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Multiple solutions to the Kirchhoff fractional equation involving Hardy–Littlewood–Sobolev critical exponent

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Abstract
In this paper, we study a fractional Kirchhoff type equation with Hardy–Littlewood–Sobolev critical exponent. By using variational methods, we obtain the existence of mountain-pass type solution and negative energy solutions. Also, we prove some further properties of solutions.

Keywords: Fractional equation; Kirchhoff type; Hardy–Littlewood–Sobolev critical exponent; Multiple solution

1 Introduction

In this paper, we study the following fractional Kirchhoff type equation with Hardy–Littlewood–Sobolev critical exponent:

$$\begin{cases} (a+b\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y)(-\Delta)^\alpha u \\ \quad = (\int_\Omega \frac{\beta F(u(y))+|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y)\times(\beta f(u)+2^*_\mu|u|^{2^*_\mu-2}u)+\gamma|u|^{q-2}u \quad \text{in } \Omega, \\ u\in H_0^\alpha(\Omega), \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N (N\geq 3)$ is a smooth bounded domain, a, $b>0$ are constants, $\alpha\in(0,1)$, $(-\Delta)^\alpha$ is the fractional Laplace operator, $\mu\in(0,N)$, $2^*_\mu=\frac{2N-\mu}{N-2\alpha}$ is the critical exponent of the Hardy–Littlewood–Sobolev inequality, F is the primitive function of f, $q\in(1,2)$, β, $\gamma>0$ are parameters.

The investigation of (1.1) is motivated by the following fractional Kirchhoff type equation:

$$\begin{cases} (a+b\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y)(-\Delta)^\alpha u = h(u) \quad \text{in } \Omega, \\ u\in H_0^\alpha(\Omega), \end{cases} \tag{1.2}$$

where h is a nonlinearity with subcritical growth, or involving the critical exponent. When $b=0$, problem (1.2) reduces to the standard fractional equation. The fractional equation appears in various areas such as plasma physics, optimization, finance, free boundary obstacle problems, population dynamics, and minimal surfaces. For more background, we refer to [4] and the references therein. In recent years, many papers have focused on fractional problems on bounded or unbounded domains.

The Kirchhoff equation occurs in various branches of mathematical physics. For example, it can be used to model suspension bridges. Also, it appears in other fields like biological systems, such as population density. Because of the presence of the nonlocal term, the problem is not a pointwise identity, which causes additional mathematical difficulties. In [12], the authors established a stationary Dirichlet problem of Kirchhoff type and proved the existence and asymptotic behavior to solutions. In [3], the authors extended the results in [12] to a more general case. In [11], the author obtained infinitely many solutions to a critical Kirchhoff type fractional problem. There are also papers on problems in the whole space. In [22], the authors obtained the existence and multiplicity of solutions to a fractional Kirchhoff type eigenvalue problem. In [23], the authors studied a nonhomogeneous fractional p-Laplacian equation of Schrödinger–Kirchhoff type. In [17], the authors considered ground states to a fractional Kirchhoff type problem with Sobolev critical exponent:

$$\left(a + b\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\alpha}{2}} u\right|^2 \mathrm{d}x\right)(-\Delta)^{\alpha} u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $N = 3$ with $\alpha \in (\frac{3}{4}, 1)$. In [29], we continued the studies in [17] and considered the equation

$$\left(1 + b\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\right)(-\Delta)^{\alpha} u + u = \beta f(u) + u^{2^*_\alpha - 1} \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

where $2^*_\alpha = \frac{6}{3-2\alpha}$ is the Sobolev critical exponent. Under some conditions on b, β, f, we obtained the existence of ground state solutions when $\alpha = \frac{3}{4}$ and the non-existence of nontrivial solutions when $\alpha \in (0, \frac{3}{4}]$. All the critical problems mentioned above contain only the Sobolev critical exponent. Also, when we consider the problem on a bounded domain, we infer from [6] that the results may be quite different. Thus, it is natural to ask what happens when we study the fractional Kirchhoff type equation with Hardy–Littlewood–Sobolev critical exponent on a bounded domain?

The problem involving Hardy–Littlewood–Sobolev critical exponent is closely related to the Choquard type equation, which has been well studied recently. The Choquard equation can be used to describe the quantum mechanics of a polaron at rest. Also, it is known as the stationary Hartree equation, or the Schrödinger–Newton equation. In [15], Lieb first proved the existence and uniqueness of radial ground state solutions to the following equation:

$$-\Delta u + u = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|^{\mu}}\,\mathrm{d}y\right)|u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{1.5}$$

Later, Lions [16] obtained the existence of infinitely many radial solutions. The authors in [18, 19] studied the existence, qualitative properties, and decay asymptotics of ground state solutions to a more general Choquard equation. For other related results, we refer the readers to [1, 7, 20, 21] for the subcritical case. There are also papers studying Choquard equations involving the Hardy–Littlewood–Sobolev critical exponent. In [13], the authors proved the existence and non-existence of solutions to a Brezis–Nirenberg type Choquard equation. In [27], the authors studied multiple solutions for a nonhomogeneous Choquard equation with Dirichlet boundary condition. In [30], we obtained multiplicity and concentration behavior of positive solutions to a singularly perturbed Choquard problem with critical growth.

In this paper, we study multiplicity of solutions to the fractional Dirichlet problem (1.1). By using the Ekeland variational principle and the mountain pass theorem, we obtain nontrivial solutions to (1.1) with positive or negative energy in a certain range of parameters. Moreover, we show some further properties of the set of solutions. Our results are new even in the case $\gamma = 0$. Recall that in [25], Servadei and Valdinoci first used the mountain pass theorem to solve the fractional problem. In this paper, since problem (1.1) includes the Kirchhoff type nonlocal term and the nonlocal critical term, it is not easy to check the geometric structure of the functional associated with the equation, the boundedness and convergence of the corresponding Palais–Smale sequence. Also, we have to distinguish between different solutions. Now we state the results. We first consider the case $\mu \in (0, 4\alpha)$. For this purpose, we assume f satisfies the following conditions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{u\to 0} \frac{f(u)}{u^{\frac{N-\mu}{N}}} = \lim_{u\to\infty} \frac{f(u)}{|u|^{2^*_\mu-2}u} = 0$, where $2^*_\mu = \frac{2N-\mu}{N-2\alpha}$.

(f_2) $F(u) = \int_0^u f(s)\,\mathrm{d}s \geq 0$ for $u \in \mathbb{R}$. Moreover, there exists $\xi > 0$ such that $F(\xi) = \int_0^\xi f(s)\,\mathrm{d}s > 0$.

(f_3) $\frac{1}{2} f(u)u - F(u) \geq 0$ for $u \in \mathbb{R}$, where $F(u) = \int_0^u f(s)\,\mathrm{d}s$.

Theorem 1.1 *Let $\mu \in (0, 4\alpha)$, $a, b, \beta > 0$. When $N \geq 4$, or $N = 3$ with $\alpha \in (0, \frac{3}{4}]$, we assume ($f_1$)–($f_3$); when $N = 3$ with $\alpha \in (\frac{3}{4}, 1)$, we assume ($f_1$)–($f_3$) and $\lim_{u\to+\infty} \frac{F(u)}{u^{\frac{4\alpha-\mu}{3-2\alpha}}} = +\infty$. Then there exists $\gamma_1 > 0$ such that, for $\gamma \in (0, \gamma_1)$, problem (1.1) has a negative energy solution $u_{1,\gamma}$ and a mountain-pass type solution $u_{2,\gamma}$ with positive energy. Moreover,*

(i) *$u_{1,\gamma} \to 0$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$;*

(ii) *$u_{2,\gamma} \to u_0$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$, where u_0 is the nontrivial solution of (1.1) with $\gamma = 0$.*

Remark 1.1 A typical example satisfying (f_1)–(f_3) is the function $f(u) = |u|^{q-2}u$, where $q \in (2, 2^*_\mu)$, $u \in \mathbb{R}$.

Now we consider the case $\mu \geq 4\alpha$. Define the best fractional Sobolev constant:

$$S_{\alpha,\mu} := \inf_{D^{\alpha,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y}{(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y)^{\frac{1}{2^*_\mu}}}, \tag{1.6}$$

where $2^*_\mu = \frac{2N-\mu}{N-2\alpha}$ is the critical exponent for the Hardy–Littlewood–Sobolev inequality.

Theorem 1.2 *Let $N \geq 3$, $\mu = 4\alpha$, $a > 0$, $b > \frac{2}{S^2_{\alpha,\mu}}$, $\beta > 0$. Assume that (f_1)–(f_2). Then there exist $\beta_0, \gamma_2 > 0$ such that, for $\beta > \beta_0$ and $\gamma \in (0, \gamma_2)$, problem (1.1) has a mountain-pass type solution $v_{0,\gamma}$ with positive energy and two negative energy solutions $v_{1,\gamma}$, $v_{2,\gamma}$. Moreover,*

(i) *$v_{0,\gamma} \to v_0$ in $H_0^\alpha(\Omega)$ and $v_{1,\gamma} \to v_1$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$, where $v_0 \neq v_1$ are nontrivial solutions of (1.1) with $\gamma = 0$;*

(ii) *$v_{2,\gamma} \to 0$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$.*

Let

$$a(b) = \frac{\mu - 4\alpha}{N - 2\alpha}\left(\frac{2_\mu^*}{S_{\alpha,\mu}^{2_\mu^*}}\right)^{\frac{1}{2-2_\mu^*}}\left(\frac{2_\mu^* - 1}{b}\right)^{\frac{N-\mu+2\alpha}{\mu-4\alpha}}. \tag{1.7}$$

Theorem 1.3 *Let $N \geq 3$, $\mu > 4\alpha$, $b > 0$, $a > a(b)$, $\beta > 0$. Assume that (f_1). Moreover, $F(u) = \int_0^u f(s)\,\mathrm{d}s \geq 0$ for $u \in \mathbb{R}$. Then there exist $b_0, \gamma_3 > 0$ such that, for $b \in (0, b_0)$ and $\gamma \in (0, \gamma_3)$, problem (1.1) has a mountain-pass type solution $w_{0,\gamma}$ with positive energy and two negative energy solutions $w_{1,\gamma}$, $w_{2,\gamma}$. Moreover,*

(i) *$w_{0,\gamma} \to w_0$ in $H_0^\alpha(\Omega)$ and $w_{1,\gamma} \to w_1$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$, where $w_0 \neq w_1$ are nontrivial solutions of (1.1) with $\gamma = 0$;*
(ii) *$w_{2,\gamma} \to 0$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$.*

2 Preliminary lemmas

In this section, we give some definitions and lemmas. Let $\alpha \in (0,1)$. Define $H_0^\alpha(\Omega) = \{u \in H^\alpha(\mathbb{R}^N) : u(x) = 0 \text{ a.e. } x \in \mathbb{R}^N \setminus \Omega\}$. For $u \in H_0^\alpha(\Omega)$, define the norm $\|u\|_\alpha = (\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y)^{\frac{1}{2}}$. By [10], we get $\|u\|_\alpha = (\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2\,\mathrm{d}x)^{\frac{1}{2}}$. Then $(H_0^\alpha(\Omega), \|\cdot\|_\alpha)$ is the fractional Hilbert space. Define $\|u\|_t = (\int_\Omega |u|^t\,\mathrm{d}x)^{\frac{1}{t}}$, where $t \geq 1$. Let $2_\alpha^* = \frac{2N}{N-2\alpha}$ be the fractional Sobolev critical exponent. By [10, 24], the embedding $H^\alpha(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ is continuous for $t \in [2, 2_\alpha^*]$, and is locally compact for $t \in [2, 2_\alpha^*)$. So the embedding $H^\alpha(\mathbb{R}^N) \hookrightarrow L^{2_\alpha^*}(\mathbb{R}^N)$ is not compact. In this case, we define the best fractional Sobolev constant:

$$S_\alpha := \inf_{u \in D^{\alpha,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y}{(\int_{\mathbb{R}^N}|u(x)|^{2_\alpha^*}\,\mathrm{d}x)^{\frac{2}{2_\alpha^*}}}. \tag{2.1}$$

By [8, 26], we know that S_α can be attained by

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2\alpha}{2}}\frac{\kappa_0}{(\mu^2 + |\frac{x}{\varepsilon S_\alpha^{\frac{1}{2\alpha}}}|^2)^{\frac{N-2\alpha}{2}}}, \tag{2.2}$$

where $\varepsilon > 0$, $\kappa_0 \in \mathbb{R}\setminus\{0\}$, $\mu > 0$. Moreover,

$$\|U_\varepsilon\|_\alpha^2 = \int_{\mathbb{R}^N}|U_\varepsilon|^{2_\alpha^*}\,\mathrm{d}x = S_\alpha^{\frac{N}{2\alpha}}. \tag{2.3}$$

Choose $r_0 > 0$ such that $B_{2r_0}(0) \subset \Omega$. Define $u_\varepsilon(x) = \psi(x)U_\varepsilon(x)$, where $\psi \in C_0^\infty(B_{2r_0}(0))$ such that $\psi(x) = 1$ for $|x| < r_0$, $0 \leq \psi \leq 1$ and $|\nabla\psi| \leq 2$. By [26], we get

$$\|u_\varepsilon\|_\alpha^2 \leq S_\alpha^{\frac{N}{2\alpha}} + O(\varepsilon^{N-2\alpha}). \tag{2.4}$$

We introduce the following Hardy–Littlewood–Sobolev inequality, which leads to a new type of critical problem.

Lemma 2.1 (Hardy–Littlewood–Sobolev inequality [14]) *Let $s, t > 1$ and $\mu \in (0, N)$ with $\frac{1}{s} + \frac{1}{t} + \frac{\mu}{N} = 2$. Let $f \in L^s(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Then there exists a constant $C(N, s, t, \mu)$*

independent of f, h such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^{\mu}}\,dx\,dy \le C(N,s,t,\mu)\|f\|_s\|h\|_t.$$

If $s=t=\frac{2N}{2N-\mu}$, then $C(N,s,t,\mu)=C(N,\mu)=\pi^{\frac{\mu}{2}}\frac{\Gamma(\frac{N}{2}-\frac{\mu}{2})}{\Gamma(N-\frac{\mu}{2})}\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\}^{-1+\frac{\mu}{N}}$.

By Lemma 2.1, we get, for all $u\in D^{\alpha,2}(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^{\mu}}\,dx\,dy\right)^{\frac{1}{2_\mu^*}} \le \bigl[C(N,\mu)\bigr]^{\frac{1}{2_\mu^*}}\|u\|_{2_\alpha^*}^2. \tag{2.5}$$

Here $2_\mu^*=\frac{2N-\mu}{N-2\alpha}$ is called the critical exponent for the Hardy–Littlewood–Sobolev inequality. In order to deal with the term $(\int_\Omega\frac{|u(y)|^{2_\mu^*}}{|x-y|^{\mu}}\,dy)|u|^{2_\mu^*-2}u$, we define the constant $S_{\alpha,\mu}$ in (1.6). Since (2.5) holds, by the definitions of $S_{\alpha,\mu}$ and S_α, we have the following result. The proof can be found in [9].

Lemma 2.2 $S_{\alpha,\mu}=\frac{S_\alpha}{[C(N,\mu)]^{\frac{N-2\alpha}{2N-\mu}}}$. *Moreover, the constant $S_{\alpha,\mu}$ is achieved if and only if*

$$u(x)=C\left(\frac{c}{c^2+|x-d|^2}\right)^{\frac{N-2\alpha}{2}},$$

where $C>0$, $d\in\mathbb{R}^N$, and $c>0$.

We establish the following Lemmas 2.3–2.5, which is crucial for estimating upper boundedness for the functional of (1.1).

Lemma 2.3 *Let $\varepsilon\in(0,\frac{r_0}{\mu S_\alpha^{\frac{1}{2\alpha}}})$. Then there exists $\sigma>0$ such that $\int_\Omega|u_\varepsilon|^q\,dx\ge\sigma\varepsilon^{\frac{(N-2\alpha)q}{2}}$.*

Proof By $\varepsilon\in(0,\frac{r_0}{\mu S_\alpha^{\frac{1}{2\alpha}}})$, we get $\mu^2\varepsilon^2\le\frac{r_0^2}{S_\alpha^{\frac{1}{\alpha}}}$. Then

$$\begin{aligned}\int_\Omega|u_\varepsilon|^q\,dx &\ge \int_{B_{r_0}(0)}|U_\varepsilon|^q\,dx = \int_{B_{r_0}(0)}\frac{\kappa_0^q\varepsilon^{\frac{(N-2\alpha)q}{2}}}{(\mu^2\varepsilon^2+|\frac{x}{S_\alpha^{\frac{1}{2\alpha}}}|^2)^{\frac{(N-2\alpha)q}{2}}}\,dx\\ &\ge \frac{\kappa_0^q\varepsilon^{\frac{(N-2\alpha)q}{2}}}{(\frac{2r_0^2}{S_\alpha^{\frac{1}{\alpha}}})^{\frac{(N-2\alpha)q}{2}}}\int_{B_{r_0}(0)}\,dx.\end{aligned}$$

So Lemma 2.3 holds. □

The relationship between $S_{\alpha,\mu}$ and S_α is crucial for proving the following estimate.

Lemma 2.4

$$\int_\Omega\int_\Omega\frac{|u_\varepsilon(y)|^{2_\mu^*}|u_\varepsilon(x)|^{2_\mu^*}}{|x-y|^{\mu}}\,dx\,dy \ge \bigl[C(N,\mu)\bigr]^{\frac{N}{2\alpha}}S_{\alpha,\mu}^{\frac{2N-\mu}{2\alpha}}-O\bigl(\varepsilon^{\frac{2N-\mu}{2}}\bigr). \tag{2.6}$$

Proof By a direct calculation,

$$\begin{aligned}
&\int_{\Omega}\int_{\Omega}\frac{|u_\varepsilon(y)|^{2_\mu^*}|u_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\geq\int_{B_{r_0}(0)}\int_{B_{r_0}(0)}\frac{|U_\varepsilon(y)|^{2_\mu^*}|U_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|U_\varepsilon(y)|^{2_\mu^*}|U_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y-2\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\int_{B_{r_0}(0)}\frac{|U_\varepsilon(y)|^{2_\mu^*}|U_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad-\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\frac{|U_\varepsilon(y)|^{2_\mu^*}|U_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{2.7}$$

By Lemma 2.2, we know that $S_{\alpha,\mu}$ is attained by U_ε. Together with $\|U_\varepsilon\|_\alpha^2=S_\alpha^{\frac{N}{2\alpha}}$ and $S_{\alpha,\mu}=\frac{S_\alpha}{[C(N,\mu)]^{\frac{N-2\alpha}{2N-\mu}}}$, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|U_\varepsilon(y)|^{2_\mu^*}|U_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y=\bigl[C(N,\mu)\bigr]^{\frac{N}{2\alpha}}S_{\alpha,\mu}^{\frac{2N-\mu}{2\alpha}}.\tag{2.8}$$

By Lemma 2.1,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\int_{B_{r_0}(0)}\frac{|U_\varepsilon(y)|^{2_\mu^*}|U_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leq\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\int_{B_{r_0}(0)}\frac{C\varepsilon^{2N-\mu}}{(\varepsilon^2+|x|^2)^{\frac{2N-\mu}{2}}(\varepsilon^2+|y|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leq C\varepsilon^{2N-\mu}\biggl(\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\frac{1}{(\varepsilon^2+|y|^2)^N}\,\mathrm{d}y\biggr)^{\frac{2N-\mu}{2N}}\biggl(\int_{B_{r_0}(0)}\frac{1}{(\varepsilon^2+|x|^2)^N}\,\mathrm{d}x\biggr)^{\frac{2N-\mu}{2N}}\\
&\quad=O\bigl(\varepsilon^{\frac{2N-\mu}{2}}\bigr).
\end{aligned}\tag{2.9}$$

Also,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\frac{|U_\varepsilon(y)|^{2_\mu^*}|U_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leq\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\frac{C\varepsilon^{2N-\mu}}{(\varepsilon^2+|x|^2)^{\frac{2N-\mu}{2}}(\varepsilon^2+|y|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leq C\varepsilon^{2N-\mu}\biggl(\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\frac{1}{(\varepsilon^2+|y|^2)^N}\,\mathrm{d}y\biggr)^{\frac{2N-\mu}{2N}}\biggl(\int_{\mathbb{R}^N\setminus B_{r_0}(0)}\frac{1}{(\varepsilon^2+|x|^2)^N}\,\mathrm{d}x\biggr)^{\frac{2N-\mu}{2N}}\\
&\quad=O\bigl(\varepsilon^{2N-\mu}\bigr).
\end{aligned}\tag{2.10}$$

By (2.7)–(2.10), we get (2.6). □

Lemma 2.5 *Let $\beta>0$, $\mu\in(0,4\alpha)$, and $t>0$. When $N\geq4$, or $N=3$ with $\alpha\in(0,\frac{3}{4}]$, we assume (f_1)–(f_3); when $N=3$ with $\alpha\in(\frac{3}{4},1)$, we assume (f_1)–(f_2) and $\lim_{u\to+\infty}\frac{F(u)}{u^{\frac{4\alpha-\mu}{3-2\alpha}}}=$*

$+\infty$. *Then there exists* $C_0 > 0$ *such that, for all* $L > 0$, *there exists* $\varepsilon(t, L) > 0$ *such that, for* $\varepsilon \in (0, \varepsilon(t, L))$,

$$\int_\Omega \int_\Omega \frac{\beta F(tu_\varepsilon(y))|tu_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \geq C_0 t^{\frac{2(N+2\alpha-\mu)}{N-2\alpha}} L \varepsilon^{N-2\alpha}.$$

Proof When $N \geq 4$, or $N = 3$ with $\alpha \in (0, \frac{3}{4}]$, by (f_3), we get $\frac{F(u)}{u^2}$ is increasing for $u > 0$. Since $\frac{4\alpha-\mu}{N-2\alpha} < 2$ when $N \geq 4$, or $N = 3$ with $\alpha \in (0, \frac{3}{4}]$, by (f_2), we get $\lim_{u\to+\infty} \frac{F(u)}{u^{\frac{4\alpha-\mu}{N-2\alpha}}} = +\infty$. So, for all $L > 0$, there exists $R_L > 0$ such that $F(u) \geq L|u|^{\frac{4\alpha-\mu}{N-2\alpha}}$ for $u \geq R_L$. Let $\varepsilon(t, L) = \min\{1, \frac{r_0}{\mu S_\alpha^{\frac{1}{2\alpha}}}, \frac{1}{2\mu^2}(\frac{t\kappa_0}{R_L})^{\frac{2}{N-2\alpha}}\}$ and $\varepsilon \in (0, \varepsilon(t, L))$. Since $u_\varepsilon(x) \geq \frac{\kappa_0 \varepsilon^{\frac{2\alpha-N}{2}}}{(2\mu^2)^{\frac{N-2\alpha}{2}}}$ for $|x| \leq \mu S_\alpha^{\frac{1}{2\alpha}} \varepsilon$, we obtain that

$$F(tu_\varepsilon) \geq \frac{L t^{\frac{4\alpha-\mu}{N-2\alpha}} \kappa_0^{\frac{4\alpha-\mu}{N-2\alpha}} \varepsilon^{\frac{\mu-4\alpha}{2}}}{(2\mu^2)^{\frac{4\alpha-\mu}{2}}}, \quad |x| \leq \mu S_\alpha^{\frac{1}{2\alpha}} \varepsilon.$$

Then, by (f_2), we derive that there exist C_0', $C_0 > 0$ such that

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{\beta F(tu_\varepsilon(y))|tu_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad\geq \int_{B_{\mu S_\alpha^{\frac{1}{2\alpha}}\varepsilon}(0)} \int_{B_{\mu S_\alpha^{\frac{1}{2\alpha}}\varepsilon}(0)} \frac{C_0' F(tu_\varepsilon(y))|tu_\varepsilon(x)|^{2^*_\mu}}{|x|^\mu + |y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad\geq \frac{C_0'}{2(\mu S_\alpha^{\frac{1}{2\alpha}}\varepsilon)^\mu} \int_{B_{\mu S_\alpha^{\frac{1}{2\alpha}}\varepsilon}(0)} \int_{B_{\mu S_\alpha^{\frac{1}{2\alpha}}\varepsilon}(0)} F\bigl(tu_\varepsilon(y)\bigr)\bigl|tu_\varepsilon(x)\bigr|^{2^*_\mu} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad\geq C_0 t^{\frac{2(N+2\alpha-\mu)}{N-2\alpha}} L \varepsilon^{N-2\alpha}.
\end{aligned}$$

So Lemma 2.5 holds. □

Let $H(u) = \beta F(u) + |u|^{2^*_\mu}$ and $h(u) = \frac{\partial H(u)}{\partial u}$. Define the functional on $H_0^\alpha(\Omega)$ by

$$I_\gamma(u) = \frac{a}{2}\|u\|_\alpha^2 + \frac{b}{4}\|u\|_\alpha^4 - \frac{1}{2}\int_\Omega \int_\Omega \frac{H(u(y))H(u(x))}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y - \frac{\gamma}{q}\int_\Omega |u|^q \,\mathrm{d}x. \tag{2.11}$$

Then $I_\gamma : H_0^\alpha(\Omega) \mapsto \mathbb{R}$ is of class C^1 and critical points of I_γ are solutions of (1.1).

Similar to the well-known Brezis–Lieb lemma in [28], we have the following Brezis–Lieb splitting.

Lemma 2.6 *Assume that* (f_1). *If* $u_n \rightharpoonup u$ *weakly in* $H_0^\alpha(\Omega)$, *let* $v_n = u_n - u$, *we have*

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{H(u_n(y))H(u_n(x))}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega \int_\Omega \frac{H(u(y))H(u(x))}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_\Omega \int_\Omega \frac{|v_n(y)|^{2^*_\mu}|v_n(x)|^{2^*_\mu}}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y + o_n(1),
\end{aligned} \tag{2.12}$$

and

$$\int_\Omega\int_\Omega \frac{H(u_n(y))h(u_n(x))u_n(x)}{|x-y|^\mu}\,dx\,dy - \int_\Omega\int_\Omega \frac{H(u(y))h(u(x))u(x)}{|x-y|^\mu}\,dx\,dy$$
$$= 2_\mu^* \int_\Omega\int_\Omega \frac{|v_n(y)|^{2_\mu^*}|v_n(x)|^{2_\mu^*}}{|x-y|^\mu} + o_n(1). \tag{2.13}$$

Proof By Lemmas 2.2 and 2.4 in [5], we can prove that

$$\int_\Omega\int_\Omega \frac{H(u_n(y))H(u_n(x))}{|x-y|^\mu}\,dx\,dy - \int_\Omega\int_\Omega \frac{H(u(y))H(u(x))}{|x-y|^\mu}\,dx\,dy$$
$$= \int_\Omega\int_\Omega \frac{H(v_n(y))H(v_n(x))}{|x-y|^\mu}\,dx\,dy + o_n(1). \tag{2.14}$$

Also,

$$\int_\Omega\int_\Omega \frac{H(v_n(y))h(v_n(x))v_n(x)}{|x-y|^\mu}\,dx\,dy + o_n(1)$$
$$= \int_\Omega\int_\Omega \frac{H(u_n(y))h(u_n(x))u_n(x)}{|x-y|^\mu}\,dx\,dy - \int_\Omega\int_\Omega \frac{H(u(y))h(u(x))u(x)}{|x-y|^\mu}\,dx\,dy. \tag{2.15}$$

By Lemma 2.1,

$$\int_\Omega\int_\Omega \frac{|F(v_n(y))||F(v_n(x))|}{|x-y|^\mu}\,dx\,dy \le C(N,\mu)\|F(v_n)\|^2_{\frac{2N}{2N-\mu}},$$
$$\int_\Omega\int_\Omega \frac{|F(v_n(y))||v_n(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy \le C(N,\mu)\|F(v_n)\|_{\frac{2N}{2N-\mu}}\||v_n|^{2_\mu^*}\|_{\frac{2N}{2N-\mu}},$$
$$\int_\Omega\int_\Omega \frac{|F(v_n(y))||f(v_n(x))||v_n(x)|}{|x-y|^\mu}\,dx\,dy \le C(N,\mu)\|F(v_n)\|_{\frac{2N}{2N-\mu}}\|f(v_n)v_n\|_{\frac{2N}{2N-\mu}},$$
$$\int_\Omega\int_\Omega \frac{|v_n(y)|^{2_\mu^*}|f(v_n(x))v_n(x)|}{|x-y|^\mu}\,dx\,dy \le C(N,\mu)\||v_n|^{2_\mu^*}\|_{\frac{2N}{2N-\mu}}\|f(v_n)v_n\|_{\frac{2N}{2N-\mu}}. \tag{2.16}$$

Since $\lim_{n\to\infty}\|v_n\|_t = 0$ for all $t \in (2, 2_\alpha^*)$, by (f_1), we get $\|F(v_n)\|_{\frac{2N}{2N-\mu}} = \|f(v_n)v_n\|_{\frac{2N}{2N-\mu}} = o_n(1)$. Together with (2.14)–(2.16), we get the results. □

3 The case $\mu < 4\alpha$

In this section, we study (1.1) for the case $\mu < 4\alpha$ and prove Theorem 1.1. Since $\mu < 4\alpha$, we get $2_\mu^* = \frac{2N-\mu}{N-2\alpha} > 2$. So $22_\mu^* > 4$. Let

$$h(t) = \frac{aS_\alpha^{\frac{N}{2\alpha}}}{2}t^2 + \frac{bS_\alpha^{\frac{N}{\alpha}}}{4}t^4 - \frac{[C(N,\mu)]^{\frac{N}{2\alpha}}S_{\alpha,\mu}^{\frac{2N-\mu}{2\alpha}}}{2}t^{22_\mu^*},$$

where $t > 0$. By the structure of h, there exists $T \in (0,+\infty)$ such that $h(T) = \sup_{t\ge 0} h(t)$ and $h'(T) = 0$. Moreover, $h'(t) > 0$ for $t \in (0,T)$ and $h'(t) < 0$ for $t \in (T,+\infty)$. Let $\lambda_1 = \inf_{u\in H_0^\alpha(\Omega)\setminus\{0\}} \frac{\|u\|_\alpha^2}{\int_\Omega |u|^2\,dx}$. Then

$$\lambda_1 \int_\Omega |u|^2\,dx \le \|u\|_\alpha^2, \quad \forall u \in H_0^\alpha(\Omega). \tag{3.1}$$

By the Sobolev embedding theorem, there exists $S_q > 0$ such that

$$\left(\int_\Omega |u|^q \,\mathrm{d}x\right)^{\frac{2}{q}} \le \frac{\|u\|_\alpha^2}{S_q}, \quad \forall u \in H_0^\alpha(\Omega). \tag{3.2}$$

Let $\eta_0 = \frac{a(2-q)}{4q}(\frac{4-q}{2a})^{\frac{2}{2-q}}\frac{1}{S_q^{\frac{q}{2-q}}}$. We establish the following local compactness result for I_γ, which plays an important role in applying the critical point theorems.

Lemma 3.1 *Let $\beta > 0$, $\gamma \ge 0$. Assume that (f_1) and (f_3). If $\{u_n\} \subset H_0^\alpha(\Omega)$ is a sequence such that $I_\gamma(u_n) \to c \in (0, \sup_{t\ge 0} h(t) - \eta_0\gamma^{\frac{2}{2-q}})$ and $I'_\gamma(u_n) \to 0$, then $\{u_n\}$ converges strongly in $H_0^\alpha(\Omega)$ up to a subsequence.*

Proof By $I_\gamma(u_n) \to c$, $I'_\gamma(u_n) \to 0$, (f_3), and (3.2),

$$\begin{aligned} &c_\gamma + o_n(1) + o_n(1)\|u_n\|_\alpha \\ &\quad= I_\gamma(u_n) - \frac{1}{4}\bigl(I'_\gamma(u_n), u_n\bigr) \\ &\quad= \frac{a}{4}\|u_n\|_\alpha^2 + \frac{1}{2}\int_\Omega\int_\Omega \frac{H(u_n(y))(\frac{1}{2}h(u_n(x))u_n(x) - H(u_n(x)))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad - \gamma\left(\frac{1}{q} - \frac{1}{4}\right)\int_\Omega |u_n|^q\,\mathrm{d}x \ge \frac{a}{4}\|u_n\|_\alpha^2 - \gamma\left(\frac{1}{q} - \frac{1}{4}\right)\frac{\|u_n\|_\alpha^q}{S_q^{\frac{q}{2}}}. \end{aligned} \tag{3.3}$$

So $\|u_n\|_\alpha$ is bounded. Assume that $u_n \rightharpoonup u$ weakly in $H_0^\alpha(\Omega)$. Let $A = \lim_{n\to\infty}\|u_n\|_\alpha^2$. Define the functionals $\hat{I}_\gamma$, $\tilde{I}_\gamma$, $\hat{J}$, $\tilde{J}$ on $H_0^\alpha(\Omega)$ by

$$\begin{aligned} \hat{I}_\gamma(u) &= \frac{a}{2}\|u\|_\alpha^2 + \frac{bA}{4}\|u\|_\alpha^2 - \frac{\gamma}{q}\int_\Omega |u|^q\,\mathrm{d}x - \frac{1}{2}\int_\Omega\int_\Omega \frac{H(u(y))H(u(x))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y, \\ \tilde{I}_\gamma(u) &= \frac{a}{2}\|u\|_\alpha^2 + \frac{bA}{2}\|u\|_\alpha^2 - \frac{\gamma}{q}\int_\Omega |u|^q\,\mathrm{d}x - \frac{1}{2}\int_\Omega\int_\Omega \frac{H(u(y))H(u(x))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y, \\ \hat{J}(u) &= \frac{a}{2}\|u\|_\alpha^2 + \frac{bA}{4}\|u\|_\alpha^2 - \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(y)|^{2_\mu^*}|u(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y, \\ \tilde{J}(u) &= \frac{a}{2}\|u\|_\alpha^2 + \frac{bA}{2}\|u\|_\alpha^2 - \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(y)|^{2_\mu^*}|u(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

By $I_\gamma(u_n) \to c$, $I'_\gamma(u_n) \to 0$, we get $\hat{I}_\gamma(u_n) \to c$, $\tilde{I}'_\gamma(u_n) \to 0$. Then $\tilde{I}'_\gamma(u) = 0$. Let $v_n = u_n - u$. By Lemma 2.6,

$$c - \hat{I}_\gamma(u) = \hat{I}_\gamma(u_n) - \hat{I}_\gamma(u) + o_n(1) = \hat{J}(v_n) + o_n(1). \tag{3.4}$$

Also,

$$o_n(1) = \bigl(\tilde{I}'_\gamma(u_n), u_n\bigr) - \bigl(\tilde{I}'_\gamma(u), u\bigr) = \bigl(\tilde{J}'(v_n), v_n\bigr) + o_n(1). \tag{3.5}$$

By (2.1), (2.5), and (3.5),

$$a\lim_{n\to\infty}\|v_n\|_\alpha^2+b\lim_{n\to\infty}\|v_n\|_\alpha^4\le\frac{2_\mu^*C(N,\mu)}{S_\alpha^{\frac{2N-\mu}{N-2\alpha}}}\lim_{n\to\infty}\|v_n\|_\alpha^{22_\mu^*}. \tag{3.6}$$

Assume that $\lim_{n\to\infty}\|v_n\|_\alpha^2=l$. If $l>0$, by (3.6), we get

$$al+bl^2\le\frac{2_\mu^*C(N,\mu)}{S_\alpha^{\frac{2N-\mu}{N-2\alpha}}}l^{2_\mu^*}. \tag{3.7}$$

By (3.7) and Lemma 2.2, we have $h'(\frac{l^{\frac{1}{2}}}{S_\alpha^{\frac{N}{4\alpha}}})\le 0$. Then $\frac{l^{\frac{1}{2}}}{S_\alpha^{\frac{N}{4\alpha}}}\ge T$. So, by (3.4)–(3.5) and Lemma 2.2,

$$\begin{aligned}
c&=\lim_{n\to\infty}\left(\hat{J}(v_n)-\frac{1}{22_\mu^*}\big(\tilde{J}'(v_n),v_n\big)\right)+\hat{I}_\gamma(u)\\
&\ge\frac{a(N+2\alpha-\mu)}{2(2N-\mu)}\lim_{n\to\infty}\|v_n\|_\alpha^2+\frac{b(4\alpha-\mu)}{4(2N-\mu)}\lim_{n\to\infty}\|v_n\|_\alpha^4+\hat{I}_\gamma(u)\\
&\ge\frac{a(N+2\alpha-\mu)}{2(2N-\mu)}S_\alpha^{\frac{N}{2\alpha}}T^2+\frac{b(4\alpha-\mu)}{4(2N-\mu)}S_\alpha^{\frac{N}{\alpha}}T^4+\hat{I}_\gamma(u)\\
&=h(T)-\frac{1}{22_\mu^*}\big(h'(T),T\big)+\hat{I}_\gamma(u)=\sup_{t\ge0}h(t)+\hat{I}_\gamma(u).
\end{aligned} \tag{3.8}$$

Since $\tilde{I}_\gamma'(u)=0$, by (f_3),

$$\begin{aligned}
\hat{I}_\gamma(u)&=\hat{I}_\gamma(u)-\frac{1}{4}\big(\tilde{I}_\gamma'(u),u\big)\ge\frac{a}{4}\|u\|_\alpha^2-\gamma\left(\frac{1}{q}-\frac{1}{4}\right)\frac{\|u\|_\alpha^q}{S_q^{\frac{q}{2}}}\\
&\ge\inf_{t\ge0}\left[\frac{a}{4}t^2-\gamma\left(\frac{1}{q}-\frac{1}{4}\right)\frac{1}{S_q^{\frac{q}{2}}}t^q\right]=-\eta_0\gamma^{\frac{2}{2-q}}.
\end{aligned} \tag{3.9}$$

By (3.8)–(3.9), we get a contradiction. So $l=0$. By (2.1) and (2.5), we have $\lim_{n\to\infty}\int_\Omega\int_\Omega\frac{|v_n(y)|^{2_\mu^*}|v_n(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy=0$. Then, by (3.5), we get $u_n\to u_\gamma$ in $H_0^\alpha(\Omega)$. □

From Lemma 3.1, we know that it is crucial to prove the estimate of upper boundedness for I_γ. Now we obtain the following result.

Lemma 3.2 *Let $\beta>0$. Assume that (f_1)–(f_3). Then there exists $\gamma_1'>0$ such that $\sup_{t\ge0}I_\gamma(tu_\varepsilon)<\sup_{t\ge0}h(t)-\eta_0\gamma^{\frac{2}{2-q}}$ for $\gamma\in[0,\gamma_1')$.*

Proof By (2.4) and Lemma 2.4, there exists $\varepsilon_1\in(0,1)$ such that $\|u_\varepsilon\|_\alpha^2\le\frac{3S_\alpha^{\frac{N}{2\alpha}}}{2}$ and $\int_\Omega\int_\Omega\frac{|u_\varepsilon(y)|^{2_\mu^*}|u_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\ge\frac{[C(N,\mu)]^{\frac{N}{2\alpha}}S_{\alpha,\mu}^{\frac{2N-\mu}{2\alpha}}}{2}$ for $\varepsilon\in(0,\varepsilon_1)$. Let $\varepsilon\in(0,\varepsilon_1)$. Then, by (f_2), there exist small $t_1\in(0,1)$ and large $t_2>1$ independent of ε and γ such that $\sup_{t\in[0,t_1]\cup[t_2,+\infty)}I_\gamma(tu_\varepsilon)\le\frac{1}{2}\sup_{t\ge0}h(t)$. Let

$$y(t)=\frac{at^2}{2}\|u_\varepsilon\|_\alpha^2+\frac{bt^4}{4}\|u_\varepsilon\|_\alpha^4-\frac{t^{22_\mu^*}}{2}\int_\Omega\int_\Omega\frac{|u_\varepsilon(y)|^{2_\mu^*}|u_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy.$$

By (2.4) and Lemma 2.4, there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for $\varepsilon \in (0, \varepsilon_2)$,

$$\max_{t\in[t_1,t_2]} y(t) \le \sup_{t\ge 0} h(t) + C\varepsilon^{N-2\alpha} + C\varepsilon^{\frac{2N-\mu}{2}}. \tag{3.10}$$

By (3.10), Lemmas 2.3 and 2.5, there exists $\varepsilon_3 \in (0, \varepsilon_2)$ such that, for $\varepsilon \in (0, \varepsilon_3)$,

$$\max_{t\in[t_1,t_2]} I_\gamma(tu_\varepsilon) < \sup_{t\ge 0} h(t) - \frac{\gamma\sigma t_1^q \varepsilon^{\frac{(N-2\alpha)q}{2}}}{q}. \tag{3.11}$$

So $\sup_{t\ge 0} I_\gamma(tu_\varepsilon) < \sup_{t\ge 0} h(t)$ for $\gamma = 0$. For $\gamma > 0$, we let $\varepsilon = \gamma^{\frac{1}{(N-2\alpha)(2-q)}}$. Then $\max_{t\in[t_1,t_2]} I_\gamma(tu_\varepsilon) < \sup_{t\ge 0} h(t) - \frac{\sigma t_1^q \gamma^{\frac{4-q}{2(2-q)}}}{q}$. Thus, there exists $\gamma_1'' > 0$ independent of t such that $\max_{t\in[t_1,t_2]} I_\gamma(tu_\varepsilon) < \sup_{t\ge 0} h(t) - \eta_0\gamma^{\frac{2}{2-q}}$ for $\gamma \in (0, \gamma_1'')$. Recall that $\sup_{t\in[0,t_1]\cup[t_2,+\infty)} I_\gamma(tu_\varepsilon) \le \frac{1}{2}\sup_{t\ge 0} h(t)$. Then there exists $\gamma_1' \in (0, \gamma_1'')$ such that $\sup_{t\ge 0} I_\gamma(tu_\varepsilon) < \sup_{t\ge 0} h(t) - \eta_0\gamma^{\frac{2}{2-q}}$ for $\gamma \in (0, \gamma_1')$. □

Since Lemmas 3.1–3.2 hold, by using the Ekeland variational principle and the mountain pass theorem, we prove that (1.1) has two different nontrivial solutions. Moreover, we obtain some further properties of solutions.

Proof of Theorem 1.1 By (f_1), Lemma 2.1, (2.1), and (3.1), there exists $C_1 > 0$ such that

$$\begin{aligned} &\int_\Omega\int_\Omega \frac{H(u(y))H(u(x))}{|x-y|^\mu}\,dx\,dy \\ &\quad\le C_1\left(\int_\Omega |u|^2\,dx\right)^{\frac{2N-\mu}{N}} + C_1\left(\int_\Omega |u|^{2_\alpha^*}\,dx\right)^{\frac{2N-\mu}{N}} \\ &\quad\le \frac{C_1}{\lambda_1^{\frac{2N-\mu}{N}}}\|u\|_\alpha^{\frac{2(2N-\mu)}{N}} + \frac{C_1}{S_\alpha^{2_\mu^*}}\|u\|_\alpha^{\frac{2(2N-\mu)}{N-2\alpha}}. \end{aligned} \tag{3.12}$$

By (3.2) and (3.12),

$$I_\gamma(u) \ge \frac{a}{2}\|u\|_\alpha^2 - \frac{C_1}{\lambda_1^{\frac{2N-\mu}{N}}}\|u\|_\alpha^{\frac{2(2N-\mu)}{N}} - \frac{C_1}{S_\alpha^{2_\mu^*}}\|u\|_\alpha^{\frac{2(2N-\mu)}{N-2\alpha}} - \frac{\gamma}{qS_q^{\frac{q}{2}}}\|u\|_\alpha^q. \tag{3.13}$$

Let

$$L_0 = \min\left\{\left(\frac{a}{8C_1}\right)^{\frac{N}{2(N-\mu)}}\lambda_1^{\frac{2N-\mu}{2(N-\mu)}}, \left(\frac{a}{8C_1}\right)^{\frac{N-2\alpha}{2(N-\mu+2\alpha)}} S_\alpha^{\frac{2N-\mu}{2(N-\mu+2\alpha)}}\right\}.$$

Let $\gamma \in (0, \frac{aqS_q^{\frac{q}{2}}}{8}(\frac{L_0}{2})^{2-q})$. Choose $\rho_0 \in (\frac{L_0}{2}, L_0)$. Then, by (3.13), we get $I_\gamma(u) \ge \frac{a}{8}\|u\|_\alpha^2 \ge \frac{a}{8}\rho_0^2$ for $\|u\|_\alpha = \rho_0$. Let $\gamma \in (0, \min\{\gamma_1', \frac{aqS_q^{\frac{q}{2}}}{8}(\frac{L_0}{2})^{2-q}\})$ with γ_1' given in Lemma 3.2. Choose $u_0 \in H_0^\alpha(\Omega)\setminus\{0\}$. By (f_2), we have $I_\gamma(tu_0) \le \frac{at^2}{2}\|u_0\|_\alpha^2 + \frac{bt^4}{4}\|u_0\|_\alpha^4 - \frac{\gamma t^q}{q}\int_\Omega |u_0|^q\,dx$. Then $I_\gamma(tu_0) < 0$ for $t > 0$ sufficiently small. So $\inf_{\|u\|_\alpha\le\rho_0} I_\gamma(u) < 0$. By the Ekeland variational principle, we derive that there exists a sequence $\{u_n\} \subset H_0^\alpha(\Omega)$ such that $I_\gamma(u_n) \to \inf_{\|u\|_\alpha\le\rho_0} I_\gamma(u) < 0$

and $I'_\gamma(u_n) \to 0$. By Lemma 3.1, there exists $u_{1,\gamma} \in H_0^\alpha(\Omega)$ such that $u_n \to u_{1,\gamma}$ in $H_0^\alpha(\Omega)$. Then $I_\gamma(u_{1,\gamma}) < 0$, $I'_\gamma(u_{1,\gamma}) = 0$. We note that

$$\begin{aligned} 0 &> I_\gamma(u_{1,\gamma}) - \frac{1}{4}\bigl(I'_\gamma(u_{1,\gamma}), u_{1,\gamma}\bigr) \\ &\geq \frac{a}{4}\|u_{1,\gamma}\|_\alpha^2 - \gamma\left(\frac{1}{q}-\frac{1}{4}\right)\int_\Omega |u_{1,\gamma}|^q\,\mathrm{d}x \geq \frac{a}{4}\|u_{1,\gamma}\|_\alpha^2 - \gamma\left(\frac{1}{q}-\frac{1}{4}\right)\frac{\|u_{1,\gamma}\|_\alpha^q}{S_q^{\frac{q}{2}}}, \end{aligned}$$

from which we derive that $\|u_{1,\gamma}\|_\alpha \to 0$ as $\gamma \to 0$.

By (f_2), we get $\lim_{t\to+\infty} I_\gamma(tu_0) \to -\infty$. Also, $I_\gamma(0) = 0$. By the mountain pass theorem in [2], there exists a sequence $\{u_n\} \subset H_0^\alpha(\Omega)$ such that $I_\gamma(u_n) \to c_\gamma$ and $I'_\gamma(u_n) \to 0$, where $c_\gamma = \inf_{g\in G_\gamma}\max_{0\leq t\leq 1} I_\gamma(g(t))$ with $G_\gamma = \{g \in C([0,1], H_0^\alpha(\Omega)) : g(0) = 0, I_\gamma(g(1)) < 0\}$. Recall that $c_\gamma \geq \frac{a}{8}\rho_0^2$. By the definition of c_γ and Lemma 3.2, we get $c_\gamma \leq \sup_{t\geq 0} I_\gamma(tu_\varepsilon) < \sup_{t\geq 0} h(t) - \eta_0\gamma^{\frac{2}{2-q}}$. By Lemma 3.1, there exists $u_{2,\gamma} \in H_0^\alpha(\Omega)$ such that $u_n \to u_{2,\gamma}$ in $H_0^\alpha(\Omega)$. So $I_\gamma(u_{2,\gamma}) = c_\gamma \geq \frac{a}{8}\rho_0^2$ and $I'_\gamma(u_{2,\gamma}) = 0$.

Let $\gamma \geq 0$. For all $g \in G_0$, we have $g \in G_\gamma$. Then $c_\gamma \leq \max_{t\in[0,1]} I_\gamma(g(t)) \leq \max_{t\in[0,1]} I_0(g(t))$ for all $g \in G_0$, from which we derive that $c_\gamma \leq c_0$. Then $I_\gamma(u_{2,\gamma}) = c_\gamma \in [\frac{a}{8}\rho_0^2, c_0]$, $I'_\gamma(u_{2,\gamma}) = 0$. By (3.3), we know that $\|u_{2,\gamma}\|_\alpha$ is bounded. Then $\lim_{\gamma\to 0} I_0(u_{2,\gamma}) = \lim_{\gamma\to 0} I_\gamma(u_{2,\gamma}) \in [\frac{a}{8}\rho_0^2, c_0]$, $\lim_{\gamma\to 0} I'_0(u_{2,\gamma}) = 0$. By Lemma 3.2, we have $c_0 < \sup_{t\geq 0} h(t)$. Then, by Lemma 3.1, we get $u_{2,\gamma} \to u_0$ as $\gamma \to 0$, $I_0(u_0) \in [\frac{a}{8}\rho_0^2, c_0]$ and $I'_0(u_0) = 0$. □

4 The case $\mu = 4\alpha$ and $b > \frac{2}{S_{\alpha,\mu}^2}$

In this section, we study (1.1) for the case $\mu = 4\alpha$ and prove Theorem 1.2. Since $\mu = 4\alpha$, we have $2_\mu^* = \frac{2N-\mu}{N-2\alpha} = 2$. By (1.6),

$$b\|u\|_\alpha^4 - 2\int_\Omega\int_\Omega \frac{|u(y)|^2|u(x)|^2}{|x-y|^{4\alpha}}\,\mathrm{d}x\,\mathrm{d}y \geq \left(b - \frac{2}{S_{\alpha,\mu}^2}\right)\|u\|_\alpha^4. \tag{4.1}$$

We first establish the following compactness result for I_γ.

Lemma 4.1 *Let $\beta > 0$, $\gamma \geq 0$. Assume that (f_1). If $\{u_n\} \subset H_0^\alpha(\Omega)$ is a sequence such that $I_\gamma(u_n) \to c$ and $I'_\gamma(u_n) \to 0$, then $\{u_n\}$ converges strongly in $H_0^\alpha(\Omega)$ up to a subsequence.*

Proof By (f_1), for all $\delta > 0$, there exists $C_\delta > 0$ such that $|F(u)| \leq \delta|u|^{2_\mu^*} + C_\delta|u|^{\frac{2N-\mu}{N}}$ for $u \in \mathbb{R}$. Then by (2.1), (3.1), and Lemma 2.1, there exists $C_2 > 0$ such that

$$\begin{aligned} &\left|\int_\Omega\int_\Omega \frac{\beta^2 F(u(y))F(u(x)) + 2\beta F(u(y))|u(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\quad\leq \beta^2 C(N,\mu)\bigl\|F(u)\bigr\|_{\frac{2N}{2N-\mu}}^2 + 2\beta C(N,\mu)\bigl\|F(u)\bigr\|_{\frac{2N}{2N-\mu}}\bigl\||u|^{2_\mu^*}\bigr\|_{\frac{2N}{2N-\mu}} \\ &\quad\leq C_2\beta^2\left[\delta^2\left(\int_\Omega |u|^{2_\alpha^*}\,\mathrm{d}x\right)^{\frac{2N-\mu}{N}} + C_\delta^2\left(\int_\Omega |u|^2\,\mathrm{d}x\right)^{\frac{2N-\mu}{N}}\right] \\ &\qquad + C_2\beta\left[\delta\left(\int_\Omega |u|^{2_\alpha^*}\,\mathrm{d}x\right)^{\frac{2N-\mu}{N}} + C_\delta\left(\int_\Omega |u|^2\,\mathrm{d}x\right)^{\frac{2N-\mu}{2N}}\left(\int_\Omega |u|^{2_\alpha^*}\,\mathrm{d}x\right)^{\frac{2N-\mu}{2N}}\right] \end{aligned}$$

$$\leq C_2\beta^2\left(\frac{\delta^2}{S_\alpha^{\frac{2N-\mu}{N-2\alpha}}}\|u\|_\alpha^{\frac{2(2N-\mu)}{N-2\alpha}}+\frac{C_\delta^2}{\lambda_1^{\frac{2N-\mu}{N}}}\|u\|_\alpha^{\frac{2(2N-\mu)}{N}}\right)$$
$$+C_2\beta\left[\frac{\delta}{S_\alpha^{\frac{2N-\mu}{N-2\alpha}}}\|u\|_\alpha^{\frac{2(2N-\mu)}{N-2\alpha}}+\frac{C_\delta}{S_\alpha^{\frac{2N-\mu}{2(N-2\alpha)}}\lambda_1^{\frac{2N-\mu}{2N}}}\|u\|_\alpha^{\frac{2N-\mu}{N-2\alpha}+\frac{2N-\mu}{N}}\right]. \tag{4.2}$$

Since $\mu = 4\alpha$, by (3.2) and (4.1)–(4.2),

$$I_\gamma(u)\geq\frac{a}{2}\|u\|_\alpha^2+\frac{1}{4}\left(b-\frac{2}{S_{\alpha,\mu}^2}\right)\|u\|_\alpha^4-C_2\beta^2\left(\frac{\delta^2}{S_\alpha^2}\|u\|_\alpha^4+\frac{C_\delta^2}{\lambda_1^{\frac{2(N-2\alpha)}{N}}}\|u\|_\alpha^{\frac{4(N-2\alpha)}{N}}\right)$$
$$-C_2\beta\left[\frac{\delta}{S_\alpha^2}\|u\|_\alpha^4+\frac{C_\delta}{S_\alpha\lambda_1^{\frac{N-2\alpha}{N}}}\|u\|_\alpha^{2+\frac{2(N-2\alpha)}{N}}\right]-\frac{\gamma}{qS_q^{\frac{q}{2}}}\|u\|_\alpha^q. \tag{4.3}$$

By choosing $\delta > 0$ sufficiently small, we get $I_\gamma(u) \to +\infty$ as $\|u\|_\alpha \to \infty$. Since $I_\gamma(u_n) \to c$, we obtain that $\|u_n\|_\alpha$ is bounded. Assume that $u_n \rightharpoonup u$ weakly in $H_0^\alpha(\Omega)$. Let $A = \lim_{n\to\infty}\|u_n\|_\alpha^2$. Define the functionals $\tilde{I}_\gamma, \tilde{J}$ on $H_0^\alpha(\Omega)$ by

$$\tilde{I}_\gamma(u)=\frac{a}{2}\|u\|_\alpha^2+\frac{bA}{2}\|u\|_\alpha^2-\frac{\gamma}{q}\int_\Omega|u|^q\,dx-\frac{1}{2}\int_\Omega\int_\Omega\frac{H(u(y))H(u(x))}{|x-y|^{4\alpha}}\,dx\,dy,$$
$$\tilde{J}(u)=\frac{a}{2}\|u\|_\alpha^2+\frac{bA}{2}\|u\|_\alpha^2-\frac{1}{2}\int_\Omega\int_\Omega\frac{|u(y)|^2|u(x)|^2}{|x-y|^{4\alpha}}\,dx\,dy.$$

By $I'_\gamma(u_n) \to 0$, we get $\tilde{I}'_\gamma(u_n) \to 0$. Then $\tilde{I}'_\gamma(u) = 0$. Let $v_n = u_n - u$. By (4.1) and Lemma 2.6,

$$o_n(1)=\left(\tilde{J}'(v_n),v_n\right)\geq a\|v_n\|_\alpha^2+b\|v_n\|_\alpha^4-2\int_\Omega\int_\Omega\frac{|v_n(y)|^2|v_n(x)|^2}{|x-y|^{4\alpha}}\,dx\,dy$$
$$\geq a\|v_n\|_\alpha^2+\left(b-\frac{2}{S_{\alpha,\mu}^2}\right)\|v_n\|_\alpha^4. \tag{4.4}$$

So $u_n \to u$ in $H_0^\alpha(\Omega)$. □

Now we use Lemma 4.1 to prove Theorem 1.2.

Proof of Theorem 1.2 Choose $r > 0$ sufficiently small such that $B_{2r}(0) \subset \Omega$. Define $w_r \in C_0^\infty(B_{2r}(0)) \setminus \{0\}$ such that $w_r(x) = \xi$ for $|x| \leq r$, $w_r(x) \geq 0$ for $|x| \leq 2r$, and $w_r(x) = 0$ for $|x| \geq 2r$. Then $w_r \in H_0^\alpha(\Omega)$. Moreover, we have $F(w_r) = F(\xi) > 0$ for $|x| \leq r$. By (f_2),

$$I_\gamma(w_r)\leq I_0(w_r)\leq\frac{a}{2}\|w_r\|_\alpha^2+\frac{b}{4}\|w_r\|_\alpha^4-\int_{B_r(0)}\int_{B_r(0)}\frac{\beta F(w_r(y))|w_r(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy. \tag{4.5}$$

Then there exists $\beta_0 > 0$ such that $I_\gamma(w_r) < 0$ for $\beta > \beta_0$. Let $\beta > \beta_0$. By choosing $\delta > 0$ sufficiently small in (4.3), we derive that

$$I_\gamma(u)\geq\frac{a}{2}\|u\|_\alpha^2-\frac{C_2C_\delta^2\beta^2}{\lambda_1^{\frac{2(N-2\alpha)}{N}}}\|u\|_\alpha^{\frac{4(N-2\alpha)}{N}}-\frac{C_2C_\delta\beta}{S_\alpha\lambda_1^{\frac{N-2\alpha}{N}}}\|u\|_\alpha^{2+\frac{2(N-2\alpha)}{N}}-\frac{\gamma}{qS_q^{\frac{q}{2}}}\|u\|_\alpha^q. \tag{4.6}$$

Let

$$L_1 = \min\left\{ \|w_r\|_\alpha, \left(\frac{a\lambda_1^{\frac{2(N-2\alpha)}{N}}}{8C_2C_\delta^2\beta^2}\right)^{\frac{N}{2(N-4\alpha)}}, \left(\frac{aS_\alpha\lambda_1^{\frac{2(N-2\alpha)}{2N}}}{8C_2C_\delta\beta}\right)^{\frac{N}{2(N-2\alpha)}} \right\}.$$

Let $\gamma \in (0, \frac{aqS_q^{\frac{q}{2}}}{8}(\frac{L_1}{2})^{2-q})$. Choose $\varrho \in (\frac{L_1}{2}, L_1)$. By (4.6), we get $I_\gamma(u) \geq \frac{a}{8}\|u\|_\alpha^2 = \frac{a}{8}\varrho^2$ for $\|u\|_\alpha = \varrho$. Also, $I_\gamma(0) = 0$. By the mountain pass theorem in [2], there exists a sequence $\{u_n\} \subset H_0^\alpha(\Omega)$ such that $I_\gamma(u_n) \to c'_\gamma$ and $I'_\gamma(u_n) \to 0$, where $c'_\gamma = \inf_{g\in G_\gamma}\max_{0\leq t\leq 1} I_\gamma(g(t))$ with $G_\gamma = \{g \in C([0,1], H_0^\alpha(\Omega)) : g(0) = 0, I_\gamma(g(1)) < 0\}$. Moreover, $c'_\gamma \geq \frac{a}{8}\varrho^2$. By Lemma 4.1, there exists $v_{0,\gamma} \in H_0^\alpha(\Omega)$ such that $u_n \to v_{0,\gamma}$ in $H_0^\alpha(\Omega)$. Then $I_\gamma(v_{0,\gamma}) = c'_\gamma \geq \frac{a}{8}\varrho^2$ and $I'_\gamma(v_{0,\gamma}) = 0$. By the proof of Theorem 1.1, we get $c'_\gamma \leq c'_0$. Similar to (4.2), we can derive from (f_1) that, for all $\delta > 0$, there exists $C_\delta > 0$ such that, for all $u \in H_0^\alpha(\Omega)$,

$$\begin{aligned}
&\left|\int_\Omega\int_\Omega \frac{\beta^2F(u(y))f(u(x))u(x) + \beta(2_\mu^*F(u(y)) + f(u(y))u(y))|u(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\right| \\
&\quad\leq C\beta^2\bigl(\delta^2\|u\|_\alpha^{\frac{2(2N-\mu)}{N-2\alpha}} + C_\delta^2\|u\|_\alpha^{\frac{2(2N-\mu)}{N}}\bigr) + C\beta\bigl(\delta\|u\|_\alpha^{\frac{2(2N-\mu)}{N-2\alpha}} + C_\delta\|u\|_\alpha^{\frac{2N-\mu}{N-2\alpha}+\frac{2N-\mu}{N}}\bigr). \qquad (4.7)
\end{aligned}$$

Since $I'_\gamma(v_{0,\gamma}) = 0$, by (3.2), (4.1), and (4.7),

$$\begin{aligned}
&a\|v_{0,\gamma}\|_\alpha^2 + \left(b - \frac{2}{S_{\alpha,\mu}^2}\right)\|v_{0,\gamma}\|_\alpha^4 - \frac{\gamma}{S_q^{\frac{q}{2}}}\|v_{0,\gamma}\|_\alpha^q \\
&\quad\leq C\beta^2\bigl(\delta^2\|v_{0,\gamma}\|_\alpha^4 + C_\delta^2\|v_{0,\gamma}\|_\alpha^{\frac{4(N-2\alpha)}{N}}\bigr) + C\beta\bigl(\delta\|v_{0,\gamma}\|_\alpha^4 + C_\delta\|v_{0,\gamma}\|_\alpha^{2+\frac{2(N-2\alpha)}{N}}\bigr). \qquad (4.8)
\end{aligned}$$

By choosing $\delta > 0$ sufficiently small, we derive that $\|v_{0,\gamma}\|_\alpha$ is bounded. Then we have $\lim_{\gamma\to 0} I_0(v_{0,\gamma}) \in [\frac{a}{8}\varrho^2, c'_0]$ and $\lim_{\gamma\to 0} I'_0(v_{0,\gamma}) = 0$. By Lemma 4.1, we get $v_{0,\gamma} \to v_0$ as $\gamma \to 0$, $I_0(v_0) \in [\frac{a}{8}\varrho^2, c'_0]$ and $I'_0(v_0) = 0$.

Recall that $I_\gamma(w_r) \leq I_0(w_r) < 0$ with $\|w_r\|_\alpha > \varrho$ and $I_\gamma(u) \to +\infty$ as $\|u\|_\alpha \to \infty$. Then there exists $R > 0$ independent of γ such that $\varrho < \|w_r\|_\alpha < R$ and $I_\gamma(u) > 0$ for $\|u\|_\alpha = R$. Let $m_{1,\gamma} = \inf_{\varrho\leq\|u\|_\alpha\leq R} I_\gamma(u)$. Then $m_{1,\gamma} \leq I_\gamma(w_r) \leq I_0(w_r) < 0$. By the Ekeland variational principle, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $\varrho < \|u_n\|_\alpha < R$, $I_\gamma(u_n) \to m_{1,\gamma}$, and $I'_\gamma(u_n) \to 0$. By Lemma 4.1, there exists $v_{1,\gamma} \in H_0^1(\Omega)$ such that $u_n \to v_{1,\gamma}$ in $H_0^1(\Omega)$. Then $\varrho < \|v_{1,\gamma}\|_\alpha < R$, $I_\gamma(v_{1,\gamma}) = m_{1,\gamma} \leq I_0(w_r) < 0$, and $I'_\gamma(v_{1,\gamma}) = 0$.

By (f_2), we get

$$I_\gamma(tw_r) \leq \frac{at^2}{2}\|w_r\|_\alpha^2 + \frac{bt^4}{4}\|w_r\|_\alpha^4 - \frac{\gamma t^q}{q}\int_\Omega |w_r|^q\,dx.$$

Then there exists small $t_r > 0$ such that $\|t_rw_r\|_\alpha < \varrho$ and $I_\gamma(t_rw_r) < 0$. Let $m_{2,\gamma} = \inf_{\|u\|_\alpha\leq\varrho} I_\gamma(u)$. Then $m_{2,\gamma} < 0$. By the Ekeland variational principle, we derive that there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $\|u_n\|_\alpha < \varrho$, $I_\gamma(u_n) \to m_{2,\gamma}$, and $I'_\gamma(u_n) \to 0$. By Lemma 4.1, there exists $v_{2,\gamma} \in H_0^1(\Omega)$ such that $u_n \to v_{2,\gamma}$ in $H_0^1(\Omega)$. Then $\|v_{2,\gamma}\|_\alpha < \varrho$, $I_\gamma(v_{2,\gamma}) = m_{2,\gamma} < 0$, and $I'_\gamma(v_{2,\gamma}) = 0$.

Assume that there exists $v_\gamma \in H_0^\alpha(\Omega)$ such that $\|v_\gamma\|_\alpha$ is bounded, $I_\gamma(v_\gamma) < 0$ and $I'_\gamma(v_\gamma) = 0$. Then $I_0(v_\gamma) \leq o_\gamma(1)$ and $I'_0(v_\gamma) = o_\gamma(1)$. By Lemma 4.1, we get $v_\gamma \to v$ in

$H_0^\alpha(\Omega)$ as $\gamma \to 0$. Thus, by $\varrho < \|v_{1,\gamma}\|_\alpha < R$, $I_\gamma(v_{1,\gamma}) \le I_0(w_r) < 0$, and $I'_\gamma(v_{1,\gamma}) = 0$, we obtain that $v_{1,\gamma} \to v_1$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$, $I_0(v_1) < 0$ and $I'_0(v_1) = 0$. Also, by $\|v_{2,\gamma}\|_\alpha < \varrho$, $I_\gamma(v_{2,\gamma}) = m_{2,\gamma} < 0$, and $I'_\gamma(v_{2,\gamma}) = 0$, we obtain that $v_{2,\gamma} \to v_2$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$ with $\|v_2\|_\alpha \le \varrho$, $I_0(v_2) \le 0$, and $I'_0(v_2) = 0$. By (4.6), we get $I_0(u) \ge \frac{a}{4}\|u\|_\alpha^2 - \frac{\gamma}{qS_q^{\frac{q}{2}}}\|u\|_\alpha^q$ for $\|u\|_\alpha \le \varrho$. Then $v_2 = 0$. □

5 The case $\mu > 4\alpha$

In this section, we study (1.1) for the case $\mu > 4\alpha$ and prove Theorem 1.3. Since $\mu > 4\alpha$, we have $2_\mu^* = \frac{2N-\mu}{N-2\alpha} < 2$. Then $22_\mu^* < 4$. We first establish the following compactness result for I_γ.

Lemma 5.1 *Let $b > 0$, $a > a(b)$, $\gamma \ge 0$ with $a(b)$ given in (1.7). Assume that (f_1). If $\{u_n\} \subset H_0^\alpha(\Omega)$ is a sequence such that $I_\gamma(u_n) \to c$ and $I'_\gamma(u_n) \to 0$, then $\{u_n\}$ converges strongly in $H_0^\alpha(\Omega)$ up to a subsequence.*

Proof In this case, we know that (3.12) holds. By (3.2) and (3.12),

$$I_\gamma(u) \ge \frac{a}{2}\|u\|_\alpha^2 + \frac{b}{4}\|u\|_\alpha^4 - \frac{C_1}{\lambda_1^{\frac{2N-\mu}{N}}}\|u\|_\alpha^{\frac{2(2N-\mu)}{N}} - \frac{C_1}{S_\alpha^{2_\mu^*}}\|u\|_\alpha^{22_\mu^*} - \frac{\gamma}{qS_q^{\frac{q}{2}}}\|u\|_\alpha^q. \tag{5.1}$$

By (5.1), we get $I_\gamma(u) \to +\infty$ as $\|u\|_\alpha \to \infty$. Then $\|u_n\|_\alpha$ is bounded. We assume that $u_n \rightharpoonup u$ weakly in $H_0^\alpha(\Omega)$. Let $A = \lim_{n\to\infty}\|u_n\|_\alpha^2$. Define the functionals $\tilde{I}_\gamma, \tilde{J}$ on $H_0^\alpha(\Omega)$ by

$$\tilde{I}_\gamma(u) = \frac{a}{2}\|u\|_\alpha^2 + \frac{bA}{2}\|u\|_\alpha^2 - \frac{\gamma}{q}\int_\Omega |u|^q\,dx - \frac{1}{2}\int_\Omega\int_\Omega \frac{H(u(y))H(u(x))}{|x-y|^\mu}\,dx\,dy,$$

$$\tilde{J}(u) = \frac{a}{2}\|u\|_\alpha^2 + \frac{bA}{2}\|u\|_\alpha^2 - \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(y)|^{2_\mu^*}|u(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy.$$

By $I'_\gamma(u_n) \to 0$, we get $\tilde{I}'_\gamma(u_n) \to 0$. Then $\tilde{I}'_\gamma(u) = 0$. Together with Lemma 2.6, we have $(\tilde{J}'(v_n), v_n) = o_n(1)$. Here $v_n = u_n - u$. Then, by (1.6),

$$a\lim_{n\to\infty}\|v_n\|_\alpha^2 + b\lim_{n\to\infty}\|v_n\|_\alpha^4 \le \frac{2_\mu^*}{S_{\alpha,\mu}^{\frac{2N-\mu}{N-2\alpha}}}\lim_{n\to\infty}\|v_n\|_\alpha^{22_\mu^*}. \tag{5.2}$$

By (5.2) and Young's inequality,

$$a\lim_{n\to\infty}\|v_n\|_\alpha^2 + b\lim_{n\to\infty}\|v_n\|_\alpha^4 \le a(b)\lim_{n\to\infty}\|v_n\|_\alpha^2 + b\lim_{n\to\infty}\|v_n\|_\alpha^4. \tag{5.3}$$

Then $u_n \to u$ in $H_0^\alpha(\Omega)$. □

Now we use Lemma 5.1 to prove Theorem 1.3.

Proof of Theorem 1.3 Let $w_0 \in H_0^\alpha(\Omega) \setminus \{0\}$. By (f_2),

$$I_\gamma(tw_0) \le \frac{at^2}{2}\|w_0\|_\alpha^2 + \frac{bt^4}{4}\|w_0\|_\alpha^4 - \frac{t^{22_\mu^*}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_0(y)|^{2_\mu^*}|w_0(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy.$$

Obviously, there exists large $t_0 > 0$ such that

$$\frac{at_0^2}{2}\|w_0\|_\alpha^2 - \frac{t_0^{22_\mu^*}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_0(y)|^{2_\mu^*}|w_0(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y < 0.$$

Choose $b_0 > 0$ sufficiently small such that $I_\gamma(t_0w_0) < 0$ for $b \in (0, b_0)$. Let $b \in (0, b_0)$ and $a > a(b)$. Recall that (5.1) holds. Let

$$L_2 = \min\left\{\|t_0w_0\|_\alpha, \left(\frac{a\lambda_1^{\frac{2N-\mu}{N}}}{8C_1}\right)^{\frac{N}{2(N-\mu)}}, \left(\frac{aS_\alpha^{2_\mu^*}}{8C_1}\right)^{\frac{N-2\alpha}{2(N-\mu+2\alpha)}}\right\}.$$

Let $\gamma \in (0, \frac{aqS_q^{\frac{q}{2}}}{8}(\frac{L_2}{2})^{2-q})$. Choose $\varrho \in (\frac{L_2}{2}, L_2)$. By (5.1), we get $I_\gamma(u) \geq \frac{a}{8}\|u\|_\alpha^2 = \frac{a}{8}\varrho^2$ for $\|u\|_\alpha = \varrho$. Also, $I_\gamma(0) = 0$. By the mountain pass theorem in [2], there exists a sequence $\{u_n\} \subset H_0^\alpha(\Omega)$ such that $I_\gamma(u_n) \to c_\gamma'' > 0$ and $I_\gamma'(u_n) \to 0$. Then, by Lemma 5.1, there exists $w_{0,\gamma} \in H_0^\alpha(\Omega)$ such that $u_n \to w_{0,\gamma}$ in $H_0^\alpha(\Omega)$, $I_\gamma(w_{0,\gamma}) = c_\gamma'' > 0$, and $I_\gamma'(w_{0,\gamma}) = 0$. Similar to the proof of Theorem 1.2, we can derive that $w_{0,\gamma} \to w_0$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$, where w_0 is a nontrivial solution of (1.1) with $\gamma = 0$ and $I_0(w_0) > 0$. Recall that $I_\gamma(u) \to +\infty$ as $\|u\|_\alpha \to \infty$. Then there exists $R > 0$ such that $\varrho < \|t_0w_0\|_\alpha < R$ and $I_\gamma(u) > 0$ for $\|u\|_\alpha = R$. We note that $I_\gamma(u) \geq \frac{a}{8}\varrho^2$ for $\|u\|_\alpha = \varrho$, $I_\gamma(t_0w_0) < 0$, and $I_\gamma(tw_0) < 0$ for $t > 0$ sufficiently small. Similar to the proof of Theorem 1.2, we can derive that $\inf_{\varrho\leq\|u\|_\alpha\leq R} I_\gamma(u)$ is attained by a function $w_{1,\gamma}$. Moreover, $I_\gamma(w_{1,\gamma}) < 0$, $I_\gamma'(w_{1,\gamma}) = 0$, and $w_{1,\gamma} \to w_1$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$, where w_1 is a nontrivial solution of (1.1) with $\gamma = 0$ and $I_0(w_1) < 0$. Also, $\inf_{\|u\|_\alpha\leq\varrho} I_\gamma(u)$ is attained by a function $w_{2,\gamma}$. Moreover, $I_\gamma(w_{2,\gamma}) < 0$, $I_\gamma'(w_{2,\gamma}) = 0$, and $w_{2,\gamma} \to 0$ in $H_0^\alpha(\Omega)$ as $\gamma \to 0$. □

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