Haixia and Wei *Boundary Value Problems* (2019) 2019:127
https://doi.org/10.1186/s13661-019-1240-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence and uniqueness for some equation of a mixed elliptic-hyperbolic type

Zhang Haixia[1] and Xiao Wei[2*]

*Correspondence:
xiaowei1802002@163.com
[2]School of Sciences, Chang'an University, Xi'an, P.R. China
Full list of author information is available at the end of the article

Abstract
In the paper, we prove results on existence and uniqueness of weak solutions for closed Dirichlet problem for a transonic flow model, and interior regularity results are also given in the important special case of the Tricomi equation. The method employed consists in variants of the a-b-c integral method of Friedrichs in Sobolev spaces with suitable weights. Particular attention is paid to the problem of attaining results with a minimum of restrictions on the boundary geometry and the form of the type change function.

Keywords: Mixed elliptic-hyperbolic type equation; Tomotika–Tamada model; a-b-c integral method; Star-shaped

1 Introduction and results

In this paper, we consider the following initial boundary value problem:

$$\begin{cases} Lu \equiv A(1-e^{-2By})u_{xx}(x,y)+u_{yy}(x,y)=f(x,y) & in\ \Omega, \\ u=0 & on\ \partial\Omega \end{cases} \tag{1}$$

with $f(x,y)$ is a known function, and A, B are constants, $A>0$, $B>0$, $u(x,y)$ is an unknown function. Let $K(y)=A(1-e^{-2By})$, Ω is a bounded, open, and connected subset of R^2 with piecewise C^1 boundary. We assume throughout that

$$\Omega^{\pm} := \Omega \cap R^2_{\pm} = \emptyset \tag{2}$$

so that (1) is of mixed elliptic-hyperbolic type. Such an equation is of Tricomi type and it is important in the problem of transonic fluid flow (see [18]).

Such a boundary value problem will be called closed in the sense that the boundary condition (2) is imposed on the entire boundary as opposed to an open problem, in which (2) is imposed on a proper subset $\Gamma \subset \partial\Omega$. Both kinds of problems are interesting for transonic flow. Much more is known about open problems, beginning with the work of Tricomi [19].

On the other hand, much less is known about closed problems. Under mild assumptions on the function K and the geometry of the boundary, one has a uniqueness theorem for regular solutions to the Tricomi problem. Such uniqueness theorems have been proven by a variety of methods, including energy integrals as in [16] and maximum principles as in [2, 10]. In order to prove well-posedness, one must choose some reasonable function space that admits a singularity strong enough to allow for existence.

For the interest in closed problems for mixed-type equations, the literature essentially contains only two results on well-posedness. The first, due to Morawetz [11], concerns the Dirichlet problem for the Tricomi equation, and the second due to Pilant [15] concerns the natural analogue of the Neumann problem (conormal boundary conditions) for the Lavrentiev–Bitsadze equation. Inspired by [5–8, 13, 17, 20], we consider the closed Dirichlet problem in the Tomotika–Tamada model (1).

The existence and uniqueness results of Sect. 2 for the Dirichlet problem do not require some smoothness assumptions on the boundary, no geometric assumptions on the geometry of the elliptic boundary an arbitrarily small distance away from the parabolic line need to be made. Similar considerations have been exploited for the Tricomi problem and will hold for other problems with an open boundary condition, such as the Frankl problem. We also note that the arguments used in obtaining the interior regularity results also apply for problems with open boundary conditions as well. The paper is organized as follows. In Sect. 1, we investigate the introduction, notions, and results. In Sect. 2, we show the existence and uniqueness of a weak solution to a problem with Dirichlet conditions. In Sect. 3, we show the regularity of the weak solution.

We define $L^2(\Omega;|K|^{-1})$ for the given $C^1(R^2)$ function $K(x,y)$ in the same way as (see [5])

$$L^2(\Omega;|K|^{-1}) := \{f \in L^2(\Omega) : |K|^{-1/2}f \in L^2(\Omega)\}$$

equipped with its norm

$$\|f\|_{L^2(\Omega;|K|^{-1})} = \left(\int_\Omega |K|^{-1}f^2\,dx\,dy\right)^{1/2} \tag{3}$$

which is the dual space to the weighted space $L^2(\Omega;|K|)$ defined as the equivalence classes of square-integrable functions with respect to the measure $|K|\,dx\,dy$; that is, with finite norm

$$\|f\|_{L^2(\Omega;|K|)} = \left(\int_\Omega |K|f^2\,dx\,dy\right)^{1/2}. \tag{4}$$

One has the obvious chain of inclusions

$$L^2(\Omega;|K|^{-1}) \subset L^2(\Omega) \subset L^2(\Omega;|K|), \tag{5}$$

where the inclusion maps are continuous and injective.

We define $H_0^1(\Omega;K)$ as the closure of $C_0^\infty(\Omega)$ (smooth functions with compact support in Ω with respect to the weighted Sobolev norm):

$$\|u\|_{H^1(\Omega;|K|)} := \left(\int_\Omega \left(|K|u_x^2 + u_y^2 + u^2\right)dx\,dy\right)^{1/2}. \tag{6}$$

Since $u \in H^1(\Omega;|K|)$ vanishes weakly on the entire boundary, one has a Poincaré inequality: there exists $C_P = C_P(\Omega, K)$

$$\|u\|^2_{L2(\Omega)} \le C_P \int_\Omega \left(|K|u_x^2 + u_y^2\right) dx\,dy; \quad u \in H^1\left(\Omega;|K|\right). \tag{7}$$

An equivalent norm on $H_0^1(\Omega;K)$ is thus given by

$$\|u\|_{H_0^1(\Omega;K)} := \left(\int_\Omega \left(|K|u_x^2 + u_y^2\right) dx\,dy\right)^{1/2}. \tag{8}$$

We denote by $H^{-1}(\Omega;K)$ the dual space to $H_0^1(\Omega;K)$ equipped with its norm [1]

$$\|u\|_{H^{-1}(\Omega;K)} := \sup_{0\neq\phi\in C_0^\infty(\Omega)} \frac{|\langle u,\phi\rangle|}{\|\phi\|_{H_0^1(\Omega;K)}}, \tag{9}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket and one has the generalized Schwarz inequality

$$\left|\langle u,\phi\rangle\right| \le \|u\|_{H^{-1}(\Omega;K)}\|\phi\|_{H_0^1(\Omega;K)}; \quad u \in H^{-1}(\Omega;K), \phi \in H_0^1(\Omega;K). \tag{10}$$

One clearly has a rigged triple of Hilbert spaces

$$H_0^1(\Omega;K) \subset L^2(\Omega) \subset H^{-1}(\Omega;K), \tag{11}$$

where the scalar product (on L^2, for example) will be denoted by $(\cdot,\cdot)_{L^2(\Omega)}$.

It is routine to check that the second-order operator L in (1) is formally self-adjoint when acting on distributions $\mathcal{D}(\Omega)$ and gives rise to a unique continuous and self-adjoint extension

$$L : H_0^1(\Omega;K) \longrightarrow H^{-1}(\Omega;K). \tag{12}$$

Now, we give the definition and the main theorem in the paper.

Definition 1 We say Ω is star-shaped with respect to the vector field of a given (Lipschitz) continuous vector field $V = (V_1(x,y), V_2(x,y))$; that is, for every $(x_0,y_0) \in \overline{\Omega}$, (the closure of Ω) one has $\mathcal{F}_t(x_0,y_0) \in \overline{\Omega}$ for each $t \in [0,+\infty]$ where $\mathcal{F}_t(x_0,y_0)$ represents the time-t flow of (x_0,y_0) in the direction of V.

If Ω is star-shaped with respect to the flow of V, then Ω is simply connected and will have a V-starlike boundary in the sense that $V(x,y)\cdot\nu \ge 0$, where ν is the unit exterior normal (see Lemma 2.2 of [6]).

Definition 2 We say that $u \in H_0^1(\Omega;K)$ is a weak solution of the Dirichlet problem (1) with $K(y) = A(1 - e^{-2By})$ if there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$\|u_n - u\|_{H_0^1(\Omega;K)} \to 0 \quad \text{and} \quad \|Lu_n - f\|_{H^{-1}(\Omega;K)} \to 0 \quad \text{for } n \to +\infty \tag{13}$$

or, equivalently,

$$\langle Lu, \varphi\rangle = -\int_{\Omega} (Ku_x\varphi_x + u_y\varphi_y)\,dx\,dy = \langle f, \varphi\rangle, \quad \varphi \in H_0^1(\Omega, K), \tag{14}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{-1}(\Omega;K)$ and $H_0^1(\Omega;K)$, L is the continuous extension defined in (12), and the relevant norms are defined in (8) and (9).

Theorem 1 *Let Ω be a bounded mixed domain with piecewise C^1 boundary and parabolic segment CO with $O = 0$ and $2\sup_{\Omega^+} e^{2By} < 3 + 2\inf_{\Omega^+} e^{2By}$. Assume that Ω is star-shaped with respect to the vector field $V = (b_0 x, -\frac{e^{2By}-1}{B})$ with*

$$b_0 = \begin{cases} -\frac{1}{2} + \epsilon + 2\sup_{\Omega^+} e^{2By} & \Omega^+, \\ \frac{5}{2} + \epsilon + 2\sup_{\Omega^-} e^{2By} & \Omega^- \end{cases} \tag{15}$$

for some ϵ. Then, for each $f \in L^2(\Omega; |K|^{-1})$, there exists a unique weak solution $u \in H_0^1(\Omega;K)$ in the sense of Definition 1 to the Dirichlet problem (1).

Theorem 2 *Let Ω be a mixed domain and $f \in L^2(\Omega)$. If $u \in H^1(\Omega, K)$ is a weak solution to the equation $Lu = f$, then $u \in H^1_{\mathrm{loc}}(\Omega)$.*

In the following sections, we will prove the results on existence, uniqueness, and interior regularity by the a-b-c integral method in Sobolev spaces with suitable weights.

2 Existence and uniqueness of weak solution

Lemma 1 *Under the hypotheses of Theorem 1, one has the a priori estimate. Then Ω is admissible, that is, there exists a positive constant $C_1 = C_1(\Omega, K)$ such that*

$$\|u\|_{L^2(\Omega;|K|)} \le C_1 \|Lu\|_{H^{-1}(\Omega;K)}; \quad u \in C_0^\infty(\Omega). \tag{16}$$

Proof Define $Mv = av + bv_x + cv_y$, where $(a, b, c) = (-\frac{1}{4}, b_0 x, \frac{e^{2By}-1}{B})$. We claim that for every $u \in C_0^\infty(\Omega)$, there exists $v \in C^\infty(\Omega) \cap C^0(\bar{\Omega}\backslash\{0,0\})$ solving

$$\begin{cases} Mv = u & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega\backslash\{0,0\}; \end{cases} \tag{17}$$

furthermore, $\int_\Omega (|K|v_x^2 + v_y^2)\,dx\,dy < +\infty$. In fact, parameterizing the integral curve of (b, c) by $\gamma(t) = (x(t), y(t)) = (x_0 e^{b_0 t}, -\frac{\ln[1-e^{2t}(1-e^{-2By_0})]}{2B})$ for $(x_0, y_0) \in \partial\Omega$ and $t \in (-\infty, 0]$ and by the assumption that Ω is star-shaped, the method of characteristic gives the unique $C^\infty(\Omega) \cap C^0(\bar{\Omega}\backslash\{0,0\})$ solution of (17)

$$v\bigl(x(t), y(t)\bigr) = e^{-at} \int_0^t e^{as} u\bigl(x(s), y(s)\bigr)\,ds \tag{18}$$

along each flow line. The unique singularity point is $(0,0)$. Since u is with compact support, there is a constant $\varepsilon > 0$ such that the infimum distance from the points of $\operatorname{supp} u$ to the boundary of Ω is denoted by $\operatorname{dist}(\operatorname{supp} u, \partial\Omega) > \varepsilon$. Denote by $B_\varepsilon(O)$ the disk with radius ε

with center O, then $B_\varepsilon(O) \cap \operatorname{supp} u = \emptyset$. For each $(x,y) \in B_\varepsilon(O) \cap \Omega$, we can re-initialize the Cauchy problem at time $t = T < 0$ by starting from points (x_T, y_T) on $\partial B_\varepsilon \cap \Omega$, then $v(x,y) = v(x_T, y_T)e^{\frac{1}{4}(t-T)}$. Since $v \in C^1$ on Ω, the variation in $v(x_T, y_T)$ is bounded for (x_T, y_T) on the initial data surface, v satisfies (18).

We proceed to estimate the expression $(v, Lu)_{L^2(\Omega)}$.

$$\begin{aligned}
(v, Lu)_{L^2(\Omega)} &= \int_\Omega div\big(v(Ku_x, u_y)\big) - \nabla v \cdot (Ku_x, u_y)\,dx\,dy \\
&= -\int_\Omega v_x K(av + bv_x + cv_y)_x + v_y(av + bv_x + cv_y)_y + \int_{\nabla\Omega} v(Ku_x, u_y)\cdot \vec{n}\,ds \\
&= \frac{1}{2}\int_\Omega \big[\big(-2aK - b_x K + bK_x + (Kc)_y\big)v_x^2 + 2(-Kc_x - b_y)v_x v_y \\
&\quad + (-2a + b_x - c_y)v_y^2 + v^2 La\big]\,dx\,dy \\
&\quad + \frac{1}{2}\int_{\partial\Omega} \big[2v(Ku_x, u_y) - \big(Kv_x^2 + v_y^2\big)(b,c) - v^2(Ka_x, a_y)\big]\vec{n}\,ds,
\end{aligned} \tag{19}$$

where $\vec{n}$ is the unit exterior normal vector while ds is the arc length element.

Using the smoothness and compact support of u, the continuity of v, Ω is star-shaped with V, the boundary integrals vanish. Notice the value of (a,b,c).

$$-2aK - b_x K + bK_x + (Kc)_y = \frac{K}{2} - b_0 K + 2K + 2e^{2By}K = \left(\frac{5}{2} - b_0 + 2e^{2By}\right)K, \tag{20}$$

$$-Kc_x - b_y = 0, \tag{21}$$

$$-2a + b_x - c_y = \frac{1}{2} + b_0 - 2e^{2By}. \tag{22}$$

On the elliptic region Ω^+, $K > 0$, and

$$\begin{aligned}
\frac{5}{2} - b_0 + 2e^{2By} &= \frac{5}{2} - \left(-\frac{1}{2} + \epsilon + 2\sup_{\Omega_+} e^{2By}\right) + 2e^{2By} = 3 - \epsilon - 2\sup_{\Omega_+} e^{2By} + 2e^{2By} \\
&= 3 - \epsilon - \left(3 + 2\inf_{\Omega_+} e^{2By} - \bar{\epsilon}\right) + 2e^{2By} > \bar{\epsilon} - \epsilon + 2e^{2By} - 2\inf_{\Omega_+} e^{2By},
\end{aligned} \tag{23}$$

where $\bar{\epsilon} = 3 + 2\inf_{\Omega^+} e^{2By} - 2\sup_{\Omega^+} e^{2By}$

$$\frac{1}{2} + b_0 - 2e^{2By} = \frac{1}{2} - \frac{1}{2} + \epsilon + 2\sup_{\Omega^+} e^{2By} - 2e^{2By} > \epsilon. \tag{24}$$

On the hyperbolic region Ω^-, $K < 0$ and

$$-\frac{5}{2} + b_0 - 2e^{2By} = -\frac{5}{2} + \frac{5}{2} + \epsilon + 2\sup_{\Omega_-} e^{2By} - 2e^{2By} > \epsilon, \tag{25}$$

$$\frac{1}{2} + b_0 - 2e^{2By} = \frac{1}{2} + \frac{5}{2} + \epsilon + 2\sup_{\Omega^-} e^{2By} - 2e^{2By} > 3 + \epsilon. \tag{26}$$

Choose $\epsilon_0 > 0$ such that $\bar{\epsilon} - \epsilon + 2e^{2By} - 2\inf_{\Omega_+} e^{2By} > \epsilon_0$. So that

$$(v, Lu)_{L^2(\Omega)} \geq \epsilon_0 \int_\Omega \big(|K|v_x^2 + v_y^2\big)\,dx\,dy = \epsilon_0 \|v\|_{H_0^1(\Omega;K)}. \tag{27}$$

On the other hand, by the generalized Schwarz inequality,

$$(v, Lu)_{L^2(\Omega)} \le \|v\|_{H_0^1(\Omega;K)} \|Lu\|_{H^{-1}(\Omega;K)}. \tag{28}$$

Using the Cauchy–Schwarz inequality and Poincaré inequality, we have

$$\begin{aligned} \|u\|^2_{L^2(\Omega;|K|)} = \|Mv\|^2_{L^2(\Omega;|K|)} &= \int_\Omega |K|(av + bv_x + cv_y)^2\,dx\,dy \\ &\le \int_\Omega \left(|K|v_x^2 + v_y^2\right) dx\,dy = C\|v\|_{H_0^1(\Omega;K)}. \end{aligned} \tag{29}$$

Combining (27)–(29) gives the desired estimate

$$\|u\|_{L^2(\Omega;|K|)} \le C\|Lu\|_{H^{-1}(\Omega;K)}. \tag{30}$$

This competes the proof of this lemma. □

Proof of Theorem 1 Define a linear functional J_f for $\varphi \in C_0^\infty(\Omega)$ by the formula $J_f(L\varphi) = (f,\varphi)_{L^2(\Omega)}$, and the estimate together with the Cauchy–Schwarz inequality yields

$$\left|J_f(L\varphi)\right| \le \|f\|_{L^2(\Omega;|K|^{-1})}\|\varphi\|_{L^2(\Omega;|K|)} \le C_1\|f\|_{L^2(\Omega;|K|^{-1})}\|L\varphi\|_{H^{-1}(\Omega;K)}. \tag{31}$$

Hence J_f is bounded on the subspace V of $H^{-1}(\Omega;K)$ of elements of the form $L\varphi$ with $\varphi \in C_0^\infty(\Omega)$. By the Hahn–Banach theorem, J_f extends to the closure of V in $H^{-1}(\Omega;K)$ in a bounded way. Extension by zero on the orthogonal complement of $\bar{V}$ gives a bounded linear functional on all of $H^{-1}(\Omega;K)$, and so, by the Riesz representation theorem, there exists $u \in H_0^1(\Omega;K)$ such that

$$\langle u, L\varphi\rangle = (f,\varphi)_{L^2(\Omega)}, \quad \varphi \in H_0^1(\Omega;K); L\varphi \in H^{-1}(\Omega;K), \tag{32}$$

where L is the self-adjoint extension defined in (12). This distributional solution is a weak solution in the sense of Definition 2. In fact, given a sequence $u_n \in C_0^\infty(\Omega)$ that $u_n \to u$ in $H_0^1(\Omega;K)$, the continuity property (12) shows that $f_n := Lu_n \overset{H^{-1}(\Omega,K)}{\longrightarrow} \tilde{f}$. One also has

$$\langle u_n, L\varphi\rangle = (f_n,\varphi)_{L^2(\Omega)}. \tag{33}$$

Taking the difference between (32) and (33) and passing to the limit shows that $\tilde{f} = f$, and hence (13) holds.

For the uniqueness, we use estimate (16). In fact, for fixed f, let $u, v \in H_0^1(\Omega;K)$ be two weak solutions which approximate sequences $\{u_n\}$ and $\{v_n\}$ satisfying (13). From the linearity of L and (16), one has that $u_n - v_n \overset{L^2(\Omega,K)}{\longrightarrow} 0$ by the injectivity of the first inclusion in (5) and the Poincaré inequality (7). Thus $u = v$ in $H_0^1(\Omega;K)$. □

3 Regularity of the weak solution

Lemma 2 *Assume that u is a weak solution to* (1). *Define* $E := \int_\Omega (|K|u_x^2 + u_y^2)\,dx\,dy < +\infty$, $F := \int_\Omega f^2\,dx\,dy < +\infty$. *If u is smooth enough, then for each compact subdomain $G \subset \Omega$,* *there exists a constant* $C = C(K, G)$ *such that*

$$\int_G u_x^2\,dx\,dy \le C(E+F). \tag{34}$$

Proof Pick a smooth cutoff function $\zeta \in C_0^\infty(\Omega)$ such that $0 \le \zeta \le 1$ and $\zeta \equiv 1$ on G. One merely computes by a sequence of integrations by parts, which will need to be justified later. Starting from

$$AB\int_\Omega e^{-2By}\zeta u_x^2\,dx\,dy = \int_\Omega K'(y)\zeta u_x^2\,dx\,dy = \int_\Omega (K\zeta)_y u_x^2\,dx\,dy - \int_\Omega K\zeta_y u_x^2\,dx\,dy, \tag{35}$$

we find there exists a positive number ϵ_0 such that

$$2AB\int_\Omega e^{-2By}\zeta u_x^2\,dx\,dy \ge \epsilon_0 \int_G u_x^2\,dx\,dy. \tag{36}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega (K\zeta)_y u_x^2\,dx\,dy - \int_\Omega K\zeta_y u_x^2\,dx\,dy \\
&\quad = -\int_\Omega K\zeta_y u_x^2\,dx\,dy - \int_\Omega 2K\zeta u_x u_{xy}\,dx\,dy \\
&\quad \le C_1 E + 2\int_\Omega (K\zeta u_x)_x u_y\,dx\,dy \le C_2 E + 2\int_\Omega K\zeta u_{xx}u_y\,dx\,dy \\
&\quad = C_2 E + 2\int_\Omega \zeta(f - u_{yy})u_y\,dx\,dy \le C_3(E+F) - 2\int_\Omega \zeta u_{yy}u_y\,dx\,dy \\
&\quad = C_3(E+F) - \int_\Omega \zeta\left(u_y^2\right)_y\,dx\,dy \\
&\quad \le C_4(E+F).
\end{aligned} \tag{37}$$

Thus

$$\int_G u_x^2\,dx\,dy \le C(E+F). \tag{38}$$

Mollification $\eta \in C_0^\infty(R)$ such that $0 \le \eta \le 1$, $\int_R \eta(t)\,dt = 1$ for $\epsilon > 0$, define $\eta_\epsilon(t) = \epsilon^{-1}\eta(\frac{t}{\epsilon})$ so that $\eta_\epsilon \in C_0^\infty(B(0;\epsilon))$ and $\int_R \eta_\epsilon(t)\,dt = 1$. Given any $u \in L^1_{\mathrm{loc}}(\Omega)$, measurable and locally integrable with respect to Lebesgue measure, define

$$u_\epsilon(x,y) := \int_R \eta_\epsilon(x-t)u(t,y)\,dt = \int_{B(0;\epsilon)} \eta_\epsilon(t)u(x-t,y)\,dt, \tag{39}$$

where we extend u by 0 outside of Ω. □

Define $I(y) = \{x \in R \mid (x,y) \in \Omega\}$, $I_\epsilon(y) = \{x \in I(y) \mid x \pm \epsilon \in I(y)\}$.

Lemma 3 (see [4]) *Let* $u \in L^1_{\mathrm{loc}}(\Omega)$, *then one has, for almost every* $y \in \pi_2(\Omega) := \{y : \exists x \in R, (x,y) \in \Omega\}$,

(i) $u_\epsilon(\cdot,y) \in C^\infty(I_\epsilon(y))$ *and for each* $k \in N$, $D_x^k u_\epsilon(x,y) = \int_{I(y)} D^k \eta_\epsilon(x-t)u(x,t)\,dt$ *for* $\forall x \in I_\epsilon(y)$;
(ii) $u_\epsilon(\cdot,y) \longrightarrow u(\cdot,y)$ *almost everywhere on* $I(y)$ *as* $\epsilon \longrightarrow 0$;
(iii) *If* $u(\cdot,y) \in C^0(I(y))$, *then* $u_\epsilon(\cdot,y) \longrightarrow u(\cdot,y)$ *uniformly on compact subsets of* $I(y)$;
(iv) *If* $1 \le p < \infty$ *and* $u \in L^p_{\mathrm{loc}}(I(y))$, *then* $u_\epsilon(\cdot,y) \longrightarrow u(\cdot,y)$ *in* $L^p_{\mathrm{loc}}(I(y))$.

Moreover, let $u \in L^2(\Omega)$. *Then one has*:
(v) $u_\epsilon \in L^2(\Omega)$ *and* $u_\epsilon \longrightarrow u$ *in* $L^2(\Omega)$;
(vi) $D_x^k J_\epsilon : L^2(I(y)) \longrightarrow L^2(I(y))$ *is bounded for every* $k \in N$, *where* $J_\epsilon u := u_\epsilon$;
(vii) $D_x^k J_\epsilon : L^2(\Omega) \longrightarrow L^2(\Omega)$ *is bounded for every* $k \in N$;
(viii) *For every* $g \in L^2(\Omega)$, $\varphi \in C_0^\infty(\Omega)$, *one has*

$$\left(e^{-2By}g\right)_\epsilon = e^{-2By}g_\epsilon, \tag{40}$$

$$D_x(\varphi_\epsilon) = (D_x\varphi)_\epsilon, \qquad D_y(\varphi_\epsilon) = (D_y\varphi)_\epsilon, \tag{41}$$

$$\int_\Omega g\varphi_\epsilon\,dx\,dy = \int_\Omega g_\epsilon\varphi\,dx\,dy. \tag{42}$$

Lemma 4 *Let* $u \in H^1(\Omega;K)$ *be a weak solution to* (1) *with* $f \in L^2(\Omega)$. *Then* $u_\epsilon \in H^2(\Omega)$ *is a weak solution to the equation*

$$Lu_\epsilon = f_\epsilon, \tag{43}$$

where $f_\epsilon = J_\epsilon f$ *is the mollification in* x *of* f.

Proof If $u \in H^1(\Omega;K)$, then u, $|K|^{1/2}u_x$ and $u_y \in L^2(\Omega)$, and by Lemma 3, we have $u_\epsilon \in L^2(\Omega)$ and $D_x^k u_\epsilon \in L^2(\Omega)$ for each $k \ge 1$. As for $D_y u_\epsilon$, since

$$\int_\Omega u_\epsilon D_y\varphi\,dx\,dy = -\int_\Omega (u_\epsilon)_y\varphi\,dx\,dy = -\int_\Omega (u_y)_\epsilon\varphi\,dx\,dy \quad \forall\varphi \in C_0^\infty(\Omega) \tag{44}$$

and $(u_y)_\epsilon \in L^2(\Omega)$, therefore $(u_\epsilon)_y = (u_y)_\epsilon \in L^2(\Omega)$.

Next we claim that if $u \in L^2(\Omega)$ solves (1) in the sense of distributions, then u_ϵ solves (43) in the sense of distributions. Indeed, starting from

$$\int_\Omega u\big((K\varphi_x)_x + \varphi_{yy}\big)\,dx\,dy = \int_\Omega f\,dx\,dy, \quad \varphi \in C_0^\infty(\Omega), \tag{45}$$

take $\epsilon > 0$ small enough so that the support of φ_ϵ inside Ω, apply (44) to φ_ϵ, and use Lemma 3 to find

$$\int_\Omega f_\epsilon\varphi\,dx\,dy = \int_\Omega u(K\varphi_{\epsilon x})_x + (\varphi_\epsilon)_{yy}\,dx\,dy = \int_\Omega u_\epsilon(K\varphi_{xx} + \varphi_{yy})\,dx\,dy, \tag{46}$$

which gives the claim.

Rewriting (46) in the form

$$\int_\Omega u_\epsilon\varphi_{yy}\,dx\,dy = \int_\Omega \big(f_\epsilon - K(u_\epsilon)_{xx}\big)\varphi\,dx\,dy, \quad \varphi \in C_0^\infty(\Omega) \tag{47}$$

and noticing $f_\epsilon - K(u_\epsilon)_x$, we know that the weak derivative $D_y^2 u_\epsilon$ exists and belongs to $L^2(\Omega)$. Since $D_x D_y u_\epsilon = D_x(D_y u_\epsilon)$ and $D_y u_\epsilon \in L^2(\Omega)$, then by Lemma 3, we know $D_x D_y u_\epsilon \in L^2(\Omega)$. Hence we have $u \in H^2(\Omega)$, which completes the proof. □

Proof of Theorem 2 By Lemma 3, we have

$$\int_\Omega \zeta (u_\epsilon)_x^2 \, dx\, dy \le C \int_\Omega \left(|K|(u_\epsilon)_x^2 + (u_\epsilon)_y^2 + f_\epsilon^2\right) dx\, dy, \tag{48}$$

where $u \in L^2(\Omega)$, then Lemma 3 and applying Fatou's lemma gives the needed estimate

$$\int_\Omega \zeta u_x^2 \, dx\, dy \le C \int_\Omega \left(|K|u_x^2 + u_y^2 + f^2\right) dx\, dy, \tag{49}$$

and the theorem follows. □

We conclude the results on existence, uniqueness, and regularity of weak solutions for closed Dirichlet problem. One can generalize Theorems 1 and 2 to include more general type change functions $K(y)$ provided that K is sufficiently smooth.

Acknowledgements
The authors would like to thank the anonymous reviewers for their careful work and comments that helped to improve the clarity of the paper.

Funding
The first author is supported by the Natural Science Foundation of Shaanxi Province of China 2019GY-202. The second author is supported by NSFC 11401043, Natural Science Foundation of Shaanxi Province of China 2018JQ1084, and the Fundamental Research Funds for the Central Universities 310812171005.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have contributed equally to the paper. All authors read and approved the final manuscript.

Author details
[1]School of Highway Engineering, Chang'an University, Xi'an, P.R. China. [2]School of Sciences, Chang'an University, Xi'an, P.R. China.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 7 March 2019 Accepted: 5 July 2019 Published online: 15 July 2019

References

1. Adams, R.A., Fournier, J.J.: Sobolev Spaces. Elsevier, New York (2003)
2. Agmon, S., Nirenberg, L., Protter, M.H.: A maximum principle for a class of hyperbolic equations and applications to equations of mixed elliptic-hyperbolic type. Commun. Pure Appl. Math. **6**, 455–470 (1953)
3. Bers, L.: Mathematical Aspects of Subsonic and Transonic Gas Dynamics. Dover, New York (2016)
4. Evans, L.C.: Partial Differential Equations. Am. Math. Soc., Providence (2010)
5. Lupo, D., Morawetz, C.S., Payne, K.R.: On closed boundary value problems for equations of mixed elliptic-hyperbolic type. Commun. Pure Appl. Math. **60**, 1319–1348 (2007)
6. Lupo, D., Payne, K.R.: Critical exponents for semilinear equations of mixed elliptic-hyperbolic and degenerate types. Commun. Pure Appl. Math. **56**, 403–424 (2003)
7. Marin, M.: Cesaro means in thermoelasticity of dipolar bodies. Acta Mech. **122**, 155–168 (1997)
8. Marin, M., Öchsner, A.: The effect of a dipolar structure on the Hölder stability in Green–Naghdi thermoelasticity. Contin. Mech. Thermodyn. **29**, 1365–1374 (2017)
9. Morawetz, C.S.: On the non-existence of continuous transonic flows past profiles. Commun. Pure Appl. Math. **9**, 45–68 (1956)

10. Morawetz, C.S.: Note on a maximum principle and a uniqueness theorem for an elliptic-hyperbolic equation. Proc. R. Soc. Lond. **236**, 141–144 (1956)
11. Morawetz, C.S.: The Dirichlet problem for the Tricomi equation. Commun. Pure Appl. Math. **23**, 587–601 (1970)
12. Morawetz, C.S.: Mixed equations and transonic flow. J. Hyperbolic Differ. Equ. **1**, 1–26 (2004)
13. Otway, T.H.: Unique solutions to boundary value problems in the cold plasma model. SIAM J. Math. Anal. **42**, 3045–3053 (2010)
14. Picone, M.: Sulle equazioni alle derivate parziali del secondordine del tipo iperbolico in due variabili indipendenti. Rend. Circ. Mat. Palermo **30**, 349–376 (1910)
15. Pilant, M.: The Neumann problem for an equation of Lavrent'ev–Bitsadze type. J. Math. Anal. Appl. **106**, 321–359 (1985)
16. Protter, M.H.: Uniqueness theorems for the Tricomi problem. II. J. Ration. Mech. Anal. **4**, 721–732 (1955)
17. Ruziev, M.K.: A problem with conditions given on inner characteristics and on the line of degeneracy for a mixed-type equation with singular coefficients. Bound. Value Probl. **2013**, 210 (2013)
18. Tomotika, S., Tamada, K.: Studies on two-dimensional transonic flows of compressible fluid. Q. Appl. Math. **7**, 381–397 (1950)
19. Tricomi, F.: Sulle equazioni lineari alle derivate parziali di tipo miste. Rend. Atti Accad. Naz. Lincei **14**, 134–147 (1923)
20. Yuan, C., Jiang, D., Zhang, Y.: Existence and uniqueness of solutions for singular higher order continuous and discrete boundary value problems. Bound. Value Probl. **2008**, 123823 (2008)