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Existence of nonconstant periodic solutions for $p(t)$-Laplacian Hamiltonian system

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Abstract
The purpose of this paper is to consider the existence of periodic solutions for the $p(t)$-Laplacian Hamiltonian system. Some results are obtained by using the least action principle and the minimax methods.

Keywords: $p(t)$-Laplacian; Minimax principle; Least action principle; Periodic solution; Critical point

1 Introduction

In this paper, we consider the following problem:

$$\begin{cases} \frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) - \nabla F(t,u(t)) = 0, & t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $u \in R^N$. $F(t,u)$ and $p(t)$ satisfy the following conditions:

(F_0) $F : [0,T] \times R^N \to R$ is measurable and T-periodic in t for each $u \in R^N$ and continuously differentiable in u for a.e. $t \in [0,T]$, and there exist $a \in C(R^+,R^+)$ and $b \in L^1([0,T],R^+)$ such that

$$|F(t,u)| \le a(|u|)b(t), \qquad |\nabla F(t,u)| \le a(|u|)b(t)$$

for all $u \in R^N$ and a.e. $t \in [0,T]$.

(P) $p(t) \in C([0,T],R^+)$, $p(t) = p(t+T)$ and

$$1 < p^- := \min p(t) \le p^+ := \max p(t) < +\infty.$$

The nonlinear problems involving the $p(t)$-Laplace type operator are extremely attractive because they can be used to model dynamical phenomena which arise from the study of elastic mechanics. The detailed application backgrounds of the $p(x)$-Laplace type operators can be found in [1, 2] and the references therein. The $p(t)$-Laplacian system possesses more complicated nonlinearity than that of the p-Laplacian, for example, it is not homogeneous, this causes many troubles, and some classical theories and methods, such as the theory of Sobolev spaces, are not applicable.

In recent years, many researchers studied the periodic solutions of the system (1.1). Some existence results are obtained by using the least action principle and minimax methods in critical point theory. Many solvability conditions were given, such as the coercivity condition, the periodicity condition, the convexity condition, the boundedness condition, the subadditive condition and the sublinear condition. We refer to [3–13].

When $F(t,x) = G(x) + H(t,x)$, the $p(t)$-Laplacian system has also been studied by many authors. Especially, in [13], the authors supposed that $H(t,x)$ is p^--sublinear, that is, there exist $f,g \in L^1([0,T],R^+)$ and $\alpha \in [0,p^-)$ such that

$$\left|\nabla H(t,x)\right| \le f(t)|x|^\alpha + g(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, and there exist $0 \le r < 1/(p^+T^{p^-})$ and $1 \le \beta \le p^-$ such that

$$\left\langle \nabla G(x) - \nabla G(y), x-y \right\rangle \ge -r|x-y|^\beta$$

for all $x,y \in R^N$.

Moreover, they consider system (1.1) with $F(t,x)$ which is the sum of a subconvex function and another function under suitable conditions, by the least action principle and the saddle point theorem, they obtain some existence results.

Motivated by the results mentioned above, we aim in this paper to study the system (1.1) with a potential $F(t,x)$ which is also the sum of $F_1(t,x)$ and $F(x)$, where the conditions on $F_1(t,x)$ and $F(x)$ are more general and simple. By the least action principle, we obtained three existence results.

This paper is organized as follows. In Sect. 2, we give some necessary preliminary knowledge on variable exponent Sobolev spaces. In Sect. 3, we present our main results and completed the proof. In Sect. 4, we give some examples to illustrate our results.

2 Preliminary

For the convenience of readers, we first state some properties of the variable exponent Lebesgue–Sobolev spaces $L^{p(t)}$ and $W_T^{1,p(t)}$ (for details, see [14–18]). In the following, we use $|\cdot|$ to denote the Euclidean norm in R^N.

Let $p(t)$ satisfy the condition (P) and define a generalized Lebesgue space

$$L^{p(t)}\left([0,T];R^N\right) = \left\{u \in L^1\left([0,T];R^N\right) : \int_0^T |u|^{p(t)}\,dt < \infty\right\},$$

with the norm

$$|u|_{L^{p(t)}} = |u|_{p(t)} = \inf\left\{\lambda > 0 : \int_0^T \left|\frac{u}{\lambda}\right|^{p(t)} dt \le 1\right\}.$$

Define

$$C_T^\infty = C_T^\infty\left(R;R^N\right) = \left\{u \in C^\infty\left(R;R^N\right)\right) : u \text{ is } T\text{-periodic}\right\}.$$

For $u,v \in L^1([0,T];\mathbb{R}^N)$, if

$$\int_0^T \left(u(t),\varphi'(t)\right) dt = -\int_0^T \left(v(t),\varphi(t)\right) dt, \quad \forall \varphi \in C_T^\infty,$$

then $v(t)$ is called a T-weak derivative of $u(t)$ and is denoted by $\dot{u}(t)$.

It has been proved that (see [19], page 6)

$$\int_0^T \dot{u}(s)\,ds = 0,$$

and there exists $C \in \mathbb{R}^N$ such that

$$u(t) = \int_0^T v(s)\,ds + C, \quad \text{a.e. } t \in [0,T],$$

and $u(0) = u(T) = C$.

Define a generalized Sobolev space

$$W_T^{1,p(t)}\big([0,T];R^N\big) = \big\{u \in L^{p(t)}\big([0,T];R^N\big) : \dot{u} \in L^{p(t)}\big([0,T];R^N\big)\big\}$$

with the norm

$$\|u\|_{W_T^{1,p(t)}} = \|u\| = |u|_{p(t)} + |\dot{u}|_{p(t)}.$$

For $u \in W_T^{1,p(t)}([0,T];R^N)$, let

$$\bar{u} = \frac{1}{T}\int_0^T u(s)\,ds, \qquad \tilde{u}(t) = u(t) - \bar{u},$$

and

$$\widetilde{W}_T^{1,p(t)}\big([0,T];R^N\big) = \left\{u \in W_T^{1,p(t)}\big([0,T];R^N\big) : \int_0^T u(s)\,ds = 0\right\},$$

then

$$W_T^{1,p(t)}\big([0,T];R^N\big) = \widetilde{W}_T^{1,p(t)}\big([0,T];R^N\big) \oplus R^N.$$

In the following we use $L^{p(t)}$, $W_T^{1,p(t)}$, $\widetilde{W}_T^{1,p(t)}$ to denote the $L^{p(t)}([0,T];R^N)$, $W_T^{1,p(t)}([0,T];R^N)$, $\widetilde{W}_T^{1,p(t)}([0,T];R^N)$, respectively.

Lemma 2.1 ([20]) *There is a continuous embedding $W^{1,p(t)} \hookrightarrow C([0,T],\mathbb{R}^N)$, when $p^- > 1$, the embedding is compact. And for $u \in \widetilde{W}_T^{1,p(t)}$, there is a constant C independent of u such that*

$$\|u\|_\infty \le C\|u\|.$$

Lemma 2.2 ([2, 20]) *For every $u \in \widetilde{W}_T^{1,p(t)}$, there exist constants c_1', c_1 such that*

$$\|u\|_\infty \le c_1|\dot{u}|_{p(t)},$$

$$\|u\|_\infty \le 2c_1'\left[\left(\int_0^T |\dot{u}(t)|^{p(t)}\,dt\right)^{\frac{1}{p^-}} + T^{\frac{1}{p^-}}\right],$$

where $c_1, c_1' > 0$.

Lemma 2.3 ([20]) *Let $u = \bar{u} + \tilde{u} \in W_T^{1,p(t)}$, then the norm $|\widetilde{\dot{u}}|_{p(t)}$ is an equivalent norm on $\widetilde{W}_T^{1,p(t)}$ and $|\bar{u}| + |\dot{u}|_{p(t)}$ is an equivalent norm on $W_T^{1,p(t)}$. Therefore, for $u \in W_T^{1,p(t)}$,*

$$\|u\| \to \infty \quad \Rightarrow \quad |\overline{u}| + \int_0^T \left|u'(t)\right|^{p(t)} dt \to \infty.$$

Lemma 2.4 ([2])

(i) *The space $(L^{p(t)}, |\cdot|)$ is a separable, reflexive, uniform convex Banach space, and its conjugate space is $L^{q(t)}$, where $\frac{1}{p(t)} + \frac{1}{q(t)}$. For any $u \in L^{p(t)}$ and $v \in L^{q(t)}$, we have*

$$\left|\int_0^T u(t)v(t)\, dt\right| \leq 2|u|_{p(t)}|v|_{q(t)}.$$

(ii) *If $p_1(t), p_2(t) \in C([0,T], \mathbb{R}^1)$ and $1 < p_1(t) \leq p_2(t)$ for any $t \in [0,T]$, then $L^{p_2(t)} \hookrightarrow L^{p_1(t)}$, and the embedding is continuous.*

3 Main results of problem (1.1)

Definition 3.1 A function $u(t) \in W_T^{1,p(t)}$ is called a weak solution of (1.1), if $|\dot{u}(t)|^{p(t)-2}\dot{u}(t)$ has a weak derivative, still denoted by $\frac{d}{dt}(\dot{u}(t)|^{p(t)-2}\dot{u}(t))$, such that

$$\begin{cases} \frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) - \nabla F(t,u(t)) = 0, & \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

Define a functional φ on $W_T^{1,p(t)}$ by

$$\varphi(u) = \int_0^T \frac{1}{p(t)})\left|\dot{u}(t)\right|^{p(t)} dt + \int_0^T F\big(t,u(t)\big)\, dt.$$

Lemma 3.1 *Suppose that assumptions (F_0) and (P) hold, then the functional φ is continuously differentiable on $W_T^{1,p(t)}$ and*

$$\left\langle\varphi'(u),v\right\rangle = \int_0^T \left(\left|\dot{u}(t)\right|^{p(t)-2}\dot{u}(t),\dot{v}(t)\right) dt + \int_0^T \left(\nabla F\big(t,u(t)\big),v(t)\right) dt$$

for all $u, v \in W_T^{1,p(t)}$. And if there exists $u \in W_T^{1,p(t)}$ such that $\langle\varphi'(u),v\rangle = 0$ for all $v \in W_T^{1,p(t)}$, then u is a weak solution (1.1).

Proof For convenience, define functionals J and H on $W_T^{1,p(t)}$ by

$$J(u) = \int_0^T \frac{1}{p(t)})\left|\dot{u}(t)\right|^{p(t)} dt, \qquad H(u) = \int_0^T F\big(t,u(t)\big)\, dt.$$

It is easy to see that ([18])

$$\varphi(u) = J(u) + H(u), \qquad \left\langle\varphi'(u),v\right\rangle = \left\langle J'(u),v\right\rangle + \left\langle H'(u),v\right\rangle,$$

for all $v \in W_T^{1,p(t)}$ and

$$\left\langle J'(u),v\right\rangle = \int_0^T \left(\left|\dot{u}(t)\right|^{p(t)-2}\dot{u}(t),\dot{v}(t)\right) dt.$$

It is easy to see that

$$\langle H'(u), v\rangle = \int_0^T \big(\nabla F\big(t, u(t)\big), v(t)\big)\, dt.$$

In fact

$$\begin{aligned}\frac{1}{s}\big[H(u+sv) - H(u)\big] &= \frac{1}{s}\int_0^T \big[F\big(t, u(t)+sv(t)\big) - F\big(t, u(t)\big)\big]\, dt \\ &= \int_0^T \big(\nabla F\big(t, u(t)+\theta sv(t)\big), v(t)\big)\, dt,\end{aligned}$$

where $0 < \theta < 1$.

Since

$$\big|\nabla F\big(t, u(t)+\theta sv(t)\big)\big| \leq a\big(\big|u(t)+\theta sv(t)\big|\big)b(t) \leq a_0 b(t),$$

and $\nabla F(t, u(t)+\theta sv(t)) \to \nabla F(t, u(t))$, a.e. $t \in [0, T]$, as $s \to 0$.

Here a_0 is some constant.

By the Lebegue dominated convergence theorem, we have

$$\lim_{s\to 0}\frac{1}{s}\big[H(u+sv) - H(u)\big] = \int_0^T \big(\nabla F\big(t, u(t), v(t)\big)\, dt,$$

which implies that H is Gâteaux differentiable, and

$$\langle H'(u), v\rangle = \int_0^T \big(\nabla F\big(t, u(t)\big), v(t)\big)\, dt, \forall v \in W_T^{1,p(t)}.$$

Next we will prove that the functional φ is continuous differentiable, it suffices to show that both of J' and H' are continuous on $W_T^{1,p(t)}$.

Let $u_n, u \in W_T^{1,p(t)}$, such that $\|u_n - u\| \to 0$ $(n \to \infty)$. Obviously,

$$\|u_n - u\| \to 0 \quad \Leftrightarrow \quad |u_n - u|_{p(t)} \to 0 \quad \text{and} \quad \big|u_n' - u'\big|_{p(t)} \to 0. \tag{3.1}$$

Note that the mapping defined by $u \in L^{p(t)} \to |u(t)|^{p(t)-2}u \in L^{q(t)}$, where $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$, is a bounded and continuous mapping from $L^{p(t)}$ into $L^{q(t)}$ (see [15]). It follows from (3.1) that

$$2\big|\big|u_n'\big|^{p(t)-2}u_n' - \big|u'\big|^{p(t)-2}u'\big|_{q(t)} \to 0,$$

which, combined with Lemma 2.4, shows

$$\begin{aligned}\big\|J'(u_n) - J'(u)\big\| &= \sup_{\|v\|\leq 1}\left|\int_0^T \big(\big|u_n'\big|^{p(t)-2}u_n' - \big|u'\big|^{p(t)-2}u', v'(t)\big)\right| \\ &\leq 2\sup_{\|v\|\leq 1}\big|\big|u_n'\big|^{p(t)-2}u_n' - \big|u'\big|^{p(t)-2}u'\big|_{q(t)}\big|v'\big|_{p(t)} \\ &\leq 2\big|\big|u_n'\big|^{p(t)-2}u_n' - \big|u'\big|^{p(t)-2}u'\big|_{q(t)} \to 0.\end{aligned}$$

Hence, J' is continuous.

Next, we prove the continuity of $H'(u) : W_T^{1,p(t)} \to (W_T^{1,p(t)})^*$.

Let $u_n, u \in W_T^{1,p(t)}$, such that $\|u_n - u\| \to 0$ $(n \to \infty)$.

From Lemma 2.1, it follows that

$$u_n, u \in C\big([0,T],\mathbb{R}^N\big) \quad \text{and} \quad \max_{0\le t\le T}\big|u_n(t) - u(t)\big| \to 0 \quad (n \to \infty).$$

From the condition (F_0), we have

$$\big|\nabla F\big(t,u_n(t)\big) - \nabla F\big(t,u(t)\big)\big| \to 0 \quad (n \to \infty), \text{a.e. } t \in [0,T], \tag{3.2}$$

and $a(|u_n(t)|) \to a(|u(t)|)$ in $C([0,T];\mathbb{R}^N)$.

Therefore, for n large enough,

$$a\big(\big|u_n(t)\big|\big) \le a\big(\big|u(t)\big|\big) + 1.$$

Then we have

$$\begin{aligned} \big|\nabla F\big(t,u_n(t)\big) - \nabla F\big(t,u(t)\big)\big| &\le \big(a\big(\big|u(t)\big|\big) + a\big(\big|u_n(t)\big|\big)\big)b(t) \\ &\le \Big(2\max_{0\le t\le T} a\big(\big|u(t)\big|\big) + 1\Big)b(t) \\ &\le Cb(t). \end{aligned} \tag{3.3}$$

Here C is some constant.

Then from (3.2) and (3.3), together with the Lebesgue dominated convergence theorem, we get

$$\int_0^T \big|\nabla F\big(t,u_n(t)\big) - \nabla F\big(t,u(t)\big)\big|\,dt \to 0 \quad (n \to \infty).$$

Therefore,

$$\begin{aligned} &\big\|H'(u_n) - H'(u)\big\|_{(W_T^{1,p(t)})^*} \\ &\quad= \sup_{\|v\|\le 1, v\in W_T^{1,p(t)}} \left|\int_0^T \big(\nabla F\big(t,u_n(t)\big) - \nabla F\big(t,u(t)\big), v(t)\big)\,dt\right| \\ &\quad\le \sup_{\|v\|\le 1, v\in W_T^{1,p(t)}} \int_0^T \big|\nabla F\big(t,u_n(t)\big) - \nabla F\big(t,u(t)\big)\big| \cdot \big|v(t)\big|\,dt \\ &\quad\le C\int_0^T \big|\nabla F\big(t,u_n(t)\big) - \nabla F\big(t,u(t)\big)\big|\,dt \to 0 \quad (n \to \infty). \end{aligned}$$

So H' is continuous.

Suppose that $u \in W_T^{1,p(t)}$ such that

$$\big\langle \varphi'(u), v\big\rangle = 0, \quad \forall v \in W_T^{1,p(t)},$$

then

$$\int_0^T \big(|\dot{u}(t)|^{p(t)-2}\dot{u}(t), v(t)\big)\,dt = -\int_0^T \big(\nabla F\big(t,u(t)\big), v(t)\big)\,dt \tag{3.4}$$

for all $v \in W_T^{1,p(t)}$, obviously, also for all $v \in C_T^\infty$.

From (F_0) and Lemma 2.1, $\nabla F(t,u(t)) \in L^1$, therefore, (3.4) implies that $\nabla F(t,u(t))$ is the weak derivative $\frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t))$ of $|\dot{u}(t)|^{p(t)-2}\dot{u}(t)$, and

$$\begin{aligned} &\frac{d}{dt}\big(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)\big) = \nabla F\big(t,u(t)\big), v(t)), \quad \text{a.e. } t \in [0,T], \\ &|\dot{u}(0)|^{p(0)-2}\dot{u}(0) = |\dot{u}(T)|^{p(T)-2}\dot{u}(T), \end{aligned}$$

since $p(0) = p(T)$, it follows that $\dot{u}(0) = \dot{u}(T)$, so $u(t)$ is the weak solution of (1.1).

The proof is completed. □

For the reader's convenience, we give a definition.

Definition 3.2 ([21]) A function $F : R^N \to R$ is said to be (λ,μ)-subconvex if

$$F\big(\lambda(x+y)\big) \le \mu\big(F(x) + F(y)\big)$$

for some $\lambda, \mu > 0$ and all $x, y \in R^N$.

Theorem 3.1 *Let $F(t,x) = F_1(t,x) + F_2(x)$, where F_1 and F_2 satisfy (F_0) and the following conditions*:

(i) *$F_1(t,\cdot)$ is $(\lambda p^-, \mu p^-)$-subconvex for a.e. $t \in [0,T]$, where*

$$\lambda p^- > 1, \qquad 1 < 2\mu p^- < \big(\lambda p^-\big)^{p^-};$$

(ii) *there exists a constant $0 \le r < \frac{p^-}{(4c_1)^{p^-}Tp^+}$ such that*

$$\big(\nabla F_2(x) - \nabla F_2(y), x-y\big) \ge -r|x-y|^{p^-}$$

for all $x, y \in R^N$ and a.e. $t \in [0,T]$;

(iii)

$$\frac{1}{\mu p^-}\int_0^T F_1\big(t, \lambda p^- x\big)\,dt + \int_0^T F_2(x)\,dt \to +\infty$$

as $|x| \to +\infty$.

Then problem (1.1) has at least one nontrivial solution which minimizes φ on $W_T^{1,p(t)}$.

Proof Let $\alpha = \log_{\lambda p^-}(2\mu p^-)$, then $0 < \alpha < p^-$. For $|x| > 1$, there exists a positive integer n such that

$$n - 1 < \log_{\lambda p^-}|x| \le n.$$

Furthermore, we have $|x|^\alpha \geq (\lambda p^-)^{(n-1)\alpha} = (2\mu p^-)^{n-1}$ and $|x| \leq (\lambda p^-)^n$. Then from (i) and condition (F_0), it follows that

$$\begin{aligned} F_1(t,x) &= F_1\left(t, \lambda p^-\left(\frac{x}{2\lambda p^-} + \frac{x}{2\lambda p^-}\right)\right) \leq 2\mu p^- F_1\left(t, \frac{x}{2\lambda p^-}\right) \leq \cdots \\ &\leq (2\mu p^-)^n F_1\left(t, \frac{x}{(2\lambda p^-)^n}\right) \leq 2\mu p^- |x|^\alpha a_0 b(t), \end{aligned}$$

for a.e. $t \in [0,T]$ and all $|x| > 1$, where $a_0 = \max_{0\leq s\leq 1} a(s)$.

Moreover, we have

$$F_1(t,x) \leq \left(2\mu p^- |x|^\alpha + 1\right) a_0 b(t)$$

for a.e. $t \in [0,T]$ and all $x \in R^N$, $0 < \alpha < p^-$.

By (i) and Lemma 2.2, it is obvious that

$$\begin{aligned} \int_0^T F_1\big(t,u(t)\big)\,dt &\geq \frac{1}{\mu p^-}\int_0^T F_1\big(t,\lambda p^- \overline{u}\big)\,dt - \int_0^T F_1(t,-\widetilde{u})\,dt \\ &\geq \frac{1}{\mu p^-}\int_0^T F_1\big(t,\lambda p^- \overline{u}\big)\,dt - \big(2\mu p^- \|\widetilde{u}\|_\infty^\alpha + 1\big) a_0 \int_0^T b(t)\,dt \\ &\geq \frac{1}{\mu p^-}\int_0^T F_1\big(t,\lambda p^- \overline{u}\big)\,dt - c_2\left(\int_0^T \big|\dot{u}(t)\big|^{p(t)}\,dt\right)^{\frac{\alpha}{p^-}} - c_3 \end{aligned} \tag{3.5}$$

for all $u \in W_T^{1,p(t)}$ and some constants c_2, c_3.

Using (ii) and Lemma 2.2, we have

$$\begin{aligned} \int_0^T \big(F_2(u) - F_2(\overline{u})\big)\,dt &= \int_0^T \int_0^1 \big(\nabla F_2(\overline{u} + s\widetilde{u}), \widetilde{u}\big)\,ds\,dt \\ &= \int_0^T \int_0^1 \frac{1}{s}\big(\nabla F_2(\overline{u} + s\widetilde{u}) - \nabla F_2(\overline{u}), s\widetilde{u}\big)\,ds\,dt \\ &\geq -\int_0^T \int_0^1 \frac{r}{s}|s\widetilde{u}|^{p^-}\,ds\,dt \\ &= -r\int_0^T \frac{1}{p^-}|\widetilde{u}|^{p^-}\,dt \\ &\geq -\frac{rT}{p^-}\|\widetilde{u}\|_\infty^{p^-} \\ &\geq -\frac{2^{2p^-} rTc_1^{p^-}}{p^-}\int_0^T |\dot{u}|^{p(t)}\,dt + c_4 \end{aligned} \tag{3.6}$$

for all $u \in W_T^{1,p(t)}$ and a constant c_4.

Combining (iii), (3.5) and (3.6), we get

$$\begin{aligned} \varphi(u) &= \int_0^T \frac{1}{p(t)})\big|\dot{u}(t)\big|^{p(t)}\,dt + \int_0^T F\big(t,u(t)\big)\,dt \\ &\geq \frac{1}{p^+}\int_0^T \big|\dot{u}(t)\big|^{p(t)}\,dt + \int_0^T F_1\big(t,u(t)\big)\,dt + \int_0^T \big(F_2(u) - F_2(\overline{u})\big)\,dt + \int_0^T F_2(\overline{u})\,dt \end{aligned}$$

$$\geq \left(\frac{1}{p^+} - \frac{(4c_1)^{p^-} rT}{p^-}\right) \int_0^T \left|\dot{u}(t)\right|^{p(t)} dt + \left(\frac{1}{\mu p^-} \int_0^T F_1\left(t, \lambda p^- \overline{u}\right) dt + \int_0^T F_2(\overline{u})\, dt\right)$$
$$- c_2\left(\int_0^T \left|\dot{u}(t)\right|^{p(t)} dt\right)^{\frac{\alpha}{p^-}} + c_5$$

for all $u \in W_T^{1,p(t)}$ and some constants c_2, c_5, it follows from (iii), $0 < \alpha < p^-$, $0 \leq r < \frac{p^-}{(4c_1)^{p^-} Tp^+}$ and Lemma 2.3, $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$. Applying Theorem 1.1 and Corollary 1.1 in Mawhin–Willem [19], we complete the proof. □

Theorem 3.2 *Let $F(t,x) = F_1(t,x) + F_2(x)$, where F_1 and F_2 satisfy (F_0) and the following conditions*:

(i) *there exist $f(t), m(t) \in L^1([0,T]; R^+)$ and $\gamma \in [0, p^- - 1)$ such that*

$$\left|\nabla F_1(t,x)\right| \leq f(t)|x|^\gamma + m(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$;

(ii) *there exist constants $0 \leq r < \frac{p^-}{p^+ T (4c_1)^{p^-}}$, such that*

$$\left(\nabla F_2(x) - \nabla F_2(y), x - y\right) \geq -r|x-y|^{p^-}$$

for all $x, y \in R^N$ and a.e. $t \in [0,T]$;

(iii)

$$\liminf_{|x|\to\infty} \frac{1}{|x|^{\gamma q}} \int_0^T F(t,x)\, dt \geq A_1 c(\varepsilon)$$

as $|x| \to +\infty$, where $\frac{1}{p^-} + \frac{1}{q} = 1$, A_1, $c(\varepsilon)$ are defined in the proof.

Then problem (1.1) has at least one nontrivial solution which minimizes φ on $W_T^{1,p(t)}$.

Proof For convenience, we denote

$$A_1 = 2^\gamma \int_0^T f(t)\, dt, \qquad A_2 = \int_0^T m(t)\, dt.$$

By condition (i), the ε-Young inequality, Sobolev's inequality and Lemma 2.2, for any $u \in W_T^{1,p(t)}$ and some constants c_6, c_7, c_8, we have

$$\left|\int_0^T \left(F_1\left(t, u(t)\right) - F_1(t, \overline{u}\right) dt\right|$$
$$= \left|\int_0^T \int_0^1 \left(\nabla F_1\left(t, \overline{u} + s\widetilde{u}(t)\right), \widetilde{u}(t)\right) ds\, dt\right|$$
$$\leq \int_0^T \int_0^1 f(t)\left|\overline{u} + s\widetilde{u}(t)\right|^\gamma \left|\widetilde{u}(t)\right| ds\, dt + \int_0^T \int_0^1 m(t)\left|\widetilde{u}(t)\right| ds\, dt$$
$$\leq 2^\gamma \left(|\overline{u}|^\gamma + \|\widetilde{u}\|_\infty^\gamma\right)\|\widetilde{u}\|_\infty \int_0^T f(t)\, dt + \|\widetilde{u}\|_\infty \int_0^T m(t)\, dt$$
$$= A_1|\overline{u}|^\gamma \cdot \|\widetilde{u}\|_\infty + A_1\|\widetilde{u}\|_\infty^{\gamma+1} + A_2\|\widetilde{u}\|_\infty$$
$$\leq A_1\left(\varepsilon\|\widetilde{u}\|_\infty^{p^-} + c(\varepsilon)|\overline{u}|^{\gamma q}\right) + c_6\left(\int_0^T \left|\dot{u}(t)\right|^{p(t)} dt\right)^{\frac{\gamma+1}{p^-}} + c_7\left(\int_0^T \left|\dot{u}(t)\right|^{p(t)} dt\right)^{\frac{1}{p^-}} + c_8$$

$$\begin{aligned}
&\leq A_1\varepsilon(4c_1)^{p^-}\int_0^T \left|\dot{u}(t)\right|^{p(t)}\,dt + A_1c(\varepsilon)|\overline{u}|^{\gamma q} \\
&\quad + c_6\left(\int_0^T \left|\dot{u}(t)\right|^{p(t)}\,dt\right)^{\frac{\gamma+1}{p^-}} + c_7\left(\int_0^T \left|\dot{u}(t)\right|^{p(t)}\,dt\right)^{\frac{1}{p^-}} + c_8,
\end{aligned} \tag{3.7}$$

where ε satisfies $0 < \varepsilon < \frac{1}{A_1}(\frac{1}{(4c_1)^{p^-}p^+} - \frac{rT}{p^-})$, $c(\varepsilon) = (\varepsilon p^-)^{-\frac{q}{p^-}} \cdot \frac{1}{q}$.

Combining (3.6), (3.7) and (iii), we obtain

$$\begin{aligned}
\varphi(u) &= \int_0^T \frac{1}{p(t)})\left|\dot{u}(t)\right|^{p(t)}\,dt + \int_0^T F\bigl(t,u(t)\bigr)\,dt \\
&\geq \frac{1}{p^+}\int_0^T \left|\dot{u}(t)\right|^{p(t)}\,dt + \int_0^T \bigl(F_1\bigl(t,u(t)\bigr) - F_1(t,\overline{u})\bigr)\,dt \\
&\quad + \int_0^T \bigl(F_2(u) - F_2(\overline{u})\bigr)\,dt + \int_0^T F(t,\overline{u})\,dt \\
&\geq \left(\frac{1}{p^+} - \frac{(4c_1)^{p^-}rT}{p^-} - A_1\varepsilon(4c_1)^{p^-}\right)\int_0^T \left|\dot{u}(t)\right|^{p(t)}\,dt - c_6\left(\int_0^T \left|\dot{u}(t)\right|^{p(t)}\,dt\right)^{\frac{\gamma+1}{p^-}} \\
&\quad - c_7\left(\int_0^T \left|\dot{u}(t)\right|^{p(t)}\,dt\right)^{\frac{1}{p^-}} + |\overline{u}|^{\gamma q}\left(\frac{1}{|\overline{u}|^{\gamma q}}\int_0^T F(t,\overline{u})\,dt - A_1c(\varepsilon)\right) + c_8
\end{aligned}$$

for all $u \in W_T^{1,p(t)}$ and a constant c_8, it follows from (iii) and Lemma 2.3, $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$. Applying Theorem 1.1 and Corollary 1.1 in Mawhin–Willem [19], we complete the proof. □

Theorem 3.3 *Let $F(t,x) = F_1(t,x) + F_2(x)$, where F_1 and F_2 satisfy (F_0) and the following conditions:*

(i) *there exist $k(t), h(t) \in L^1([0,T];R^+)$ and $\beta \in [0,p^-)$ such that*

$$F_1(t,x) \geq -k(t)|x|^\beta + h(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$;

(ii) *there exist constants $0 \leq r < \frac{p^-}{(4c_1)^{p^-}Tp^+}$ such that*

$$\bigl(\nabla F_2(x) - \nabla F_2(y), x-y\bigr) \geq -r|x-y|^{p^-}$$

for all $x, y \in R^N$ and a.e. $t \in [0,T]$;

(iii)

$$\liminf_{|x|\to\infty} \frac{1}{|x|^\beta}\int_0^T F_2(x)\,dt \geq B$$

as $|x| \to +\infty$, where $B = 2^\beta\int_0^T k(t)\,dt$.

Then problem (1.1) has at least one nontrivial solution which minimizes φ on $W_T^{1,p(t)}$.

Proof By condition (i) and Sobolev's inequality and Lemma 2.2, we get

$$\int_0^T F_1\bigl(t,u(t)\bigr)\,dt \geq \int_0^T \bigl[-k(t)\left|u(t)\right|^\beta + h(t)\bigr]\,dt$$

$$\begin{aligned}
&\geq -2^{\beta}\left(|\overline{u}|^{\beta} + \left\|\widetilde{u}(t)\right\|_{\infty}^{\beta}\right)\int_0^T k(t)\,dt + \int_0^T h(t)\,dt \\
&= -B|\overline{u}|^{\beta} - B\left\|\widetilde{u}(t)\right\|_{\infty}^{\beta} + \int_0^T h(t)\,dt \\
&\geq -B(4c_1)^{p-}\left(\int_0^T \left|\dot{u}(t)\right|^{p(t)}\right)^{\frac{\beta}{p^-}} - B|\overline{u}|^{\beta} + c_9
\end{aligned} \tag{3.8}$$

for all $u \in W_T^{1,p(t)}$ and a constant c_9, where $B = 2^{\beta}\int_0^T k(t)\,dt$.

It follows from (3.6) and (3.8) that

$$\begin{aligned}
\varphi(u) &= \int_0^T \frac{1}{p(t)}\left|\dot{u}(t)\right|^{p(t)}dt + \int_0^T F\bigl(t,u(t)\bigr)\,dt \\
&\geq \frac{1}{p^+}\int_0^T \left|\dot{u}(t)\right|^{p(t)}dt + \int_0^T F_1\bigl(t,u(t)\bigr)\,dt + \int_0^T \bigl(F_2(u) - F_2(\overline{u})\bigr)\,dt + \int_0^T F_2(\overline{u})\,dt \\
&\geq \left(\frac{1}{p^+} - \frac{(4c_1)^{p^-}r_1 T}{p^-}\right)\int_0^T \left|\dot{u}(t)\right|^{p(t)}dt - B(4c_1)^{p^-}\left(\int_0^T \left|\dot{u}(t)\right|^{p(t)}\right)^{\frac{\beta}{p^-}} \\
&\quad + |\overline{u}|^{\beta}\left(\frac{1}{|\overline{u}|^{\beta}}\int_0^T F_2(\overline{u})\,dt - B\right) + c_{10}
\end{aligned}$$

for all $u \in W_T^{1,p(t)}$ and a constant c_{10}, it follows from (iii), $0 < \beta < p^-$, $0 \leq r < \frac{p^-}{(4c_1)^{p^-}Tp^+}$ and Lemma 2.3, $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$. Applying Theorem 1.1 and Corollary 1.1 in Mawhin–Willem [19], we complete the proof. □

4 Examples

In this section, we give some examples of $F(t,x)$ and $p(t)$ to illustrate our results.

Example 4.1 In system (1.1), let $p(t) = 3 + \cos\omega t$, $F(t,x) = F_1(t,x) + F_2(x)$, where

$$\begin{aligned}
&T = \frac{2\pi}{\omega}, \qquad p^+ = 4, \qquad p^- = 2, \qquad \lambda = 2, \qquad \mu = 3, \\
&F_1(t,x) = |x|^2|\sin\omega t|, \qquad F_2(x) = c(x) - \frac{1}{2}r|x|^2,
\end{aligned}$$

with

$$r > 0, \qquad c(x) = \frac{1}{2}r\bigl(|x_1|^3 + |x_2|^2 + |x_3|^2 + \cdots + |x_N|^2\bigr).$$

By some computation, we can take $0 \leq r < \frac{1}{32c_1^2T}$ such that

$$\bigl(\nabla F_2(x) - \nabla F_2(y), x - y\bigr) \geq -r|x-y|^2,$$

and $F_1(t,\cdot)$ is $(4,6)$-subconvex, for all $x, y \in R^N$ and a.e. $t \in [0,T]$.

It is easy to verify

$$\frac{1}{\mu p^-}\int_0^T F_1\bigl(t,\lambda p^- x\bigr)\,dt + \int_0^T F_2(x)\,dt \to +\infty$$

as $|x| \to +\infty$.

Therefore all the conditions of Theorem 3.1 are satisfied, the problem (1.1) has at least one solution.

Example 4.2 In system (1.1), let $p(t) = 5 + \cos\omega t$, $F(t,x) = F_1(t,x) + F_2(x)$, where

$$\omega = \frac{2\pi}{T}, \qquad p^+ = 6, \qquad p^- = 4, \qquad \gamma = \frac{3}{4},$$
$$F_1(t,x) = |x|^{\frac{7}{4}}|\sin\omega t| + \bigl(h(t),x\bigr), \qquad F_2(x) = c(x) - \frac{1}{4}r|x|^4,$$

with

$$r > 0, \qquad h(t) \in L^1\bigl([0,T],R^+\bigr), \qquad c(x) = \frac{1}{4}r\bigl(|x_1|^6 + |x_2|^4 + |x_3|^4 + \cdots + |x_N|^4\bigr).$$

By some computation, we can take $0 \le r \le \frac{1}{32c_1^2T}$ such that

$$\bigl(\nabla F_2(x) - \nabla F_2(y), x - y\bigr) \ge -r|x-y|^4,$$
$$\bigl|\nabla F_1(t,x)\bigr| = \frac{7}{4}|x|^{\frac{3}{4}}|\sin\omega t| + \bigl|h(t)\bigr|,$$

for all $x, y \in R^N$ and a.e. $t \in [0,T]$.

On the other hand, we can verify that condition (iii) of Theorem 3.2 is satisfied. Therefore all the conditions of Theorem 3.2 hold, the problem (1.1) has at least one solution.

Example 4.3 In system (1.1), let $p(t) = \frac{5}{2} + \cos\omega t$, $F(t,x) = F_1(t,x) + F_2(x)$, where

$$\omega = \frac{2\pi}{T}, \qquad p^+ = \frac{7}{2}, \qquad p^- = \frac{3}{2}, \qquad \beta = \frac{1}{2},$$
$$F_1(t,x) = |x|^{\frac{1}{2}}\left(\frac{1}{2}\sin\omega t + 1\right) + \bigl(h(t),x\bigr) \ge -|x|^{\frac{1}{2}}\frac{1}{2}\sin\omega t + \bigl(h(t),x\bigr),$$
$$F_2(x) = c(x) - \frac{2}{3}r|x|^{\frac{3}{2}},$$

with

$$r > 0, \qquad h(t) \in L^1\bigl([0,T],R^+\bigr), \qquad c(x) = \frac{2}{3}r\bigl(|x_1|^2 + |x_2|^{\frac{3}{2}} + |x_3|^{\frac{3}{2}} + \cdots + |x_N|^{\frac{3}{2}}\bigr).$$

Similar to Example 4.1,we can see all the conditions of Theorem 3.3 are satisfied, the problem (1.1) has at least one solution.

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