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Existence and multiplicity of solutions for second-order Hamiltonian systems satisfying generalized periodic boundary value conditions at resonance

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Abstract
We investigate the existence and multiplicity of solutions for second-order Hamiltonian systems satisfying generalized periodic boundary value conditions at resonance by means of the index theory, the critical point theory without compactness assumptions, the least action principle, the saddle point reduction theorem, and the minimax method. Applying the results to second-order HS satisfying periodic boundary value conditions, we obtain some new results.

Keywords: Generalized periodic boundary value conditions; Index theory; Critical point; Saddle point reduction theorem; The least action principle; Second-order Hamiltonian systems

1 Introduction and main results

Solutions of Hamiltonian systems are very important in applications. In recent years, the existence and multiplicity of solutions for Hamiltonian systems via critical point theory have been studied by many authors (see [2, 5–10, 12–22]). In particular, by means of critical point theory, the least action principle, and the minimax method, the existence and multiplicity of periodic solutions for second-order Hamiltonian systems with periodic boundary conditions were extensively studied in the cases where the gradient of the nonlinearity is bounded sublinearly and linearly, and many interesting results are given in [5, 9, 10, 13–19, 22]. In this paper, we discuss the existence and multiplicity of solutions for the following second-order Hamiltonian systems satisfying generalized periodic boundary value conditions:

$$\begin{cases} -x'' - B_1(t)x = \nabla_x V(t,x), & \text{a.e. } t \in [0,1], \\ x(1) = Mx(0), \qquad x'(1) = Nx'(0), \end{cases} \tag{1.1}$$

where $B_1(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n)) = \{B(t) = (b_{jk})_{n\times n} | b_{jk}(t) = b_{kj}(t), t \in [0,1], b_{jk}(t) \in L^\infty([0,1])\}$ with $\nu_M^s(B_1) \neq 0$, $M,N \in GL(n) = \{A = (a_{jk})_{n\times n} | a_{jk} \in \mathbf{R} \text{ and } \det(A) \neq 0\}$, and $MN^T = I_n$, where I_n is the unit matrix of order n, and $\nabla_x V(t,x)$ denotes the gradient of $V(t,x)$ for $x \in \mathbf{R}^n$. We suppose that $V : [0,1] \times \mathbf{R}^n \to \mathbf{R}$ satisfies the following condition:

(A) $V(t,x)$ is measurable in t for every $x \in \mathbf{R}^n$ and continuously differentiable in x for a.e. $t \in [0,1]$. Moreover, there exist $a(x) \in C(\mathbf{R}^+,\mathbf{R}^+)$ and $b(t) \in L^1([0,1],\mathbf{R}^+)$ such that

$$\left|V(t,x)\right| \le a\left(|x|\right)b(t) \quad \text{and} \quad \left|\nabla_x V(t,x)\right| \le a\left(|x|\right)b(t)$$

for all $x \in \mathbf{R}^n$ and a.e. $t \in [0,1]$, where $\mathbf{R}^+ = [0,+\infty)$.

Note that if $M = N = I_n$ and $B_1(t) \equiv 0$, then $\nu^s_{I_n}(0) \neq 0$ via (2.5) in the next section. Therefore the periodic solution problem

$$\begin{cases} -x'' = \nabla_x V(t,x) \quad \text{a.e. } t \in [0,1], \\ x(1) - x(0) = x'(1) - x'(0) = 0, \end{cases} \tag{1.2}$$

is a particular case of (1.1).

Now we use the index $(i^s_M(B), \nu^s_M(B)) \in \mathbf{Z} \times \mathbf{N}$ defined in [6, 7] (see the next section) for all $B \in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$ to reach our main results.

Theorem 1.1 *Assume that $V(t,x)$ is convex in x for a.e. $t \in [0,1]$ and satisfies* (A) *and*

(A$_1$) $i^s_M(B_1) = 0$;

(A$_2$)

$$\int_0^1 V(t,x)\,dt \to +\infty \quad \text{as } \|x\| \to \infty, x \in Z^0(B_1) = \ker(\Lambda - B_1);$$

(A$_3$) *there exists $B_2 \in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$ such that $B_2 > B_1$, $\nu^s_M(B_2) \neq 0$, and $i^s_M(B_2) = i^s_M(B_1) + \nu^s_M(B_1)$, and there exists $\gamma(t) \in L^1([0,1],\mathbf{R}^+)$ such that*

$$V(t,x) \le \frac{1}{2}\left(\left(B_2(t) - B_1(t)\right)x,x\right) + \gamma(t)$$

for all $x \in \mathbf{R}^n$ and a.e. $t \in [0,1]$, and

$$\operatorname{meas}\left\{t \in [0,1] \,\middle|\, V(t,x) - \frac{1}{2}\left(\left(B_2(t) - B_1(t)\right)x,x\right) \to -\infty \text{ as } \|\bar{x}\| \to \infty\right\} > 0,$$

where $x = \tilde{x} + \bar{x}$ and $\bar{x} \in Z^0(B_2)$.

Then problem (1.1) *has at least one solution in $Z = \{x \in H^1([0,1],\mathbf{R}^n) | x(1) = Mx(0)\}$.*

Theorem 1.2 *Assume that $V(t,x)$ satisfies* (A), (A$_1$), *and*

(A$_4$) *there exist $f, g \in L^1([0,1],\mathbf{R}^+)$ with $\nu^s_M(B_1 + f(t)I_n) = 0$ and $i^s_M(B_1 + f(t)I_n) = i^s_M(B_1) + \nu^s_M(B_1)$ such that*

$$\left|\nabla_x V(t,x)\right| \le f(t)|x| + g(t)$$

for all $x \in \mathbf{R}^n$ and a.e. $t \in [0,1]$;

(A$_5$) *there exists a function $\mu(t) \in L^1([0,1],\mathbf{R}^+)$ with $\inf_{t\in[0,1]} \mu(t) > 0$ such that $V(t,x) - \frac{1}{2}\mu(t)|x|^2$ is convex in x for a.e. $t \in [0,1]$.*

Then problem (1.1) *has at least one solution with saddle character in* Z (*i.e., the solution is a saddle point*).

Assume in addition that

(A_6) *there exist* $r > 0$, $B_{01}, B_{02} \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ *such that* $B_{02} > B_{01} > B_1$ *and* $\nu_M^s(B_{0i}) \neq 0$ $(i = 1, 2)$, $i_M^s(B_{02}) = i_M^s(B_{01}) + \nu_M^s(B_{01})$, *and for all* $\|x\| \leq r$,

$$\frac{1}{2}\big(\big(B_{01}(t) - B_1(t)\big)x, x\big) \leq V(t,x) \leq \frac{1}{2}\big(\big(B_{02}(t) - B_1(t)\big)x, x\big)$$

for a.e. $t \in [0,1]$.

Then problem (1.1) *has at least three distinct solutions in* Z.

Theorem 1.3 *Assume that* $V(t,x)$ *satisfies* (A), (A_1), *and*

(A_4') *there exist* $\alpha \in [0,1)$, *and* $f \in L^2([0,1], \mathbf{R}^+)$, *and* $g \in L^1([0,1], \mathbf{R}^+)$ *such that*

$$\big|\nabla_x V(t,x)\big| \leq f(t)|x|^\alpha + g(t)$$

for all $x \in \mathbf{R}^n$ *and a.e.* $t \in [0,1]$;

(A_7) *there exists* $c_0 > 0$ *large enough such that*

$$\liminf_{\|x\|\to\infty} \|x\|^{-2\alpha} \int_0^1 V(t,x)\,dt > c_0 \tag{1.3}$$

or

$$\limsup_{\|x\|\to\infty} \|x\|^{-2\alpha} \int_0^1 V(t,x)\,dt < -c_0 \tag{1.4}$$

for $x \in Z^0(B_1) = \ker(\Lambda - B_1)$.

Then problem (1.1) *has at least one solution in* Z.

Assume in addition that

(A_6') *there exist* $\epsilon > 0$, $r > 0$, $B_{01}, B_{02} \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ *such that* $B_{02} > B_{01} > B_1$ *and* $\nu_M^s(B_{0i}) \neq 0$ $(i = 1, 2)$, $i_M^s(B_{02}) = i_M^s(B_{01}) + \nu_M^s(B_{01})$, *and for all* $\|x\| \leq r$,

$$\frac{1}{2}\big(\big(\epsilon I_n + B_{01}(t) - B_1(t)\big)x, x\big) \leq V(t,x)$$

for a.e. $t \in [0,1]$, *whereas for all* $x \in \mathbf{R}^n$,

$$V(t,x) \leq \frac{1}{2}\big(\big(B_{02}(t) - B_1(t)\big)x, x\big)$$

for a.e. $t \in [0,1]$.

Then problem (1.1) *has at least two distinct solutions in* Z.

Theorem 1.4 *Assume that* $V(t,x)$ *satisfies* (A), (A_4'), (A_7), *and*

(A_1') $i_M^s(B_1) \neq 0$.

Then problem (1.1) *has at least one solution in* Z.

Assume in addition that

(A_8) $V(t,x)$ *is an even function in* x *for a.e.* $t \in [0,1]$;

(A$_9$) *there exist* $\epsilon > 0$, $r > 0$, $B_2 \in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$ *such that* $B_2 > B_1$ *and* $\nu_M^s(B_2) \neq 0$, $i_M^s(B_2) > i_M^s(B_1) + \nu_M^s(B_1)$, *and for all* $\|x\| \leq r$,

$$V(t,x) - V(t,0) \geq \frac{1}{2}\big(\big(\epsilon I_n + B_2(t) - B_1(t)\big)x, x\big)$$

for a.e. $t \in [0,1]$.

Then problem (1.1) *has at least* $i_M^s(B_2) - i_M^s(B_1) - \nu_M^s(B_1)$ *pairs of solutions in* Z.

We give the proofs in Sect. 3, and now we return to some discussions on problem (1.2).

Corollary 1.5 *Assume that* $V(t,x)$ *is convex in* x *for a.e.* $t \in [0,1]$ *and satisfies* (A) *and the following conditions*:

(H$_1$)

$$\int_0^1 V(t,x)\,dt \to +\infty \quad \text{as } |x| \to \infty, x \in \mathbf{R}^n;$$

(H$_2$) *there exists* $\gamma(t) \in L^1([0,1],\mathbf{R}^+)$ *such that*

$$V(t,x) \leq \frac{(2\pi)^2}{2}|x|^2 + \gamma(t)$$

for all $x \in \mathbf{R}^n$ *and a.e.* $t \in [0,1]$, *and*

$$\operatorname{meas}\left\{t \in [0,1] \,\middle|\, V(t,x) - \frac{(2\pi)^2}{2}|x|^2 \to -\infty \text{ as } |x| \to \infty\right\} > 0.$$

Then problem (1.2) *has at least one solution in* $H_0^1 = \{x \in H^1([0,1],\mathbf{R}^n) | x(1) - x(0) = 0\}$.

Remark 1.6 For the interval $[0,T]$ considered in second-order HS satisfying periodic boundary value conditions, if $T = 1$, then Corollary 1.5 reduces to Theorem 3.2 in [17]. By Remark 1.4 and Remark 3.2 in [17] we can see that Corollary 1.5 generalizes Theorem 3.5 in [10] and the corresponding theorem in [13] as $T = 1$.

Corollary 1.7 *Assume that* $V(t,x)$ *satisfies* (A), (A$_5$), *and*

(H$_3$) *there exist* $f,g \in L^1([0,1],\mathbf{R}^+)$ *with* $0 < f(t) < 4\pi^2$ *such that*

$$\big|\nabla_x V(t,x)\big| \leq f(t)|x| + g(t) \tag{1.5}$$

for all $x \in \mathbf{R}^n$ *and a.e.* $t \in [0,1]$.

Then problem (1.2) *has at least one solution with saddle character in* H_0^1.

Assume in addition that

(H$_4$) *there exist* $\delta > 0$ *and* $k \in \mathbf{N} \setminus \{0\}$ *such that, for all* $|x| \leq \delta$,

$$2(k\pi)^2|x|^2 \leq V(t,x) \leq 2\big((k+1)\pi\big)^2|x|^2$$

for a.e. $t \in [0,1]$.

Then problem (1.2) *has at least three distinct solutions in* H_0^1.

Remark 1.8 As $T = 1$, in Theorem 2.2 of [18], assume that $V(t,x)$ satisfies (A), (H_4), and

($H_{3,1}$) there exist $f, g \in L^1([0,1],\mathbf{R}^+)$ with $\int_0^1 f(t)\,dt < 12$ such that (1.5) holds;

($A_{5,1}$) there exists a function $\mu(t) \in L^1([0,1],\mathbf{R}^+)$ with $\int_0^1 \mu(t)\,dt > 0$ such that $V(t,\cdot)$ is $\mu(t)$-monotone.

Then the conclusion of Corollary 1.7 is also true.

On one hand, by Remark 1.7 in [17] we know that the $\mu(t)$-monotonicity of $V(t,\cdot)$ is equivalent to the convexity of $V(t,\cdot) - \frac{1}{2}\mu(t)$. Since $\inf_{t\in[0,1]} \mu(t) > 0 \Rightarrow \int_0^1 \mu(t)\,dt > 0$, this shows that ($A_5$) $\Rightarrow$ ($A_{5,1}$).

On the other hand, for $f \in L^1([0,1],\mathbf{R}^+)$, we have $\int_0^1 f(t)\,dt < 12 \nRightarrow 0 < f(t) < 4\pi^2$ and $0 < f(t) < 4\pi^2 \nRightarrow \int_0^1 f(t)\,dt < 12$. Indeed, if $f(t) = \begin{cases} 4\pi^2, & x \in [0, \frac{1}{4\pi^2}], \\ 0, & x \in (\frac{1}{4\pi^2}, 1], \end{cases}$ then $\int_0^1 f(t)\,dt = 1$ and $f(t) \geq 4\pi^2$ for $x \in [0, \frac{1}{4\pi^2}]$; if $12 < f(t) < 4\pi^2$, then $\int_0^1 f(t)\,dt > 12$. So Corollary 1.7 is a new result and in a sense a development of Theorem 2.2 in [18].

Next, we give some examples of a potential function $V(t,x)$ satisfying the assumptions of Corollary 1.7. Let $\mu(t) = 2\pi^2$ for all $t \in [0,1]$, and let

$$V(t,x) = \begin{cases} \pi^2|x|^2 + 2\pi^2|x| - \pi^2, & |x| \geq 1, \\ 2\pi^2|x|^2, & |x| \leq 1, \end{cases}$$

for all $x \in \mathbf{R}^n$. Clearly, assumptions (A), (H_3), (H_4) hold, and $F(x) = V(t,x) - \frac{1}{2}\mu(t)|x|^2$ is convex in x because

$$F(x) = g\bigl(h(x)\bigr)$$

is convex, which follows from the facts that

$$g(s) = \begin{cases} 2\pi^2 s - \pi^2, & s \geq 1, \\ \pi^2 s^2, & 0 \leq s \leq 1, \end{cases}$$

is convex and increasing and

$$h(x) = |x|, \quad x \in \mathbf{R}^n,$$

is convex. Thus V satisfies the conditions of Corollary 1.7. Similarly, we can see that

$$V(t,x) = \begin{cases} \pi^2(1+\sin t)|x|^2 + 2\pi^2(1+\sin t)|x| - \pi^2(1+\sin t), & |x| \geq 1, \\ 2\pi^2(1+\sin t)|x|^2, & |x| \leq 1, \end{cases}$$

also satisfies the conditions of Corollary 1.7.

Corollary 1.9 *Assume that* $V(t,x)$ *satisfies* (A), (A_4'), *and*

(H_5) *there exists* $c_0 > 0$ *large enough such that*

$$\liminf_{|x|\to\infty} |x|^{-2\alpha} \int_0^1 V(t,x)\,dt > c_0 \tag{1.6}$$

or

$$\limsup_{|x|\to\infty} |x|^{-2\alpha} \int_0^1 V(t,x)\,dt < -c_0 \tag{1.7}$$

for $x \in \mathbf{R}^n$.

Then problem (1.2) *has at least one solution in* H_0^1.

Assume in addition that

(H_4') *there exist* $\epsilon > 0$, $r > 0$, *and* $k \in \mathbf{N} \setminus \{0\}$ *such that*

$$\frac{1}{2}\big(\epsilon + (2k\pi)^2\big)|x|^2 \le V(t,x)$$

for all $|x| \le r$ *and a.e.* $t \in [0,1]$, *and*

$$V(t,x) \le \frac{1}{2}\big(2(k+1)\pi\big)^2|x|^2$$

for all $x \in \mathbf{R}^n$ *and a.e.* $t \in [0,1]$.

Then problem (1.2) *has at least two distinct solutions in* H_0^1.

Remark 1.10 As $T = 1$, in Theorems 1–3 of [14] assume that $V(t,x)$ satisfies (A), (H_4'), and

(A_4'') there exist $\alpha \in [0,1)$ and $f, g \in L^1([0,1], \mathbf{R}^+)$ such that

$$\big|\nabla_x V(t,x)\big| \le f(t)|x|^\alpha + g(t)$$

for all $x \in \mathbf{R}^n$ and a.e. $t \in [0,1]$;

($H_{5,1}$)

$$|x|^{-2\alpha} \int_0^1 V(t,x)\,dt \to +\infty \quad \text{as } |x| \to \infty \tag{1.8}$$

or

$$|x|^{-2\alpha} \int_0^1 V(t,x)\,dt \to -\infty \quad \text{as } |x| \to \infty. \tag{1.9}$$

Then the conclusion of Corollary 1.9 is also true. Clearly, condition ($H_{5,1}$) is stronger than condition (H_5), and condition (A_4'') is weaker than condition (A_4'). Moreover, we can see that Examples 3.1–3.2 in [16] satisfy the conditions of Corollary 1.9 but do not satisfy Theorems 1–3 in [14]. So Corollary 1.9 is a new result and in a sense a development of Theorems 1–3 in [14].

In addition, if $T = 1$ and $c_0 = \frac{1}{8\pi^2}\int_0^1 f^2(t)\,dt$ or $c_0 = -\frac{3}{8\pi^2}\int_0^1 f^2(t)\,dt$, then Corollary 1.9 reduces to Theorems 1.1–1.2 in [16]. In particular, we need to point out that condition (A_4'') of Theorems 1.1–1.2 in [16] must be amended to condition (A_4'), because $\int_0^1 f^2(t)\,dt$ was used in the proof of Theorems 1.1–1.2 in [16].

Remark 1.11 As $T = 1$, in Theorem 1 of [22] assume that $V(t,x)$ satisfies (A), (H_4'), ($H_{3,1}$), and

($H'_{5,1}$)

$$|x|^{-2} \int_0^1 V(t,x)\,dt \to +\infty \quad \text{as } |x| \to \infty \tag{1.10}$$

or

$$|x|^{-2} \int_0^1 V(t,x)\,dt \to -\infty \quad \text{as } |x| \to \infty. \tag{1.11}$$

Then the conclusion of Corollary 1.9 holds. Unfortunately, Tang and Meng [16] pointed out that (1.5) of condition ($H_{3,1}$) and (1.10) or (1.11) of condition ($H'_{5,1}$) cannot hold together, so that Theorem 1 in [22] is also incorrect.

Corollary 1.12 *Assume that* $V(t,x)$ *satisfies* (A), (A'_4), *and*

(H'_5) *there exists* $c_0 > 0$ *large enough such that* (1.3) *or* (1.4) *hold for* $x \in \ker(\Lambda - (2k\pi)^2)$.

Then problem

$$\begin{cases} -x'' - (2k\pi)^2 x = \nabla_x V(t,x) & a.e.\ t \in [0,1], \\ x(1) - x(0) = x'(1) - x'(0) = 0, \end{cases} \tag{1.12}$$

has at least one periodic solution in H_0^1. *Further, assume that* (A_8) *is satisfied together with*

(H_6) *there exist* $\epsilon > 0$ *and* $r > 0$ *such that, for all* $|x| \le r$,

$$V(t,x) - V(t,0) \ge \frac{\epsilon + 4m(2k+m)\pi^2}{2}|x|^2$$

for a.e. $t \in [0,1]$ *and* $k, m \in \mathbf{N} \setminus \{0\}$ *with* $m > 1$.

Then problem (1.12) *has at least* $2nm - 2n$ *pairs of solutions in* H_0^1.

Remark 1.13 As $T = 1$, in Theorems 1.1–1.2 of [19] assume that $V(t,x)$ satisfies (A), (A''_4), (H_6), and

($H'_{5,2}$)

$$\|x\|^{-2\alpha} \int_0^1 V(t,x)\,dt \to +\infty \quad \text{as } \|x\| \to \infty \tag{1.13}$$

or

$$\|x\|^{-2\alpha} \int_0^1 V(t,x)\,dt \to -\infty \quad \text{as } \|x\| \to \infty \tag{1.14}$$

for $x \in \ker(\Lambda - (2k\pi)^2)$.

Then the conclusion of Corollary 1.12 is also true. Clearly, condition ($H'_{5,2}$) is stronger than condition (H'_5), and condition (A''_4) is weaker than condition (A'_4). So Corollary 1.12 is a new conclusion and in a sense a development of Theorems 1.1–1.2 in [19].

Corollary 1.14 *Assume that* $V(t,x)$ *satisfies* (A), (A'_4), *and*

(A''_1) $\ker(\Lambda - A(t)) \setminus \{\theta\} \neq \emptyset$;

(H_5') *there exists* $c_0 > 0$ *large enough such that* (1.3) *or* (1.4) *hold for* $x \in \ker(\Lambda - A(t))$.
Then problem

$$\begin{cases} -x'' - A(t)x = \nabla_x V(t,x) & a.e.\ t \in [0,1], \\ x(1) - x(0) = x'(1) - x'(0) = 0, \end{cases} \tag{1.15}$$

has at least one periodic solution in H_0^1, *where* $A(t)$ *is a continuous symmetric matrix of order* n.

Remark 1.15 As $T = 1$, in Theorems 2–3 in [15] assume that $V(t,x)$ satisfies (A), (A_1''), (A_4''), and

($H_{5,3}'$) there exists $\gamma(t) \in L^1([0,1], \mathbf{R}^+)$ such that $|x|^{-2\alpha} V(t,x) \geq -\gamma(t)$ for all $x \in \mathbf{R}^n$ and a.e. $t \in [0,1]$, and there exists a subset E of $[0,1]$ with $\text{meas}(E) > 0$ such that

$$|x|^{-2\alpha} V(t,x) \to +\infty \quad as\ |x| \to \infty$$

for a.e. $t \in E$; or there exists $\gamma(t) \in L^1([0,1], \mathbf{R}^+)$ such that $|x|^{-2\alpha} V(t,x) \leq \gamma(t)$ for all $x \in \mathbf{R}^n$ and a.e. $t \in [0,1]$, and there exists a subset E of $[0,1]$ with $\text{meas}(E) > 0$ such that

$$|x|^{-2\alpha} V(t,x) \to -\infty \quad as\ |x| \to \infty$$

for a.e. $t \in E$.

Then the conclusion of Corollary 1.14 is also true. Clearly, from the proof of Theorems 2–3 in [15] we can see that ($H_{5,3}'$) $\Rightarrow$ ($H_{5,2}'$). Moreover, we know that condition (H_5') is weaker than condition ($H_{5,2}'$). So, although condition (A_4'') is weaker than condition (A_4'), Corollary 1.14 is also a new conclusion and in a sense a development of Theorems 2–3 in [15].

The proof of Theorems 1.1–1.4 and these corollaries will be given in Sect. 3, and in Sect. 2, we recall some useful results concerning the index theory for linear second-order Hamiltonian systems satisfying generalized periodic boundary value conditions in [6, 7], which will be used in other sections.

2 Brief introduction of the index theory

Let $L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n)) = \{B(t) \in GL(n) | b_{jk}(t) = b_{kj}(t) \text{ for } t \in [0,1] \text{ and } b_{jk}(t) \in L^\infty([0,1])\}$. Index theory in [6, 7] deals with a classification of $L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ associated with the following system:

$$-x'' - B(t)x = 0, \tag{2.1}$$

$$x(1) = Mx(0), \qquad x'(1) = Nx'(0), \tag{2.2}$$

where $M, N \in GL(n)$ and $MN^T = I_n$.

Let $L = L^2([0,1], \mathbf{R}^n)$ and $Z = \{x \in H^1([0,1], \mathbf{R}^n) | x \text{ satisfies } (2.2)\}$. Define $\Lambda : D(\Lambda) \to L$ by $(\Lambda x)(t) = -x''(t)$. From the Sect. 7.1 in [6] we can check that Λ is self-adjoint in L and

$\sigma(\Lambda) = \sigma_d(\Lambda) \subset [0, +\infty)$. In particular, if $M = N = I_n$, then $\sigma(\Lambda) = \sigma_d(\Lambda) = \{(2k)^2\pi^2 | k \in \mathbf{Z}\}$, and if $M = N = -I_n$, then $\sigma(\Lambda) = \sigma_d(\Lambda) = \{(2k-1)^2\pi^2 | k \in \mathbf{Z}\}$.

For any $B(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we define

$$q_B(x,y) = \int_0^1 \bigl[(x', y') - (B(t)x, y)\bigr]\, dt, \quad x, y \in Z, \tag{2.3}$$

where $(\cdot,\cdot)$ is the usual inner product in $\mathbf{R}^n$, and Z is a Hilbert space with norm $\|x\|^2 = \int_0^1 |x'|^2\, dt + \int_0^1 |x|^2\, dt$ for each $x \in Z$. Clearly, the embeddings $Z \hookrightarrow L$ and $Z \hookrightarrow L^\infty$ are compact.

Proposition 2.1 ([7], Proposition 7.2.1) *For any $B(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, the space Z has a q_B-orthogonal decomposition*

$$Z = Z^+(B) \oplus Z^0(B) \oplus Z^-(B)$$

such that q_B is positive definite, null, and negative definite on $Z^+(B)$, $Z^0(B)$, and $Z^-(B)$, respectively. Moreover, $Z^0(B)$ and $Z^-(B)$ are finite-dimensional.

Definition 2.2 ([6], Definition 2.4.1; [7], Definition 7.1.3) For any $B(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we define

$$\nu_M^s(B) = \dim\ker(\Lambda - B), \qquad i_M^s(B) = \sum_{\lambda<0} \nu_M^s(B + \lambda I_n).$$

Definition 2.3 ([7], Definition 7.2.1) For any $B(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we define

$$\nu_q(B) = \dim Z^0(B), \qquad i_q(B) = \dim Z^-(B).$$

We call $\nu_q(B)$ and $i_q(B)$ the nullity and index of B with respect to the bilinear form $q_B(\cdot,\cdot)$, respectively.

Proposition 2.4 ([7], Proposition 7.2.2) *For any $B(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we have*

$$\nu_M^s(B) = \nu_q(B), \qquad i_M^s(B) = i_q(B).$$

For any $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we write $B_1 \le B_2$ if $B_1(t) \le B_2(t)$ for a.e. $t \in [0,1]$ and define $B_1 < B_2$ if $B_1 \le B_2$ and $B_1(t) < B_2(t)$ on a subset of $(0,1)$ of positive measure.

Proposition 2.5

(1) *For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we have that $Z^0(B)$ is the solution subspace of systems* (2.1)–(2.2), *and $\nu_M^s(B) \in \{0, 1, 2, \ldots, 2n\}$* ([6], *Proposition* 2.4.2(1); [7], *Corollary* 7.2.2(i)).

(2) *For any $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, if $B_1 \le B_2$, then $i_M^s(B_1) \le i_M^s(B_2)$ and $i_M^s(B_1) + \nu_M^s(B_1) \le i_M^s(B_2) + \nu_M^s(B_2)$; if $B_1 < B_2$, then $i_M^s(B_1) + \nu_M^s(B_1) \le i_M^s(B_2)$* ([6], *Proposition* 2.4.2(2); [7], *Corollary* 7.2.2(ii)).

(3) *For any* $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, *if* $B_1(t) < B_2(t)$ *for a.e.* $t \in [0,1]$, *then*

$$i^s_M(B_2) - i^s_M(B_1) = \sum_{\lambda\in[0,1)} \nu^s_M\big(B_1 + \lambda(B_2 - B_1)\big).$$

The summand denoted by $I^s_M(B_1, B_2)$ *is called the relative Morse index between* B_1 *and* B_2 *with respect to* $q_B(\cdot,\cdot)$ ([7], *Proposition* 7.2.2(iii)).

(4) (*Poincaré inequality*) *For any* $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, *if* $i^s_M(B) = 0$, *then*

$$q_B(x,x) \geq 0, \quad x \in Z,$$

and the equality holds if and only if $x \in Z^0(B)$ ([7], *Proposition* 7.2.2(v)).

(5) *For any* $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, *if* $B_1 < B_2$ *and* $i^s_M(B_2) = i^s_M(B_1) + \nu^s_M(B_1)$, *then* $Z = Z^-(B_1) \oplus Z^0(B_1) \oplus Z^0(B_2) \oplus Z^+(B_2)$, *and* $(-q_{B_1}(x_1,x_1))^{\frac{1}{2}} + (q_{B_2}(x_2,x_2))^{\frac{1}{2}}$ *is an equivalent norm on* Z *for* $x = x_1 + x_2$ *with* $x_1 \in Z^-(B_1)$ *and* $x_2 \in Z^+(B_2)$.

Proof We only prove (5). Let $Z_1 = Z^-(B_1) \oplus Z^0(B_1)$, $Z_2 = Z^0(B_2) \oplus Z^+(B_2)$. Noticing that $q_{B_1}(x,x) \geq q_{B_2}(x,x)$ for all $x \in Z$, $q_{B_1}(x,x) \leq 0$ for all $x \in Z_1$, and $q_{B_2}(x,x) \geq 0$ for all $x \in Z_2$, if $x \in Z_1 \cap Z_2$, we have $q_{B_2}(x,x) = 0 = q_{B_1}(x,x)$. It follows that $x \in Z^0(B_2) \cap Z^0(B_1)$ and $x(t) = 0$ on a subset of $[0,1]$ of positive measure, and hence $x = 0$ via (1). Thus $Z_1 \cap Z_2 = \{\theta\}$. It remains to prove that $Z = Z_1 + Z_2$. By Proposition 2.1 we have $Z = Z_2 \oplus Z^-(B_2)$, and for any $x \in Z$, there exists a unique pair $(x_1, x_2) \in Z_2 \times Z^-(B_2)$ such that $x = x_1 + x_2$. Let $\{e_j\}_{j=1}^k$ be a basis of $Z_1, e_j = e_j^2 + e_j^-$ with $e_j^2 \in Z_2$, $e_j^- \in Z^-(B_2)$ for $j = 1, 2, \ldots$, $k = i^s_M(B_1) + \nu^s_M(B_1)$. By $i^s_M(B_2) = i^s_M(B_1) + \nu^s_M(B_1) = k$, to prove that $\{e_j^-\}_{j=1}^k$ is a basis of $Z^-(B_2)$, we only need to show that $\{e_j^-\}_{j=1}^k$ is linearly independent. In fact, otherwise there would exist not all zero constants $c_1, \ldots, c_k$ such that $\sum_{j=1}^k c_j e_j^- = 0$. This leads to $\sum_{j=1}^k c_j e_j \in Z_1 \cap Z_2$, a contradiction. The linear independence shows that there exist constants $\{\alpha_j\}_{j=1}^k$ such that $x_2 = \sum_{j=1}^k \alpha_j e_j^-$, and hence $x = x_1 + x_2 = x = x_1 + \sum_{j=1}^k \alpha_j e_j^- = \sum_{j=1}^k \alpha_j e_j + (x_1 - \sum_{j=1}^k \alpha_j e_j^2)$.

Similarly to the proof of Proposition 7.2.2(iv) in [7], $(-q_{B_1}(x_1,x_1))^{\frac{1}{2}} + (q_{B_2}(x_2,x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z^-(B_1)$, $x_2 \in Z^+(B_2)$. □

Proposition 2.6 *For any* $B(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ *and* $Z = Z^+(B) \oplus Z^0(B) \oplus Z^-(B)$, *we have*

$$q_B(x,y) = 0, \quad x, y \in Z^0(B).$$

Proof By (2.3) and Proposition 2.1, for any $x, y \in Z^0(B)$, we have

$$0 = q_B(x+y, x+y) = 2\int_0^1 \big[\big(x', y'\big) - \big(B(t)x, y\big)\big]\,dt = 2q_B(x,y),$$

which shows that $q_B(x,y) = 0$ for all $x, y \in Z^0(B)$. □

Proposition 2.7 *For any* $B(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, *if* $\nu^s_M(B) \neq 0$, *then there exists* $\varepsilon_0 > 0$ *such that* $\nu^s_M(B + \varepsilon_0 I_n) = 0$ *and* $i^s_M(B + \varepsilon_0 I_n) = i^s_M(B) + \nu^s_M(B)$.

Proof Clearly, $B(t) < B(t) + \varepsilon I_n$ for all $t \in [0,1]$ and $\varepsilon > 0$. By (3) of Proposition 2.5 we have

$$i^s_M(B + \varepsilon I_n) - i^s_M(B) = \sum_{\lambda\in[0,1)} \nu^s_M(B + \lambda\varepsilon I_n).$$

Because $i_M^s(B+\varepsilon I_n)$ is finite, there are only finitely many λ such that $\nu_M^s(B+\lambda\varepsilon I_n)\neq 0$ via (2) of Proposition 2.5. Thus, since $\nu_M^s(B)\neq 0$, we can choose $\varepsilon_0>0$ such that $\nu_M^s(B+\varepsilon_0 I_n)=0$ and $i_M^s(B+\varepsilon_0 I_n)-i_M^s(B)=\nu_M^s(B)$. □

Remark 2.8 ([6], Example 2.4.3; [7], Remark 7.1.3) Let $\alpha_1\le\alpha_2\le\cdots\le\alpha_n$ be the eigenvalues of a constant $n\times n$ symmetric matrix B. Then

$$i_{I_n}^s(B)={}^{\#}\{k:\alpha_k>0\}+2\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:4(j\pi)^2<\alpha_k\},\tag{2.4}$$

$$\nu_{I_n}^s(B)={}^{\#}\{k:\alpha_k=0\}+2\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:4(j\pi)^2=\alpha_k\},\tag{2.5}$$

$$i_{-I_n}^s(B)=2\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:\bigl((2j-1)\pi\bigr)^2<\alpha_k\},\tag{2.6}$$

$$\nu_{-I_n}^s(B)=2\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:\bigl((2j-1)\pi\bigr)^2=\alpha_k\},\tag{2.7}$$

where ${}^{\#}A$ denotes the number of elements in a set A. For $\eta\in\mathbf{R}\setminus\{\pm1,0\}$ with $\lambda_0=\arccos\frac{2}{\eta^{-1}+\eta}$, we have

$$\begin{aligned}
i_{\eta I_n}^s(B)&=\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:(2j\pi+\lambda_0)^2<\alpha_k\}\\
&\quad+\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:(2\pi-\lambda_0+2j\pi)^2<\alpha_k\},\\
\nu_{\eta I_n}^s(B)&=\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:(2j\pi+\lambda_0)^2=\alpha_k\}\\
&\quad+\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:(2\pi-\lambda_0+2j\pi)^2=\alpha_k\}.
\end{aligned}$$

In particular, formulae (2.4) and (2.5) were given first by Mawhin and Willem in the book [10].

3 Proof of the main results

In this section, we give proofs of the main results. To this end, we define

$$I(x)=\int_0^1\left[-\frac{1}{2}|x'|^2+\frac{1}{2}\bigl(B_1(t)x,x\bigr)+V(t,x)\right]dt,\quad x\in Z.\tag{3.1}$$

From assumption (A) it is easy to check that I is continuously differentiable and weakly upper semicontinuous on Z (see [6, 7, 10]), where

$$Z=\bigl\{x\in H^1\bigl([0,1],\mathbf{R}^n\bigr)\big|x(1)=Mx(0)\bigr\}$$

is a Hilbert space with the norm

$$\|x\|^2 = \int_0^1 |x'|^2\,dt + \int_0^1 |x|^2\,dt$$

for $x \in Z$. Clearly, for $x \in Z$, we have

$$|x(t)| \le \|x(t)\|_\infty \le \|x(t)\|.$$

Moreover, we have

$$I'(x)y = \int_0^1 \bigl[-(x',y') + (B_1(t)x,y) + (\nabla_x V(t,x),y)\bigr]\,dt, \quad x,y \in Z,$$

and I' is weakly continuous. As in the proof of Proposition 2.4.2(1) in [6], we can find that the critical points of I correspond to the solutions of (1.1) and omit the details.

3.1 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following critical point theorem without the compactness assumptions.

Lemma 3.1 ([17], Theorem 1.1) *Let X_1 and X_2 be reflexive Banach spaces, and let $\varphi \in C^1(X_1 \times X_2, \mathbf{R})$ be such that $\varphi(x_1,\cdot)$ is weakly upper semicontinuous for all $x_1 \in X_1$, $\varphi(\cdot,x_2) : X_1 \to \mathbf{R}$ is convex for all $x_2 \in X_2$, and φ' is weakly continuous. Assume that*

$$\varphi(\theta,x_2) \to -\infty \tag{3.2}$$

as $\|x_2\| \to +\infty$ and, for every $M > 0$,

$$\varphi(x_1,x_2) \to +\infty \tag{3.3}$$

as $\|x_1\| \to +\infty$ uniformly for $\|x_2\| \le M$. Then φ has at least one critical point.

Proof of Theorem 1.1 By assumption (A_1), Propositions 2.1–2.4, and Definition 2.3 we have $Z = Z^0(B_1) \oplus Z^+(B_1)$. Set $X_1 = Z^0(B_1)$, $X_2 = Z^+(B_1)$, $x \in Z$, $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. Next, we divide the proof into three steps.

Step 1. It is obvious that $V(t,x_1(t) + x_2(t))$ is convex in $x_1(t) \in X_1$, so is $\int_0^1 V(t,x_1(t) + x_2(t))\,dt$. From (2.3) and Proposition 2.1 we can see that for every $x_2(t) \in X_2$,

$$I(x_1 + x_2) = \int_0^1 \left[-\frac{1}{2}|x_2'(t)|^2 + \frac{1}{2}\bigl(B_1(t)x_2(t),x_2(t)\bigr) + V\bigl(t,x_1(t) + x_2(t)\bigr)\right]dt$$

is convex in $x_1 \in X_1$.

Step 2. We prove that (3.3) of Lemma 3.1 holds. By assumption (A) and the convexity of $V(t,\cdot)$ we can see that there exists $c_1 > 0$ such that

$$\begin{aligned}&\int_0^1 V\bigl(t,x_1(t) + x_2(t)\bigr)\,dt\\ &\quad\ge 2\int_0^1 V\left(t,\frac{1}{2}x_1(t)\right)dt - \int_0^1 V\bigl(t,-x_2(t)\bigr)\,dt\end{aligned}$$

$$
\begin{aligned}
&\geq 2\int_0^1 V\left(t,\frac{1}{2}x_1(t)\right)dt-\int_0^1 a\left(\left|x_2(t)\right|\right)b(t)\,dt \\
&\geq 2\int_0^1 V\left(t,\frac{1}{2}x_1(t)\right)dt-\max_{0\leq u\leq\|x_2\|_\infty} a(u)\int_0^1 b(t)\,dt \\
&\geq 2\int_0^1 V\left(t,\frac{1}{2}x_1(t)\right)dt-\max_{0\leq u\leq c_1\|x_2\|} a(u)\int_0^1 b(t)\,dt \\
&\geq 2\int_0^1 V\left(t,\frac{1}{2}x_1(t)\right)dt-\max_{0\leq u\leq c_1 M} a(u)\int_0^1 b(t)\,dt
\end{aligned}
$$

for all $x_1\in X_1$ and $x_2\in X_2$ with $\|x_2\|\leq M$. Note that $\|x\|^2=\|x'\|_{L^2}^2+\|x\|_{L^2}^2$ and $B_1(t)\in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$. By (2.3) and Proposition 2.1 we know that there exists $c_2>0$ such that

$$
\begin{aligned}
&I(x_1+x_2) \\
&\quad\geq -\frac{1}{2}\left\|x_2'\right\|_{L^2}^2-\frac{1}{2}c_2\|x_2\|_{L^2}^2+2\int_0^1 V\left(t,\frac{1}{2}x_1(t)\right)dt-\max_{0\leq u\leq c_1 M} a(u)\int_0^1 b(t)\,dt \\
&\quad\geq -\frac{1+c_2}{2}M^2+2\int_0^1 V\left(t,\frac{1}{2}x_1(t)\right)dt-\max_{0\leq u\leq c_1 M} a(u)\int_0^1 b(t)\,dt
\end{aligned}
$$

for all $x_1\in X_1$ and $x_2\in X_2$ with $\|x_2\|\leq M$. By assumption (A_2) it is easy to see that (3.3) of Lemma 3.1 holds.

Step 3. We check (3.2) of Lemma 3.1. If not, there exist a constant c_3 and a sequence $x_{2,n}$ in X_2 such that $\|x_{2,n}\|\to+\infty$ as $n\to\infty$ and

$$
I(x_{2,n})\geq c_3 \tag{3.4}
$$

for all n. Notice that $\nu_M^s(B_2)\neq 0$ and $i_M^s(B_2)=i_M^s(B_1)+\nu_M^s(B_1)$ in (A_3). By (A_1) and (5) of Proposition 2.5 we have $Z=Z^0(B_1)\oplus Z^0(B_2)\oplus Z^+(B_2)$ and $X_2=Z^0(B_2)\oplus Z^+(B_2)$. Let $x_{2,n}=u_n+v_n$, $u_n\in Z^0(B_2)$, $v_n\in Z^+(B_2)$. Then by (A_3), (3.4), (2.3), and Proposition 2.1, we have

$$
\begin{aligned}
c_3\leq I(x_{2,n})&\leq \int_0^1\left[-\frac{1}{2}\left|x_{2,n}'\right|^2+\frac{1}{2}\left(B_2(t)x_{2,n},x_{2,n}\right)\right]dt+\int_0^1\gamma(t)\,dt \\
&=-q_{B_2}(v_n,v_n)+\int_0^1\gamma(t)\,dt,
\end{aligned}
$$

which shows that $\{v_n\}$ is bounded since $(-q_{B_1}(x_1,x_1))^{\frac{1}{2}}+(q_{B_2}(x_2,x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x=x_1+x_2$ with $x_1\in Z^-(B_1)$and $x_2\in Z^+(B_2)$, where $Z^-(B_1)=\{\theta\}$. Since $\|x_{2,n}\|\leq\|u_n\|+\|v_n\|$, we have $\|u_n\|\to\infty$ as $n\to+\infty$. Set

$$
E=\left\{t\in[0,1]\,\middle|\,V(t,x)-\frac{1}{2}\left(\left(B_2(t)-B_1(t)\right)x,x\right)\to-\infty\text{ as }\|\bar{x}\|\to\infty\right\},
$$

where $x=\tilde{x}+\bar{x}$ and $\bar{x}\in Z^0(B_2)$. Noting that $x_{2,n}\in X_2=Z^0(B_2)\oplus Z^+(B_2)$, we have $q_{B_2}(x_{2,n},x_{2,n})\geq 0$ for all n via Proposition 2.1. From the Lebesgue–Fatou lemma we have

$$
\begin{aligned}
&\limsup_{n\to\infty} I(x_{2,n}) \\
&\quad\leq \limsup_{n\to\infty}\int_0^1\left[V(t,x_{2,n})-\frac{1}{2}\left(\left(B_2(t)-B_1(t)\right)x_{2,n},x_{2,n}\right)\right]dt
\end{aligned}
$$

$$\leq \limsup_{n\to\infty} \int_E \left[V(t, x_{2,n}) - \frac{1}{2}\big(\big(B_2(t) - B_1(t)\big)x_{2,n}, x_{2,n}\big) \right] dt + \int_0^1 \gamma(t)\, dt$$
$$\to -\infty$$

via (A_3), which contradicts (3.4). Hence (3.2) of Lemma 3.1 holds.

By Lemma 3.1 I has at least one critical point. Hence problem (1.1) has at least one solution in Z. The proof is complete. □

3.2 Proof of Theorem 1.2

To prove Theorem 1.2, we need the following saddle point reduction theorem under rather general assumptions.

Lemma 3.2 ([1], Theorem 2.3) *Let Y, X_1, X_2 be Hilbert spaces, and let $\psi \in C^1(Y \times X_1 \times X_2, \mathbf{R})$. Suppose that ψ satisfies the following conditions*:

(1) *$D_1\psi(\cdot, x_1, x_2) : Y \to Y$ is μ-monotone for all $(x_1, x_2) \in X_1 \times X_2$, that is, there exists $\mu > 0$ such that*

$$\big\langle D_1\psi(y_1, x_1, x_2) - D_1\psi(y_2, x_1, x_2), y_1 - y_2\big\rangle \geq \mu \|y_1 - y_2\|^2, \quad y_1, y_2 \in Y;$$

(2) *$-D_2\psi(y, \cdot, x_2) : X_1 \to X_1$ is μ-monotone for all $(y, x_2) \in Y \times X_2$.*

Then there exists a map $\phi \in C(X_2, Y \times X_1)$ such that $\phi(x_2) = (y(x_2), x_1(x_2))$ is the unique saddle point of $\psi(\cdot, \cdot, x_2)$ for every $x_2 \in X_2$. Moreover, the map $\varphi : X_2 \to \mathbf{R}$ defined by

$$\varphi(x_2) = \psi\big(y(x_2), x_1(x_2), x_2\big) = \min_{y\in Y} \sup_{x_1\in X_1} \psi(y, x_1, x_2) \tag{3.5}$$

is continuously differentiable, and its derivative is given by

$$\varphi'(x_2) = D_3\psi\big(y(x_2), x_1(x_2), x_2\big) \quad \textit{for every } x_2 \in X_2. \tag{3.6}$$

Proof of Theorem 1.2 By assumption (A_1), Propositions 2.1–2.4, and Definition 2.3 we have $Z = Z^0(B_1) \oplus Z^+(B_1)$. Set $Y = \{\theta\}$, $X_1 = Z^0(B_1)$, $X_2 = Z^+(B_1)$. We define the functional φ as follows:

$$\varphi(x_2) = \sup_{x_1\in X_1} \psi(x_1 + x_2) = \sup_{x_1\in X_1} -I(x_1 + x_2), \quad x_2 \in X_2.$$

By assumption (A) and the convexity of $V(t,x) - \frac{1}{2}\mu(t)|x|^2$ in x for a.e. $t \in [0, 1]$ we have

$$\big(\nabla_x V(t,x) - \nabla_x V(t,y), x - y\big) \geq \mu(t)|x - y|^2, \quad x, y \in Z.$$

Thus for each fixed $x_2 \in X_2$ and any $x_{1,1}, x_{1,2} \in X_1$, we have

$$\begin{aligned} &\int_0^1 \big(\nabla_x V(t, x_{1,1} + x_2) - \nabla_x V(t, x_{1,2} + x_2), x_{1,1} - x_{1,2}\big)\, dt \\ &\quad \geq \mu \int_0^1 |x_{1,1} - x_{1,2}|^2\, dt \end{aligned} \tag{3.7}$$

for all $x, y \in Z$, where $\mu = \inf_{t\in[0,1]} \mu(t) > 0$. Since $Z = X_1 \oplus X_2 = Z^0(B_1) \oplus Z^+(B_1)$, from (2.3) and Propositions 2.1, and 2.6 we know that

$$\begin{aligned}
&\bigl\langle -\psi'(x_{1,1}+x_2) - \bigl(-\psi'(x_{1,2}+x_2)\bigr), x_{1,1}-x_{1,2}\bigr\rangle \\
&\quad = \bigl\langle I'(x_{1,1}+x_2) - I'(x_{1,2}+x_2), x_{1,1}-x_{1,2}\bigr\rangle \\
&\quad = \int_0^1 \bigl(\nabla_x V(t, x_{1,1}+x_2) - \nabla_x V(t, x_{1,2}+x_2), x_{1,1}-x_{1,2}\bigr)\,dt \\
&\quad \geq \mu \int_0^1 |x_{1,1}-x_{1,2}|^2\,dt.
\end{aligned}$$

Noticing that $X_1 = Z^0(B_1)$ is finite-dimensional, we can see that there exists $c_4 > 0$ such that

$$\bigl\langle -\psi'(x_{1,1}+x_2) - \bigl(-\psi'(x_{1,2}+x_2)\bigr), x_{1,1}-x_{1,2}\bigr\rangle \geq c_4\mu\|x_{1,1}-x_{1,2}\|^2.$$

By Lemma 3.2 there exists a continuous mapping $\phi : X_2 \to X_1$ such that $\varphi(x_2) = \psi(\phi(x_2) + x_2)$ for all $x_2 \in X_2$, $\varphi : X_2 \to \mathbf{R}$ is continuously differentiable, and $\varphi'(x_2) = \psi'(\phi(x_2)+x_2)|_{X_2}$ for $x_2 \in X_2$. Hence $x_2 \in X_2$ is a critical point of φ, which shows that $\phi(x_2) + x_2$ is a critical point of ψ and I.

Further, for every $x_2 \in X_2$, by assumption (A_4) we have

$$\begin{aligned}
\left|\int_0^1 \bigl(V(t,x_2) - V(t,\theta)\bigr)\,dt\right| &= \left|\int_0^1\int_0^1 \bigl(\nabla_x V(t, sx_2), x_2\bigr)\,ds\,dt\right| \\
&\leq \frac{1}{2}\int_0^1 f(t)|x_2|^2\,dt + \int_0^1 g(t)|x_2|\,dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
\varphi(x_2) &\geq \psi(x_2) = -I(x_2) \\
&\geq \frac{1}{2}\int_0^1 \bigl[|x_2'|^2 - \bigl(B_1(t)x_2, x_2\bigr) - f(t)|x_2|^2\bigr]\,dt \\
&\quad - \int_0^1 g(t)|x_2|\,dt - \int_0^1 V(t,\theta)\,dt.
\end{aligned}$$

Since $\nu_M^s(B_1 + f(t)I_n) = 0$ and $i_M^s(B_1 + f(t)I_n) = i_M^s(B_1) + \nu_M^s(B_1)$, by (5) of Proposition 2.5 we know that $(q_{B_1+fI_n}(x_2,x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_2$ with $x_2 \in X_2 = Z^+(B_1) = Z^+(B_1 + fI_n)$. Hence there exist $c_5, c_6, c_7 > 0$ such that

$$\begin{aligned}
\varphi(x_2) &\geq \frac{c_5}{2}\|x_2\|^2 - \|x_2\|_\infty \int_0^1 g(t)\,dt - \int_0^1 V(t,\theta)\,dt \\
&\geq \frac{c_5}{2}\|x_2\|^2 - c_6\|x_2\| - c_7 \\
&\to +\infty
\end{aligned}$$

as $\|x_2\| \to +\infty$. Consequently, there exists $x_{2,0} \in X_2$ such that $\varphi(x_{2,0}) = \min_{x_2\in X_2}\varphi(x_2)$, and hence $x_{2,0} + \phi(x_{2,0})$ is a solution with saddle point character of problem (1.1) in Z.

If condition (A_6) holds, then $Z = Z^-(B_{01}) \oplus Z^0(B_{01}) \oplus Z^0(B_{02}) \oplus Z^+(B_{02})$ via (5) of Proposition 2.5. Since $B_{01} > B_1$, there exists $E \subset [0,1]$ with meas $E > 0$ such that $B_{01}(t) > B_1(t)$ for all $t \in E$. Hence from (2.3) we have

$$\begin{aligned} q_{B_1}(x,x) &= \int_0^1 |x'|^2\,dt - \int_E \big(B_1(t)x,x\big)\,dt - \int_{[0,1]\setminus E} \big(B_1(t)x,x\big)\,dt \\ &> \int_0^1 |x'|^2\,dt - \int_E \big(B_{01}(t)x,x\big)\,dt - \int_{[0,1]\setminus E} \big(B_{01}(t)x,x\big)\,dt \\ &= q_{B_{01}}(x,x) \end{aligned}$$

for all $x \in Z \setminus \{\theta\}$, which implies that $q_{B_1}(x,x) > q_{B_{01}}(x,x)$ for all $x \in Z \setminus \{\theta\}$ and $q_{B_1}(x,x) > 0$ for all $x \in Z^0(B_{01}) \setminus \{\theta\}$ with $Z^0(B_{01}) \subset Z^+(B_1)$. Let $X_{2,1} = (Z^-(B_{01}) \oplus Z^0(B_{01})) \cap Z^+(B_1)$. Then we can suppose that $X_{2,2}$ is the orthogonal complement of $X_{2,1}$ in X_2. We claim that $\phi(\theta) = \theta$. Indeed, (A_6) implies $V(t,\theta) = 0$ and $\nabla_x V(t,\theta) = \theta$ for a.e. $t \in [0,1]$. From condition (A_5) and (3.7) we have

$$\begin{aligned} 0 &= \big\langle \psi'\big(\phi(\theta)\big), -\phi(\theta)\big\rangle = \big\langle -I'\big(\phi(\theta)\big), -\phi(\theta)\big\rangle \\ &= \int_0^1 \big(-\nabla_x V\big(t,\phi(\theta)\big), -\phi(\theta)\big)\,dt \\ &= \int_0^1 \big(\nabla_x V(t,\theta) - \nabla_x V\big(t,\phi(\theta)\big), -\phi(\theta)\big)\,dt \\ &\geq \mu \int_0^1 \big|\phi(\theta)\big|^2\,dt \\ &\geq 0, \end{aligned}$$

which shows that $\phi(\theta) = \theta$. From the continuity of ϕ, we know that there exists $0 < \delta < r$ such that $\|\phi(x_2)\| < r$ as $\|x_2\| \leq \delta$. Consequently, from (A_6) and (2.3) we obtain

$$\begin{aligned} \varphi(x_{2,1}) &= \psi\big(x_{2,1} + \phi(x_{2,1})\big) = -I\big(x_{2,1} + \phi(x_{2,1})\big) \\ &\leq \frac{1}{2}\int_0^1 \Big[\big|x'_{2,1} + \big(\phi(x_{2,1})\big)'\big|^2 - \big(B_{01}(t)\big(x_{2,1} + \phi(x_{2,1})\big), x_{2,1} + \phi(x_{2,1})\big)\Big]\,dt \\ &= \frac{1}{2} q_{B_{01}}\big(x_{2,1} + \phi(x_{2,1}), x_{2,1} + \phi(x_{2,1})\big) \\ &\leq 0 \end{aligned}$$

for all $x_{2,1} \in X_{2,1}$ with $\|x_{2,1}\| \leq \delta$ via $B_{01} > B_1$ and $x_{2,1} + \phi(x_{2,1}) \in (Z^-(B_{01}) \oplus Z^0(B_{01})) \cup Z^0(B_1)$, and

$$\begin{aligned} \varphi(x_{2,2}) &\geq \psi(x_{2,2}) = -I(x_{2,2}) \\ &\geq \frac{1}{2}\int_0^1 \Big[\big|x'_{2,2}\big|^2 - \big(B_{02}(t)x_{2,2}, x_{2,2}\big)\Big]\,dt \\ &= \frac{1}{2} q_{B_{02}}(x_{2,2}, x_{2,2}) \\ &\geq 0 \end{aligned}$$

for all $x_{2,2} \in X_{2,2}$ with $\|x_{2,2}\| \le \delta$ via $X_{2,2} = Z^+(B_{01}) \cap Z^+(B_1)$ and $Z^+(B_{01}) = Z^+(B_{02}) \oplus Z^0(B_{02})$.

Since I is weakly upper semicontinuous on Z, φ id weakly lower semicontinuous on X_2. By the coerciveness and weak lower semicontinuity of φ we see that satisfies (PS)-condition and is bounded below.

If $\inf\{\varphi(x_2) : x_2 \in X_2\} = 0$, then all $x_{2,1} \in X_{2,1}$ with $\|x_{2,1}\| \le \delta$ are minima of φ, which shows that φ has infinitely many critical points. If $\inf\{\varphi(x_2) : x_2 \in X_2\} < 0$, then φ has at least two nonzero critical points via Theorem 4 in [2]. Thus problem (1.1) has at least two nontrivial solutions in Z. In addition, since $V(t,\theta) = 0$ for a.e. $t \in [0,1]$, we know that problem (1.1) has trivial solution θ. Hence problem (1.1) has three distinct solutions in Z. The proof is complete. □

3.3 Proof of Theorem 1.3

In the section, we use the saddle point theorem (see Theorem 4.6, [12] or [10]) and a generalization of the mountain pass theorem (see Theorem 5.29 and Example 5.26 in [12]) to prove Theorem 1.3.

Proof of Theorem 1.3 First, we verify that I satisfies the (PS)-condition. Suppose that $I'(x_n) \to 0$ as $n \to +\infty$ and $I(x_n)$ is bounded. From condition (A_1) we have $Z = Z^0(B_1) \oplus Z^+(B_1)$. Set $x_n = \bar{x}_n + \tilde{x}_n$ and $\bar{x}_n \in Z^0(B_1)$, $\tilde{x}_n \in Z^+(B_1)$. By assumption (A_4') we have

$$\begin{aligned}
&\left|\int_0^1 \bigl(\nabla_x V(t,x_n),\tilde{x}_n\bigr)\,dt\right| \\
&\quad\le \int_0^1 f(t)|\bar{x}_n+\tilde{x}_n|^\alpha|\tilde{x}_n|\,dt + \int_0^1 g(t)|\tilde{x}_n|\,dt \\
&\quad\le \int_0^1 f(t)2\bigl(|\bar{x}_n|^\alpha + |\tilde{x}_n|^\alpha\bigr)|\tilde{x}_n|\,dt + \int_0^1 g(t)|\tilde{x}_n|\,dt \\
&\quad\le 2\left(\int_0^1 f^2(t)\,dt\right)^{\frac12}\left(\int_0^1 |\tilde{x}_n|^2|\bar{x}_n|^{2\alpha}\,dt\right)^{\frac12} + 2\int_0^1 f(t)|\tilde{x}_n|^{1+\alpha}\,dt \\
&\qquad+ \int_0^1 g(t)|\tilde{x}_n|\,dt \\
&\quad\le 2\beta_0\|\bar{x}_n\|_\infty^\alpha\left(\int_0^1 |\tilde{x}_n|^2\,dt\right)^{\frac12} + 2\int_0^1 f(t)|\tilde{x}_n|^{1+\alpha}\,dt + \int_0^1 g(t)|\tilde{x}_n|\,dt \\
&\quad\le \varepsilon\beta_0\int_0^1 |\tilde{x}_n|^2\,dt + \frac{\beta_0}{\varepsilon}\|\bar{x}_n\|_\infty^{2\alpha} + 2\|\tilde{x}_n\|_\infty^{1+\alpha}\int_0^1 f(t)\,dt \\
&\qquad+ \|\tilde{x}_n\|_\infty\int_0^1 g(t)\,dt
\end{aligned} \tag{3.8}$$

for all n, where $\beta_0 = (\int_0^1 f^2(t)\,dt)^{\frac12}$ and $\varepsilon > 0$. Thus, from $\bar{x}_n \in Z^0(B_1)$, $\tilde{x}_n \in Z^+(B_1)$, (2.3), and Proposition 2.1 we have

$$\begin{aligned}
\|\tilde{x}_n\| &\ge \bigl\langle -I'(x_n),\tilde{x}_n\bigr\rangle \\
&\ge \int_0^1 \bigl[\bigl(x_n',\tilde{x}_n'\bigr) - \bigl(B_1(t)x_n,\tilde{x}_n\bigr) - \varepsilon\beta_0|\tilde{x}_n|^2\bigr]\,dt - \frac{\beta_0}{\varepsilon}\|\bar{x}_n\|_\infty^{2\alpha}
\end{aligned}$$

$$
\begin{aligned}
&-2\|\tilde{x}_n\|_\infty^{1+\alpha}\int_0^1 f(t)\,dt-\|\tilde{x}_n\|_\infty\int_0^1 g(t)\,dt\\
&=\int_0^1\bigl[|\tilde{x}_n'|^2-\bigl(B_1(t)\tilde{x}_n,\tilde{x}_n\bigr)-\varepsilon\beta_0|\tilde{x}_n|^2\bigr]\,dt-\frac{\beta_0}{\varepsilon}\|\bar{x}_n\|_\infty^{2\alpha}\\
&\quad-2\|\tilde{x}_n\|_\infty^{1+\alpha}\int_0^1 f(t)\,dt-\|\tilde{x}_n\|_\infty\int_0^1 g(t)\,dt
\end{aligned}
$$

for n large enough. By Proposition 2.7 we can choose $\varepsilon_0>0$ such that $\nu_M^s(B_1+\varepsilon_0\beta_0 I_n)=0$ and $i_M^s(B_1+\varepsilon_0\beta_0 I_n)=i_M^s(B_1)+\nu_M^s(B_1)$. From (5) of Proposition 2.5 we know that $(q_{B_1+\varepsilon_0\beta_0 I_n}(x_2,x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x=x_2$ with $x_2\in Z^+(B_1)=Z^+(B_1+\varepsilon_0\beta_0 I_n)$. Hence there exist $c_8,c_9,c_{10},c_{11}>0$ such that

$$
\|\tilde{x}_n\|+c_9\|\bar{x}_n\|^{2\alpha}+c_{10}\|\tilde{x}_n\|^{1+\alpha}+c_{11}\|\tilde{x}_n\|\ge c_8\|\tilde{x}_n\|^2,
$$

which implies that there are $k_1>0$ and $k_2>0$ such that

$$
k_1\|\bar{x}_n\|^{2\alpha}+k_2\ge\|\tilde{x}_n\|^2. \tag{3.9}
$$

In a way similar to (3.8), for all n, we obtain

$$
\begin{aligned}
&\left|\int_0^1\bigl(V(t,x_n)-V(t,\bar{x}_n)\bigr)\,dt\right|\\
&\quad=\left|\int_0^1\int_0^1\bigl(\nabla_x V(t,\bar{x}_n+s\tilde{x}_n),\tilde{x}_n\bigr)\,ds\,dt\right|\\
&\quad\le\int_0^1\int_0^1 f(t)|\bar{x}_n+s\tilde{x}_n|^\alpha|\tilde{x}_n|\,ds\,dt+\int_0^1\int_0^1 g(t)|\tilde{x}_n|\,ds\,dt\\
&\quad\le\int_0^1 2f(t)\left(|\bar{x}_n|^\alpha+\frac{1}{1+\alpha}|\tilde{x}_n|^\alpha\right)|\tilde{x}_n|\,dt+\int_0^1 g(t)|\tilde{x}_n|\,dt\\
&\quad\le 2\left(\int_0^1 f^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^1|\tilde{x}_n|^2|\bar{x}_n|^{2\alpha}\,dt\right)^{\frac{1}{2}}+2\int_0^1 f(t)|\tilde{x}_n|^{1+\alpha}\,dt\\
&\qquad+\int_0^1 g(t)|\tilde{x}_n|\,dt\\
&\quad\le 2\beta_0\|\bar{x}_n\|_\infty^\alpha\left(\int_0^1|\tilde{x}_n|^2\,dt\right)^{\frac{1}{2}}+2\int_0^1 f(t)|\tilde{x}_n|^{1+\alpha}\,dt+\int_0^1 g(t)|\tilde{x}_n|\,dt\\
&\quad\le\frac{\varepsilon_0\beta_0}{2}\int_0^1|\tilde{x}_n|^2\,dt+\frac{2\beta_0}{\varepsilon_0}\|\bar{x}_n\|_\infty^{2\alpha}+2\|\tilde{x}_n\|_\infty^{1+\alpha}\int_0^1 f(t)\,dt\\
&\qquad+\|\tilde{x}_n\|_\infty\int_0^1 g(t)\,dt.
\end{aligned} \tag{3.10}
$$

Notice that by the boundedness of $\{I(x_n)\}$ and $\alpha\in[0,1)$, the equivalence of the norm $(q_{B_1+\varepsilon_0\beta_0 I_n}(x_2,x_2))^{\frac{1}{2}}$ on Z for $x=x_2$ with $x_2\in Z^+(B_1)=Z^+(B_1+\varepsilon_0\beta_0 I_n)$, and (3.9) we can see that there exist $c_{12}\in\mathbf{R}$ and $c_{13},c_{14},c_{15},c_{16}>0$ such that

$$
\begin{aligned}
c_{12}&\le -I(x_n)\\
&=\int_0^1\frac{1}{2}\bigl[|x_n'|^2-\bigl(B_1(t)x_n,x_n\bigr)\bigr]\,dt
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^1 \bigl(V(t,x_n) - V(t,\bar{x}_n)\bigr)\,dt - \int_0^1 V(t,\bar{x}_n)\,dt \\
&\le \int_0^1 \frac{1}{2}\bigl[|\tilde{x}_n'|^2 - \bigl(B_1(t)\tilde{x}_n,\tilde{x}_n\bigr) - \varepsilon_0\beta_0|\tilde{x}_n|^2\bigr]\,dt + \varepsilon_0\beta_0\int_0^1 |\tilde{x}_n|^2\,dt \\
&\quad + \frac{2\beta_0}{\varepsilon_0}\|\bar{x}_n\|_\infty^{2\alpha} + 2\|\tilde{x}_n\|_\infty^{1+\alpha}\int_0^1 f(t)\,dt + \|\tilde{x}_n\|_\infty \int_0^1 g(t)\,dt \\
&\quad - \int_0^1 V(t,\bar{x}_n)\,dt \\
&\le c_{13}\|\tilde{x}_n\|^2 + c_{14}\|\tilde{x}_n\|^2 + 2c_9\|\bar{x}_n\|^{2\alpha} + c_{10}\|\tilde{x}_n\|^{1+\alpha} + c_{11}\|\tilde{x}_n\| \\
&\quad - \int_0^1 V(t,\bar{x}_n)\,dt \\
&\le c_{15}\|\tilde{x}_n\|^2 + c_{16} + 2c_9\|\bar{x}_n\|^{2\alpha} - \int_0^1 V(t,\bar{x}_n)\,dt \\
&\le (c_{15}k_1 + 2c_9)\|\bar{x}_n\|^{2\alpha} + c_{15}k_2 + c_{16} - \int_0^1 V(t,\bar{x}_n)\,dt \\
&\le \|\bar{x}_n\|^{2\alpha}\left((c_{15}k_1 + 2c_9) - \|\bar{x}_n\|^{-2\alpha}\int_0^1 V(t,\bar{x}_n)\,dt\right) + c_{15}k_2 + c_{16}
\end{aligned}
$$

for n large enough. Taking $c_0 > c_{15}k_1 + 2c_9$, by this inequality and (1.3) of condition (A_7) we obtain that $\{\|\bar{x}_n\|\}$ is bounded. If (1.4) of condition (A_7) holds, similarly to this inequality, by (3.9) and (3.10) we have

$$
\begin{aligned}
-I(x_n) &\ge -2c_9\|\bar{x}_n\|^{2\alpha} - c_{10}\|\tilde{x}_n\|^{1+\alpha} - c_{11}\|\tilde{x}_n\| - \int_0^1 V(t,\bar{x}_n)\,dt \\
&\ge -2c_9\|\bar{x}_n\|^{2\alpha} - (c_{10} + c_{11})\|\tilde{x}_n\|^2 - \int_0^1 V(t,\bar{x}_n)\,dt - (c_{10} + c_{11}) \\
&\ge \|\bar{x}_n\|^{2\alpha}\left[-\bigl(k_1(c_{10} + c_{11}) + 2c_9\bigr) - \|\bar{x}_n\|^{-2\alpha}\int_0^1 V(t,\bar{x}_n)\,dt\right] \\
&\quad - (k_2 + 1)(c_{10} + c_{11}).
\end{aligned}
$$

Taking $c_0 > k_1(c_{10} + c_{11}) + 2c_9$, by this inequality and (1.4) of condition (A_7) we also obtain that $\{\|\bar{x}_n\|\}$ is bounded. Hence $\{\|x_n\|\}$ is bounded by (3.9). Arguing then as in Proposition 4.1 of [10], we easily conclude that the (PS)-condition is satisfied.

Next, we will check that

$$
-I(x) \to +\infty \tag{3.11}
$$

as $\|x\| \to +\infty$ in $Z^+(B_1)$. In fact, by the proof of (3.10) we have

$$
\begin{aligned}
&\left|\int_0^1 \bigl(V(t,x) - V(t,\theta)\bigr)\,dt\right| \\
&\quad \le \frac{1}{1+\alpha}\int_0^1 f(t)|x|^{1+\alpha}\,dt + \int_0^1 g(t)|x|\,dt
\end{aligned}
$$

$$\begin{aligned}
&\le 2\left(\int_0^1 f^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^1 |x|^{2(1+\alpha)}\,dt\right)^{\frac{1}{2}} + \|x\|\int_0^1 g(t)\,dt \\
&\le \frac{\varepsilon_0\beta_0}{2}\int_0^1 |x|^2\,dt + \frac{2\beta_0}{\varepsilon_0}\|x\|_\infty^{2\alpha} + \|x\|\int_0^1 g(t)\,dt
\end{aligned}$$

for all $x \in Z^+(B_1)$. It follows that

$$\begin{aligned}
-I(x) &= \int_0^1 \frac{1}{2}\left[\left|x'\right|^2 - \left(B_1(t)x,x\right)\right]dt \\
&\quad - \int_0^1 \left(V(t,x) - V(t,\theta)\right)dt - \int_0^1 V(t,\theta)\,dt \\
&\ge \int_0^1 \frac{1}{2}\left[\left|x'\right|^2 - \left(B_1(t)x,x\right) - \varepsilon_0\beta_0|x|^2\right]dt - \frac{2\beta_0}{\varepsilon_0}\|x\|_\infty^{2\alpha} \\
&\quad - \|x\|\int_0^1 g(t)\,dt - \int_0^1 V(t,\theta)\,dt \\
&\ge c_8\|x\|^2 - \frac{2\beta_0}{\varepsilon_0}\|x\|^{2\alpha} - \|x\|\int_0^1 g(t)\,dt - \int_0^1 V(t,\theta)\,dt \\
&\to +\infty
\end{aligned}$$

as $\|x\| \to +\infty$ in $Z^+(B_1)$, which shows (3.11).

On the other hand, if (1.3) of condition (A_7) holds, then we clearly have

$$-I(x) \to -\infty \tag{3.12}$$

as $\|x\| \to +\infty$ in $Z^0(B_1)$. Thus by (3.11), (3.12), and the saddle point theorem (see Theorem 4.6 in [12] or [10]) we obtain that problem (1.1) has at least one solution in Z. If (1.4) of condition (A_7) holds, then we have

$$-I(x) \to +\infty$$

as $\|x\| \to +\infty$ in $Z^0(B_1)$. Thus by (3.11) we can see that $-I(x) \to +\infty$ as $\|x\| \to +\infty$ in Z. From Theorem 1.1 and Corollary 1.1 in [10] we know that problem (1.1) also has at least one solution in Z.

Further, if condition (A_6') holds, then $Z = Z^-(B_{01}) \oplus Z^0(B_{01}) \oplus Z^0(B_{02}) \oplus Z^+(B_{02})$ via (5) of Proposition 2.5. Let $X_1 = Z^-(B_{01}) \oplus Z^0(B_{01})$ and $X_2 = Z^-(B_{01})$. Then $X_1^\perp = Z^0(B_{02}) \oplus Z^+(B_{02})$ and $X_2^\perp = Z^0(B_{01}) \oplus Z^0(B_{02}) \oplus Z^+(B_{02})$. Note that $I \in C^1(Z, \mathbf{R})$ satisfies the (PS)-condition. By Theorem 5.29 and Example 5.26 in [12] we only need to verify that

(I_1) $\liminf \|x\|^{-2}I(x) > 0$ as $\|x\| \to 0$ in X_1,

(I_2) $I(x) \le 0$ for all $x \in X_1^\perp$, and

(I_3) $I(x) \to -\infty$ as $\|x\| \to +\infty$ in $X_2^\perp$.

By condition (A_6') we can see that $V(t,\theta) = 0$. Since

$$V(t,x) - V(t,\theta) = \int_0^1 \left(\nabla_x V(t,sx),x\right)ds$$

for all $x \in \mathbf{R}^n$ and a.e. $t \in [0,1]$, from condition (A_4') we obtain

$$\left|V(t,x)\right| \le \frac{1}{1+\alpha}f(t)|x|^{1+\alpha} + g(t)|x|$$

for all $x \in \mathbf{R}^n$ and a.e. $t \in [0,1]$, and there exist $c_{17}, c_{18} > 0$ such that

$$\begin{aligned}\left|\int_0^1 V(t,x)\,dt\right| &\le \frac{1}{1+\alpha}\int_0^1 f(t)|x|^{1+\alpha}\,dt + \int_0^1 g(t)|x|\,dt \\ &\le \frac{1}{1+\alpha}\|x\|_\infty^{1+\alpha}\int_0^1 f(t)\,dt + \|x\|_\infty\int_0^1 g(t)\,dt \\ &\le c_{17}\|x\|^{1+\alpha} + c_{18}\|x\| \le k_3\|x\|^3\end{aligned}$$

for all $\|x\| \ge r$ and $k_3 > 0$ given by $k_3 = c_{17}r^{\alpha-2} + c_{18}r^{-2}$. Now it follows from condition (A_6') that

$$\int_0^1 V(t,x)\,dt \ge \int_0^1 \frac{1}{2}\big(\big(\epsilon I_n + B_{01}(t) - B_1(t)\big)x, x\big)\,dt - k_3\|x\|^3$$

for all $x \in Z$. Hence by (3.1) we have

$$\begin{aligned}I(x) &\ge -\int_0^1 \frac{1}{2}\big[\big|x'\big|^2 - \big(B_{01}(t)x, x\big)\big]\,dt + \frac{1}{2}\epsilon\int_0^1 |x|^2\,dt - k_3\|x\|^3 \\ &= -\frac{1}{2}q_{B_{01}}(x,x) + \frac{1}{2}\epsilon\|x\|_{L^2}^2 - k_3\|x\|^3.\end{aligned}$$

Noting that $X_1 = Z^-(B_{01}) \oplus Z^0(B_{01})$ is finite-dimensional, we can see that there exists $k_4 > 0$ such that

$$I(x) \ge \frac{1}{2}\epsilon k_4\|x\|^2 - k_3\|x\|^3$$

for all $x \in X_1$, from which (I_1) follows.

For $x \in X_1^\perp$, again by condition (A_6') we have

$$I(x) \le -\int_0^1 \frac{1}{2}\big[\big|x'\big|^2 - \big(B_{02}(t)x, x\big)\big]\,dt \le 0$$

via $X_1^\perp = Z^0(B_{02}) \oplus Z^+(B_{02})$ and Proposition 2.1, which shows that (I_2) holds.

Since $B_{02} > B_{01} > B_1$, by (2.3) we have

$$q_{B_1}(x,x) > q_{B_{01}}(x,x) > q_{B_{02}}(x,x), \quad x \in Z \setminus \{\theta\}.$$

Noticing that $X_2^\perp = Z^0(B_{01}) \oplus Z^0(B_{02}) \oplus Z^+(B_{02})$, we have $X_2^\perp \subset Z^+(B_1)$. Finally, ($I_3$) follows from (3.11). Hence the proof is completed. □

3.4 Proof of Theorem 1.4

In the section, we first use the saddle point theorem (see Theorem 4.6, [12] or [10]) to prove that problem (1.1) has at least one solution. Then, to prove that problem (1.1) has multiple periodic solutions, we need the following abstract critical point theorem developed recently in [3].

Lemma 3.3 ([3], Theorem 5.2.23) *Let X be a Banach space, and let $\varphi \in C^1(X,\mathbf{R})$ be an even function satisfying the (PS)-condition. Assume that $a < b$ and $\varphi(\theta) \ge b$. Further, suppose that*

(1) *there are an m-dimensional linear subspace G and $\rho > 0$ such that*

$$\sup_{x\in G\cap\partial B_\rho(\theta)} \varphi(x) < b,$$

where $\partial B_\rho(\theta) = \{x \in X \mid \|x\| = \rho\}$;

(2) *there is a j-dimensional linear subspace F such that*

$$\inf_{x\in F^\perp} \varphi(x) > a,$$

where $F^\perp$ is the orthogonal complementary space of F;

(3) $m > j$.

Then φ has at least $m - j$ pairs of distinct critical points.

Proof of Theorem 1.4 By assumption (A_1'), Propositions 2.1–2.4 and Definition 2.3 we have $Z = Z^-(B_1) \oplus Z^0(B_1) \oplus Z^+(B_1)$. Set $X_0 = Z^-(B_1)$, $X_1 = Z^0(B_1)$, $X_2 = Z^+(B_1)$, $x \in Z$, $x = x_0 + x_1 + x_2$ with $x_0 \in X_0$, $x_1 \in X_1$, $x_2 \in X_2$. Next, we divide the proof into four steps.

Step 1. We verify that I satisfies the (PS)-condition. Suppose that $I'(x_n) \to 0$ as $n \to \infty$ and $I(x_n)$ is bounded. Let $x_n = x_{n0} + x_{n1} + x_{n2}$ with $x_{n0} \in Z^-(B_1)$, $x_{n1} \in Z^0(B_1)$ and $x_{n2} \in Z^+(B_1)$. In a way similar to (3.8), by assumption (A_4') we have

$$\begin{aligned}
&\left|\int_0^1 \bigl(\nabla_x V(t,x_n), x_{n2} - x_{n0}\bigr)\,dt\right| \\
&\quad\le \int_0^1 f(t)|x_{n2} + x_{n1} + x_{n0}|^\alpha |x_{n2} - x_{n0}|\,dt + \int_0^1 g(t)|x_{n2} - x_{n0}|\,dt \\
&\quad\le \int_0^1 2f(t)\bigl(|x_{n2} + x_{n0}|^\alpha + |x_{n1}|^\alpha\bigr)|x_{n2} - x_{n0}|\,dt + \int_0^1 g(t)|x_{n2} - x_{n0}|\,dt \\
&\quad\le \int_0^1 2f(t)|x_{n1}|^\alpha|x_{n2}|\,dt + \int_0^1 2f(t)|x_{n1}|^\alpha|x_{n0}|\,dt \\
&\qquad + 2\bigl(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\bigr)^{1+\alpha}\int_0^1 f(t)\,dt + \bigl(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\bigr)\int_0^1 g(t)\,dt \\
&\quad\le 2\left(\int_0^1 f^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^1 |x_{n1}|^{2\alpha}|x_{n2}|^2\,dt\right)^{\frac{1}{2}} \\
&\qquad + 2\|x_{n1}\|_\infty^\alpha\bigl(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\bigr)\int_0^1 f(t)\,dt \\
&\qquad + 2\bigl(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\bigr)^{1+\alpha}\int_0^1 f(t)\,dt + \bigl(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\bigr)\int_0^1 g(t)\,dt \\
&\quad\le 2\beta_0\|x_{n1}\|_\infty^\alpha\left(\int_0^1 |x_{n2}|^2\,dt\right)^{\frac{1}{2}} + 2\|x_{n1}\|_\infty^\alpha\bigl(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\bigr)\int_0^1 f(t)\,dt \\
&\qquad + 2\bigl(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\bigr)^{1+\alpha}\int_0^1 f(t)\,dt + \bigl(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\bigr)\int_0^1 g(t)\,dt \\
&\quad\le \varepsilon\beta_0\int_0^1 |x_{n2}|^2\,dt + \frac{\beta_0}{\varepsilon}\|x_{n1}\|_\infty^{2\alpha} + 2\|x_{n1}\|_\infty^\alpha\bigl(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\bigr)
\end{aligned}$$

$$\cdot \int_0^1 f(t)\,dt + 2\big(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\big)^{1+\alpha} \int_0^1 f(t)\,dt$$
$$+ \big(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\big) \int_0^1 g(t)\,dt \tag{3.13}$$

for all n, where $\beta_0 = (\int_0^1 f^2(t)\,dt)^{\frac{1}{2}}$ and $\varepsilon > 0$. Thus from $x_{n0} \in Z^-(B_1)$, $x_{n1} \in Z^0(B_1)$, $x_{n2} \in Z^+(B_1)$, (3.13), (2.3), and Proposition 2.1 we have

$$\begin{aligned}
\|x_{n2}\| + \|x_{n0}\| \geq \|x_{n2} - x_{n0}\| &\geq \big\langle -I'(x_n), x_{n2} - x_{n0}\big\rangle \\
&= \int_0^1 \big[\big(x_n', x_{n2}' - x_{n0}'\big) - \big(B_1(t)x_n, x_{n2} - x_{n0}\big)\big]\,dt \\
&\quad - \int_0^1 \big(\nabla_x V(t, x_n), x_{n2} - x_{n0}\big)\,dt \\
&\geq \int_0^1 \big[\big|x_{n2}'\big|^2 - \big(B_1(t)x_{n2}, x_{n2}\big) - \varepsilon\beta_0 |x_{n2}|^2\big]\,dt \\
&\quad - \int_0^1 \big[\big|x_{n0}'\big|^2 - \big(B_1(t)x_{n0}, x_{n0}\big)\big]\,dt - \frac{\beta_0}{\varepsilon}\|x_{n1}\|_\infty^{2\alpha} \\
&\quad - 2\|x_{n1}\|_\infty^{\alpha}\big(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\big) \int_0^1 f(t)\,dt \\
&\quad - 2\big(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\big)^{1+\alpha} \int_0^1 f(t)\,dt - \big(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\big) \int_0^1 g(t)\,dt \\
&= q_{B_1+\varepsilon\beta_0 I_n}(x_{n2}, x_{n2}) - q_{B_1}(x_{n0}, x_{n0}) - \frac{\beta_0}{\varepsilon}\|x_{n1}\|_\infty^{2\alpha} - 2\|x_{n1}\|_\infty^{\alpha} \\
&\quad \cdot \big(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\big) \int_0^1 f(t)\,dt - 2\big(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\big)^{1+\alpha} \int_0^1 f(t)\,dt \\
&\quad - \big(\|x_{n2}\|_\infty + \|x_{n0}\|_\infty\big) \int_0^1 g(t)\,dt
\end{aligned}$$

for n large enough. By Proposition 2.7 we can choose $\varepsilon_0 > 0$ such that $\nu_M^s(B_1 + \varepsilon_0\beta_0 I_n) = 0$ and $i_M^s(B_1 + \varepsilon_0\beta_0 I_n) = i_M^s(B_1) + \nu_M^s(B_1)$. From (5) of Proposition 2.5 we know that $(-q_{B_1}(x_0, x_0))^{\frac{1}{2}} + (q_{B_1+\varepsilon_0\beta_0 I_n}(x_2, x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_0 + x_2$ with $x_0 \in Z^-(B_1)$ and $x_2 \in Z^+(B_1) = Z^+(B_1 + \varepsilon_0\beta_0 I_n)$. Hence there exist $c_{19}, c_{20}, c_{21}, c_{22} > 0$ such that

$$\begin{aligned}
&\big(\|x_{n2}\| + \|x_{n0}\|\big)^2 \\
&\quad \leq c_{19}\|x_{n1}\|^{2\alpha} + c_{20}\|x_{n1}\|^{\alpha}\big(\|x_{n2}\| + \|x_{n0}\|\big) + c_{21}\big(\|x_{n2}\| + \|x_{n0}\|\big)^{1+\alpha} \\
&\qquad + c_{22}\big(\|x_{n2}\| + \|x_{n0}\|\big).
\end{aligned} \tag{3.14}$$

From (3.14) we claim that there exist n large enough and $k_5, k_6 > 0$ such that

$$k_5\|x_{n1}\|^{2\alpha} + k_6 \geq \big(\|x_{n2}\| + \|x_{n0}\|\big)^2. \tag{3.15}$$

In fact, we only need to consider two cases: $\|x_{n2}\| + \|x_{n0}\|$ is bounded, or $\|x_{n2}\| + \|x_{n0}\|$ is unbounded.

(i) If $\|x_{n2}\| + \|x_{n0}\|$ is bounded, then $\|x_{n2}\| + \|x_{n0}\| \leq c_{23}$. By (3.14) we have

$$(c_{19} + c_{20} + c_{23})\|x_{n1}\|^{2\alpha} + c_{21}c_{23}^{1+\alpha} + 2c_{22}c_{23} \geq \left(\|x_{n2}\| + \|x_{n0}\|\right)^2.$$

Thus (3.15) follows.

(ii) If $\|x_{n2}\| + \|x_{n0}\|$ is unbounded, then there is n large enough such that

$$\begin{aligned} & c_{19}\frac{\|x_{n1}\|^{2\alpha}}{(\|x_{n2}\| + \|x_{n0}\|)^2} + c_{20}\frac{\|x_{n1}\|^{\alpha}}{\|x_{n2}\| + \|x_{n0}\|} \\ & \quad \geq 1 - c_{21}\frac{1}{(\|x_{n2}\| + \|x_{n0}\|)^{1-\alpha}} - c_{22}\frac{1}{\|x_{n2}\| + \|x_{n0}\|} \geq \frac{1}{2}, \end{aligned}$$

which implies that there is $c_{24} > 0$ such that $\frac{\|x_{n1}\|^{\alpha}}{\|x_{n2}\|+\|x_{n0}\|} \geq c_{24}$. From (i) and (ii) we get that (3.15) holds.

To prove the boundedness of $\{x_n\}$, by (3.15) it suffices to prove that $\{x_{n1}\}$ is bounded. In a way similar to (3.8), for all n, we have

$$\begin{aligned} & \left|\int_0^1 \left(V(t,x_n) - V(t,x_{n1})\right) dt\right| \\ & \quad = \left|\int_0^1 \int_0^1 \left(\nabla_x V\left(t, x_{n1} + s(x_{n0} + x_{n2})\right), x_{n0} + x_{n2}\right) ds\, dt\right| \\ & \quad \leq \int_0^1 \int_0^1 f(t)\left|x_{n1} + s(x_{n0} + x_{n2})\right|^{\alpha} |x_{n0} + x_{n2}|\, ds\, dt \\ & \qquad + \int_0^1 \int_0^1 g(t)|x_{n0} + x_{n2}|\, ds\, dt \\ & \quad \leq \int_0^1 2f(t)\left(|x_{n1}|^{\alpha} + \frac{1}{1+\alpha}\left(|x_{n0}| + |x_{n2}|\right)^{\alpha}\right)\left(|x_{n0}| + |x_{n2}|\right) dt \\ & \qquad + \int_0^1 g(t)\left(|x_{n0}| + |x_{n2}|\right) dt \\ & \quad \leq 2\beta_0\|x_{n1}\|_{\infty}^{\alpha}\left(\int_0^1 \left(|x_{n0}| + |x_{n2}|\right)^2 dt\right)^{\frac{1}{2}} + 2\left(\|x_{n0}\|_{\infty} + \|x_{n2}\|_{\infty}\right)^{1+\alpha} \\ & \qquad \cdot \int_0^1 f(t)\, dt + \left(\|x_{n0}\|_{\infty} + \|x_{n2}\|_{\infty}\right)\int_0^1 g(t)\, dt \\ & \quad \leq \frac{\varepsilon_0\beta_0}{4}\int_0^1 \left(|x_{n0}| + |x_{n2}|\right)^2 dt + \frac{4\beta_0}{\varepsilon_0}\|x_{n1}\|_{\infty}^{2\alpha} + 2\int_0^1 f(t)\, dt \\ & \qquad \cdot \left(\|x_{n0}\|_{\infty} + \|x_{n2}\|_{\infty}\right)^{1+\alpha} + \left(\|x_{n0}\|_{\infty} + \|x_{n2}\|_{\infty}\right)\int_0^1 g(t)\, dt \\ & \quad \leq -\frac{\varepsilon_0\beta_0}{2}\int_0^1 |x_{n2}|^2\, dt + \varepsilon_0\beta_0\left(\|x_{n0}\|_{\infty} + \|x_{n2}\|_{\infty}\right)^2 + \frac{4\beta_0}{\varepsilon_0}\|x_{n1}\|_{\infty}^{2\alpha} \\ & \qquad + 2\left(\|x_{n0}\|_{\infty} + \|x_{n2}\|_{\infty}\right)^{1+\alpha}\int_0^1 f(t)\, dt \\ & \qquad + \left(\|x_{n0}\|_{\infty} + \|x_{n2}\|_{\infty}\right)\int_0^1 g(t)\, dt. \end{aligned} \tag{3.16}$$

Notice that by the boundedness of $\{I(x_n)\}$ and $\alpha \in [0,1)$, the equivalence of the norm $(q_{B_1+\varepsilon_0\beta_0 I_n}(x_2,x_2))^{\frac{1}{2}}$ on Z for $x = x_2$ with $x_2 \in Z^+(B_1) = Z^+(B_1 + \varepsilon_0\beta_0 I_n)$, $q_{B_1}(x_1,x_1) < 0$ on Z and for $x = x_0$ with $x_0 \in Z^-(B_1)$, (3.15), and (3.16) we obtain that there exist $c_{25} \in \mathbf{R}$ and $c_{26}, c_{27} > 0$ such that

$$\begin{aligned}
c_{25} &\le -I(x_n)\\
&= \int_0^1 \frac{1}{2}\big[|x_n'|^2 - (B_1(t)x_n, x_n)\big]\,dt - \int_0^1 \big(V(t,x_n) - V(t,x_{n1})\big)\,dt\\
&\quad - \int_0^1 V(t,x_{n1})\,dt\\
&\le \int_0^1 \frac{1}{2}\big[|x_{n2}'|^2 - (B_1(t)x_{n2}, x_{n2}) - \varepsilon_0\beta_0|x_{n2}|^2\big]\,dt\\
&\quad + \int_0^1 \frac{1}{2}\big[|x_{n1}'|^2 - (B_1(t)x_{n1}, x_{n1})\big]\,dt + \varepsilon_0\beta_0\big(\|x_{n0}\|_\infty + \|x_{n2}\|_\infty\big)^2\\
&\quad + \frac{4\beta_0}{\varepsilon_0}\|x_{n1}\|_\infty^{2\alpha} + 2\big(\|x_{n0}\|_\infty + \|x_{n2}\|_\infty\big)^{1+\alpha}\int_0^1 f(t)\,dt\\
&\quad + \big(\|x_{n0}\|_\infty + \|x_{n2}\|_\infty\big)\int_0^1 g(t)\,dt - \int_0^1 V(t,x_{n1})\,dt\\
&\le c_{26}\|x_{n2}\|^2 + c_{26}\big(\|x_{n0}\| + \|x_{n2}\|\big)^2 + c_{26}\|x_{n1}\|^{2\alpha}\\
&\quad + c_{26}\big(\|x_{n0}\| + \|x_{n2}\|\big)^{1+\alpha} + c_{26}\big(\|x_{n0}\| + \|x_{n2}\|\big) - \int_0^1 V(t,x_{n1})\,dt\\
&\le 4c_{26}\big(\|x_{n0}\| + \|x_{n2}\|\big)^2 + c_{26}\|x_{n1}\|^{2\alpha} + c_{27} - \int_0^1 V(t,x_{n1})\,dt\\
&\le (4c_{26}k_5 + c_{26})\|x_{n1}\|^{2\alpha} + 4c_{26}k_6 + c_{27} - \int_0^1 V(t,x_{n1})\,dt\\
&\le \|x_{n1}\|^{2\alpha}\left((4c_{26}k_5 + c_{26}) - \|x_{n1}\|^{-2\alpha}\int_0^1 V(t,x_{n1})\,dt\right) + 4c_{26}k_6 + c_{27}
\end{aligned}$$

for n large enough. Taking $c_0 > 4c_{26}k_5 + c_{26}$ in this inequality, by (1.3) of condition (A_7) we get that $\{\|x_{n1}\|\}$ is bounded. If (1.4) of condition (A_7) holds, then similarly to the proof of Theorem 1.3, by this inequality, (3.15), and (3.16) we also get that $\{\|x_{n1}\|\}$ is bounded. Hence $\{\|x_n\|\}$ is bounded by (3.15). Arguing then as in Proposition 4.1 in [10], we easily conclude that the (PS)-condition is satisfied.

Step 2. We prove that $-I(x_2) \to +\infty$ as $\|x_2\| \to +\infty$ with $x_2 \in X_2 = Z^+(B_1)$ and $-I(x_0) \to -\infty$ as $\|x_0\| \to +\infty$ with $x_0 \in X_2 = Z^-(B_1)$.

For $x_2 \in Z^+(B_1)$, from condition (A_4') we have

$$\begin{aligned}
&\left|\int_0^1 \big(V(t,x_2) - V(t,\theta)\big)\,dt\right|\\
&\quad\le \frac{1}{1+\alpha}\int_0^1 f(t)|x_2|^{1+\alpha}\,dt + \int_0^1 g(t)|x_2|\,dt\\
&\quad\le 2\beta_0\left(\int_0^1 |x_2|^{2(1+\alpha)}\,dt\right)^{\frac{1}{2}} + \|x_2\|\int_0^1 g(t)\,dt\\
&\quad\le \frac{\varepsilon_0\beta_0}{2}\int_0^1 |x_2|^2\,dt + \frac{2\beta_0}{\varepsilon_0}\|x_2\|^{2\alpha} + \|x_2\|\int_0^1 g(t)\,dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
-I(x_2) &= \int_0^1 \frac{1}{2}\bigl[|x_2'|^2 - \bigl(B_1(t)x_2, x_2\bigr)\bigr]\,dt - \int_0^1 \bigl(V(t,x_2) - V(t,\theta)\bigr)\,dt \\
&\quad - \int_0^1 V(t,\theta)\,dt \\
&\geq \int_0^1 \frac{1}{2}\bigl[|x_2'|^2 - \bigl(B_1(t)x_2, x_2\bigr) - \varepsilon_0\beta_0|x_2|^2\bigr]\,dt \\
&\quad - \frac{2\beta_0}{\varepsilon_0}\|x_2\|^{2\alpha} - \|x_2\|\int_0^1 g(t)\,dt - \int_0^1 V(t,\theta)\,dt \\
&\geq c_{28}\|x_2\|^2 - \frac{2\beta_0}{\varepsilon_0}\|x_2\|^{2\alpha} - \|x_2\|\int_0^1 g(t)\,dt - \int_0^1 V(t,\theta)\,dt \\
&\to +\infty
\end{aligned} \tag{3.17}$$

as $\|x\| \to +\infty$ in $Z^+(B_1)$, where $c_{28} > 0$.

Similarly, for $x_0 \in Z^-(B_1)$, from condition (A_4') we have

$$\begin{aligned}
&\left|\int_0^1 \bigl(V(t,x_0) - V(t,\theta)\bigr)\,dt\right| \\
&\quad\leq \frac{1}{1+\alpha}\int_0^1 f(t)|x_0|^{1+\alpha}\,dt + \int_0^1 g(t)|x_0|\,dt \\
&\quad\leq \frac{1}{1+\alpha}\|x_0\|^{1+\alpha}\int_0^1 f(t)\,dt + \|x_0\|\int_0^1 g(t)\,dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
-I(x_0) &\leq \int_0^1 \frac{1}{2}\bigl[|x_0'|^2 - \bigl(B_1(t)x_0, x_0\bigr)\bigr]\,dt + \frac{1}{1+\alpha}\|x_0\|^{1+\alpha}\int_0^1 f(t)\,dt \\
&\quad + \|x_0\|\int_0^1 g(t)\,dt - \int_0^1 V(t,\theta)\,dt \\
&= -\frac{1}{2}\bigl(-q_{B_1}(x_0,x_0)\bigr) + \frac{1}{1+\alpha}\|x_0\|^{1+\alpha}\int_0^1 f(t)\,dt \\
&\quad + \|x_0\|\int_0^1 g(t)\,dt - \int_0^1 V(t,\theta)\,dt \\
&\leq -\frac{c_{29}}{2}\|x_0\|^2 + \frac{1}{1+\alpha}\|x_0\|^{1+\alpha}\int_0^1 f(t)\,dt \\
&\quad + \|x_0\|\int_0^1 g(t)\,dt - \int_0^1 V(t,\theta)\,dt \\
&\to -\infty
\end{aligned} \tag{3.18}$$

as $\|x_0\| \to +\infty$ in $Z^-(B_1)$, where $c_{29} > 0$.

Step 3. Next, we prove that problem (1.1) has at least one solution in Z. If (1.3) of condition (A_7) holds, then we let $X^- = Z^-(B_1) \oplus Z^0(B_1)$ and $X^+ = Z^+(B_1)$. For $x = x_0 + x_1 \in X^-$

with $x_0 \in Z^-(B_1)$ and $x_1 \in Z^0(B_1)$, by condition (A_4') we have

$$\begin{aligned}
&\left|\int_0^1 \big(V(t,x) - V(t,x_1)\big)\,dt\right| \\
&\quad = \left|\int_0^1 \int_0^1 \big(\nabla_x V(t, sx_0 + x_1), x_0\big)\,ds\,dt\right| \\
&\quad \le \int_0^1 \int_0^1 f(t)|sx_0 + x_1|^\alpha |x_0|\,ds\,dt + \int_0^1 \int_0^1 g(t)|x_0|\,ds\,dt \\
&\quad \le 2\int_0^1 f(t)|x_1|^\alpha|x_0|\,dt + 2\|x_0\|^{1+\alpha}\int_0^1 f(t)\,dt + \|x_0\|\int_0^1 g(t)\,dt.
\end{aligned}$$

Thus, there is $\varepsilon > 0$ with $2\varepsilon\int_0^1 f(t)\,dt < c_{29}$ such that

$$\begin{aligned}
-I(x) &\le \int_0^1 \frac{1}{2}\big[|x_0'|^2 - \big(B_1(t)x_0, x_0\big)\big]\,dt + 2\int_0^1 f(t)|x_1|^\alpha|x_0|\,dt \\
&\quad + 2\|x_0\|^{1+\alpha}\int_0^1 f(t)\,dt + \|x_0\|\int_0^1 g(t)\,dt - \int_0^1 V(t,x_1)\,dt \\
&\le -\frac{c_{29}}{2}\|x_0\|^2 + \varepsilon\|x_0\|^2\int_0^1 f(t)\,dt + \frac{1}{\varepsilon}\|x_1\|^{2\alpha}\int_0^1 f(t)\,dt \\
&\quad + 2\|x_0\|^{1+\alpha}\int_0^1 f(t)\,dt + \|x_0\|\int_0^1 g(t)\,dt - \int_0^1 V(t,x_1)\,dt \\
&\le \left[-\left(\frac{c_{29}}{2} - \varepsilon\int_0^1 f(t)\,dt\right)\|x_0\|^2 + 2\|x_0\|^{1+\alpha}\int_0^1 f(t)\,dt + \|x_0\|\int_0^1 g(t)\,dt\right] \\
&\quad + \|x_1\|^{2\alpha}\left(\frac{1}{\varepsilon}\int_0^1 f(t)\,dt - \|x_1\|^{-2\alpha}\int_0^1 V(t,x_1)\,dt\right).
\end{aligned}$$

Taking $c_0 > \frac{1}{\varepsilon}\int_0^1 f(t)\,dt$, by (1.3) of condition (A_7) we see that $-I(x) \to -\infty$ as $\|x\| \to +\infty$ in X^-.

If (1.4) of condition (A_7) holds, then we let $X^- = Z^-(B_1)$ and $X^+ = Z^0(B_1) \oplus Z^+(B_1)$. As before, we easily get that $-I(x) \to +\infty$ as $\|x\| \to +\infty$ in X^+. Together, from (3.17) and (3.18) we have $-I(x) \to +\infty$ as $\|x\| \to +\infty$ in X^+ and $-I(x) \to -\infty$ as $\|x\| \to +\infty$ in X^-. By the saddle point theorem (see Theorem 4.6 in [12] or [10]) we see that problem (1.1) has at least one solution in Z.

Step 4. Finally, we prove that problem (1.1) has at least $i_M^s(B_2) - i_M^s(B_1) - \nu_M^s(B_1)$ pairs of solutions in Z. Since $B_2 > B_1$ and $\nu_M^s(B_2) \neq 0$, we have $Z = Z^-(B_1) \oplus Z^0(B_1) \oplus Z^+(B_1) = Z^-(B_2) \oplus Z^0(B_2) \oplus Z^+(B_2)$ and $Z^+(B_2) \subset Z^+(B_1)$. Set $G = Z^-(B_2)$, $F = Z^-(B_1) \oplus Z^0(B_1)$, $b = 0$. We define $\varphi(x) = -I(x) + \int_0^1 V(t,\theta)\,dt$ for all $x \in Z$. Then $\varphi(\theta) = 0 \ge b$. Noting that $F^\perp = Z^+(B_1)$, we get that (2) of Lemma 3.3 holds. By condition (A_8) and the proof of *Step* 1 it suffices to show that (1) of Lemma 3.3 holds.

By condition (A_9), for any $x \in G \cap B_r(\theta)$, we have

$$\begin{aligned}
\varphi(x) &= \int_0^1 \frac{1}{2}\big[|x'|^2 - \big(B_1(t)x, x\big)\big]\,dt - \int_0^1 V(t,x)\,dt + \int_0^1 V(t,\theta)\,dt \\
&\le \int_0^1 \frac{1}{2}\big[|x'|^2 - \big(B_2(t)x, x\big)\big]\,dt - \frac{\epsilon}{2}\int_0^1 |x|^2\,dt = \frac{1}{2}q_{B_2}(x,x) - \frac{\epsilon}{2}\|x\|_{L^2}^2.
\end{aligned}$$

Noticing that $E = Z^-(B_2)$ is finite-dimensional, we get that there exists $k_7 > 0$ such that

$$\varphi(x) \leq -\frac{k_7\epsilon}{2}\|x\|^2,$$

which implies that, for all $x \in G \cap \partial B_r(\theta)$,

$$\varphi(x) \leq -\frac{rk_7\epsilon}{2} < 0.$$

Thus φ has at least $i^s_M(B_2) - i^s_M(B_1) - \nu^s_M(B_1)$ pairs of distinct critical points, which implies that problem (1.1) has at least $i^s_M(B_2) - i^s_M(B_1) - \nu^s_M(B_1)$ pairs of solutions in Z. □

3.5 Proof of the corollaries

In the section, we use Theorems 1.1–1.4 to prove that the corollaries.

Proof of Corollary 1.5 Letting $M = N = I_n$ and $B_1(t) \equiv 0$, from the index theory of Sect. 2 we easily see that $Z = H^1_0$, $\nu^s_{I_n}(0) \neq 0$, $\ker(\Lambda) = \mathbf{R}^n$, and $i^s_{I_n}(0) = 0$, that is, (A_1) holds. By $\ker(\Lambda) = \mathbf{R}^n$ we know that $|x| = \|x\|$ for all $x \in \ker(\Lambda)$. Again setting $B_2(t) \equiv (2\pi)^2$, from (2.4) and (2.5) of Remark 2.8 we have $\nu^s_{I_n}((2\pi)^2) \neq 0$ and $i^s_{I_n}((2\pi)^2) = i^s_{I_n}(0) + \nu^s_{I_n}(0)$. Thus (A_2) and (A_3) follow from (H_1) and (H_2). The proof is complete. □

Proof of Corollary 1.7 Letting $M = N = I_n$ and $B_1(t) \equiv 0$, we have $Z = H^1_0$, $\nu^s_{I_n}(0) \neq 0$, $\ker(\Lambda) = \mathbf{R}^n$, and $i^s_{I_n}(0) = 0$, that is, (A_1) holds. We need only to show that (A_4) follows from (H_3) and (A_6) follows from (H_4). In fact, from (2.4) and (2.5) of Remark 2.8 we obtain that $\nu^s_{I_n}(f(t)I_n) = 0$ and $i^s_{I_n}(f(t)I_n) = i^s_{I_n}(0) + \nu^s_{I_n}(0)$, which shows that (A_4) holds.

Moreover, setting $B_{01}(t) \equiv (2k\pi)^2$ and $B_{02}(t) \equiv (2(k+1)\pi)^2$, from (2.4) and (2.5) of Remark 2.8 we see that $\nu^s_{I_n}(B_{0i}) \neq 0$ $(i = 1,2)$ and $i^s_{I_n}(B_{02}) = i^s_{I_n}(B_{01}) + \nu^s_{I_n}(B_{01})$. Noting that $|x| \leq \|x\|_\infty \leq \|x\|$ for $x \in H^1_0$, we have $|x| \leq \delta$ as $\|x\| \leq \delta$, which implies that (A_6) holds. The proof is complete. □

Proof of Corollary 1.9 Letting $M = N = I_n$ and $B_1(t) \equiv 0$, we have $Z = H^1_0$, $\nu^s_{I_n}(0) \neq 0$, $\ker(\Lambda) = \mathbf{R}^n$, and $i^s_{I_n}(0) = 0$, that is, (A_1) holds. Similarly to the proof of Corollary 1.7, (A'_6) follows from (H'_4). Since $\ker(\Lambda) = \mathbf{R}^n$, we have that $|x| = \|x\|$ for all $x \in \ker(\Lambda)$. So (A_7) follows from (H_5). The proof is complete. □

Proof of Corollary 1.12 Letting $M = N = I_n$, $B_1(t) \equiv (2k\pi)^2$, and $B_2(t) \equiv (2(k+m)\pi)^2$, from (2.4) and (2.5) of Remark 2.8 we obtain $Z = H^1_0$, $\nu^s_{I_n}(B_i) \neq 0$ $(i = 1,2)$, $\nu^s_{I_n}(B_1 + f(t)I_n) = 0$, $i^s_{I_n}(B_1 + f(t)I_n) = i^s_{I_n}(B_1) + \nu^s_{I_n}(B_1)$, and $i^s_{I_n}(B_2) - i^s_{I_n}(B_1) - \nu^s_{I_n}(B_1) = 2nm - 2n > 0$ via some simple calculation. Similarly to the proof of Corollary 1.7, we see that the conditions of Theorem 1.4 hold. The proof is complete. □

Proof of Corollary 1.14 Letting $M = N = I_n$ and $B_1(t) = A(t)$, from (2.4) and (2.5) of Remark 2.8 we obtain that $Z = H^1_0$, $\nu^s_{I_n}(B_1) \neq 0$, and $i^s_{I_n}(A(t))$ is at most finite-dimensional. If $i^s_{I_n}(A(t)) = 0$, then we see that the conditions of Theorem 1.3 hold, and if $i^s_{I_n}(A(t)) \neq 0$, then we also see that the conditions of Theorem 1.4 hold. The proof is complete. □

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