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Direct and inverse problem for geometric perturbation of the Laplace operator in a strip

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Abstract

This paper is concerned with the inverse problem of determining geometric shape of a part γ of the boundary of a perturbed strip Ω from a pair of Cauchy data of a harmonic function u in Ω. This leads to the study of the direct problem. Using the variational method, we show that is well posed, and by the integral equation method we seek the solution in the form of combined double- and single-layer potential. For the identification of γ we prove a uniqueness result, that is, a pair of Cauchy data on the accessible part Γ_0 uniquely determines the missing part γ of the boundary, and we derive a system of nonlinear integral equations equivalent to our inverse problem. We present numerical examples for both the direct and inverse problems.

MSC: 65R32; 65N21; 35J05; 45A05

Keywords: Corrosion detection; Geometric inverse problem; Boundary integral equation

1 Introduction

We consider the perturbed strip $\Omega \subset \mathrm{R}^2$ as follows:

$$\Omega = \{(x,y) \in \mathrm{R}^2 : 0 < y < 1 - h(x)\}, \tag{1}$$

where $h : \mathrm{R} \to [0,1]$ is a continuous function which is a parametrization of a local perturbation of a strip $\Omega_0 = \{(x,y) \in \mathrm{R}^2 : 0 < y < 1\}$. We assume that there exist $a > 0$ and $0 < b < 1$ such that $h \in C^2[-a,a]$ and it satisfies

$$(1) \quad h(x) = 0 \quad \text{if } |x| \ge a; \qquad (2) \quad 0 \le h(x) \le b \quad \text{if } |x| \le a.$$

The boundary Γ of Ω is decomposed as $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\Gamma_1 = \Gamma_1^- \cup \gamma \cup \Gamma_1^+$ (see Fig. 1) where γ is the arc

$$\gamma = \{(x,y) \in \mathrm{R}^2 : y = 1 - h(x), |x| \le a\}.$$

For a given function $f \in H^{3/2}(\mathrm{R})$ consider the Dirichlet problem for the Laplace equation:

$$\Delta u = 0 \quad \text{in } \Omega \tag{2}$$

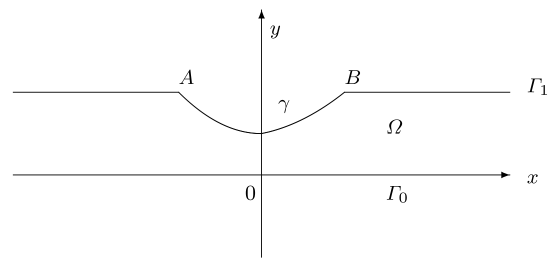


Figure 1 Domain Ω

subject to the boundary condition

$$u = f \quad \text{on } \Gamma_0, \qquad u = 0 \quad \text{on } \Gamma_1. \tag{3}$$

The inverse problem we are concerned with consists in recovering the shape of $\gamma : y = 1 - h(x)$, $-a \leq x \leq a$, from the Cauchy data $f := u|_{\Gamma_0}$ and $g := \frac{\partial u}{\partial y}|_{\Gamma_0}$.

This problem arises in electrostatic or thermal imaging methods, detecting a corrosion surface in nondestructive testing. This could be the case for an electrically conducting specimen, which is subject to wear by corrosion, causing material loss or cracks. In practice, it often happens that the unknown part of the boundary that has suffered corrosion is not accessible to direct inspection. The aim is to detect the presence of such defects by nondestructive methods from the knowledge of the imposed voltage $f := u|_{\Gamma_0}$ and the measured resulting current $g := \frac{\partial u}{\partial y}|_{\Gamma_0}$ on the accessible part Γ_0 of the boundary Γ. Various application related to the above model problem are described, for example, in [1, 11] (see also the references therein). Let us mention the articles [2, 5, 10, 15, 17] on the inverse boundary value problems for the Laplace equation in a bounded domain. Our problem presents additional difficulties related to the unbounded nature of the domain and, to our knowledge, it was not considered in the literature. In this work we generalize the potential method used in the precursor works of Kress and Rundel [9, 10].

The paper is organized as follows. In Sect. 2 we will be concerned with the direct problem, we proceed by showing the existence and uniqueness of the solution. Using Green's function, we obtain a representation of the solution in the form of potentials and we give a numerical test. Section 3 is devoted to the study of the inverse problem, a uniqueness theorem is obtained, we derive the two-by-two system of integral equations and describe the proposed iteration scheme. The paper concludes with some numerical examples.

2 The direct problem

2.1 Existence and uniqueness

The following notations are used throughout this text. For $s \geq 0$, we say that $f \in H^s(\mathbb{R})$ if the norm

$$\|f\|_{H^s(\mathbb{R})} := \left(\int_{-\infty}^{+\infty} \left(1 + |\xi|^2\right)^s \left|\widehat{f}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}}$$

is finite. For $s = 0$, we denote $\|f\|_{L^2(\mathbb{R})} := \|f\|_0$. Here the Fourier transform pair of f is defined by the formulae

$$\begin{aligned} \widehat{f}(\xi) \equiv \mathcal{F}(f)(\xi) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,dx, \\ f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi} \widehat{f}(\xi)\,d\xi. \end{aligned} \tag{4}$$

We will show that the direct problem (2)–(3) is well posed.

Theorem 1 *Suppose that* $f \in H^{3/2}(\mathbb{R})$. *Then*

1. *There exists a recovery function* $u_f \in H^2(\Omega)$ *such that* $u_f(x,0) = f(x)$.
2. *The problem* (2)–(3) *has a unique solution* $u \in H^1(\Omega)$.
3. *The solution* $u \in H^2_{\mathrm{loc}}(\Omega)$.

Proof 1. Using Fourier transform of function f, we define u_f by

$$u_f(x,y) = w_0(x,y)\varphi_0(y),$$

where

$$w_0(x,y) = \mathcal{F}^{-1}\bigl(e^{-y|\xi|}\hat{f}(\xi)\bigr)$$

and φ_0 is a truncation function of class $C^2[0,1]$ such that

$$\varphi_0(y) = \begin{cases} 1 & \text{if } 0 < y < b/2, \\ 0 & \text{if } b < y < 1. \end{cases} \tag{5}$$

It is easy to see that $u_f \in H^2(\Omega)$ and $u_f(x,0) = f(x)$.

2. We put $v = u - u_f$ and $F = -\Delta u_f \in L^2(\Omega)$. Then v solves the homogeneous Dirichlet problem:

$$\begin{cases} \triangle v = F & \text{in } \Omega; \\ v = 0 & \text{on } \Gamma. \end{cases} \tag{6}$$

The bilinear form $a(v,w) = \int_\Omega \nabla v \nabla w\,dx\,dy$, $v, w \in H^1_0(\Omega)$, is coercive since Ω is bounded in the y direction (see [3]). By the Lax–Milgram theorem, there exists a unique weak solution $v \in H^1_0(\Omega)$.

3. Now we show that $v \in H^2_{\mathrm{loc}}(\Omega)$. The corners A and B of Γ_1 have an angle less that π, then from the regularity theory it follows that $v \in H^2(\Omega_R)$, $\Omega_R = \{(x,y) \in \Omega; |x| \le R\}$, for any $R \ge a$ (see [6]).

This completes the proof. □

Theorem 2 *The Dirichlet-to-Neumann map* $f \mapsto g = \frac{\partial u}{\partial y}(x,0)$ *where* u *is the solution of the Dirichlet problem* (2)–(3) *is continuous from* $H^{3/2}(\mathbb{R})$ *to* $L^2(\mathbb{R})$.

Proof From the previous theorem, $g \in L^2_{\mathrm{loc}}(\mathrm{R})$. It suffices to prove that g is in $L^2(R_0,+\infty)$ for $R_0 \geq a+1$. Let $v \in H^1_0(\Omega)$ be the solution of (6). We put $w = \theta(x)v$ where $\theta \in C^2(\mathrm{R})$ is a truncation function such that $\theta(x) = 0$ for $x \leq a$ and $\theta(x) = 1$ for $x \geq R_0$. Then w satisfies the equation

$$\begin{cases} \triangle w = F_1 & \text{in } \Omega_a^+ =]a,+\infty[\times]0,1[; \\ w = 0 & \text{on } \Gamma_a^+ = \partial\Omega_a^+, \end{cases} \tag{7}$$

with $F_1 = \theta F + v\theta'' + 2\frac{\partial v}{\partial x}\theta' \in L^2(\Omega_a^+)$. We can assume in (7) that $a = 0$ (after a translation in the direction of x) and use sine-Fourier transform

$$\widehat{w}(\xi,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty w(x,y)\sin(\xi x)\,dx, \qquad \widehat{F}_1(\xi,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_1(x,y)\sin(\xi x)\,dx,$$

to obtain the second order differential equation in y:

$$\begin{cases} \widehat{w}_{yy} - \xi^2\widehat{w} = \widehat{F}_1(\xi,y), & y \in]0,1[, \\ \widehat{w}(\xi,0) = 0, \quad \widehat{w}(\xi,1) = 0, \end{cases} \tag{8}$$

with the solution

$$\widehat{w}(\xi,y) = \int_0^1 G(\xi,y,z)\widehat{F}_1(\xi,z)\,dz$$

and $G(\xi,y,z)$ being the Green's function

$$G(\xi,y,z) = \frac{1}{\xi\cosh\xi}\begin{cases} \sinh(\xi z)\sinh(\xi(1-y)), & z \leq y; \\ \sinh(\xi y)\sinh(\xi(1-z)), & z \geq y. \end{cases} \tag{9}$$

Then

$$\frac{\partial\widehat{w}}{\partial y}(\xi,0) = \frac{-1}{\cosh\xi}\int_0^1 \sinh\bigl(\xi(1-z)\bigr)\widehat{F}_1(\xi,z)\,dz,$$

which leads to

$$\forall\xi \geq 0, \quad \left|\frac{\partial\widehat{w}}{\partial y}(\xi,0)\right|^2 \leq \int_0^1 \bigl|\widehat{F}_1(\xi,z)\bigr|^2\,dz$$

and then

$$\left\|\frac{\partial w}{\partial y}(\cdot,0)\right\|_{L^2(0,+\infty)} \leq \|F_1\|_{L^2((0,+\infty)\times]0,1[)}.$$

On the other hand, we have

$$\begin{aligned} \|F_1\|_{L^2((R_0,+\infty)\times]0,1[)} &\leq 2\|\theta\|_{C^2}\bigl(\|F\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega)}\bigr) \\ &\leq C_1\|F\|_{L^2(\Omega)} \leq C_2\|f\|_{H^{3/2}(\mathrm{R})}, \end{aligned}$$

since $w = v$, for $x \geq R_0$, and $v = u - u_f$, so we get

$$g(x) = \frac{\partial w}{\partial y}(x,0) + \frac{\partial u_f}{\partial y}(x,0), \quad x \geq R_0,$$

and deduce that

$$\|g\|_{L^2} \leq C_2\|f\|_{3/2} + \|f'\|_0 \leq C_3\|f\|_{3/2},$$

ending the proof. □

Theorem 3 (Uniqueness) *Suppose that* $u \in H^1_{\mathrm{loc}}(\Omega)$ *satisfies*

$$\begin{cases} \triangle u = 0 & \text{in } \Omega, \\ \gamma_0(u) = 0 & \text{on } \Gamma, \\ \int_0^1 |u(x,y)|^2\,dy \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{10}$$

Then $u = 0$.

Proof 1. In the semi-strip $\Omega_a^+ =]a,+\infty[\times]0,1[$, u has the representation

$$u(x,y) = \sum_{n=1}^{\infty} \sin(n\pi y)\big(A_n e^{-n\pi(x-a)} + B_n e^{n\pi(x-a)}\big),$$

and then, for $R > a$,

$$\int_0^1 \big|u(R,y)\big|^2\,dy = \frac{2}{\pi}\sum_{n=1}^{\infty}\big(A_n e^{-n\pi(R-a)} + B_n e^{n\pi(R-a)}\big)^2.$$

If $R \to +\infty$, we deduce that for all $n \in \mathbb{N}^*$,

$$\lim_{R\to+\infty}\big[A_n e^{-n\pi(R-a)} + B_n e^{n\pi(R-a)}\big] = 0,$$

thus $\forall n \geq 1$, $B_n = 0$ and $A_n = \frac{2}{\pi}\int_0^1 u(a,y)\sin(n\pi y)\,dy$.

Then we deduce the following estimate, for all $|x| \geq R$ and $y \in [0,1]$:

$$\big|u(x,y)\big| + \left|\frac{\partial u}{\partial x}(x,y)\right| \leq C(a)\|u\|_{H^1(\Omega_a)}e^{-\pi R}. \tag{11}$$

2. Suppose (10) holds. Applying the first Green's identity in the set $\Omega_R = \{(x,y) \in \Omega; |x| < R\}$, for $R \geq a$, we have

$$\int_{\Omega_R} |\nabla u|^2\,dx\,dy = \int_0^1 \left(u\frac{\partial u}{\partial x}\bigg|_{x=R} + u\frac{\partial u}{\partial x}\bigg|_{x=-R}\right)dy.$$

Inequality (11) implies that

$$\int_{\Omega_R} |\nabla u|^2\,dx\,dy = O\big(e^{-2\pi R}\big).$$

Letting $R \to +\infty$, we deduce that $\nabla u = 0$ in Ω, and thus u is constant in Ω. Since $\gamma_0(u) = 0$, it follows that $u = 0$ in Ω. □

2.2 Unperturbed problem

By an unperturbed problem we mean the problem on the strip $\Omega_0 = \mathrm{R} \times]0,1[$:

$$(P_0) \quad \begin{cases} \triangle u_0 = 0 & \text{in } \Omega_0, \\ u_0(x,0) = f(x), \quad u_0(x,1) = 0, & \forall x \in \mathrm{R}. \end{cases}$$

Using the partial Fourier transform with respect to x, we obtain the solution

$$u_0(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(\xi,y)\hat{f}(\xi)e^{ix\xi}\, d\xi, \tag{12}$$

where

$$a(\xi,y) = \frac{\sinh(\xi(1-y))}{\sinh\xi}. \tag{13}$$

Proposition 1 *Assume that $f \in H^{3/2}(\mathrm{R})$. Then $u_0 \in H^2(\Omega_0)$ and it satisfies*

$$\text{(i)} \quad \|u_0\|_{H^2(\Omega_0)} \le C\|f\|_{H^{3/2}(\mathrm{R})}, \qquad \text{(ii)} \quad \left\| \frac{\partial u_0}{\partial y}(x,0) \right\|_{L^2(\mathrm{R})} \le C\|f\|_{L^2(\mathrm{R})}.$$

Proof (i) We use the asymptotic behavior of $a(\xi,y)$, that is,

$$a(\xi,y) \simeq 1-y \quad \text{if } \xi \to 0 \quad \text{and} \quad a(\xi,y) \simeq e^{-y|\xi|} \quad \text{if } |\xi| \to +\infty.$$

Indeed, for example, we have

$$\int_{\Omega_0} \left| \frac{\partial^2 u_0}{\partial x^2} \right|^2 dx\,dy = \int_{\mathrm{R}} \xi^4 \left|\hat{f}(\xi)\right|^2 \left(\int_0^1 \left|a(\xi,y)\right|^2 dy \right) d\xi$$

and

$$\int_0^1 \left|a(\xi,y)\right|^2 dy \le C \int_0^1 e^{-2y|\xi|}\, dy \le \frac{1}{2|\xi|}.$$

Then

$$\int_{\Omega_0} \left| \frac{\partial^2 u_0}{\partial x^2} \right|^2 dx\,dy \le \int_{\mathrm{R}} |\xi|^3 \left|\hat{f}(\xi)\right|^2 d\xi \le \|f\|_{H^{3/2}(\mathrm{R})}.$$

(ii) We have $\widehat{\frac{\partial u_0}{\partial y}}(\xi,0) = -(\xi \coth\xi)\widehat{f}(\xi)$. Since the function $\xi \mapsto \xi\coth\xi$ is bounded on R,

$$\left\| \frac{\partial u_0}{\partial y}(x,0) \right\|_{L^2(\mathrm{R})} \le C\|f\|_0.$$

□

2.3 Green's function of the strip Ω_0

The Green's function of the Dirichlet problem for the Laplace equation in the strip Ω_0 is defined by (see [13]) by

$$G\big((x,y),(\xi,\eta)\big)=\frac{1}{4\pi}\log\left[\frac{\cosh\pi(x-\xi)-\cos\pi(y+\eta)}{\cosh\pi(x-\xi)-\cos\pi(y-\eta)}\right],$$

which satisfies in the sense of the distribution

$$\begin{cases}\Delta G=\delta(x-\xi,y-\eta) & \text{in } \Omega_0,\\ G|_{y=0}=0,\\ G|_{y=1}=0.\end{cases}\tag{14}$$

For a fixed point source $(\xi,\eta)\in\Omega$, G exhibits the following asymptotic behavior:

$$G\big((x,y),(\xi,\eta)\big)=-\frac{1}{2\pi}\log r+\frac{1}{4\pi}\log\left(\frac{4\sin\pi\eta}{\pi}\right)+O(r)\tag{15}$$

as $r=\sqrt{(x-\xi)^2+(y-\eta)^2}\to 0$. Also

$$G\big((x,y),(\xi,\eta)\big)=O\left(\frac{1}{|x|}\right)\quad \text{as } |x|\to+\infty \ \big(\text{uniformly in } y\in[0,1]\big)\tag{16}$$

and

$$\frac{\partial G}{\partial x}\big((x,y),(\xi,\eta)\big)=O\big(e^{-\pi|x|}\big)\quad \text{as } |x|\to+\infty \ \big(\text{uniformly in } y\in[0,1]\big).\tag{17}$$

2.4 Integral equation method

Let u be the solution of problem (2)–(3), and u_0 the solution of (P_0). We put $v_0=u-u_0$, and then $v_0\in H^1(\Omega)\cap H^2_{loc}(\Omega)$ is a solution of the following problem:

$$\begin{cases}\triangle v_0=0 & \text{in } \Omega,\\ v_0=0 & \text{on } \Gamma_0,\\ v_0=-u_0 & \text{on } \gamma,\\ v_0=0 & \text{on } \Gamma_1\setminus\gamma.\end{cases}\tag{18}$$

We introduce the boundary integral equation formulation for (18) as follows. For this we apply Green's second theorem to the harmonic functions v_0 and $G(\cdot,(\xi,\eta))$ in the domain $\Omega_{R,\epsilon}=\Omega_R\setminus B((\xi,\eta),\epsilon)$, $\Omega_R=\{(x,y)\in\Omega:|x|<R\}$ to obtain

$$\int_{\partial\Omega_{R,\epsilon}}\left(v_0\frac{\partial G}{\partial n}-G\frac{\partial v_0}{\partial n}\right)ds=0.$$

The function $v_0\in H^1(\Omega)$ satisfies the decay property in (10). Then from the proof of Theorem 3 (part 1, estimate (11)), we deduce that

$$\lim_{R\to+\infty}\int_0^1\left(\big|v_0(\pm R,y)\big|^2+\left|\frac{\partial v_0}{\partial x}(\pm R,y)\right|^2\right)dy=0,$$

and (16)–(17) lead to the limit

$$\lim_{R\to+\infty}\int_0^1\left(v_0\frac{\partial G}{\partial x}-G\frac{\partial v_0}{\partial x}\right)\bigg|_{x=\pm R}dy=0. \tag{19}$$

Then, as $R\to+\infty$ and $\epsilon\to 0$, we obtain the integral representation formula

$$v_0(\xi,\eta)=\int_\gamma\left(v_0\frac{\partial G}{\partial n}-G\frac{\partial v_0}{\partial n}\right)ds,\quad (\xi,\eta)\in\Omega. \tag{20}$$

We put $\psi=\frac{\partial v_0}{\partial n}|_\gamma$, which is unknown, and $q=-v_0|_\gamma=u_0|_\gamma$, which is known through (12). When (ξ,η) tends to a point $M=(x,1-h(x))$ of γ, and utilizing the properties of single and double layer potentials (see [9]), we obtain the integral equation of the first kind

$$S\psi=\frac{1}{2}q+Dq\quad\text{on }\gamma, \tag{21}$$

where S and D are respectively the single- and double-layer potentials defined for $M=(x,y)\in\gamma$ by

$$S\psi(M)=\int_\gamma G(M,M')\psi(M')\,ds(M'),\qquad Dq(M)=\int_\gamma\frac{\partial G}{\partial n}(M,M')q(M')\,ds(M').$$

The solvability of (21) follows from the potential theory developed in the framework of Sobolev spaces $\tilde{H}^s(\gamma)$, $s>1/2$ (see [9, 10, 12, 14]). Once equation (21) is solved, we have the solution

$$v_0(M)=\int_\gamma\left(q(M')\frac{\partial G}{\partial n}(M,M')-G(M,M')\psi(M')\right)ds(M'),\quad M\in\Omega. \tag{22}$$

For the numerical solution of the integral equation (21), a parametrization of γ is required.

2.5 Parametrization of the integral equation

In order to use the results of integral equation on a contour, we consider the extension $\tilde{\gamma}$ of γ defined by $\tilde{\gamma}=\gamma\cup\gamma^+$, where $\gamma^+=\{(x,1+h(x)),-a<x<a\}$. Extending the data q on γ^+ by symmetry $q(x,1+h(x))=-q(x,1-h(x))$ and using the parametrization of $\tilde{\gamma}$, $z(t)=(x(t),y(t))$, $0\le t\le 2\pi$, we transform (21) as follows:

$$\frac{1}{2\pi}\int_0^{2\pi}L(t,s)\varphi(s)\,ds=\frac{1}{2}q(t)+\int_0^{2\pi}M(t,s)q(s)\,ds,\quad 0\le t\le 2\pi, \tag{23}$$

where

$$\varphi(t)=\psi\bigl(z(t)\bigr)\bigl|z'(t)\bigr|,\qquad q(t)=q\bigl(z(t)\bigr), \tag{24}$$

and

$$L(t,s)=\log\left[\frac{\cosh(x(t)-x(s))-\cos(y(t)-y(s))}{\cosh(x(t)-x(s))-\cos(y(t)+y(s))}\right], \tag{25}$$

$$M(t,s)=\frac{\sinh(x(t)-x(s))y'(s)+\sin(y(t)+y(s))x'(s)}{\cosh(x(t)-x(s))-\cos(y(t)+y(s))}+\frac{-\sinh(x(t)-x(s))y'(s)+\sin(y(t)-y(s))x'(s)}{\cosh(x(t)-x(s))-\cos(y(t)-y(s))}. \tag{26}$$

For the discretization of the integral operators, we note that kernel $L(t,s)$ can be decomposed in the form

$$L(t,s)=\log\left(4\sin^2\left(\frac{t-s}{2}\right)\right)+k(t,s), \tag{27}$$

where k and M are smooth with diagonal values

$$k(t,t)=\log\left(\frac{16\sin^2 y(t)}{|z'(t)|^2}\right) \tag{28}$$

and

$$M(t,t)=\frac{-y''(t)}{|z'(t)|^2}+\pi\cot\bigl(\pi y(t)\bigr). \tag{29}$$

Using the decomposition (27), we write equation (23) in operational form

$$(A+K)\varphi=\frac{1}{2}q+Bq, \tag{30}$$

where

$$A\varphi(t)=\frac{1}{2\pi}\int_0^{2\pi}\left[\log\left(4\sin^2\left(\frac{t-s}{2}\right)\right)-2\right]\varphi(s)\,ds,\quad 0\le t\le 2\pi \tag{31}$$

and

$$K\varphi(t)=\frac{1}{2\pi}\int_0^{2\pi}\bigl[k(t,s)+2\bigr]\varphi(s)\,ds,\qquad Bq(t)=\int_0^{2\pi}M(t,s)q(s)\,ds,\quad 0\le t\le 2\pi. \tag{32}$$

The properties of the operators A, K and B are given in the following lemma [9].

Lemma 1 *For $r\ge 0$, we have:*

- *The operator $A: H^r[0,2\pi]\to H^{r+1}[0,2\pi]$ is bounded and bijective.*
- *The operator K is compact from $H^r[0,2\pi]$ to $H^{r+1}[0,2\pi]$.*
- *B is bounded in $H^{r+1}[0,2\pi]$.*

The potentials S and D defined in (22) have the following properties (see [9]).

Lemma 2 *The single-layer potential defines a bounded linear operator from $\tilde{H}^{-1/2}(\gamma)$ into $H^1_{\mathrm{loc}}(\Omega)$. The double-layer potential defines a bounded linear operator from $\tilde{H}^{1/2}(\gamma)$ into $H^1_{\mathrm{loc}}(\Omega)$.*

Theorem 4 *The integral equation (30) has a unique solution, i.e., $N(A+K)=\{0\}$.*

Proof Assume that $(A + K)\varphi = 0$. We associate to φ the potential $v_0(M)$ defined by the integral (22) (with $q = 0$). Using Lemma 2 and the asymptotic behavior of $G(M,M')$ when $|x| \to +\infty$ (see (16)–(17)), we can prove that v_0 satisfies the uniqueness Theorem 3. Then $v_0 = 0$ in Ω, it follows that $\psi = \frac{\partial v_0}{\partial n}|_\gamma = 0$. Therefore $\varphi(t) = \psi(z(t))|z'(t)| = 0$. □

2.6 Nyström's method

In this section we use Nyström's method for the numerical approximation of the integral equation (30) of the first kind with weakly singular kernels.

Following Kress [9], we construct numerical quadratures for the improper integral

$$(Q\varphi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \log\left[4\sin^2\left(\frac{t-s}{2}\right)\right]\varphi(s)\,ds \tag{33}$$

by replacing the continuous periodic function φ by its trigonometric interpolation polynomial described in [9, Sect. 11.3]. Using the Lagrange basis (L_j), we obtain

$$(Q_n\varphi)(t) \simeq \sum_{j=0}^{2n-1} R_j^n(t)\varphi(t_j), \qquad t_j = \frac{j\pi}{n}, \tag{34}$$

with the quadrature weights

$$R_j^n(t) = \frac{1}{2\pi}\int_0^{2\pi} \log\left[4\sin^2\left(\frac{t-s}{2}\right)\right]L_j(s)\,ds, \qquad j = 0,\dots,2n-1. \tag{35}$$

More precisely, we have $R_j^n(t_k) = R_{|j-k|}^n$ such that

$$R_j^n = -\frac{1}{n}\left\{\sum_{m=1}^{n-1}\frac{1}{m}\cos\left(\frac{mj}{\pi}\right) + \frac{(-1)^j}{2n}, j = 0,\dots,2n-1\right\}. \tag{36}$$

Thus, we have the algebraic system

$$\sum_{j=0}^{2n-1}\left(R_{|i-j|}^n + \frac{1}{n}k(t_i,t_j)\right)\varphi(t_j) = \frac{1}{2}g(t_i) + \frac{1}{n}\sum_{j=0}^{2n-1} M(t_i,t_j)g(t_j), \qquad i = 0,\dots,2n-1. \tag{37}$$

Remark 1 In the numerical computation, we approach the Fourier integral (12) by the trigonometric polynomial

$$u_{0n}(x,y) = \pi\sum_{j=-n}^{n} a(\xi_j,y)\hat{f}(\xi_j)e^{ix\xi_j}, \qquad \xi_j = j\pi. \tag{38}$$

In particular, if $f(x) = q_a(x - x_0)$, where q_a is an even function (wavelet), then $\hat{f}(\xi) = e^{i\xi x_0}\hat{q}_a(\xi)$, and we have

$$u_{0n}(x,y) = \pi\sum_{j=1}^{n+1} H_j(y)\cos\bigl(\pi j(x-x_0)\bigr), \tag{39}$$

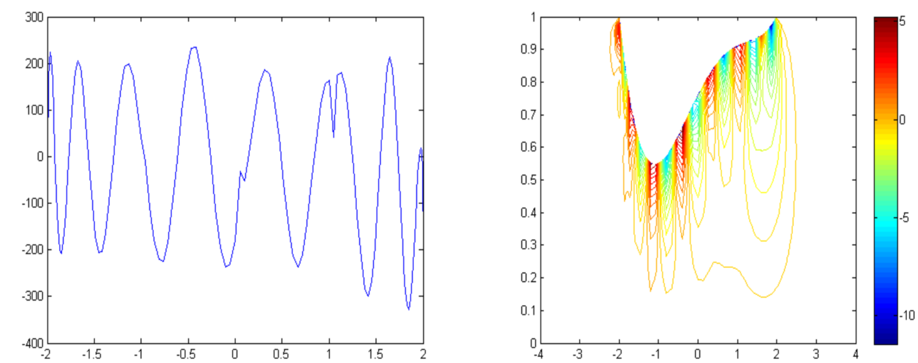


Figure 2 (left) Solution ψ of the integral equation (30); (right) Solution v_0 given by (22) in the domain Ω_R, $R = 4$

where

$$H_j(y) = a(\xi_j, y)\hat{q}_a(\xi_j). \tag{40}$$

2.7 Numerical implementation and examples

To illustrate our method, we show a numerical test with the boundary condition $f(t)$ and the parametrization of the arc $h(t)$ such that

$$f(t) = a_1 \exp\bigl(-c_1(t-t_1)^2\bigr) + a_2 \exp\bigl(-c_2(t-t_2)^2\bigr),$$

with $a_1 = 3$, $c_1 = 2$, $a_2 = 1$, $c_2 = 1$, $t_1 = -1$, $t_2 = 1$, and

$$h(t) = H\bigl(a^2 - t^2\bigr) \times \bigl[(t-t_0)^2 + b\bigr],$$

with $a = 2$, $b = 1$, $H = 0.03$, $t_0 = 1$.

In the discretization we use the parameters $a = 2$, $n = 120$, which is the number of points in $[0, \pi]$, and $x_j = a\cos(\frac{j\pi}{n})$, $j = 1, \dots, n$, the points of $[-a, a]$.

In the approximation of the Fourier integral (38), we use the interval $\xi \in [0, \xi_{\max}]$ with $\xi_{\max} = M\pi$, $M = n/2$ and the points $\xi_j = \frac{j\pi}{2}$, $j = 1, \dots, n$. Figures 2 and 3 show the results of the simulation.

3 The inverse problem

It is shown (in Sect. 2) that, for $f \in H^{3/2}(\mathrm{R})$, there exists a unique solution $u \in H^1(\Omega) \cap H^2_{loc}(\Omega)$ of (2)–(3). Recall that our inverse problem can be formulated as follows: given $f := u|_{\Gamma_0}$ and $g := \frac{\partial u}{\partial y}|_{\Gamma_0}$, $g \in L^2(\mathrm{R})$, determine $h(x)$ the parametrization of γ, i.e.,

$$\gamma : y = 1 - h(x), \quad -a \le x \le a.$$

We proceed with giving a uniqueness theorem.

3.1 Uniqueness for the inverse problem

Theorem 5 *Let Ω_1 and Ω_2 be two perturbed strips with two boundary arcs γ_1 and γ_2 of class C^2, respectively. Denote by u_1 and u_2 the solutions to the problem (2)–(3) for the*

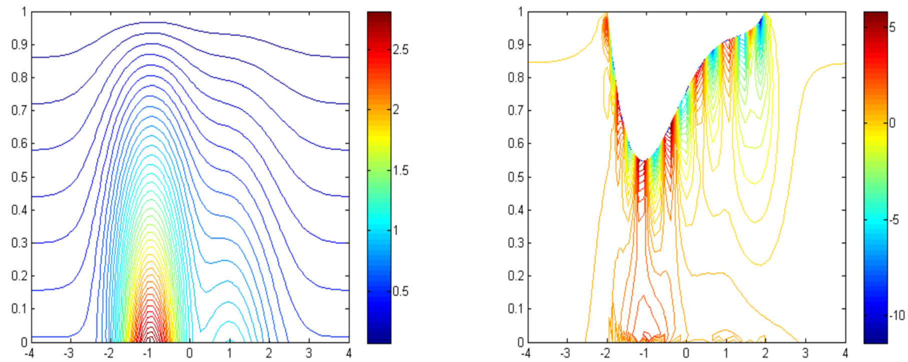
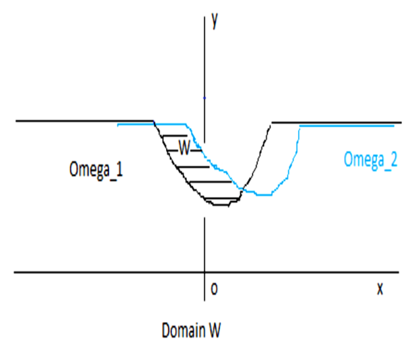


Figure 3 (left) Solution u_0 of the unperturbed problem; (right) Solution $u = v_0 + u_0$ of the direct problem (2)–(3)

Figure 4 Domain W



domains Ω_1 and Ω_2, respectively, and assume that

$$u_1 = u_2 \quad \text{and} \quad \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} \quad \text{on } \Gamma_0. \tag{41}$$

Then $\gamma_1 = \gamma_2$.

Proof (1) Suppose that $\Omega_1 \neq \Omega_2$ and put $u = u_1 - u_2$, then $\Delta u = 0$ in $\Omega_{12} = \Omega_1 \cap \Omega_2$ and $u|_{y=0} = \frac{\partial u}{\partial y}|_{y=0} = 0$. Holmgren's uniqueness theorem (see [16]) implies that $u = 0$ in a neighbourhood of the axis $y = 0$, and by analyticity $u_1 = u_2$ in Ω_{12}.

(2) Without loss of generality, we may assume that the open set $W = \Omega_2 \setminus \bar{\Omega}_1$ is connected and not empty (see Fig. 4). Then from the boundary conditions we can deduce that $u_2 = 0$ on the boundary of W. Now by the maximum–minimum principle for harmonic functions, we can conclude that $u_2 = 0$ in W, and consequently, by analyticity it follows that $u_2 = 0$ in Ω_2. However, this contradicts the fact that f is not identically zero, and the proof is complete. □

3.2 Nonlinear integral equations

Our approach for solving the inverse problem is based on a system of nonlinear and ill-posed integral equations. We shall use identity (22) to derive two nonlinear integral equations for the unknowns (ψ, h), hence we obtain

$$\mathcal{H}'q - \mathcal{H}\psi = g, \tag{42}$$

where

$$\mathcal{H}\psi(x) = \int_{-a}^{a} \psi(x')H(x,x',1-h(x'))\omega(x')\,dx', \quad \omega(x') = \sqrt{1+h'^2(x')}, \tag{43}$$

with

$$H(x,x',y') = \frac{\partial G}{\partial y}\big((x,y),(x',y')\big)\Big|_{y=0} \tag{44}$$

and

$$\mathcal{H}'q(x) = \int_{-a}^{a} q(x')H'(x,x',h(x'))\,dx', \tag{45}$$

with

$$H'(x,x',h(x')) = \frac{\partial H}{\partial x'}(x,x',1-h(x'))h'(x') + \frac{\partial H}{\partial y'}(x,x',1-h(x')). \tag{46}$$

Then we obtain the system:

$$S(h)\psi = \frac{1}{2}q + D(h)q, \qquad q(x) = u_0(x,1-h(x)), \tag{47}$$

$$\mathcal{H}'(h)q - \mathcal{H}(h)\psi = g. \tag{48}$$

The kernels of the integral operators are given in Appendix 2.

System (47)–(48) is linear in ψ and nonlinear in h. We remark that H and H' are C^∞ in x and decay faster than $e^{-\pi|x|}$ (we assume that $h \in C^1[-a,a]$). Hence $g \in C^\infty(\mathrm{R})$ and $|g(x)| = O(e^{-\pi|x|})$.

3.3 The iterative procedure

We suggested the following iterative method (Newton-type method) for approximately solving system (47)–(48). It involves partial linearization of the system with respect to the variable h, the boundary parametrization. Given an approximation h_0, we first solve the linear equation

$$S(h_0)\psi = \frac{1}{2}q_0 + D(h_0)q_0, \qquad q_0(x) = u_0(x,1-h_0(x)) \tag{49}$$

for ψ. Then, keeping ψ fixed, we replace (48) by the linearized equation

$$D\mathcal{H}'(h_0;wq_0) - D\mathcal{H}(h_0,w\psi) = \mathcal{H}(h_0)\psi - \mathcal{H}'(h_0)q_0 + g, \tag{50}$$

which we have to solve for w in order to improve an approximate boundary given by the parametrization h into the new approximation given by $h = h_0 + w$. The method consists in iterating this procedure. For a theoretical foundation of such a regularized Newton-type method for nonlinear ill-posed problems, in general, we refer to [4].

The Fréchet derivatives $D\mathcal{H}(h;k)$ and $D\mathcal{H}'(h;k)$ can be found by differentiating their kernels with respect to h, their representations are given in Appendix 2.

Equation (47) is solved by Nyström's method. Since the integral operators $D\mathcal{H}'$ and $D\mathcal{H}$ have kernels in $L^2(\mathrm{R} \times]-a,a[)$, they are compact from $L^2(]-a,a[)$ to $L^2(\mathrm{R})$, and therefore the integral equation (50) is ill-posed and requires a regularization. We use a regularization based on the least squares method [8]. All the computational codes are written in MATLAB.11 and in particular the MATLAB codes developed by Hansen [7] for solving the discrete ill-posed equation (50) has been adopted in our computations.

Our algorithm consists in the following steps:

Step 1: Given exact h and f, we compute Cauchy data g by solving integral equation (23) and using equation (42).

Step 2: Perturb the data $g_\delta = g + \delta\varepsilon(t)$, where $\varepsilon(t)$ is a Gaussian noise.

Step 3: Initialize $h = h_0$ and fix the parameters m, n of the discrete problem.

Step 4: For *it* = 1 to *itermax* do

- compute $q_0(x) = u_0(x, 1 - h(x))$ and solve equation (49) by Nyström's method
- solve the linearized equation (50) for w using a least squares algorithm.
- put $h = h_0 + w$

end do

3.4 Numerical examples and discussion

In this final section we present some numerical results to illustrate the accuracy and effectiveness of the reconstruction method as described in the previous section. We choose three profiles of γ to test our recovery algorithm.

Example 1 $h_1(x) = 0.2(1 - \frac{x^2}{4})$, $-2 \le x \le 2$.

Example 2 $h_2(x) = 0.03(4 - x^2)(1 + x^2)$, $-2 \le x \le 2$.

Example 3 $h_3(x) = 0.01(4 - x^2)(2 - 2x + x^2)$, $-2 \le x \le 2$.

The initial guess was taken to be $h_0(x) = 0.1(1 - \frac{x^2}{4})$ for all examples.

The synthetic Cauchy data (f,g) were obtained by solving the Dirichlet problem in Ω (see Fig. 5), with boundary condition $u = f$ with

$$f(t) = 3e^{-2t^2}.$$

In the examples we approximate function h for the unknown boundary curve $\gamma(t) = (a\cos t, 1 - h(t))$, $t \in [0,\pi]$, by a trigonometric polynomial

$$h(t) \approx \sin t\left(a_0 + \sum_{k=1}^{m} a_k \cos kt + b_k \sin kt\right).$$

In all the tests we used $n = 50$ grid points $y_j = a\cos(\frac{j\pi}{n})$, $j = 0,\ldots,n$, for the arc $\gamma(y) = (y, 1 - h(y))$ ($n = 150$ for the simulated data) and $x_i = -2a + i\frac{2a}{n}$, $i = 0,\ldots,2n$, points of collocation. Linear equation (50) is reduced to a $((2n + 1) \times (2m + 1))$ system. Then the problem becomes determining $(2m + 1)$ coefficients (a_0, a_k, b_k) of the approximation. In view of the ill-posedness, this approximating linear system is solved via the least squares algorithm *lsqr_b.m* from Hansen's package "Regularization tools" [7].

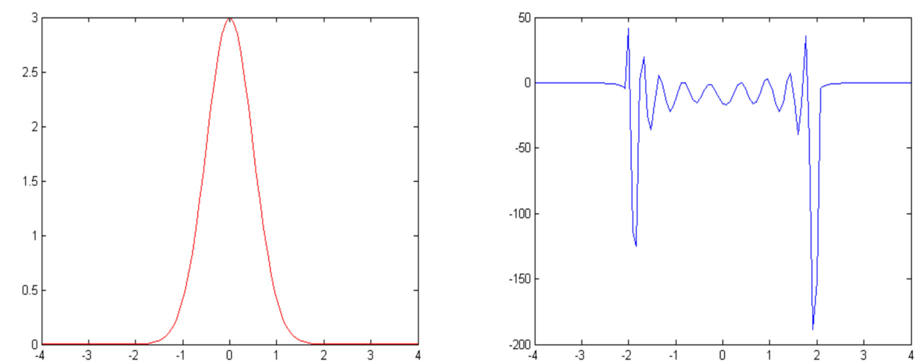


Figure 5 Cauchy data calculated by the direct algorithm, function f (left), function g for Example 1 (right)

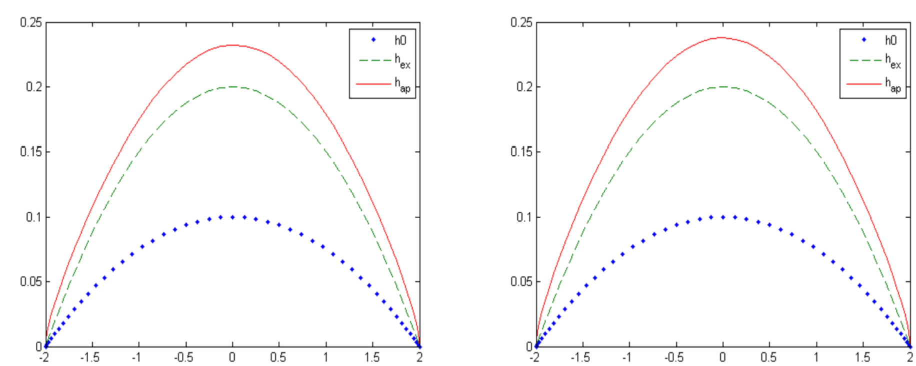


Figure 6 Reconstruction of h_1 after one iteration for exact data (left), and noisy data with $\delta = 0.05$ (right)

To obtain data with noise, we add random noise of given level δ (relative to l_2-discreet norm $\|g\|_2$) to the simulated data g as

$$g_\delta(t) = g(t) + \delta\|g\|_2\sigma(t),$$

where $\sigma(t)$ represents random numbers uniformly distributed on the interval $(-1; 1)$.

In Figs. 6–9, we present the numerical results for Examples 1, 2 and 3, respectively, involving both exact and noisy data. The reconstructions are obtained after one iteration for all examples. The numerical experiments show rather satisfying reconstructions under the restrictions:

- $0 \leq h(x) \leq 0.2$,
- with adequate initial guess h_0,
- $m = p$ the degree of the polynomial which interpolates h ($p = 2$ for h_1 and $p = 4$ for h_2 and h_3).
- $0 \leq \delta \leq 0.1$.

We remark that with arbitrary initial guess (in particular, for $h_0 = 0$) the algorithm fails to recover the boundary. In general, the quality of the reconstruction is affected by the shape of γ (see Figs. 8–9). As expected, the first iteration is stable with respect to the addition of random noise.

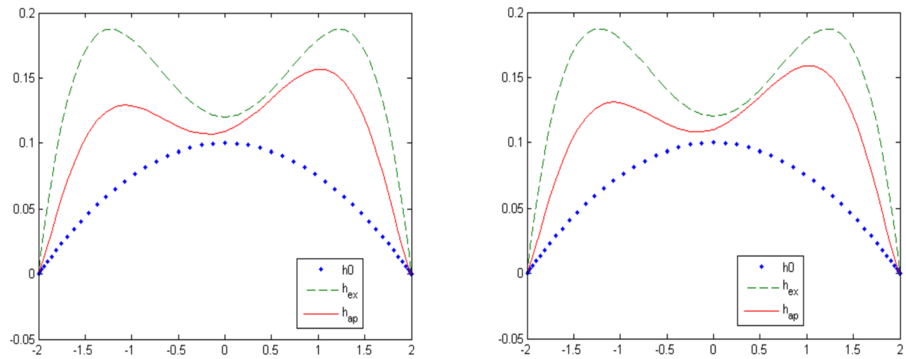


Figure 7 Reconstruction of h_2 after one iteration for exact data (left), and noisy data with $\delta = 0.01$ (right)

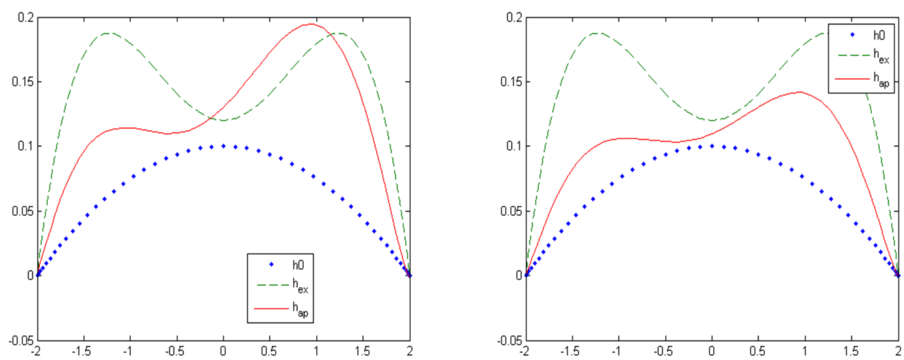


Figure 8 Reconstruction of h_2 for $\delta = 0.05$ (left), for $\delta = 0.1$ (right)

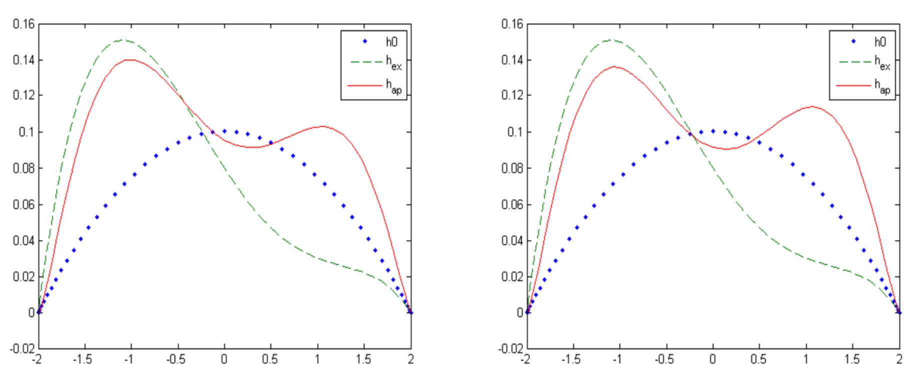


Figure 9 Reconstruction of h_3 after one iteration for exact data (left), and noisy data with $\delta = 0.01$ (right)

4 Conclusion

In this paper, direct and inverse boundary value problems associated with the two-dimensional harmonic equation have been investigated. We obtained the following results:

– The direct problem was studied systematically by employing the boundary integral method.

– The inverse problem was reduced to a nonlinear system which was approximated by a Gauss–Newton method (partially).
– The unknown arc of the boundary was approximated by a trigonometric polynomial. The resulting ill-conditioned system of linear algebraic equations has been regularized by using a least squares method.
– The numerical results obtained showed satisfying reconstructions for exact and noisy data with suitable initial guess.

Further research is required in order to improve the performance of the algorithm for the reconstruction of the boundary. It is necessary to introduce additional regularization treatments such as the choice of the parametrization of h, initial guess h_0 and a stopping rule for terminating the iterations.

Appendix 1: Derivative of nonlinear integral operator

In the following we state a useful formula. Let $f(x,y,z)$ be a real function defined and of class $C^1(D)$ in an open set D of R^3. To a function $g \in C^1([a,b])$ we associated a function F defined by an integral

$$F(x) = \int_a^b f\bigl(x,y,g(y)\bigr)\,dy, \quad x \in [c,d].$$

We consider the nonlinear operator

$$\mathcal{F} : V \to W$$

defined by the mapping $g \to F$ between the Banach spaces $V = C_0^1[a,b]$ and $W = C[c,d]$. We can prove the theorem.

Theorem 6 *$\mathcal{F}$ is Fréchet differentiable. The derivative, in the direction w, $Aw = D\mathcal{F}(g;w)$ is given by*

$$(Aw)(x) = \int_a^b \frac{\partial f}{\partial z}\bigl(x,y,g(y)\bigr)w(y)\,dy, \quad x \in [c,d]. \tag{51}$$

Now we consider the case of the function F_1 defined by

$$F_1(x) = \int_a^b f_1\bigl(x,y,g(y)\bigr)g'(y)\,dy, \quad x \in [c,d],$$

then $A_1 w = D\mathcal{F}_1(g;w)$ is given by

$$(A_1 w)(x) = \int_a^b \biggl[\frac{\partial f_1}{\partial z}\bigl(x,y,g(y)\bigr)\bigl(1 - g'(y)\bigr) - \frac{\partial f_1}{\partial y}\bigl(x,y,g(y)\bigr)\biggr]w(y)\,dy, \quad x \in [c,d]. \tag{52}$$

Appendix 2: Kernels of integral operators

We proceed with giving the expressions of the kernels of the nonlinear integral operators $\mathcal{H}$ and $\mathcal{H}'$ and their derivatives. Following the definitions (44)–(46), we deduce the formulas

$$H(x,x',y') = \frac{1}{2}\left[\frac{\sin\pi y'}{\cosh\pi(x-x') - \cos\pi y'}\right], \tag{53}$$

$$\begin{aligned} &\left(\frac{\partial H}{\partial x'}, \frac{\partial H}{\partial y'}\right)(x,x',y') \\ &\quad = \frac{\pi}{2}\left(\frac{\sin\pi y' \sinh\pi(x-x')}{(\cosh\pi(x-x') - \cos\pi y')^2}, \frac{\cos\pi y' \cosh\pi(x-x') - 1}{(\cosh\pi(x-x') - \cos\pi y')^2}\right), \end{aligned} \tag{54}$$

and

$$H'(x,x',h(x')) = \frac{\partial H}{\partial x'}(x,x',1-h(x'))h'(x') + \frac{\partial H}{\partial y'}(x,x',1-h(x')). \tag{55}$$

The Fréchet derivatives of the integral operators $\mathcal{H}$ and $\mathcal{H}'$ with respect to h can be obtained by differentiating their kernels with respect to h (see (51)–(52)). More precisely, we have

$$\begin{aligned} &D\mathcal{H}(h;k)(x) = \frac{\pi}{2}\int_{-a}^{a}\left[\frac{\cos\pi h(x')\cosh\pi(x-x') - 1}{(\cosh\pi(x-x') + \cos\pi h(x'))^2}\right]k(x')\omega(x')\,dx', \\ &\omega(x') = \sqrt{1 + h'^2(x')}, \end{aligned} \tag{56}$$

and

$$\begin{aligned} D\mathcal{H}'(h;k)(x) &= \int_{-a}^{a}\left[\left(\frac{\partial^2 H}{\partial y'^2}(x,x',1-h(x')) - \frac{\partial^2 H}{\partial x'^2}(x,x',1-h(x'))\right)q(x')\right. \\ &\quad \left. - \frac{\partial H}{\partial x'}(x,x',1-h(x'))q'(x')\right]k(x')\,dx' \\ &= \int_{-a}^{a}\left[K_1(x,x')q(x') - K_2(x,x')q'(x')\right]k(x')\,dx', \end{aligned} \tag{57}$$

with

$$K_1(x,x') = -2\pi^2 \times \frac{\sin\pi h(x')\sinh^2\pi(x-x')}{(\cosh\pi(x-x') + \cos\pi h(x'))^3} \tag{58}$$

and

$$K_2(x,x') = \frac{\pi}{2} \times \frac{\sin\pi h(x')\sinh\pi(x-x')}{(\cosh\pi(x-x') + \cos\pi h(x'))^2}. \tag{59}$$

In the previous equations we assume that $h \in C^1[-a,a]$ and $h(\pm a) = 0$, hence $q \in C^1[-a,a]$. For simplicity, we suppose in the differentiation that $\omega(x')$ is independent of h, i.e., depends on h_0 of the previous step.

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References

1. Alessandrini, G., Beretta, E., Rosset, E., Vessella, S.: Inverse boundary value problems with unknown boundaries: optimal stability. C. R. Acad. Sci. Paris **328**, 607–611 (2000)
2. Cakoni, F., Kress, R.: Integral equations for inverse problems in corrosion detection from partial Cauchy data. Inverse Probl. Imaging **1**(2), 229–245 (2007)
3. Chipot, M.: Elliptic Equations: An Introductory Course. Birkhäuser, Basel (2009)
4. Engl, H.W., Hanke, M., Neubauer, A.: Regularization of Inverse Problems. Kluwer Academic, Dordrecht (1996)
5. Fang, W., Yan, L., Zen, S.: Numerical recovery of Robin boundary from boundary measurements for the Laplace equation. J. Comput. Appl. Math. **224**, 573–580 (2009)
6. Grisvard, P.: Elliptic Problem in Nonsmooth Domains. Classics in Applied Mathematics, vol. 69. SIAM, Philadelphia (1985)
7. Hansen, P.C.: Regularization tools version 4.0 for Matlab 7.3. Numer. Algorithms **46**, 189–194 (2007)
8. Kirsch, A.: An Introduction to the Mathematical Theory of Inverse Problems. Springer, New York (2011)
9. Kress, R.: Linear Integral Equations. Applied Mathematical Sciences, vol. 82. Springer, New York (2014)
10. Kress, R., Rundel, W.: Nonlinear integral equations and the iterative solution for an inverse boundary value problem. Inverse Probl. **21**, 1207 (2005)
11. Kurt, B., Caudill, L.: An inverse problem in thermal imaging. SIAM J. Appl. Math. **56**(3), 715–735 (1996)
12. McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge (2000)
13. Melnikov, M.Y.: Green's Functions: Construction and Application. de Gruyter, Berlin (2012)
14. Monch, L.: On the inverse acoustic scattering problem by an open arc: the sound-hard case. Inverse Probl. **13**, 1379 (1997)
15. Nachaoui, A.: An improved implementation of an iterative method in boundary identification problems. Numer. Algorithms **33**, 381–398 (2003)
16. Smoller, J.: Shock Waves and Reaction–Diffusion Equations. Springer, Berlin (1983)
17. Yang, F., Yan, L., Wei, T.: Reconstruction of the corrosion boundary for the Laplace equation by using a boundary collocation method. Math. Comput. Simul. **79**, 2148–2156 (2009)