \left\| h(x) \right\|_{L^\infty(\Omega)} \left\| |u_n|^{p^*(x)-1} \right\|_{L^{\frac{p^*(x)}{p^*(x)-1}}(\Omega)} \|u_n - u\|_{L^{p^*(x)}} \to 0.$$

Therefore, $\lim_{n\to\infty} \int_\Omega a(x,\nabla u_n)\, dx = 0$, by Lemma 3.4, a is a S_+ type operator, then $u_n \to u$ strongly in $L^{p^*(x)}(\Omega)$. □

Next, in order to complete Theorem 3.1, we prove the following lemma.

Lemma 3.5 *Let the assumptions of Theorem 3.1 be satisfied. If the* (*PS*) *sequence* $\{u_n\} \subset W^{1,p(x)}(\Omega)$ *is bounded, then there exists* $M > 0$ *such that whenever* $h(x) \le M$, $u_n \to u$ *strongly in* $L^{p^*(x)}(\Omega)$.

Proof As $u_n \to u$ strongly in $L^{p(x)}(\Omega)$, there exists subsequence (still denoted by $\{u_n\}$), $u_n \to u$ a.e. on Ω. Note that $\{u_n\} \subset W^{1,p(x)}(\Omega)$ is bounded, by Borel measure theory, we may assume that

$$|\nabla u_n|^{p(x)} \to \mu \quad \text{weak-}* \text{ in } M(\bar{\Omega}),$$
$$|u_n|^{p^*(x)} \to \nu \quad \text{weak-}* \text{ in } M(\bar{\Omega}).$$

$M(\overline{\Omega})$ is the space of finite nonnegative Borel measures on Ω.

From the principle of concentration compactness,

$$\mu = |\nabla u|^{p(x)} + \sum_{j\in J} \mu_j \delta_{x_j} + \widetilde{\mu},$$
$$\nu = |u|^{p^*(x)} + \sum_{j\in J} \nu_j \delta_{x_j},$$

where J is a countable set, $\{x_j, j \in J\} \subset \overline{\Omega}$, $\{\nu_j\} \subset [0, +\infty)$, δ_{x_j} is a measure concentrating upon x_j, $\widetilde{\mu}$ is a nonnegative non-atomic measure.

a. First, we show that $\mu(\{x_j\}) = \nu(\{x_j\}) = 0$ for any $j \in J$.

As $\overline{\Omega}$ is compact, so we only need to verify, for any $x \in \overline{\Omega}$, there exists $r_0 > 0$ such that $\mu(\{x_j\}) = \nu(\{x_j\}) = 0$ for $x_j \in \overline{\Omega} \cap B_{r_0}(x)$.

Note that $p(x)$ is Lipschitz continuous and $p(x) \ll p^*(x)$, there exists $r_0 > 0$ such that

$$p_x^+ = \sup_{y \in \overline{\Omega} \cap B_{r_0}(x)} p(y) < p_x^{*-} = \inf_{y \in \overline{\Omega} \cap B_{r_0}(x)} p^*(y).$$

For any $\varepsilon > 0$, let $\phi_\varepsilon \in C_0^\infty(B_{2\varepsilon}(x_j))$ such that $\phi \equiv 1$ for any $x \in B_{2\varepsilon}(x_j)$, $0 \le \phi_\varepsilon \le 1$ and $|\nabla \phi_\varepsilon| \le \frac{2}{\varepsilon}$. Note that

$$\begin{aligned} \int_\Omega |u_n \phi_\varepsilon|^{p(x)}\, dx &\le \int_\Omega |u_n|^{p(x)}\, dx, \\ \int_\Omega \bigl|\nabla(u_n\phi_\varepsilon)\bigr|^{p(x)}\, dx &= \int_\Omega |\nabla u_n \cdot \phi_\varepsilon + \nabla\phi_\varepsilon \cdot u_n|^{p(x)}\, dx \\ &\le \int_\Omega 2^{p_2}(|\nabla u_n|^{p(x)} + |\nabla\phi_\varepsilon|^{p(x)}|u_n|^{p(x)}\, dx. \end{aligned}$$

Since $u_n \in W^{1,p(x)}(\Omega)$, $\{u_n\phi_\varepsilon\}$ is bounded on $W^{1,p(x)}(\Omega)$, we have $\langle I'(u_n), u_n\phi_\varepsilon\rangle \to 0$ as $n \to \infty$. Note that

$$\begin{aligned} \bigl\langle I'(u_n), u_n\phi_\varepsilon\bigr\rangle &= \int_\Omega a(x,\nabla u_n)\nabla(u_n\phi_\varepsilon)\, dx + \int_\Omega |u_n|^{p(x)}\phi_\varepsilon\, dx - \int_\Omega h(x)|u_n|^{p^*(x)}\phi_\varepsilon\, dx \\ &\quad - \int_\Omega f(x,u_n)u_n\phi_\varepsilon\, dx - \int_{\partial\Omega} b(x,u_n)u_n\phi_\varepsilon\, dx \\ &\ge \int_\Omega a(x,\nabla u_n)\nabla\phi_\varepsilon \cdot u_n\, dx + \int_\Omega |\nabla u_n|^{p(x)}\phi_\varepsilon\, dx + \int_\Omega |u_n|^{p(x)}\phi_\varepsilon\, dx \\ &\quad - \int_\Omega h(x)|u_n|^{p^*(x)}\phi_\varepsilon\, dx - \int_\Omega f(x,u_n)u_n\phi_\varepsilon\, dx - \int_{\partial\Omega} b(x,u_n)u_n\phi_\varepsilon\, dx. \end{aligned}$$

Since $|f(x,u_n)| \le C_1(1 + |u_n|^{\alpha_1(x)-1})$, $|f(x,u_n)u_n| \le C_1(1 + |u_n|^{\alpha_1(x)})$. So there exists $\delta > 0$ such that, for $mE < \delta$,

$$\int_E f(x,u_n)u_n\phi_\varepsilon\, dx \le C\bigl\|1 + |u_n|^{\alpha_1(x)}\bigr\|_{L^{\frac{p^*(x)}{\alpha_1(x)}}(E)} \|\phi_\varepsilon\|_{L^{(\frac{p^*(x)}{\alpha_1(x)})'}(E)} \to 0.$$

From the Vitali theorem, $\int_\Omega f(x,u_n)u_n\phi_\varepsilon\, dx \to \int_\Omega f(x,u)u\phi_\varepsilon\, dx$. In the same way, $\int_{\partial\Omega} b(x, u_n)u_n\phi_\varepsilon\, dx \to \int_{\partial\Omega} b(x,u)u\phi_\varepsilon\, dx$. Then

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega a(x,\nabla u_n) \cdot u_n \cdot \nabla\phi_\varepsilon\, dx \\ &\quad \le -\int_\Omega \phi_\varepsilon\, d\mu - \int_\Omega |u|^{p(x)}\phi_\varepsilon\, d\mu + \int_\Omega h(x)\phi_\varepsilon\, d\nu \\ &\qquad - \int_\Omega f(x,u)u\phi_\varepsilon\, dx - \int_{\partial\Omega} b(x,u)u\phi_\varepsilon\, dx, \end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_\Omega a(x,\nabla u_n)\cdot u_n\cdot\nabla\phi_\varepsilon\,dx\right| \\
&\quad\le\lim_{n\to\infty}\left|\int_\Omega c_0\bigl(1+|\nabla u_n|^{p(x)-1}\bigr)\cdot u_n\cdot\nabla\phi_\varepsilon\,dx\right| \\
&\quad\le\lim_{n\to\infty}c_0\int_\Omega u_n\nabla\phi_\varepsilon\,dx+\lim_{n\to\infty}c_0\int_\Omega|\nabla u_n|^{p(x)-1}|u_n\nabla\phi_\varepsilon|\,dx.
\end{aligned}$$

Note that $u_n\to u$ strongly in $L^{p(x)}(B_{2\varepsilon}(x_j))$, thus, as $n\to\infty$, $\|\nabla\phi_\varepsilon\cdot u_n\|_{L^{p(x)}}\to\|\nabla\phi_\varepsilon\cdot u\|_{L^{p(x)}}$. Then

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega|\nabla u_n|^{p(x)-1}|u_n\nabla\phi_\varepsilon|\,dx&\le\lim_{n\to\infty}\sup\int_\Omega|\nabla u_n|^{p(x)-1}|u_n\nabla\phi_\varepsilon|\,dx \\
&\le 2\lim_{n\to\infty}\sup\bigl\||\nabla u_n|^{p(x)-1}\bigr\|_{L^{p'(x)}}\||u_n\nabla\phi_\varepsilon|\|_{L^{p(x)}} \\
&\le C\||u\nabla\phi_\varepsilon|\|_{L^{p(x)}}.
\end{aligned}$$

Note that

$$\int_\Omega|\nabla\phi_\varepsilon u|^{p(x)}\,dx\le 2\bigl\||\nabla\phi_\varepsilon|^{p(x)}\bigr\|_{(\frac{p^*(x)}{p(x)})',\Omega\cap B_{2\varepsilon}(x_j)}\bigl\||u|^{p(x)}\bigr\|_{\frac{p^*(x)}{p(x)},\Omega\cap B_{2\varepsilon}(x_j)}$$

and

$$\int_{B_{2\varepsilon}(x_j)}\bigl(|\nabla\phi_\varepsilon|^{p(x)}\bigr)^{(\frac{p^*(x)}{p(x)})'}\,dx=\int_{B_{2\varepsilon}(x_j)}|\nabla\phi_\varepsilon|^N\,dx\le 2^{2N}\omega_N.$$

From absolute continuity of the integral, we have $\int_{B_{2\varepsilon}(x_j)\cap\Omega}(|u|^{p(x)})^{\frac{p^*(x)}{p(x)}}\,dx\to 0$, then $\||u\cdot\nabla\phi_\varepsilon|\|_{L^{p(x)}}\to 0$ as $\varepsilon\to 0$. Therefore

$$\left|\int_{B_{2\varepsilon}(x_j)\cap\Omega}\nabla\phi_\varepsilon\cdot u_n\right|dx\to 0.$$

Similarly, we can also obtain

$$\begin{aligned}
&\left|\int_\Omega f(x,u)u\phi_\varepsilon\,dx\right|\le\int_{B_{2\varepsilon}(x_j)\cap\Omega}\bigl|f(x,u)u\bigr|\,dx\to 0, \\
&\left|\int_{\partial\Omega} b(x,u)u\phi_\varepsilon\,dx\right|\le\int_{B_{2\varepsilon}(x_j)\cap\Omega}\bigl|b(x,u)u\bigr|\,dx\to 0, \\
&\left|\int_\Omega|u|^{p(x)}\phi_\varepsilon\,dx\right|\le\int_{B_{2\varepsilon}(x_j)\cap\Omega}\bigl||u|^{p(x)}\bigr|\,dx\to 0.
\end{aligned}$$

Thus

$$0\le-\mu\bigl(\{x_j\}\bigr)+h(x_j)\nu(x_j).$$

Similarly, by the principle of concentration compactness

$$\nu_j\le C_*\max\bigl\{\mu_j^{\frac{p_x^{*+}}{p_x^-}},\mu_j^{\frac{p_x^{*-}}{p_x^+}}\bigr\}.$$

Denote $h_2 = \sup_{x\in\Omega} h(x)$. For any $j \in J$, we have $\mu_j \le h_2\nu_j$. Suppose there exists $j_0 \in J$ such that $\mu_{j_0} = \mu_{x_{j_0}} > 0$. If $\mu_{j_0} \ge 1$, then $\nu_{j_0} \le C_*(h_2\nu_j)^{\frac{p_x^{*+}}{p_x^-}}$, and further

$$\nu_{j_0} \ge \left[C_*^{-1} h_2^{-\frac{p_x^{*+}}{p_x^-}}\right]^{\frac{p_x^-}{p_x^{*+}-p_x^-}}.$$

If $\mu_{j_0} < 1$, then

$$\nu_{j_0} \ge \left[C_*^{-1} h_2^{\frac{p_x^{*-}}{p_x^+}}\right]^{\frac{p_x^+}{p_x^*-p_x^+}}.$$

Note that $\int_\Omega |u_n|^{p^*(x)}\,dx$ is bounded and $\int_\Omega |u_n|^{p^*(x)}\,dx \to \int_\Omega 1\,d\nu = \nu(\overline{\Omega})$ as $n \to \infty$, so $\nu_{j_0} = \nu(\{x_{j_0}\}) \le \nu(\overline{\Omega}) < \infty$. Since $p_x^- \le p_x^+ < p_x^{*-} \le p_x^{*+}$, there exists $M > 0$ such that, for $h_2 \le M$,

$$\nu(\overline{\Omega}) < \left[C_*^{-1} h_2^{-\frac{p_x^{*-}}{p_x^+}}\right]^{\frac{p_x^+}{p_x^{*+}-p_x^-}},$$
$$\nu(\overline{\Omega}) < \left[C_*^{-1} h_2^{-\frac{p_x^{*+}}{p_x^-}}\right]^{\frac{p_x^-}{p_x^{*+}-p_x^-}},$$

which is a contradiction. So there exists $M > 0$ such that, for $h(x) \le M$, $\nu_j = 0$, $\mu_j = 0$, where any $j \in J$.

b. Next, we show that $u_n \to u$ strongly in $L^{p^{*(x)}}(\Omega)$ as $n \to \infty$. From the discussion above, we know if $h(x) \le M$, then $\nu = |u|^{p^*(x)}$. Thus

$$\int_\Omega |u_n|^{p^*(x)}\,dx \to \int_\Omega 1\,d\nu = \int_\Omega |u|^{p^*(x)}\,dx.$$

As $|u_n - u|^{p^{*(x)}} \le 2^{p_2^*}(|u_n|^{p^{(x)}} + |u|^{p^{*(x)}})$, by the Fatou lemma, we have

$$\begin{aligned}
\int_\Omega 2^{p_2^*+1}|u|^{p^*(x)}\,dx &= \int_\Omega \lim_{n\to\infty}\inf\left(2^{p_2^*}|u_n|^{p^*(x)} + 2^{p_2^*}|u_n|^{p^*(x)} - |u_n - u|^{p^{*(x)}}\right)dx \\
&\le \lim_{n\to\infty}\inf\int_\Omega \left(2^{p_2^*}|u_n|^{p^*(x)} + 2^{p_2^*}|u_n|^{p^*(x)} - |u_n - u|^{p^{*(x)}}\right) \\
&\le 2^{p_2^*+1}\int_\Omega |u|^{p^*(x)}\,dx - \lim_{n\to\infty}\sup\int_\Omega |u_n - u|^{p^*(x)}\,dx,
\end{aligned}$$

then $\lim_{n\to\infty}\sup\int_\Omega |u_n - u|^{p^*(x)}\,dx = 0$, and further $\int_\Omega |u_n - u|^{p^*(x)}\,dx \to 0$. So $u_n \to u$ strongly in $L^{p^*(x)}(\Omega)$ as $n \to \infty$. □

4 Multiple solutions for the problems

First, let us introduce some notation. Let $O(N)$ be the group of orthogonal linear transformations in $\mathbb{R}^N$, and G be a subgroup of $O(N)$. For $x \neq 0$, we denote the cardinality of $G_x = \{gx : g \in G\}$ by $|G_x|$ and set $|G| = \inf_{x\in\Omega,\bar{x}\neq 0} |G_x|$. An open subset Ω of $\mathbb{R}^N$ is G-invariant if $g\Omega = \Omega$ for any $g \in G$.

Definition 4.1 Let Ω be a G-invariant open subset of $\mathbb{R}^N$. The action of G on $W^{1,p(x)}(\Omega)$ is defined $gu(x) = u(g^{-1}x)$ for any $u \in W^{1,p(x)}(\Omega)$. The subspace of invariant functions is defined by

$$W_G^{1,p(x)}(\Omega) = \{u \in W^{1,p(x)}(\Omega) : gu = u, \text{for any } g \in G\}.$$

A functional $\varphi : W^{1,p(x)}(\Omega) \to \mathbb{R}$ is G-invariant if $\varphi \circ g = \varphi$ for any $g \in G$.

If the space X is a separable and reflexive Banach space, there exist $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_{n=1}^{\infty}\} \subset X^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$$

and

$$X = \overline{\operatorname{span}}\{e_n : n = 1, 2, \ldots\}, \qquad X^* = \overline{\operatorname{span}}\{f_n : n = 1, 2, \ldots\}.$$

For $k = 1, 2, \ldots$ we denote

$$X_k = \operatorname{span}\{e_k\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=1}^{\infty} X_j}.$$

In order to obtain the multiple solutions for the equation, we need the following hypotheses.

Let Ω be a G-invariant subset of $\mathbb{R}^N$, $p(x)$ is Lipschitz continuous and G-invariant, and it satisfies $1 < p_1 \leq p(x) \leq p_2 < N$. We have:

(F5) $f(gx, t) = f(x, t)$ for any $g \in G$, $x \in \Omega$, $t \in \mathbb{R}$.

(B5) $b(gx, t) = b(x, t)$ for any $g \in G$, $x \in \Omega$, $t \in \mathbb{R}$.

(A6) $A(x, \nabla gu) = A(x, \nabla u)$ for any $g \in G$, $x \in \Omega$.

In the following, denote $G = O(N)$. It is immediate that $I(u) \in C^1(X, \mathbb{R})$ is G-invariant. Then, by the principle of symmetric criticality, we know that u is a critical point of I if and only if u is a critical point of $I|_{W_G^{1,p(x)}}$. Therefore, it suffices to prove the existence of a sequence of critical points for I on $W_G^{1,p(x)}$.

In the following, we prove the existence of a sequence of critical points for I by the fountain theorem, and we take $X = W_G^{1,p(x)}(\Omega)$.

Lemma 4.1 ([21], Lemma 3.3) *For any $x \in \bar{\Omega}$, denote $\psi_k = \sup_{u \in Z_k, |||u|||=1} \int_\Omega |u|^{p^*(x)}\, dx$, then* $\lim_{k\to\infty} \psi_k = 0$.

Lemma 4.2 *If $\alpha(x) \in C(\bar{\Omega})$, $\alpha(x) > 1$ and $\alpha(x) \ll p_*(x)$ for any $x \in \bar{\Omega}$, denote $\gamma_k =$* $\sup_{u \in Z_k, |||u|||=1} \int_{\partial\Omega} |u|^{\alpha(x)}\, dx$*, then* $\lim_{k\to\infty} \gamma_k = 0$.

Proof Because $0 < \gamma_{k+1} \leq \gamma_k$, $\gamma_k \to \gamma \geq 0$, there exists $u_k \in Z_k$ such that $|||u_k||| = 1$ and

$$0 \leq \gamma_k - \int_\Omega |u_k|^{p^*(x)}\, dx < \frac{1}{k}.$$

As $W_G^{1,p(x)}(\Omega)$ is reflexive, passing to a subsequence (still denoted by $\{u_k\}$), we may assume that there exists $u \in W_G^{1,p(x)}(\Omega)$ such that $u_k \to u$ weakly in $W_G^{1,p(x)}(\Omega)$. For any $f_m \in \{f_n, n =$

$1, 2, \ldots\}$, we have $f_m(u_k) = 0$ when $m < k$, then $\lim_{k\to\infty} f_m(u_k) = f_m(u) = 0$. So for any $m \in N$, $f_m(u) = 0$, which implies that $u = 0$, and further $u_k \to 0$ weakly in $W_G^{1,p(x)}(\Omega)$. According to Theorem 2.6, the embedding $W^{1,p(x)}(\Omega) \to L^{\alpha(x)}(\partial\Omega)$ is compact, so $u_k \to 0$ strongly in $L^{\alpha(x)}(\partial\Omega)$, that is, $\|u_k\|_{L^{\alpha(x)}(\partial\Omega)} \to 0$. Thus $\gamma_k \to 0$ as $k \to \infty$. □

Theorem 4.1 *Assume hypotheses* (F1), (F2), (F3) *and* (F5) *or* $(\widetilde{\mathrm{F}}1)$, (F2), (F5), $(\widetilde{\mathrm{B}}1)$, (B2), (B3), (B5) *and* (H1) *are fulfilled, $p(x)$ is a Lipschitz continuous function on $\bar{\Omega}$ and G-invariant. Then there exists $M > 0$ such that, whenever $h(x) \le M$, the problem has a sequence of weak solutions $\{u_n\}$ such that $I(u_n) \to \infty$, as $n \to \infty$.*

The theorem will be verified by the fountain theorem in three steps.

Proof (1) For every $k \in N$, there exists $\gamma_k > 0$, such that $\inf_{u\in Z_k, \|\|u\|\|=\gamma_k} I(u) \to \infty$ as $k \to \infty$.

From $(\widetilde{\mathrm{F}}1)$ and $(\widetilde{\mathrm{B}}1)$

$$\begin{aligned} \bigl|F(x,u)\bigr| &\le C_1\left(|u| + \frac{1}{\alpha_1(x)}|u|^{\alpha_1(x)}\right), \\ \bigl|B(x,u)\bigr| &\le C_2\left(|u| + \frac{1}{\alpha_2(x)}|u|^{\alpha_2(x)}\right), \end{aligned} \tag{4.1}$$

then

$$\begin{aligned} I(u) \ge{}& \frac{1}{p_2}\int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)}\,dx - \int_\Omega C_1\left(|u| + \frac{1}{\alpha_1(x)}|u|^{\alpha_1(x)}\right)dx \\ &- \int_{\partial\Omega} C_2\left(|u| + \frac{1}{\alpha_2(x)}|u|^{\alpha_2(x)}\right)dx - \int_\Omega \frac{h(x)}{p^*(x)}|u|^{p^*(x)}\,dx. \end{aligned}$$

By the Young inequalities

$$\begin{aligned} &\int_\Omega |u|\,dx \le \int_\Omega \frac{1}{p^*(x)}|u|^{p^*(x)} + \frac{p^*(x)-1}{p^*(x)}\,dx \le \int_\Omega \frac{1}{p^*(x)}|u|^{p^*(x)}\,dx + C, \\ &\int_\Omega |u|^{\alpha_1(x)}\,dx \le \int_\Omega \frac{\alpha_1(x)}{p^*(x)}|u|^{p^*(x)} + \frac{p^*(x)-\alpha_1(x)}{p^*(x)}\,dx \le \int_\Omega \frac{\alpha_1(x)}{p^*(x)}|u|^{p^*(x)}\,dx + C, \\ &\int_{\partial\Omega} |u|\,dx \le \int_{\partial\Omega} \frac{1}{\alpha_2(x)}|u|^{\alpha_2(x)} + \frac{\alpha_2(x)-1}{\alpha_2(x)}\,dx \le \int_{\partial\Omega} \frac{1}{\alpha_2(x)}|u|^{\alpha_2(x)}\,dx + C, \end{aligned}$$

then

$$I(u) \ge \frac{1}{p_2}\int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)}\,dx - C\int_\Omega |u|^{p^*(x)}\,dx - C\int_{\partial\Omega} |u|^{\alpha_2(x)}\,dx - C.$$

Denote

$$\beta_k = \sup_{u\in Z_k, \|\|u\|\|=1} \int_\Omega |u|^{p^*(x)}\,dx, \qquad \omega_k = \sup_{u\in Z_k, \|\|u\|\|=1} \int_{\partial\Omega} |u|^{\alpha_2(x)}\,dx.$$

As $\alpha_2(x) \ll p_*(x)$, so by Lemma 4.2 and Lemma 4.1, we obtain $\omega_k \to 0$ and $\beta_k \to 0$ as $k \to \infty$. Take

$$\alpha_2^+ = \sup_{x\in\Omega} \alpha_2(x), \qquad p_2^* = \sup_{x\in\Omega} p^*(x).$$

Then, for $|||u||| > 1$, we have

$$I(u) \geq \frac{1}{p_2}|||u|||^{p_1} - C|||u|||^{p_2^*}\beta_k - C|||u|||^{\alpha_2^+}\omega_k - C.$$

From the Young inequality

$$|||u|||^{\alpha_2^+} \leq \frac{\alpha_2^+}{p_2^*}|||u|||^{p_2^*} + C,$$

then

$$I(u) \geq \frac{1}{p_2}|||u|||^{p_1} - C(\beta_k + \omega_k)|||u|||^{p_2^*} - C.$$

Next, we consider the following equation:

$$\frac{1}{2p_2}t^{p_1} - C(\beta_k + \omega_k)t^{p_2^*} = 0. \tag{4.2}$$

Let t_k be the solution of (4.2),

$$t_k = \left[\frac{1}{2p_2C(\beta_k + \omega_k)}\right]^{\frac{1}{p_2^*-p_1}},$$

$t_k \to \infty$ as $k \to \infty$. We choose $\gamma_k = t_k$, thus, for $|||u||| = \gamma_k$, $k \to \infty$, we have

$$I(u) \geq \frac{1}{2p_2}\gamma_k^{p_1} - C \to \infty.$$

(2) For all $k \in N$, there exists $\rho_k > \gamma_k$ such that $\max_{u\in Y_k, |||u|||=\rho_k} I(u) \leq 0$ as $k \to \infty$, where γ_k is given by (1).

From (4.1), we have

$$\begin{aligned}
I(u) \leq{}& \int_\Omega c_0|\nabla u|\,dx + \int_\Omega \frac{c_0}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|u|^{p(x)}\,dx - \int_\Omega \frac{h(x)}{p^*(x)}|u|^{p^*(x)}\,dx \\
& - \int_\Omega F(x,u)\,dx - \int_{\partial\Omega} B(x,u)\,dS \\
\leq{}& \int_\Omega c_0|\nabla u|\,dx + \int_\Omega \frac{c_0}{p_1}|\nabla u|^{p(x)} + \frac{1}{p_1}|u|^{p(x)}\,dx - \int_\Omega \frac{h_1}{p_2^*}|u|^{p^*(x)}\,dx \\
& + \int_\Omega C_1\left(|u| + \frac{1}{\alpha_1(x)}|u|^{\alpha_1(x)}\right)dx.
\end{aligned}$$

By the Young inequality

$$\begin{aligned}
&\int_\Omega |u|\,dx \leq \int_\Omega \frac{\varepsilon_1}{p^*(x)}|u|^{p^*(x)} + \frac{p^*(x)-1}{p^*(x)}\varepsilon_1^{\frac{1}{1-p^*(x)}}\,dx \leq \varepsilon_1\int_\Omega |u|^{p^*(x)}\,dx + C(\varepsilon_1), \\
&\int_\Omega |u|^{\alpha_1(x)}\,dx \leq \int_\Omega \varepsilon_2\frac{\alpha_1(x)}{p^*(x)}|u|^{p^*(x)} + \frac{p^*(x)-\alpha_1(x)}{p^*(x)}\varepsilon_1^{\frac{1}{\alpha_1(x)-p^*(x)}}\,dx \\
&\qquad \leq \varepsilon_2\int_\Omega |u|^{p^*(x)}\,dx + C(\varepsilon_2).
\end{aligned}$$

We choose $\varepsilon_1 = \min\{1, \frac{h_1}{4C_1p_2^*}\}$, $\varepsilon_2 = \min\{1, \frac{\alpha_1^- h_1}{4C_1p_2^*}\}$, then $C_1\varepsilon_1 \le \frac{h_1}{4p_2^*}$, $\frac{C_1\varepsilon_2}{\alpha_1^-} \le \frac{h_1}{4p_2^*}$. Thus

$$I(u) \le \int_\Omega c_0|\nabla u|\,dx + \frac{\max\{c_0,1\}}{p_1}\int_\Omega |\nabla u|^{p(x)}+|u|^{p(x)}\,dx - \frac{h_1}{2p_2^*}\int_\Omega |u|^{p^*(x)}\,dx + C.$$

As $p(x)$, $p^*(x)$ are continuous on $\bar{\Omega}$, and $p(x) \ll p^*(x)$. Similarly to Theorem 3.1 we can get hypercubes $\{\Omega_i\}_{i=1}^m$ which mutually have no common points and $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega_i}$. On Ω_i,

$$p_i^+ = \sup_{x\in\bar{\Omega}_i} p(x) < p_i^{*-} = \inf_{x\in\bar{\Omega}_i} p^*(x), \tag{4.3}$$

then

$$\begin{aligned} I(u) \le{}& \sum_{i=1}^m \int_{\Omega_i} c_0|\nabla u|\,dx + \frac{\max\{c_0,1\}}{p_1}\sum_{i=1}^m \int_{\Omega_i} |\nabla u|^{p(x)}+|u|^{p(x)}\,dx \\ &- \frac{h_1}{2p_2^*}\sum_{i=1}^m \int_{\Omega_i} |u|^{p^*(x)}\,dx + C. \end{aligned}$$

Since $p(x) > 1$, from the continuous embedding $L^{p(x)}(\Omega) \to L^1(\Omega)$, there exists $C > 0$ such that

$$\|\nabla u\|_{L^1(\Omega_i)} \le C\|\nabla u\|_{L^{p(x)}(\Omega_i)} \le 2C|||u|||_{\Omega_i}.$$

Because Y_k is a finite dimensional space, $|||u|||$ and $\|u\|_{L^{p^*(x)}}$ are equivalent. Thus, for any $i \in \{1,2,\dots,m\}$, $|||u|||_{\Omega_i} \ge 1$,

$$I(u) \le \sum_{i=1}^m \left(2c_0C|||u|||_{\Omega_i} + \frac{\max\{c_0,1\}}{p_1}|||u|||_{\Omega_i}^{p_i^+} - \frac{h_1}{2p_2^*}|||u|||_{\Omega_i}^{p_i^{*-}}\right) + C.$$

Let

$$g_i(t) = 2c_0Ct - \frac{\max\{c_0,1\}}{p_1}t^{p_i^+} - \frac{h_1}{2p_2^*}t^{p_i^{*-}}.$$

Due to (4.3), there exist $M_i > 0$, $g_i(t)$ negative and monotone decreasing for any $t \in [M_i, +\infty)$, and $g_i(t) \to -\infty$ as $t \to \infty$. Denote $t_0 = \max\{1, M_i, i = 1,2,\dots,m\}$, when $t > t_0$, we have $g_j(t) \le 0$ for $j \in \{i = 1,2,\dots,m\}$.

For any $i \in \{1,2,\dots,m\}$, $|||u|||_{\Omega_i} \ge t_0$ when $|||u|||_{\Omega_i}$ sufficiently large. It is easy to check that $I(u) \le 0$. So when $|||u|||$ is large enough, we can find that $|||u|||_{\Omega_i}$ is sufficiently large for any $i \in \{1,2,\dots,m\}$. Thus $I(u) \le 0$ when $|||u||| = \rho_k > \gamma_k$.

(3) The functional I satisfies the (PS) condition.

If the function $f(x)$ satisfies (F1), (F2), (F3) and (F5), the proof is similar to (3) of Theorem 3.1. We only need to change the space $W^{1,p(x)}$ to $W_G^{1,p(x)}$.

If the function $f(x)$ satisfies ($\widetilde{\text{F}}$1), (F2) and (F5), we choose

$$\nu(x) = p(x) + \min\left\{\inf_{x\in\Omega}\big(\mu_2(x) - p(x)\big), \inf_{x\in\Omega}\big(p^*(x) - p(x)\big)\right\}$$

in the proof of (3) of Theorem 3.1. Then

$$\begin{aligned}
&I(u_n)-\left\langle I'(u_n),\frac{u_n}{\nu(x)}\right\rangle\\
&\quad\geq\int_\Omega\left(\frac{1}{p(x)}a(x,\nabla u_n)\nabla u_n-\frac{1}{\nu(x)}a(x,\nabla u_n)\nabla u_n+a(x,\nabla u_n)\frac{u_n}{\nu(x)^2}\nabla\nu(x)\right)dx\\
&\qquad+\int_\Omega\left(\frac{1}{p(x)}-\frac{1}{\nu(x)}\right)|u_n|^{p(x)}\,dx-\int_\Omega\left(\frac{1}{p^*(x)}-\frac{1}{\nu(x)}\right)h(x)|u_n|^{p^*(x)}\,dx\\
&\qquad-\int_\Omega\left(F(x,u_n)-\frac{1}{\nu(x)}f(x,u_n)u_n\right)dx\\
&\quad\geq\frac{l_1}{2\nu_2p_2}\int_\Omega|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\,dx-\int_\Omega\left(F(x,u_n)-\frac{1}{\nu(x)}f(x,u_n)u_n\right)dx-C.
\end{aligned}$$

From (4.1) and ($\widetilde{\mathrm{F}}1$), we obtain

$$\begin{aligned}
&I(u_n)-\left\langle I'(u_n),\frac{u_n}{\nu(x)}\right\rangle\\
&\quad\geq\frac{l_1}{2\nu_2p_2}\int_\Omega|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\,dx-\int_\Omega C_1\left(|u_n|+\frac{1}{\alpha_1(x)}|u_n|^{\alpha_1(x)}\right)dx\\
&\qquad-\int_\Omega\left(\frac{C_1}{\nu(x)}|u_n|+\frac{1}{\nu(x)}|u_n|^{\alpha_1(x)}\right)dx-C\\
&\quad\geq\frac{l_1}{2\nu_2p_2}\int_\Omega|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\,dx-\frac{\nu_1C_1+C_1}{\nu_1}\int_\Omega|u_n|\,dx\\
&\qquad-\frac{\alpha_1^-+\nu_1}{\alpha_1^-\nu_1}\int_\Omega|u_n|^{\alpha_1(x)}\,dx-C,
\end{aligned}$$

where $\alpha_1^+=\sup_{x\in\Omega}\alpha_1(x)$, $\alpha_1^-=\inf_{x\in\Omega}\alpha_1(x)$.

Since $\alpha_1(x)\ll p(x)$, by the Young inequality, we have

$$\begin{aligned}
\int_\Omega|u_n|^{\alpha_1(x)}\,dx&\leq\int_\Omega\varepsilon\frac{\alpha_1(x)}{p(x)}|u_n|^{p(x)}+\frac{p(x)-\alpha_1(x)}{p(x)}\varepsilon^{\frac{\alpha_1(x)}{\alpha_1(x)-p(x)}}\,dx\\
&\leq\frac{\alpha_1^+}{p_1}\varepsilon\int_\Omega|u_n|^{p(x)}\,dx+C(\varepsilon).
\end{aligned}\tag{4.4}$$

In (3.7) and (4.4), we choose

$$\varepsilon_1=\min\left\{1\frac{l_1\nu_1p_1}{4\nu_2p_2(\nu_1C_1+C_1)}\right\},\qquad\varepsilon=\min\left\{1\frac{l_1p_1\nu_1\alpha_1^-}{4\nu_2p_2\alpha_1^+(\alpha_1^-+\nu_1)}\right\}.$$

Then

$$I(u_n)-\left\langle I'(u_n),\frac{u_n}{\nu(x)}\right\rangle\geq\frac{l_1}{4\nu_2p_2}\int_\Omega|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\,dx-C.$$

Similarly to (3) of Theorem 3.1, we find that $\{u_n\}$ is bounded. By Lemma 3.5, we find that the functional I satisfies the (PS) condition.

From the fountain theorem, the proof of Theorem 4.1 follows immediately from (1), (2) and (3). □

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