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On an electrorheological fluid equation with orientated convection term

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Abstract

A kind of electrorheological fluid equations with orientated convection terms is considered. If the diffusion coefficient $a(x,t) \in C^1(\overline{Q_T})$ is degenerate on the boundary $\partial\Omega$, not only the uniqueness of weak solution is proved, but also the stability of the solutions can be proved without any boundary condition, provided that there are some restrictions on the diffusion coefficient $a(x,t)$ and the convective coefficient $\vec{b}(x,t)$. Moreover, the large time behavior of weak solution is studied.

MSC: 35K65; 35K55; 35R92

Keywords: The electrorheological fluid equation; Orientated convection term; Partial boundary value condition; Stability; Large time behavior

1 Introduction

The initial boundary value problem of an electrorheological fluids equation with orientated convection term

$$u_t = \operatorname{div}\big(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\big) + \vec{f}(x,t)\cdot\nabla u^q, \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.3}$$

is studied in this paper, where $1 < p(x,t) \in C(\overline{Q_T})$, $q > 0$, $a(x,t) \in C^1(\overline{Q_T})$, $\vec{f} = \{f^i(x,t)\}$, $f^i(x,t) \in C^1(\overline{Q_T})$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$.

When $p(x,t) > 1$ is a measurable function on Q_T, equation (1.1) arises in electrorheological fluids theory [1]. If $\vec{f}(x,t) = 0$, $a(x,t) = 1$ for all $(x,t) \in \overline{Q_T}$, the existence and uniqueness results of equation (1.1) have been obtained in [2–6] etc. If $p(x,t) = p > 1$ is a constant, $a(x,t) = 1$ and $\vec{f}(x,t) = 0$, equation (1.1) is well known as non-Newtonian fluid equation and has been studied by many mathematicians, one can refer to [7–15] and the references therein. From these papers, we know that the uniqueness and the stability of weak solutions can be proved if the Dirichlet boundary value condition (1.3) is imposed. In recent years, the equations with the type

$$u_t = \operatorname{div}\big(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\big) + f(x,t,u,\nabla u), \quad (x,t) \in Q_T \tag{1.4}$$

have drawn wide public attention [16–23] etc. When $p(x,t) = p > 1$ is a constant, $f(x,t,u,\nabla u)$ is a linear function, the well-posedness problem of equation (1.4) was studied in [24–26]. In addition, the non-Newtonian polytropic filtration equation with orientated convection

$$u_t = \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) + \vec{b}(x)\cdot\nabla u^q, \quad x\in\mathbb{R}^N, t>0,$$

was studied in [27], where $m>0$, $p>2$ and $\vec{b}=\{b^i(x)\}$, $b_i(x)\in C^1(\mathbb{R})^N$. The author has been interested in the stability of weak solutions to equation (1.4) for a long time. When $a(x,t)=a(x)$, $a(x)>0$ in Ω and

$$a(x) = 0, \quad x\in\partial\Omega, \tag{1.5}$$

some progresses have been made in [22, 23]. If $a(x,t)=a(x)$, $p(x,t)=p$, the stability of weak solutions to equation (1.1) has been studied in [28, 29]. We have found that condition (1.5) may replace the usual Dirichlet boundary value condition (1.3) for some special $f(x,t,u,\nabla u)$, the stability of solutions can be established without any boundary value condition (1.3), provided that there are some other restrictions on $f(x,t,u,\nabla u)$.

In this paper, we first generalize the results contained in [15, 28, 29] to equation (1.1), since there is time variable t in the exponents, there are some essential difficulties that should be overcome. Secondly, we will used some ideas [3, 4, 30] to prove the uniqueness of weak solution. Thirdly, the large time behavior of weak solutions is studied free from the limitations of the boundary value condition.

We denote that

$$p_+ = \max_{(x,t)\in\overline{Q_T}} p(x,t), \qquad p_- = \min_{(x,t)\in\overline{Q_T}} p(x,t),$$

assume that $p_->1$, and the constants c appearing in different places represent different constants, $a(x,t)$ is a nonnegative function in $C^1(\overline{Q_T})$, and for every $t\in[0,T]$,

$$a(x,t)=0, \quad x\in\partial\Omega \quad \text{and} \quad a(x,t)>0, \quad x\in\Omega. \tag{1.6}$$

We give the basic definitions and the main results now.

Definition 1.1 If a nonnegative function $u(x,t)$ satisfies

$$u\in L^\infty(Q_T), \qquad u_t\in L^{p_+'}\left(0,T;W^{-1,p_+'}(\Omega)\right), \qquad a(x,t)|\nabla u|^{p(x,t)}\in L^\infty\left(0,T;L^1(\Omega)\right),$$

and for any $\varphi(x,t)\in C_0^1(\overline{Q_T})$,

$$\begin{aligned}&\iint_{Q_T}\left[\frac{\partial u}{\partial t}\varphi + a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\cdot\nabla\varphi\right]dx\,dt\\ &\quad+\sum_{i=1}^N\iint_{Q_T}u^q\left[f^i_{x_i}(x,t)\varphi + f^i(x,t)\varphi_{x_i}\right]dx\,dt\\ &=0,\end{aligned}$$

then we say $u(x,t)$ is a solution of equation (1.1) with the initial value (1.2) which is satisfied in the sense

$$\lim_{t\to 0}\int_{\Omega} u(x,t)\phi(x)\,dx = \int_{\Omega} u_0(x)\phi(x)\,dx \tag{1.7}$$

for any $\phi(x)\in C_0^{\infty}(\Omega)$.

Here, $p'_+ = \frac{p_+}{p_+-1}$, $b^i_{x_i} = \frac{\partial b^i(x)}{\partial x_i}$, $g_{x_i} = \frac{\partial g}{\partial x_i}$ as usual, $i = 1,2,\ldots,N$. In this paper, the existence of the nonnegative solution is proved.

Theorem 1.2 *If $p_- > 1$, $1\le q < p_+$, $a(x,t)\ge 0$ satisfies (1.6),*

$$0\le u_0\in L^{\infty}(\Omega), \qquad a(x,0)|\nabla u_0|^{p(x,0)}\in L^1(\Omega), \quad i = 1,2,\ldots,N, \tag{1.8}$$

then equation (1.1) with initial value (1.2) has a nonnegative weak solution u.

If $\int_{\Omega} a(x,t)^{-\frac{1}{p(x,t)-1}}\,dx < \infty$ for any $t\in[0,T]$, similar as the proof of Theorem 1.1 in [12], the weak solution u in Theorem 1.2 satisfies

$$\begin{aligned}
&\int_{\Omega}|\nabla u|\,dx\\
&\quad= \int_{\{x\in\Omega: a(x,t)^{\frac{1}{p(x,t)-1}}|\nabla u|\le 1\}}|\nabla u|\,dx + \int_{\{x\in\Omega: a(x,t)^{\frac{1}{p(x,t)-1}}|\nabla u|> 1\}}|\nabla u|\,dx\\
&\quad\le \int_{\Omega} a(x,t)^{-\frac{1}{p(x,t)-1}}\,dx + \int_{\Omega} a(x,t)|\nabla u|^{p(x,t)}\,dx\\
&\quad\le c.
\end{aligned} \tag{1.9}$$

Then the boundary value condition (1.3) is valid in the sense of the trace. However, in general, $u(x,t)$ is in $W^{1,p(x,t)}_{\mathrm{loc}}(\Omega)$ and cannot be defined the trace on the boundary. Accordingly, instead of considering the boundary value condition itself, we would pay a close attention to finding some other conditions to replace the boundary value condition and prove the corresponding stability of weak solutions (or uniqueness of weak solution).

Theorem 1.3 *Let $q\ge 1$, $a(x,t)\ge 0$ satisfy (1.6), $p(x,t)\ge p_- > 1$, $u(x,t)$ and $v(x,t)$ be two nonnegative weak solutions of equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$. If*

$$\bigl|f^i(x,t)\bigr|\ge ca(x,t), \quad i = 1,2,\ldots,N, t\in[0,T], \tag{1.10}$$

$$\int_{\Omega} a(x,t)^{-(p(x,t)-1)}\,dx < \infty, \quad t\in[0,T], \tag{1.11}$$

then

$$\int_{\Omega}\bigl|u(x,t)-v(x,t)\bigr|\,dx \le c\int_{\Omega}\bigl|u_0(x)-v_0(x)\bigr|\,dx, \quad a.e.\ t\in[0,T). \tag{1.12}$$

Theorem 1.4 *If $q\ge 1$, $a(x,t)\ge 0$ satisfies (1.6), $p(x,t)\ge p_- > 1$, $u(x,t)$ and $v(x,t)$ are two nonnegative weak solutions of equation (1.1) with the initial values $u_0(x) = v_0(x)$,*

$$\int_{\Omega} a(x,t)^{-\frac{1}{p(x)-1}}\,dx < \infty, \quad t\in[0,T], \tag{1.13}$$

$$\operatorname{div}\vec{f}(x,t) \geq 0, \tag{1.14}$$

then

$$u(x,t) = v(x,t), \quad (x,t) \in Q_T. \tag{1.15}$$

One can see that condition (1.11) in Theorem 1.3 and condition (1.13) in Theorem 1.4 are complementary to each other. In this paper, ∇u represents the gradient of u on the spatial variable x, $\operatorname{div}\vec{f}(x,t)$ represents the divergence of $\vec{f}$ on the spatial variable x.

By the uniqueness of weak solutions, we will study the large time behavior of weak solutions without the boundary value condition in the last section.

2 The existence of weak solutions

In this section, we use the parabolically regularized method to prove Theorem 1.2. Consider the initial boundary value problem

$$\begin{aligned} &u_{\varepsilon t} - \varepsilon \operatorname{div}\big(|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon\big) - \operatorname{div}\big(a(x,t)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\big) - \vec{f}(x,t)\cdot \nabla u_\varepsilon^q \\ &\quad = 0, \quad (x,t) \in Q_T, \end{aligned} \tag{2.1}$$

$$u_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.2}$$

$$u_\varepsilon(x,0) = u_{\varepsilon,0}(x), \quad x \in \Omega, \tag{2.3}$$

where $0 \leq u_{\varepsilon,0} \in C_0^\infty(\Omega)$, $\|u_{\varepsilon,0}\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$, $u_{\varepsilon,0} \to u_0(x)$ in $W_0^{1,p_+}(\Omega)$. Then there is a unique nonnegative solution $u_\varepsilon \in L^{p_+}(0,T;W_0^{1,p_+}(\Omega))$ [5], which satisfies

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq c. \tag{2.4}$$

By multiplying (2.1) with u_ε, integrating it over $Q_t = \Omega \times [0,t]$, we achieve

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_\varepsilon^2(x,t)\,dx + \frac{\varepsilon}{2}\iint_{Q_t} |\nabla u_\varepsilon|^{p_+}\,dx\,dt \\ &\quad + \iint_{Q_t} a(x,t)|\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \leq c, \quad \forall t \in [0,T). \end{aligned} \tag{2.5}$$

Here, we have used the following fact:

$$\begin{aligned} \iint_{Q_T} \big|u_\varepsilon \vec{f}(x,t)\cdot \nabla u_\varepsilon^q\big|\,dx\,dt &= q\iint_{Q_T} \big|u_\varepsilon^q \vec{f}(x,t)\cdot\nabla u_\varepsilon\big| \\ &\leq \frac{\varepsilon}{2}\iint_{Q_T} |\nabla u_\varepsilon|^{p_+}\,dx\,dt + c(\varepsilon). \end{aligned}$$

By (2.5), we achieve

$$\iint_{Q_T} a(x,t)|\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \leq c \tag{2.6}$$

and

$$\varepsilon\iint_{Q_T} |\nabla u_\varepsilon|^{p_+}\,dx\,dt \leq c. \tag{2.7}$$

Let $v \in L^{p_+}(0,T;W_0^{1,p_+}(\Omega))$, $\|v\|_{L^{p_+}(0,T;W_0^{1,p_+}(\Omega))} = 1$. Since $u_\varepsilon \in L^{p_+}(0,T;W_0^{1,p_+}(\Omega)) \cap L^\infty(Q_T)$, we have

$$\begin{aligned}
&\langle u_{\varepsilon t}, v\rangle + \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon \nabla v\,dx\,dt \\
&\quad + \iint_{Q_T} a(x,t)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla v\,dx\,dt \\
&\quad + \sum_{i=1}^{N}\iint_{Q_T} u^q\bigl[f^i_{x_i}(x,t)v + f^i(x,t)v_{x_i}\bigr]\,dx\,dt \\
&= 0.
\end{aligned} \tag{2.8}$$

By Young's inequality, we extrapolate that

$$\|u_{\varepsilon t}\|_{L^{p'_+}(0,T;W^{-1,p'_+}(\Omega))} \le c, \tag{2.9}$$

where $p'_+ = \frac{p_+}{p_+-1}$ as before.

For any $\phi \in C_0^1(\Omega)$, $0 \le \phi \le 1$, it is not difficult to show that

$$\bigl\|(\phi u_\varepsilon)_t\bigr\|_{L^{p'_+}(0,T;W^{-1,p'_+}(\Omega))} \le c \tag{2.10}$$

by (2.9).

Since $H_0^s(\Omega) \hookrightarrow W^{1,p_+}(\Omega)$ when $s > \frac{N}{2}+1$, we have $W^{-1,p'_+}(\Omega) \hookrightarrow H^{-s}(\Omega)$. Then

$$\bigl\|(\phi u_\varepsilon)_t\bigr\|_{L^{p'_+}(0,T;H^{-s}(\Omega))} \le c. \tag{2.11}$$

In addition, we have

$$\iint_{Q_T} \bigl|\nabla(\phi u_\varepsilon)\bigr|^{p_-}\,dx\,dt \le c(\phi)\left(1 + \int_0^T\int_{\Omega_\phi} |\nabla u_\varepsilon|^{p_-}\,dx\,dt\right) \le c(\phi),$$

where $\Omega_\phi = \operatorname{supp}\phi$. Thus,

$$\|\phi u_\varepsilon\|_{L^{p'_+}(0,T;W_0^{1,p_-}(\Omega))} \le c. \tag{2.12}$$

Since $W_0^{1,p_-}(\Omega) \hookrightarrow L^{p_-}(\Omega) \hookrightarrow H^{-s}(\Omega)$, Aubin's compactness theorem in [33] yields $\phi u_\varepsilon \to \phi u$ strongly in $L^{p'_+}(0,T;L^{p_-}(\Omega))$. Then $\phi u_\varepsilon \to \phi u$ a.e. in Q_T, and so $u_\varepsilon \to u$ a.e. in Q_T.

Combining (2.4), (2.5), (2.6), and (2.7), there exist a function u and an n-dimensional vector function $\overrightarrow{\zeta} = (\zeta_1,\ldots,\zeta_n)$ such that

$$\varepsilon|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon \rightharpoonup 0, \quad \text{in } L^{\frac{p_+}{p_+-1}}(Q_T),$$

$$u \in L^\infty(Q_T), \quad |\zeta_i| \in L^1\bigl(0T;L^{\frac{p(x,t)}{p(x,t)-1}}(\Omega)\bigr),$$

and

$$u_\varepsilon \rightharpoonup u, \quad \text{weakly star in } L^\infty(Q_T), \qquad u_\varepsilon \to u, \quad \text{a.e. in } Q_T,$$

$$u_\varepsilon^q \to u^q, \quad \text{a.e. in } Q_T,$$

$$u_{\varepsilon x_i} \rightharpoonup u_{x_i} \quad \text{in } L^1\big(0,T;L_{\mathrm{loc}}^{p(x,t)}(\Omega)\big),$$

$$a(x,t)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \rightharpoonup \overrightarrow{\zeta}, \quad \text{in } \big\{L^1\big(0,T;L^{\frac{p(x,t)}{p(x,t)-1}}(\Omega)\big)\big\}^N.$$

Meanwhile, similar as the proof of Lemma 2.6 in [4], we can prove that

$$\iint_{Q_T} a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\cdot\nabla\varphi\,dx\,dt = \iint_{Q_T}\overrightarrow{\zeta}\cdot\nabla\varphi\,dx\,dt$$

for any given function $\varphi\in C_0^1(Q_T)$. Then, for any $\varphi\in C_0^1(Q_T)$,

$$\langle u_t,\varphi\rangle + \iint_{Q_T}\left[a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\nabla\varphi + \sum_{i=1}^N u^q\big(f^i(x,t)\varphi_{x_i} + f^i_{x_i}(x,t)\varphi\big)\right]dx\,dt$$
$$= 0.$$

Moreover, one can prove the initial value condition in the sense of (1.7) as in [2], thus u is a solution of equation (1.1) with the initial value (1.2) in the sense of Definition 1.1. The proof is complete.

3 The proof of Theorem 1.3

The following lemma can be found in [30, 31].

Lemma 3.1 *The variable exponent spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$ are reflexive Banach spaces. The following hold*:

(i) *Let $p_1(x)$ and $p_2(x)$ be real functions with $\frac{1}{p_1(x)}+\frac{1}{p_2(x)}=1$. Then the conjugate space of $L^{p_1(x)}(\Omega)$ is $L^{p_2(x)}(\Omega)$. For any $u\in L^{p_1(x)}(\Omega)$ and $v\in L^{p_2(x)}(\Omega)$, there holds*

$$\left|\int_\Omega uv\,dx\right| \le 2\|u\|_{L^{p_1(x)}(\Omega)}\|v\|_{L^{p_2(x)}(\Omega)}.$$

(ii) *Let $p_{1+}=\max_{x\in\overline{\Omega}}p_1(x)$, $p_{1-}=\min_{x\in\overline{\Omega}}p_1(x)$.*

$$\text{If } \|u\|_{L^{p_1(x)}(\Omega)}=1, \quad \text{then } \int_\Omega |u|^{p_1(x)}\,dx = 1.$$

$$\text{If } \|u\|_{L^{p_1(x)}}(\Omega)>1, \quad \text{then } |u|^{p_{1-}}_{L^{p_1(x)}} \le \int_\Omega |u|^{p_1(x)}\,dx \le |u|^{p_{1+}}_{L^{p_1(x)}}.$$

$$\text{If } \|u\|_{L^{p_1(x)}}(\Omega)<1, \quad \text{then } |u|^{p_{1+}}_{L^{p_1(x)}} \le \int_\Omega |u|^{p_1(x)}\,dx \le |u|^{p_{1-}}_{L^{p_1(x)}}.$$

Lemma 3.2 (see [2]) *Let $v\in L^{p_+}(0,T;W_0^{1,p_+}(\Omega))$, $v_t\in L^{p'_+}(0,T;W^{-1,p_+{}'}(\Omega))$. For any continuous function $h(s)$, $H(s)=\int_0^s h(s)\,ds$, a.e. $t_1,t_2\in[0,T)$,*

$$\int_{t_1}^{t_2}\big\langle v_t,h(v)\big\rangle\,dt = \int_\Omega\big(H(v)(x,t_2)-H(v)(x,t_1)\big)\,dx. \tag{3.1}$$

For small $r>0$, let

$$S_\sigma(s)=\int_0^s h_\sigma(\tau)\,d\tau,\qquad h_\sigma(s)=\frac{2}{\sigma}\left(1-\frac{|s|}{\sigma}\right)_+.$$

Then

$$\lim_{\sigma\to 0}S_\sigma(s)=\operatorname{sgn}s,\qquad \lim_{\sigma\to 0}H_\sigma(s)=\int_0^s S_\sigma(\tau)\,d\tau=|s| \tag{3.2}$$

and

$$\lim_{\sigma\to 0}sh_\sigma(s)=0. \tag{3.3}$$

Let $\varphi(x,t)$ be a nonnegative function in $C^1(\overline{Q_T})$, and for every $t\in[0,T]$,

$$\varphi(x,t)=0,\quad x\in\partial\Omega\quad\text{and}\quad\varphi(x,t)>0,\quad x\in\Omega. \tag{3.4}$$

Theorem 3.3 *If $q\geq 1$, $a(x,t)\geq 0$ satisfies (1.6), $p(x,t)\geq p_->1$, $u(x,t)$ and $v(x,t)$ are two nonnegative weak solutions of equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$ respectively, and there is a nonnegative function $\varphi\in C^1(\overline{Q_T})$ satisfying (3.4) such that*

$$\int_\Omega a(x,t)\left|\frac{\nabla\varphi}{\varphi}\right|^{p(x,t)}dx<\infty,\qquad \int_\Omega\frac{|\sum_{i=1}^N f^i(x,t)\varphi_{x_i}|}{\varphi}\,dx<\infty,\quad t\in[0,T], \tag{3.5}$$

$$\int_\Omega a(x,t)^{-\frac{1}{p(x,t)-1}}\left|\sum_{i=1}^N f^i(x,t)\right|^{\frac{p(x,t)}{p(x,t)-1}}dx<\infty,\quad t\in[0,T],i=1,2,\ldots,N, \tag{3.6}$$

then

$$\int_\Omega\bigl|u(x,t)-v(x,t)\bigr|\,dx\leq c\int_\Omega\bigl|u_0(x)-v_0(x)\bigr|\,dx,\quad a.e.\ t\in[0,T). \tag{3.7}$$

Proof For two solutions $u(x,t)$, $v(x,t)$, the test function can be chosen as $S_\sigma(\varphi(u-v))$, where $\varphi(x,t)$ satisfies (3.4). Then

$$\begin{aligned}
&\int_0^t\int_\Omega S_\sigma\bigl(\varphi(u-v)\bigr)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\
&\quad+\int_0^t\int_\Omega a(x,t)\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\cdot\nabla(u-v)S_\sigma'\bigl(\varphi(u-v)\bigr)\,dx\,dt\\
&\quad+\int_0^t\int_\Omega a(x,t)\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\cdot\nabla\varphi(u-v)S_\sigma'\bigl(\varphi(u-v)\bigr)\,dx\,dt\\
&\quad+\sum_{i=1}^N\int_0^t\int_\Omega f^i_{x_i}(x,t)\bigl(u^q-v^q\bigr)S_\sigma\bigl(\varphi(u-v)\bigr)\,dx\,dt\\
&\quad+\sum_{i=1}^N\int_0^t\int_\Omega\varphi f^i(x,t)\bigl(u^q-v^q\bigr)\cdot(u-v)_{x_i}S_\sigma'\bigl(\varphi(u-v)\bigr)\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\int_0^t\int_\Omega f^i(x,t)\bigl(u^q-v^q\bigr)\cdot\varphi_{x_i}(u-v)S'_\sigma\bigl(\varphi(u-v)\bigr)\,dx\,dt\\
&=0. \qquad (3.8)
\end{aligned}
$$

Since $a(x,t)\geq 0$, $S'_\sigma(s)\geq 0$, obviously,

$$
\int_\Omega a(x,t)\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\cdot\nabla(u-v)S'_\sigma\bigl(\varphi(u-v)\bigr)\,dx\geq 0. \qquad (3.9)
$$

Since $\int_\Omega a(x,t)|\frac{\nabla\varphi}{\varphi}|^{p(x,t)}\,dx<\infty$,

$$
\begin{aligned}
&\left|\int_\Omega a(x,t)(u-v)S'_\sigma\bigl(\varphi(u-v)\bigr)\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\nabla\varphi\,dx\right|\\
&=\left|\int_\Omega a(x,t)^{-\frac{p(x,t)-1}{p(x,t)}}a(x,t)(u-v)S'_\sigma\bigl(\varphi(u-v)\bigr)\right.\\
&\quad\left.\cdot\,a(x,t)^{\frac{p(x,t)-1}{p(x,t)}}\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\nabla\varphi\,dx\right|\\
&\leq\left(\int_\Omega\left|a(x,t)^{\frac{1}{p(x,t)}}\frac{\nabla\varphi}{\varphi}\varphi(u-v)S'_\sigma\bigl(\varphi(u-v)\bigr)\right|^{p(x,t)}dx\right)^{\frac{1}{p_1(t)}}\\
&\quad\cdot\left(\int_\Omega a(x,t)\bigl(|\nabla u|^{p(x,t)}+|\nabla v|^{p(x,t)}\bigr)\,dx\right)^{\frac{1}{p_1'(t)}}\\
&\to 0 \qquad (3.10)
\end{aligned}
$$

as $\sigma\to 0$, where $p'(x,t)=\frac{p(x,t)}{p(x,t)-1}$. $p_1(t)=\max_{x\in\overline{\Omega}}p(x,t)$, or $p_1(t)=\min_{x\in\overline{\Omega}}p(x,t)$ according to

$$
\int_\Omega\left|a(x,t)^{\frac{1}{p(x,t)}}\frac{\nabla\varphi}{\varphi}\varphi(u-v)S'_\sigma\bigl(\varphi(u-v)\bigr)\right|^{p(x,t)}dx\geq 1
$$

or

$$
\int_\Omega\left|a(x,t)^{\frac{1}{p(x,t)}}\frac{\nabla\varphi}{\varphi}\varphi(u-v)S'_\sigma\bigl(\varphi(u-v)\bigr)\right|^{p(x,t)}dx< 1
$$

by Lemma 3.1. $p_1'(t)$ has a similar meaning.

In addition, since $u,v\in L^\infty(Q_T)$ and $\int_\Omega\frac{|\sum_{i=1}^N b^i(x,t)\varphi_{x_i}|}{\varphi}\,dx<\infty$, the dominated convergence theorem yields

$$
\begin{aligned}
&\lim_{\sigma\to 0}\left|\int_\Omega\sum_{i=1}^{N}f^i(x,t)\bigl(u^q-v^q\bigr)\cdot\varphi_{x_i}(u-v)S'_\sigma\bigl(\varphi(u-v)\bigr)\,dx\right|\\
&\leq\lim_{\sigma\to 0}\int_\Omega\left|\sum_{i=1}^{N}f^i(x,t)\bigl(u^q-v^q\bigr)S'_\sigma\bigl(\varphi(u-v)\bigr)\varphi(u-v)\frac{\varphi_{x_i}}{\varphi}\right|dx\\
&\leq c\lim_{\sigma\to 0}\int_\Omega\bigl|S'_\sigma\bigl(\varphi(u-v)\bigr)\varphi(u-v)\bigr|\frac{|\sum_{i=1}^N f^i(x,t)\varphi_{x_i}|}{\varphi}\,dx\\
&=0. \qquad (3.11)
\end{aligned}
$$

Since

$$\left|u^q - v^q\right| \le c|u-v|, \quad q \ge 1,$$

and by (3.6)

$$\int_\Omega a(x,t)^{-\frac{1}{p(x,t)-1}} \left|\sum_{i=1}^N f^i(x,t)\right|^{\frac{p(x,t)}{p(x,t)-1}} dx \le c,$$

we have

$$\begin{aligned}
&\lim_{\sigma\to 0}\left|\int_\Omega \sum_{i=1}^N f^i(x,t)\varphi\left(u^q - v^q\right) S_\sigma{}'\left(\varphi(u-v)\right)(u-v)_{x_i}\,dx\right| \\
&\quad\le c\lim_{\sigma\to 0}\left(\int_\Omega a(x,t)\left(|\nabla u|^{p(x,t)} + |\nabla v|^{p(x,t)}\right)dx\right)^{\frac{1}{p_1(t)}} \\
&\qquad\cdot\left(\int_\Omega \left|\sum_{i=1}^N a(x,t)^{-\frac{1}{p(x,t)}} f^i(x,t)\right|^{\frac{p(x,t)}{p(x,t)-1}} \left|\varphi(u-v)S_\sigma{}'\left(\varphi(u-v)\right)\right|^{\frac{p(x,t)}{p(x,t)-1}} dx\right)^{\frac{1}{p_1'(t)}} \\
&\quad\le c\lim_{\sigma\to 0}\left(\int_\Omega a(x,t)^{-\frac{1}{p(x,t)}}\left|\sum_{i=1}^N f^i(x,t)\right|^{\frac{p(x,t)}{p(x,t)-1}} \left|\varphi(u-v)S_\sigma{}'\left(\varphi(u-v)\right)\right|^{\frac{p(x,t)}{p(x,t)-1}} dx\right)^{\frac{1}{p_1'(t)}} \\
&\quad= 0.
\end{aligned} \tag{3.12}$$

Here $p_1(t) = \max_{x\in\overline{\Omega}} p(x,t)$ or $p_1(t) = \min_{x\in\overline{\Omega}} p(x,t)$ according to

$$\int_\Omega a(x,t)\left(|\nabla u|^{p(x,t)} + |\nabla v|^{p(x,t)}\right)dx \ge 1$$

or

$$\int_\Omega a(x,t)\left(|\nabla u|^{p(x,t)} + |\nabla v|^{p(x,t)}\right)dx < 1$$

by Lemma 3.1 for any $t \in [0,T)$, $p_1'(t)$ has a similar meaning.

At the same time,

$$\begin{aligned}
&\lim_{\sigma\to 0}\left|\int_\Omega \sum_{i=1}^N f^i_{x_i}(x,t)\left(u^q - v^q\right) S_\sigma\left(\varphi(u-v)\right)dx\right| \\
&\quad\le c\int_\Omega |u-v|\,dx, \quad i = 1,2,\ldots,N,
\end{aligned} \tag{3.13}$$

is obvious by the assumption that $f^i(x,t) \in C^1(\overline{Q_T})$, $i = 1,2,\ldots,N$.

By the definition of the weak characteristic function $\varphi(x,t)$, we can employ Lemma 3.2 to deduce that

$$\begin{aligned}
&\lim_{\sigma\to 0}\int_0^t\int_\Omega S_\sigma\left(\varphi(u-v)\right)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&\quad= \int_\Omega\int_0^t \operatorname{sign}(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\sigma\to 0}\int_0^t\int_\Omega S_\sigma(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\
&= \lim_{\sigma\to 0}\int_0^t \big(H_\sigma(u-v)(t)-H_\sigma(u_0-v_0)\,dt\\
&= \int_\Omega \big|u(x,t)-v(x,t)\big|\,dx-\int_\Omega \big|u_0(x)-v_0(x)\big|\,dx. \qquad (3.14)
\end{aligned}
$$

Let $\sigma\to 0$ in (3.8). By (3.9), (3.11), (3.12), (3.13), and (3.14), we have

$$
\int_\Omega \big|u(x,t)-v(x,t)\big|\,dx\le c\int_\Omega \big|u_0(x)-v_0(x)\big|\,dx,\quad \forall t\in[0,T). \qquad \square
$$

Proof of Theorem 1.3 By conditions (1.10) and (1.11), only if we choose $\phi(x,t)=a(x,t)$, we know conditions (4.1) (4.2) in Theorem 3.3 are true, the conclusion follows easily. $\square$

4 The proof of Theorem 1.4

Theorem 4.1 *If* $q\ge 1$, $a(x,t)\ge 0$ *satisfies* (1.6), $p(x,t)\ge p_->1$, $u(x,t)$ *and* $v(x,t)$ *are two nonnegative weak solutions of equation* (1.1) *and with the same initial value* $u_0(x)=v_0(x)$, *and*

$$
\int_\Omega \left|\sum_{i=1}^N f^i(x,t)\right|^{\frac{p(x,t)}{p(x,t)-1}} a(x,t)^{-\frac{1}{p(x,t)-1}}\,dx<c,\quad i=1,2,\dots,N, t\in[0,T], \qquad (4.1)
$$

$$
\operatorname{div}\vec{f}(x,t)\ge 0, \qquad (4.2)
$$

then

$$
u(x,t)=v(x,t),\quad (x,t)\in Q_T. \qquad (4.3)
$$

Proof For a small positive constant $\delta>0$, denoting $D_\delta=\{x\in\Omega: w=u-v>\delta\}$, we suppose that the measure $\mu(D_\delta)>0$. Let

$$
F_\lambda(\xi)=\begin{cases}\frac{1}{1-\beta}\lambda^{\beta-1}-\frac{1}{1-\beta}\xi^{\beta-1}, & \text{if } \xi>\lambda,\\ 0, & \text{if } \xi\le\lambda,\end{cases} \qquad (4.4)
$$

where $\delta>2\lambda>0$, $1>\beta>0$.

Now, by a process of limit, we can choose $F_\lambda(w)=F_\lambda(u-v)$ and integrate it over Q_t, $0\le t<T$, accordingly,

$$
\begin{aligned}
0 &= \int_0^t\int_\Omega \big[w_tF_\lambda(w)+a(x,t)\big(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\big)\nabla F_\lambda(w)\big]\,dx\,dt\\
&\quad+\sum_{i=1}^N\int_0^t\int_\Omega \big(u^q-v^q\big)\big[f^i(x,t)\big(F_\lambda(u-v)\big)_{x_i}\big]\,dx\,dt\\
&\quad+\sum_{i=1}^N\int_0^t\int_\Omega \big(u^q-v^q\big)\big[f^i_{x_i}(x,t)F_\lambda(u-v)\big]\,dx\,dt. \qquad (4.5)
\end{aligned}
$$

In the first place,

$$\begin{aligned}
&\int_0^t\int_\Omega a(x,t)\big(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\big)\cdot\nabla(u-v)F_\lambda'(u-v)\,dx\,dt\\
&\quad\ge\int_0^t\int_\Omega a(x,t)(u-v)^{2-\beta}2^{-p(x,t)}|\nabla w|^{p(x,t)}\,dx\,dt\\
&\quad\ge 2^{-p_+}\int_0^t\int_\Omega a(x,t)w^{2-\beta}|\nabla w|^{p(x,t)}\,dx\,dt\\
&\quad\ge 0. \qquad (4.6)
\end{aligned}$$

By that $q>0$ and condition (4.1),

$$\int_\Omega\sum_{i=1}^N f^i(x,t)^{\frac{p(x,t)}{p(x,t)-1}}a^{-\frac{1}{p(x,t)-1}}(x)\,dx<c,\quad i=1,2,\dots,N,t\in[0,T],$$

using the last formula of (3.3) and the dominated convergence theorem, we have

$$\begin{aligned}
&\left|\sum_{i=1}^N\int_0^t\int_\Omega f^i(x,t)\big(u^q-v^q\big)F_\lambda'(u-v)(u-v)_{x_i}\,dx\,dt\right|\\
&\quad=\left|\int_0^t\int_\Omega\sum_{i=1}^N f^i(x,t)a(x,t)^{-\frac{1}{p(x,t)}}\big(u^q-v^q\big)F_\lambda'(u-v)a^{\frac{1}{p(x,t)}}(u-v)_{x_i}\,dx\,dt\right|\\
&\quad\le c\int_0^t\int_\Omega w^{2-\beta}\left(\left|\sum_{i=1}^N f^i(x,t)\right|a(x,t)^{-\frac{1}{p(x,t)}}\big(u^q-v^q\big)\right)^{\frac{p(x,t)}{p(x,t)-1}}\\
&\qquad+2^{-p_+-1}\int_0^t\int_\Omega a(x,t)w^{2-\beta}|\nabla w|^{p(x,t)}\,dx\,dt\\
&\quad\le c+\frac{2^{-p_+}}{2}\int_0^T\int_\Omega a(x,t)w^{2-\beta}|\nabla w|^{p(x,t)}\,dx\,dt. \qquad (4.7)
\end{aligned}$$

Since $\operatorname{div}\vec{f}(x)\ge 0$,

$$\begin{aligned}
&\int_0^t\int_\Omega\sum_{i=1}^N f^i_{x_i}(x,t)\big(u^q-v^q\big)F_\lambda(u-v)\,dx\,dt\\
&\quad=\int_0^t\int_\Omega\operatorname{div}\vec{f}(x,t)\big(u^q-v^q\big)\frac{1}{\beta-1}(u-v)^{\beta-1}\,dx\,dt\\
&\qquad-\int_0^t\int_\Omega\operatorname{div}\vec{f}(x,t)\big(u^q-v^q\big)\frac{1}{\beta-1}\lambda^{\beta-1}\,dx\,dt\\
&\quad\ge-\int_0^t\int_{D_\lambda}\operatorname{div}\vec{f}(x,t)\big(u^q-v^q\big)\frac{1}{\beta-1}\lambda^{\beta-1}\,dx\,dt\\
&\quad\ge -c_1, \qquad (4.8)
\end{aligned}$$

where c_1 is independent of λ.

Moreover, let $t_0 = \inf\{\tau \in (0,t] : w > \lambda\}$. Then

$$\begin{aligned}\int_0^t \int_{D_\lambda} w_t F_\lambda(w)\,dx\,dt &= \int_{D_\lambda}\left(\int_0^{t_0} w_t F_\lambda(w)\,dt + \int_{t_0}^t w_t F_\lambda(w)\,dt\right)dx \\ &\geq \int_{D_\lambda}\int_\lambda^{w(x,t)} F_\lambda(s)\,ds\,dx \\ &\geq \int_{D_\lambda}(w-2\lambda)F_\lambda(2\lambda)\,dx \geq (\delta-2\lambda)F_\lambda(2\lambda)\mu(D_\lambda). \end{aligned} \tag{4.9}$$

Thus, we have

$$(\delta-2\lambda)\frac{1-2^{\beta-1}}{1-\beta} \leq c_2,$$

where c_1 is independent of λ. Letting $\lambda \to 0$, we get the contradiction. □

Proof of Theorem 1.4 By conditions (1.13) and (1.14), only if we choose $\phi(x,t) = a(x,t)$, we know conditions (3.5) (3.6) in Theorem 3.3 are true, the conclusion follows easily. □

5 Asymptotic behavior of weak solutions

In what follows, $p(x,t) = p(x)$, $p_+ = \max_{x\in\overline{\Omega}} p(x)$, $p_- = \min_{x\in\overline{\Omega}} p(x)$, $q(x) = \frac{p(x)}{p(x)-1}$.

Lemma 5.1 *Let $p, s \in C_+(\overline{\Omega})$ and $a(x)$ satisfy*

(w1) $a \in L^1_{\mathrm{loc}}(\Omega)$ *and* $a^{-\frac{1}{p(x)-1}} \in L^1_{\mathrm{loc}}(\Omega)$;

(w2) $a^{-s(x)} \in L^1(\Omega)$ *with* $s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty)$. *Then we have the following compact embedding*:

$$W^{1,p(x)}(a,\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$$

provided that $r \in C_+(\overline{\Omega})$ and $1 \leq r(x) < p_s^*(x)$ for all $x \in \Omega$. Here,

$$p_s(x) = \frac{p(x)s(x)}{1+s(x)},$$

and

$$p_s^*(x) = \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N-p(x)s(x)}, & \text{if } p_s(x) < N, \\ +\infty, & \text{if } p_s(x) \geq N. \end{cases}$$

Lemma 5.2 *Let $p \in C_+(\overline{\Omega})$. If* (w1) *and* (w2) *hold, then the estimate*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(a,\Omega)}$$

holds for every $u \in C_0^\infty(\Omega)$ with a positive constant C independent of u.

These two lemmas and the definitions about the weighted variable exponent Sobolev space $W^{1,p(x)}(a,\Omega)$ can be found in [32].

Theorem 5.3 *Suppose that $a(x,t) = a(x)$ satisfying* (1.6), (w1) *and* (w2), $p(x,t) = p(x) \geq p_- > 1$ *and $f_i(x,t) = f_i(x)$ satisfies* (4.1) *and* (4.2), $2 < \frac{Np_+}{N-p_+}$. *If there is $0 < \alpha < \frac{p_-}{p_+(p_+-p_-)}$ such that*

$$\int_\Omega a(x)|\nabla u|^{p(x)}\,dx \geq M(u_0) \geq 1, \qquad \int_\Omega a(x)|\nabla u|^{p(x)}\,dx \leq ct^\alpha, \quad t > 1, \tag{5.1}$$

$$\lim_{T\to\infty}\int_0^T \frac{1}{1+t^{\alpha\beta}}\,dx = \infty, \tag{5.2}$$

$$\left(\int_\Omega \left|\sum_{i=1}^N f^i(x)a(x)^{-\frac{1}{p(x)}}u^q\right|^{\frac{p(x)}{p(x)-1}}dx\right)^{\frac{1}{q^+}} < 1, \tag{5.3}$$

where $\beta^{-1} = \frac{p_-}{p_+(p_+-p_-)}$, then

$$\lim_{T\to\infty}\|u(x,T)\|_{L^2(\Omega)} = 0. \tag{5.4}$$

Proof Let $G(u) = \frac{1}{2}\int_\Omega |u|^2\,dx$. Then it is well known that G is a convex functional on $L^2(\Omega)$. For any $t \in (0,T)$ and $h > 0$, δ represents Gâteaux differential, i.e.,

$$\frac{\delta G(u)}{\delta u} = u.$$

By the convexity of G, we have

$$G\bigl(u(t+h)\bigr) - G\bigl(u(t)\bigr) \geq \int_\Omega \bigl[u(x,t+h) - u(x,t)\bigr]u(x,t)\,dx. \tag{5.5}$$

For any $t_1, t_2 \in [0,T]$, $t_1 < t_2$,

$$\begin{aligned}
&\int_{t_1}^{t_2} G\bigl(u(t+h)\bigr)\,dt - \int_{t_1}^{t_2} G\bigl(u(t)\bigr)\,dt \\
&\quad= \int_{t_1+h}^{t_2+h} G\bigl(u(t)\bigr)\,dt - \int_{t_1}^{t_2} G\bigl(u(t)\bigr)\,dt \\
&\quad= \int_{t_2}^{t_2+h} G\bigl(u(t+h)\bigr)\,dt - \int_{t_1}^{t_+h} G\bigl(u(t)\bigr)\,dt \\
&\quad\geq \int_{t_1}^{t_2}\int_\Omega \bigl[u(x,t+h) - u(x,t)\bigr]u(x,t)\,dx.
\end{aligned} \tag{5.6}$$

Dividing both sides of (5.6) by h, we let $h \to 0$. Then

$$G\bigl(u(t_2)\bigr) - G\bigl(u(t_1)\bigr) \geq \int_{t_1}^{t_2}\int_\Omega \frac{\partial u}{\partial t}u(x,t)\,dx\,dt.$$

In a similar way, we have

$$G\bigl(u(t)\bigr) - G\bigl(u(t-h)\bigr) \leq \int_\Omega \bigl[u(x,t) - u(x,t-h)\bigr]u(x,t)\,dx, \tag{5.7}$$

accordingly,

$$G\big(u(t_2)\big) - G\big(u(t_1)\big) \le \int_{t_1}^{t_2} \int_\Omega \frac{\partial u}{\partial t} u(x,t)\,dx\,dt.$$

Then

$$G\big(u(t_2)\big) - G\big(u(t_1)\big) = \int_{t_1}^{t_2} \int_\Omega \frac{\partial u}{\partial t} u(x,t)\,dx\,dt. \tag{5.8}$$

In particular, by the definition of weak solution, we have

$$G\big(u(t)\big) - G\big(u(0)\big) = -\int_0^t \int_\Omega a(x)|\nabla u|^{p(x)}\,dx\,dt + \int_0^t \int_\Omega \vec{f}(x)\cdot\nabla u^q\,dx\,dt. \tag{5.9}$$

By Theorem 4.1, the solution of equation (1.1) with initial (1.2) is unique, then we can regard it as the limit

$$u(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t),$$

where u_ε is the solution of the initial boundary value problem (2.1)–(2.3). Thus,

$$u_\varepsilon \in L^{p_+}\big(0,T;W_0^{1,p_+}(\Omega)\big), \qquad u_{\varepsilon t} \in L^{p'_+}\big(0,T;W^{-1,p'_+}(\Omega)\big) \tag{5.10}$$

by that $2 < \frac{Np_+}{N-p_+}$, we have

$$W_0^{1,p_+}(\Omega) \hookrightarrow (\text{compact})\ L^2(\Omega) \hookrightarrow W^{-1,p'_+}(\Omega). \tag{5.11}$$

Thus, $u(x,t)\in C(0,T;L^2(\Omega))$.

Let $G(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2\,dx$. Then $G(t)$ is continuous in $[0,T)$, and by (5.1)(5.3), we have

$$\begin{aligned} G'(t) &= -\int_\Omega a(x)|\nabla u|^{p(x)}\,dx - \sum_{i=1}^N \int_\Omega \big[f^i(x)u^q u_{x_i} + f^i_{x_i}(x,t)u^{q+1}\big]\,dx \\ &\le -\int_\Omega a(x)|\nabla u|^{p(x)}\,dx - \sum_{i=1}^N \int_\Omega f^i(x)u^q u_{x_i}\,dx \\ &\le -\int_\Omega a(x)|\nabla u|^{p(x)}\,dx \\ &\quad + \left(\int_\Omega \left|\sum_{i=1}^N f^i(x)a(x)^{-\frac{1}{p(x)}}u^q\right|^{\frac{p(x)}{p(x)-1}} dx\right)^{\frac{1}{q_1}} \left(\int_\Omega a(x)|\nabla u|^{p(x)}\,dx\right)^{\frac{1}{p_1}} \\ &\le 0. \end{aligned} \tag{5.12}$$

Here, $p_1 = p_+$ or p_- according to $\int_\Omega a(x)|\nabla u|^{p(x)}\,dx \ge 1$ or $\int_\Omega a(x)|\nabla u|^{p(x)}\,dx < 1$, q_1 has a similar meaning.

We choose $s(x) = 2$ in Lemma 5.1, $p_s(x) = \frac{2p(x)}{3}$. If $p_s(x) < N$ and $3N - 2p(x) < Np(x)$, then $2 < p_s^*(x) = \frac{2Np(x)}{3N-2p(x)}$. If $p_s(x)\ge N$, $p_s^* = \infty$, $2 < p_s^*$ is naturally. By Lemma 5.1,

$$W^{1,p(x)}(a,\Omega) \hookrightarrow\hookrightarrow L^2(\Omega).$$

Accordingly, by Lemma 5.2 and assumption (5.1), we are able to show that

$$\begin{aligned}\int_\Omega |u|^2\,dx &\le c\|u\|_{W^{1,p(x)}(a,\Omega)} \le c\|\nabla u\|_{L^{p(x)}(w,\Omega)} \\ &\le ct^{\frac{2\alpha(p_+-p_-)}{p_-}}\left|G'(t)\right|^{\frac{2}{p_+}}, \quad t\ge 1. \end{aligned} \tag{5.13}$$

By (5.13), we can extrapolate that

$$\int_\Omega \left|u(x,T)\right|^2 dx \le \frac{1}{(c_1\int_0^T(1+t^{\alpha\beta})^{-1}+c_2)^\delta}, \quad \delta = \frac{2}{p_+-2}, c_1>0, c_2>0.$$

This accomplishes the proof of the theorem. □

6 Conclusion

The initial boundary value problem of an electrorheological fluid equation with orientated convection term is considered. The diffusion coefficient $a(x,t)$, the variable exponent $p(x,t)$, and the oriented convection coefficient $b_i(x,t)$ are all dependent on time variable t. If, for any t, $a(x,t)=0$, $x\in\partial\Omega$, then the stability of weak solutions may be true without boundary value condition. This conclusion generalizes our previous works [28, 29]. The essential improvement lies in that only if $a(x,t)|_{x\in\partial\Omega}=0$ for any t, the uniqueness of weak solution is always true, no other conditions are required. Just by this important result, we can study the large time behavior of weak solutions without the boundary value condition. To the best knowledge, this is the first paper to study the large time behavior for an initial boundary value problem but independent of the boundary value condition.

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