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Positive solutions for a class of nonhomogeneous Kirchhoff–Schrödinger–Poisson systems

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Abstract

This paper deals with the following generalized nonhomogeneous Kirchhoff–Schrödinger–Poisson system:

$$\begin{cases} (a + \int_{\mathbb{R}^3} |\nabla u|^2 + b \int_{\mathbb{R}^3} u^2)(-\Delta u + bu) + q\phi f(u) = g(u) + h(x), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 2qF(u), & \text{in } \mathbb{R}^3, \end{cases}$$

where $a > 0$, $b \geq 0$ are constants, $q \geq 0$ is a parameter, and $F(t) = \int_0^t f(s) \, ds$. Under some appropriate assumptions on $g(u)$ and $h(x)$, the existence of two positive radial solutions is proved by applying Ekeland's variational principle and the mountain pass theorem.

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1 Introduction and main results

In this paper, we consider the existence of multiple positive solutions for the following generalized nonhomogeneous Kirchhoff–Schrödinger–Poisson system:

$$\begin{cases} (a + \int_{\mathbb{R}^3} |\nabla u|^2 + b \int_{\mathbb{R}^3} u^2)(-\Delta u + bu) + q\phi f(u) = g(u) + h(x), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 2qF(u), & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $a > 0$, $b \geq 0$ are constants, $q \geq 0$ is a parameter, $F(t) = \int_0^t f(s) \, ds$, f and h satisfy the following hypotheses:

- (f) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and there exists $C > 0$ such that $f(t) \leq C(|t| + |t|^\alpha)$ for all $t \in \mathbb{R}$, where $\alpha \in (2, 4)$;
- (h₁) $0 \leq h(x) \in L^2(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$ and $0 \neq h(x) = h(|x|)$;
- (h₂) $\|h\|_2 \leq m_p$ with $m_p = \frac{a(p-2)}{4\gamma_2(p-1)} \left(\frac{ap}{4m(p-1)\gamma_p} \right)^{1/(p-2)}$, where $p \in (2, 6)$ and γ_p is the Sobolev embedding constant of $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$, m is a constant depending on γ_2 which will be introduced in Lemma 3.1;
- (h₃) $\langle \nabla h(x), x \rangle \in L^2(\mathbb{R}^3)$ and $\langle \nabla h(x), x \rangle \geq 0$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product.

When $q = 0$, system (1.1) reduces to the following Kirchhoff-type equation:

$$\left(a + \int_{\mathbb{R}^3} |\nabla u|^2 + b \int_{\mathbb{R}^3} u^2\right)(-\Delta u + bu) = g(u) + h(x), \quad \text{in } \mathbb{R}^3. \quad (1.2)$$

In recent years, the following Dirichlet problem of Kirchhoff-type on a bounded domain $\Omega \subset \mathbb{R}^N$

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2) \Delta u + V(x)u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

has been studied extensively by many researchers. With the aid of variational methods, for various conditions of the potential $V(x)$ and the nonlinearity $f(x, u)$, the existence, nonexistence, and multiplicity results for problem (1.3) have been investigated in the literature, one can see [1, 2, 6, 8, 12, 17, 22] and the references therein. There are also many works on the existence and multiplicity results for system (1.3) on unbounded domains. More precisely, Duan and Huang [7] dealt with problem (1.3) with sublinear case, and the existence of infinitely many solutions for the problem has been established by using the genus properties in critical point theory. Wu [27] studied the existence of nontrivial solutions and infinitely many high energy solutions for problem (1.3) by using a symmetric mountain pass theorem. Liu and He [15] also studied the existence of infinitely many high energy solutions for superlinear Kirchhoff problem (1.3) by a variant version of the fountain theorem. For more related topics, we refer the readers to [9, 14, 23, 28–30] and the references therein.

Very recently, Li et al. [14] studied the existence of at least one positive radial solution to the following nonlinear homogeneous Kirchhoff-type equation:

$$\left(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda b \int_{\mathbb{R}^N} u^2\right)(-\Delta u + bu) = f(u), \quad \text{in } \mathbb{R}^N, \quad (1.4)$$

where $N \geq 3$, $\lambda \geq 0$ is a parameter. Their result can be regarded as an extension of a classical result for the semilinear equation

$$-\Delta u + bu = f(u), \quad \text{in } \mathbb{R}^N, \quad (1.5)$$

to the nonlinear Kirchhoff-type equation (1.4). The more general semilinear Schrödinger equation (1.5) with $b = V(x)$ has been studied by many researchers under various stipulations, one can see [16, 24, 25] and the references therein.

On the other hand, the well-known Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + q\phi u = f(x, u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = qu^2, & \text{in } \mathbb{R}^3, \end{cases} \quad (1.6)$$

also known as the nonlinear Schrödinger–Maxwell system, and the similar systems arise in many mathematical physics contexts, such as in quantum electrodynamics, to describe the interaction between a charge particle interacting with the electromagnetic field, and also in semiconductor theory, in nonlinear optics, and in plasma physics (see [5] for more

details on the physics aspects). Many papers deal with problem (1.6) by using the modern variational method and the critical point theory under various assumptions on the potential $V(x)$ and the nonlinearity $f(x, u)$; see [19–21] and the references therein. Very recently, there have also been many authors who investigated the generalized Schrödinger–Poisson system on bounded domains or on unbounded domains, see [3, 4, 13, 31]. In [13] the authors considered the following system:

$$\begin{cases} -\Delta u + u + q\phi f(u) = g(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 2qF(u), & \text{in } \mathbb{R}^3. \end{cases} \quad (1.7)$$

They proved that there exists $q_0 > 0$ such that the system has at least a positive radial solution for $q \in [0, q_0)$.

By using the symmetric mountain pass theorem, the authors in [32] studied the following Kirchhoff–Schrödinger–Poisson system:

$$\begin{cases} (a + b \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2])[-\Delta u + V(x)u] + \lambda l(x)\phi u = f(x, u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = \lambda l(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \quad (1.8)$$

where constants $a > 0$, $b \geq 0$, and $\lambda \geq 0$. They proved the existence of infinitely many solutions assuming that f has sublinear growth in u .

Motivated by the above facts, in this paper, we consider the more general nonhomogeneous Kirchhoff–Schrödinger–Poisson system (1.1). The aim of this paper is to prove the existence and multiplicity of positive radial solutions for system (1.1) when f is subcritical and g is superlinear at infinity. To the best of our knowledge, there have been no works concerning this case up to now. Compared to the aforementioned results, our result extends these results to some extent.

In this paper, since we are concerned with the existence of positive solutions to (1.1), we assume that $f(t) = g(t) = 0$ for $t < 0$. And the following assumptions will be used in this paper.

(g₁) $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ and there exists $C > 0$ such that $g(t) \leq C(1 + |t|^{p-1})$ for all $t \in \mathbb{R}$, where $p \in (2, 6)$;

(g₂) $\lim_{t \rightarrow 0} g(t)/t = 0$;

(g₃) $\lim_{t \rightarrow \infty} g(t)/t^3 = \infty$.

The main results of the present paper can be described as follows.

Theorem 1.1 *Assume that (f) , (h_1) – (h_3) , and (g_1) – (g_3) hold. Then there exists $q_0 > 0$ such that problem (1.1) has at least two positive radial solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ for all $0 \leq q < q_0$ and $0 \leq b \leq 1/4\gamma_2^2$.*

According to Theorem 3.4 in Sect. 3, we have the following result.

Remark 1.2 *Assume that (f) , (h_1) – (h_2) , and (g_1) – (g_3) hold. Then, for any $q \geq 0$, problem (1.1) has at least one positive radial solution $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.*

This paper is organized as follows. In Sect. 2, we give the variational framework to our problem. Section 3 is devoted to proving the existence of a local minimum around the

origin with negative energy. In Sect. 4, the existence of mountain-pass type critical point is obtained.

2 Variational setting and preliminaries

In this section, we assume that (f) , (h_1) , (g_1) , and (g_2) hold, then we study the variational framework of (1.1). Firstly, we give the following notations:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space equipped with the inner product and norm

$$\langle u, v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + buv) \, dx, \quad \|u\| = \langle u, u \rangle^{1/2}.$$

- $L^s(\mathbb{R}^3)$ ($1 \leq s \leq +\infty$) is a Lebesgue space whose norm is denoted by $\|\cdot\|_s$.
- Let $\mathcal{D}^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$ be the Sobolev space with the norm $\|u\|_{\mathcal{D}^{1,2}}^2 := \int_{\mathbb{R}^3} |\nabla u|^2 \, dx$.
- C and C_i denote various positive constants, which may vary from line to line.
- The strong (respectively weak) convergence is denoted by \rightarrow (respectively \rightharpoonup).
- $B_\rho(0)$ denotes a ball centered at the origin with radius $\rho > 0$.
- Let S be the best constant of the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, which is given by

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|_{\mathcal{D}^{1,2}}^2}{\|u\|_6^2}.$$

Now we have that the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [2, 6]$, and there exists $\gamma_s > 0$ such that

$$\|u\|_s \leq \gamma_s \|u\|, \quad \forall u \in H^1(\mathbb{R}^3).$$

Let $H = H_r^1(\mathbb{R}^3)$ be the subspace of $H^1(\mathbb{R}^3)$ containing only the radial functions. Then, due to the result of [26], the embedding $H \hookrightarrow L^s(\mathbb{R}^3)$ is compact for $s \in (2, 6)$.

By virtue of the variational nature of problem (1.1), its weak solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ are critical points of the C^1 functional $J : H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} J(u, \phi) = & \frac{a}{2} \|u\|^2 + \frac{1}{4} \|u\|^4 - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx + q \int_{\mathbb{R}^3} F(u) \phi \, dx \\ & - \int_{\mathbb{R}^3} G(u) \, dx - \int_{\mathbb{R}^3} hu \, dx, \end{aligned} \quad (2.1)$$

where $G(u) = \int_0^u g(s) \, ds$. By condition (f) and the Lax–Milgram theorem, for every $u \in H^1(\mathbb{R}^3)$, there exists unique $\phi = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ satisfying $-\Delta \phi = 2qF(u)$. Similar to the argument in [13], we can derive that the function ϕ_u has the following properties.

Lemma 2.1 ([13]) *For every $u \in H^1(\mathbb{R}^3)$, we have*

- $\|\phi_u\|_{\mathcal{D}^{1,2}}^2 = 2q \int_{\mathbb{R}^3} F(u) \phi_u \, dx$;
- $\phi_u \geq 0$;
- $\|\phi_u\|_{\mathcal{D}^{1,2}} \leq qC(\|u\|^2 + \|u\|^{\alpha+1})$;
- $\int_{\mathbb{R}^3} F(u) \phi_u \, dx \leq qC_1(\|u\|^4 + \|u\|^{2(\alpha+1)})$;
- if u is a radial function, then ϕ_u is radial too.

According to Lemma 2.1 and (2.1), we can prove that $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the one variable functional defined as

$$J_q(u) = \frac{a}{2} \|u\|^2 + \frac{1}{4} \|u\|^4 + \frac{q}{2} \int_{\mathbb{R}^3} F(u) \phi_u \, dx - \int_{\mathbb{R}^3} G(u) \, dx - \int_{\mathbb{R}^3} hu \, dx. \quad (2.2)$$

It is easy to check that $J_q : H \rightarrow \mathbb{R}$ is well defined and is class of C^1 , and we have

$$\begin{aligned} \langle J'_q(u), v \rangle &= a \langle u, v \rangle + \|u\|^2 \langle u, v \rangle + q \int_{\mathbb{R}^3} f(u) \phi_u v \, dx \\ &\quad - \int_{\mathbb{R}^3} g(u) v \, dx - \int_{\mathbb{R}^3} hv \, dx, \quad \forall u, v \in H. \end{aligned} \quad (2.3)$$

In order to obtain the existence of mountain-pass type critical point in Sect. 4, the boundedness of Palais–Smale sequences is crucial. But the standard arguments used to prove the boundedness of Palais–Smale sequences do not work. To overcome the difficulty, following [11], we use a cut-off function $\chi \in C(\mathbb{R}_+, [0, 1])$ satisfying

$$\begin{cases} \chi(t) = 1, & t \in [0, 1/2], \\ \chi(t) = 0, & t \in [1, \infty], \\ \|\chi'\|_\infty \leq 4, \end{cases}$$

and study the following modified functional $J_q^T : H \rightarrow \mathbb{R}$ defined by

$$J_q^T(u) = \frac{a}{2} \|u\|^2 + \frac{1}{4} \|u\|^4 + \frac{q}{2} h_T(u) \int_{\mathbb{R}^3} F(u) \phi_u \, dx - \int_{\mathbb{R}^3} G(u) \, dx - \int_{\mathbb{R}^3} hu \, dx, \quad \forall u \in H,$$

where $h_T(u) = \chi(\frac{\|u\|^2}{T^2})$. In the following, we discuss the existence of critical points of J_q^T . In fact, for $T > 0$ sufficiently large and q sufficiently small, we can find a critical point of J_q^T such that $\|u\| \leq T/\sqrt{2}$, hence u is also a critical point of J_q .

In Sect. 4, we use the method based on the “monotonicity trick” introduced by Struwe in [18], which has been successfully used to handle many homogeneous elliptic problems. Now we recall the following result.

Theorem 2.2 ([10]) *Let $(X, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ be an interval. Consider the family of C^1 functionals on X :*

$$J_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in I,$$

with B nonnegative and either $A(u) \rightarrow \infty$ or $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$. We assume that there are two points $v_1, v_2 \in X$ such that

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(v_1), J_\lambda(v_2)\}, \quad \forall \lambda \in I,$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in I$, there is a sequence $\{u_n\} \subset X$ such that

- (i) $\{u_n\}$ is bounded;
- (ii) $J_\lambda(u_n) \rightarrow c_\lambda$;
- (iii) $J'_\lambda(u_n) \rightarrow 0$ in the dual space X^{-1} of X .

3 Existence of solution u_0 with negative energy

In this section, we prove that, for any $q \geq 0$, system (1.1) has a positive radial solution with negative energy. With the aid of Ekeland's variational principle, this solution is obtained by seeking a local minimum of the energy functional J_q . Now we give some useful lemmas that will be used later.

Lemma 3.1 Assume that (f) , (h_1) , (h_2) , (g_1) , (g_2) , and (g_3) hold. Then there exist $\rho, \alpha > 0$ such that, for all $q \geq 0$, $J_q|_{\|u\|=\rho}(u) \geq \alpha$.

Proof By conditions (g_1) and (g_2) , for $\frac{a}{2\gamma_2^2}$, there exists $m > 0$ such that

$$g(u) \leq \frac{a}{2\gamma_2^2}|u| + m|u|^{p-1}, \quad \forall u \in \mathbb{R}, \quad (3.1)$$

and

$$G(u) \leq \frac{a}{4\gamma_2^2}u^2 + \frac{m}{p}|u|^p, \quad \forall u \in \mathbb{R}. \quad (3.2)$$

Then, for all $q \geq 0$ and $u \in H$, by Lemma 2.1, Hölder's inequality, and Sobolev's embedding theorem, we have

$$\begin{aligned} J_q(u) &= \frac{a}{2}\|u\|^2 + \frac{1}{4}\|u\|^4 + \frac{q}{2} \int_{\mathbb{R}^3} F(u)\phi_u \, dx - \int_{\mathbb{R}^3} G(u) \, dx - \int_{\mathbb{R}^3} hu \, dx \\ &\geq \frac{a}{2}\|u\|^2 - \int_{\mathbb{R}^3} G(u) \, dx - \int_{\mathbb{R}^3} hu \, dx \\ &\geq \frac{a}{2}\|u\|^2 - \int_{\mathbb{R}^3} \left(\frac{a}{4\gamma_2^2}u^2 + \frac{m}{p}|u|^p \right) dx - \int_{\mathbb{R}^3} hu \, dx \\ &\geq \|u\| \left(\frac{a}{4}\|u\| - \frac{m}{p}\gamma_p^p\|u\|^{p-1} - \gamma_2\|h\|_2 \right). \end{aligned} \quad (3.3)$$

Set

$$\eta(t) = \frac{a}{4}t - \frac{m}{p}\gamma_p^p t^{p-1}, \quad t \geq 0.$$

Since $p \in (2, 6)$, by direct calculation, we see that

$$\max_{t \geq 0} \eta(t) = \eta(\rho) = \frac{a(p-2)}{4(p-1)}\rho,$$

where

$$\rho = \left(\frac{ap}{4m(p-1)\gamma_p^p} \right)^{1/(p-2)}.$$

Then it follows from (3.3) that, if $\gamma_2 \|h\|_2 < \eta(\rho)$, i.e., $\|h\|_2 < \gamma_2^{-1} \eta(\rho) := m_p$, there exists $\alpha = \rho(\eta(\rho) - \gamma_2 \|h\|_2) > 0$ such that $J_q|_{\|u\|=\rho}(u) \geq \alpha$, where $m_p = \frac{a(p-2)}{4\gamma_2(p-1)} \left(\frac{ap}{4m(p-1)\gamma_p^p} \right)^{1/(p-2)}$. \square

Lemma 3.2 Assume that (f) , (h_1) , and (g_1) hold. Then, for any $q \geq 0$,

$$c_0 = \inf\{J_q(u) : u \in \overline{B}_\rho\} < 0,$$

where ρ is given by Lemma 3.1 and $\overline{B}_\rho = \{u \in H \text{ and } \|u\| \leq \rho\}$.

Proof By (h_1) , we can choose a function $\varphi \in H$ such that $\int_{\mathbb{R}^3} h(x)\varphi \, dx > 0$. By Lemma 2.1, for $t > 0$ small enough, we obtain

$$\begin{aligned} J_q(t\varphi) &= \frac{at^2}{2} \|u\|^2 + \frac{t^4}{4} \|\varphi\|^4 + \frac{q}{2} \int_{\mathbb{R}^3} F(t\varphi)\phi_{t\varphi} \, dx - \int_{\mathbb{R}^3} G(t\varphi) \, dx - t \int_{\mathbb{R}^3} h\varphi \, dx \\ &\leq \frac{at^2}{2} \|u\|^2 + \frac{t^4}{4} \|\varphi\|^4 + \frac{q^2}{2} C_1 (t^4 \|\varphi\|^4 + t^{2(1+\alpha)} \|\varphi\|^{2(1+\alpha)}) - t \int_{\mathbb{R}^3} h\varphi \, dx \\ &< 0, \end{aligned}$$

which shows that $c_0 = \inf\{J_q(u) : u \in \overline{B}_\rho\} < 0$. \square

Lemma 3.3 Assume that (f) , (h_1) , (g_1) , (g_2) , and (g_3) hold. If $\{u_n\}$ is a bounded Palais–Smale sequence of J_q , then $\{u_n\}$ has a convergent subsequence in H .

Proof Since $\{u_n\}$ is a bounded (PS) sequence of J_q , then $J_q(u_n)$ is bounded, $J'_q(u_n) \rightarrow 0$ in H^{-1} , where H^{-1} is the dual space of H . We may assume that, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H, \\ u_n &\rightarrow u \quad \text{in } L^s(\mathbb{R}^3), s \in (2, 6), \\ u_n &\rightarrow u \quad \text{a.e. on } \mathbb{R}^3. \end{aligned}$$

By (g_1) and (g_2) , for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$g(u) \leq \varepsilon |u| + C_\varepsilon |u|^{p-1}, \quad \forall u \in \mathbb{R},$$

then

$$\begin{aligned} \left| \int_{\mathbb{R}^3} g(u_n)(u_n - u) \, dx \right| &\leq \int_{\mathbb{R}^3} [\varepsilon |u_n| + C_\varepsilon |u_n|^{p-1}] |u_n - u| \, dx \\ &\leq \varepsilon \|u_n\|_2 \|u_n - u\|_2 + C_\varepsilon \|u_n\|_p^{p-1} \|u_n - u\|_p \\ &\leq \varepsilon \|u_n\| \|u_n - u\| + C_\varepsilon \gamma_p^{p-1} \|u_n\|^{p-1} \|u_n - u\|_p. \end{aligned}$$

It follows that

$$\int_{\mathbb{R}^3} g(u_n)(u_n - u) \, dx \rightarrow 0, \quad n \rightarrow \infty.$$

By a similar argument, we have

$$\int_{\mathbb{R}^3} g(u)(u_n - u) \, dx \rightarrow 0, \quad n \rightarrow \infty.$$

By condition (f) and Hölder's inequality, we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} f(u_n) \phi_{u_n} (u_n - u) \, dx \right| \\ & \leq C \int_{\mathbb{R}^3} [|u_n| + |u_n|^\alpha] \phi_{u_n} |u_n - u| \, dx \\ & \leq C [\|\phi_{u_n}\|_6 \|u_n\|_{12/5} \|u_n - u\|_{12/5} + \|\phi_{u_n}\|_6 \|u_n\|_6^\alpha \|u_n - u\|_\beta], \end{aligned}$$

where $\beta = 6/(5 - \alpha) \in (2, 6)$. Then, by Lemma 2.1 and Sobolev's embedding theorem, we have

$$\int_{\mathbb{R}^3} f(u_n) \phi_{u_n} (u_n - u) \, dx \rightarrow 0, \quad n \rightarrow \infty.$$

Similarly,

$$\int_{\mathbb{R}^3} f(u) \phi_u (u_n - u) \, dx \rightarrow 0, \quad n \rightarrow \infty.$$

Noting that $u_n \rightharpoonup u$ in E , we have that

$$\langle u, u_n - u \rangle \rightarrow 0.$$

Thus,

$$\begin{aligned} & \langle J'_q(u_n) - J'_q(u), u_n - u \rangle \\ & = (a + \|u_n\|^2) \langle u_n, u_n - u \rangle - (a + \|u\|^2) \langle u, u_n - u \rangle \\ & \quad + q \int_{\mathbb{R}^3} (f(u_n) \phi_{u_n} - f(u) \phi_u) (u_n - u) \, dx \\ & \quad - \int_{\mathbb{R}^3} (g(u_n) - g(u)) (u_n - u) \, dx \\ & = (a + \|u_n\|^2) \|u_n - u\|^2 + (\|u_n\|^2 - \|u\|^2) \langle u, u_n - u \rangle \\ & \quad + q \int_{\mathbb{R}^3} (f(u_n) \phi_{u_n} - f(u) \phi_u) (u_n - u) \, dx \\ & \quad - \int_{\mathbb{R}^3} (g(u_n) - g(u)) (u_n - u) \, dx \\ & = (a + \|u_n\|^2) \|u_n - u\|^2 + o_n(1), \quad n \rightarrow \infty. \end{aligned}$$

Consequently, we have $\|u_n - u\| \rightarrow 0$, that is, $u_n \rightarrow u$ in E , and the proof is complete. \square

Theorem 3.4 Assume that (f) , (h_1) , (h_2) , (g_1) , (g_2) , and (g_3) hold. Then, for any $q \geq 0$, there exists $u_0 \in H$ such that

$$J'_q(u_0) = 0 \quad \text{and} \quad J_q(u_0) = c_0,$$

where c_0 is given by Lemma 3.2

Proof Since $c_0 = \inf\{J_q(u) : u \in \overline{B}_\rho\} < 0$, by Ekeland's variational principle [18], there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \leq J_q(u_n) \leq c_0 + \frac{1}{n}, \quad J_q(w) \geq J_q(u_n) - \frac{1}{n}\|w - u_n\|, \quad \forall w \in \overline{B}_\rho.$$

By the standard procedure, we can see that $\{u_n\}$ is a bounded (PS) sequence of J_q . By Lemma 3.3, $\{u_n\}$ possesses a convergent subsequence. We may assume that, up to a subsequence, $u_n \rightarrow u_0$ in H . Hence $J_q(u_0) = c_0$ and $J'_q(u_0) = 0$. \square

4 Proof of Theorem 1.1

In this section, we aim to prove that system (1.1) has a positive energy solution. It is difficult to prove the boundedness of a (PS) sequence of J_q . Here we consider the following perturbed functional. For $u \in H$,

$$\begin{aligned} J_{q,\lambda}^T(u) &= \frac{a}{2}\|u\|^2 + \frac{1}{4}\|u\|^4 + \frac{q}{2}h_T(u) \int_{\mathbb{R}^3} F(u)\phi_u \, dx - \lambda \int_{\mathbb{R}^3} G(u) \, dx - \int_{\mathbb{R}^3} hu \, dx \\ &= A(u) - \lambda B(u), \end{aligned}$$

where $\lambda \in [1/2, 1]$ and

$$\begin{aligned} A(u) &= \frac{a}{2}\|u\|^2 + \frac{1}{4}\|u\|^4 + \frac{q}{2}h_T(u) \int_{\mathbb{R}^3} F(u)\phi_u \, dx - \int_{\mathbb{R}^3} hu \, dx, \\ B(u) &= \int_{\mathbb{R}^3} G(u) \, dx. \end{aligned}$$

Then $J_{q,\lambda}^T$ is of C^1 functionals on $X = H$, and for any $u, v \in H$, we have

$$\begin{aligned} \langle (J_{q,\lambda}^T)'(u), v \rangle &= (a + \|u\|^2 + a_q^T(u)) \langle u, v \rangle + qh_T(u) \int_{\mathbb{R}^3} f(u)\phi_u v \, dx \\ &\quad - \lambda \int_{\mathbb{R}^3} g(u)v \, dx - \int_{\mathbb{R}^3} hv \, dx, \end{aligned} \quad (4.1)$$

where

$$a_q^T(u) = qT^{-2}\chi'(T^{-2}\|u\|^2) \int_{\mathbb{R}^3} F(u)\phi_u \, dx. \quad (4.2)$$

The following lemmas imply that $J_{q,\lambda}^T$ satisfies the conditions of Theorem 2.2.

Lemma 4.1 Assume that (f) , (h_1) , (h_2) , (g_1) , (g_2) , and (g_3) hold. Then the following results hold:

- (i) There exist $r, a > 0$ such that, for all $\lambda \in I$, $J_{q,\lambda}^T|_{\|u\|=r} \geq a$.
- (ii) There exists $e \in H \setminus \{0\}$ with $\|e\| \geq r$ such that, for all $\lambda \in I$, $J_{q,\lambda}^T(e) < 0$.
- (iii) For all $\lambda \in I$,

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_{q,\lambda}^T(\gamma(t)) > \max\{J_{q,\lambda}^T(0), J_\lambda(e)\} > 0,$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = e\}.$$

Proof (i) By conditions (g_1) and (g_2) , for any $u \in H$ and $\lambda \in I$,

$$\begin{aligned} J_{q,\lambda}^T(u) &= \frac{a}{2} \|u\|^2 + \frac{1}{4} \|u\|^4 + \frac{q}{2} h_T(u) \int_{\mathbb{R}^3} F(u) \phi_u \, dx - \lambda \int_{\mathbb{R}^3} G(u) \, dx - \int_{\mathbb{R}^3} hu \, dx \\ &\geq \frac{a}{2} \|u\|^2 - \int_{\mathbb{R}^3} G(u) \, dx - \int_{\mathbb{R}^3} hu \, dx. \end{aligned}$$

So, by (3.3) and Lemma 3.1, there exists $r > 0$ such that, for all $\lambda \in I$ and $u \in H$ with $\|u\| = r$, we have $J_{q,\lambda}^T(u) \geq a > 0$.

(ii) For any $\lambda \in I$, we choose a radial function $\psi \in H$ with $\psi \geq 0$ and $\|\psi\| = 1$. By (g_3) , we have that, for any $C_2 > 0$ with $C_2 \int_{\mathbb{R}^3} \psi^4 \, dx > 1/2$, there exists $C_3 > 0$ such that

$$G(u) \geq C_2 |u|^4 - C_3, \quad \forall u \in \mathbb{R}.$$

Then, for $t > T$,

$$\begin{aligned} J_{q,\lambda}^T(t\psi) &= \frac{at^2}{2} \|\psi\|^2 + \frac{t^4}{4} \|\psi\|^4 + \frac{q}{2} \chi\left(\frac{t^2}{T^2}\right) \int_{\mathbb{R}^3} F(t\psi) \phi_{t\psi} \, dx \\ &\quad - \lambda \int_{\mathbb{R}^3} G(t\psi) \, dx - t \int_{\mathbb{R}^3} h\psi \, dx \\ &= \frac{at^2}{2} + \frac{t^4}{4} - \lambda \int_{\mathbb{R}^3} G(t\psi) \, dx - t \int_{\mathbb{R}^3} h\psi \, dx \\ &\leq \frac{at^2}{2} + \frac{t^4}{4} - \frac{1}{2} C_2 t^4 \int_{\mathbb{R}^3} \psi^4 \, dx + C_4. \end{aligned} \tag{4.3}$$

Then we can choose $t > 0$ large enough such that $J_{q,\lambda}^T(t\psi) < 0$. Taking $e = t\psi$, then (ii) holds.

(iii) Now fix $\lambda \in I$ and $\gamma \in \Gamma_\lambda$ with $\gamma(1) = e$. By the continuity of γ , there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\| = r$. Therefore, for any $\lambda \in I$, we have

$$c_\lambda \geq \inf_{\gamma \in \Gamma_\lambda} J_{q,\lambda}^T(\gamma(t_\gamma)) \geq a > 0.$$

The proof is complete. \square

Lemma 4.2 For any $\lambda \in I$ and $4q^2\tilde{T} < a$, each bounded Palais–Smale sequence of the functional $J_{q,\lambda}^T$ admits a convergent subsequence, where $\tilde{T} = C_1(T^2 + T^{2\alpha})$.

Proof Let $\lambda \in I$ and $\{u_n\}$ be a bounded (PS) sequence of $J_{q,\lambda}^T$, that is, $\{u_n\}$ and $J_{q,\lambda}^T(u_n)$ are bounded, $(J_{q,\lambda}^T)'(u_n) \rightarrow 0$ in H^{-1} . We may assume that, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H, \\ u_n &\rightarrow u \quad \text{in } L^s(\mathbb{R}^3), s \in (2, 6), \\ u_n &\rightarrow u \quad \text{a.e. on } \mathbb{R}^3. \end{aligned}$$

By (4.1) and the proof of Lemma 3.3,

$$\begin{aligned} & \langle (J_{q,\lambda}^T)'(u_n) - (J_{q,\lambda}^T)'(u), u_n - u \rangle \\ &= (a + \|u_n\|^2 + a_q^T(u_n)) \langle u_n, u_n - u \rangle - (a + \|u\|^2 + a_q^T(u)) \langle u, u_n - u \rangle \\ & \quad + qh_T(u_n) \int_{\mathbb{R}^3} f(u_n) \phi_{u_n}(u_n - u) \, dx - qh_T(u) \int_{\mathbb{R}^3} f(u) \phi_u(u_n - u) \, dx \\ & \quad - \lambda \int_{\mathbb{R}^3} (g(u_n) - g(u))(u_n - u) \, dx \\ &= (a + \|u_n\|^2 + a_q^T(u_n)) \langle u_n, u_n - u \rangle + o_n(1), \end{aligned}$$

and then $(a + \|u_n\|^2 + a_q^T(u_n)) \langle u_n, u_n - u \rangle \rightarrow 0$, $n \rightarrow \infty$.

When $\|u_n\| \leq T$, by Lemma 2.1(iv), we obtain that

$$\left| \int_{\mathbb{R}^3} F(u_n) \phi_{u_n} \, dx \right| \leq qC_1 (\|u_n\|^4 + \|u_n\|^{2(1+\alpha)}) = qT^2 \tilde{T}. \quad (4.4)$$

By (4.2),

$$|a_q^T(u_n)| \leq qT^{-2} |\chi'(T^{-2} \|u_n\|^2)| \left| \int_{\mathbb{R}^3} F(u_n) \phi_{u_n} \, dx \right| \leq 4q^2 \tilde{T}. \quad (4.5)$$

It follows from the assumption $4q^2 \tilde{T} < a$ that $a + \|u_n\|^2 + a_q^T(u_n) \geq a - 4q^2 \tilde{T} > 0$, hence $\langle u_n, u_n - u \rangle \rightarrow 0$. This together with $u_n \rightharpoonup u$ shows that $u_n \rightarrow u$ in H . The proof is complete. \square

Lemma 4.3 *Let $4q^2 \tilde{T} < a$, then for almost every $\lambda \in I$, there exists $u^\lambda \in H \setminus \{0\}$ such that $(J_{q,\lambda}^T)'(u^\lambda) = 0$ and $J_{q,\lambda}^T(u^\lambda) = c_\lambda$.*

Proof Firstly, it is easy to see that $B(u) \geq 0$ and $A(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$. Then, by Lemma 4.1 and Theorem 2.2, for almost every $\lambda \in I$, there exists a bounded sequence $\{u_n^\lambda\} \subset H$ such that

$$(J_{q,\lambda}^T)'(u_n^\lambda) \rightarrow 0, \quad J_{q,\lambda}^T(u_n^\lambda) \rightarrow c_\lambda.$$

By Lemma 4.2, we can obtain that there exists $u^\lambda \in H$ such that $u_n^\lambda \rightarrow u^\lambda$. Hence $(J_{q,\lambda}^T)'(u^\lambda) = 0$ and $J_{q,\lambda}^T(u^\lambda) = c_\lambda$. Furthermore, it follows from $f(t) = g(t) = 0$ for $t < 0$ and Lemma 4.1(i) that $u^\lambda \in H \setminus \{0\}$. \square

By virtue of Lemma 4.3, we have that there exist $\{\lambda_n\} \subset I$ with $\{\lambda_n\} \rightarrow 1^-$ and a nonnegative sequence $\{u^{\lambda_n}\}$ (denoted by $\{u_n\}$ for simplicity) satisfying

$$(J_{q,\lambda_n}^T)'(u_n) = 0, \quad J_{q,\lambda_n}^T(u_n) = c_{\lambda_n}.$$

In order to obtain that $\|u_n\| \leq T/\sqrt{2}$, the following Pohozaev identity is important, the proof can be obtained as Lemma 2.2 in [13], and the details are omitted here.

Lemma 4.4 *If $u \in H$ is a weak solution of*

$$\begin{cases} [a + \|u\|^2 + a_q^T(u)](-\Delta u + bu) + qh_T(u)\phi f(u) = \lambda g(u) + h(x), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 2qF(u), & \text{in } \mathbb{R}^3, \end{cases} \quad (4.6)$$

then the following Pohozaev identity holds:

$$\begin{aligned} & [a + \|u\|^2 + a_q^T(u)] \left(\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2} b \int_{\mathbb{R}^3} u^2 dx \right) + \frac{5}{2} qh_T(u) \int_{\mathbb{R}^3} F(u)\phi_u dx \\ & = 3\lambda \int_{\mathbb{R}^3} G(u) dx + \int_{\mathbb{R}^3} (3h(x) + \langle \nabla h(x), x \rangle) u dx. \end{aligned} \quad (4.7)$$

The following lemma shows that $\|u_n\| \leq T/\sqrt{2}$.

Lemma 4.5 *Let u_n be a critical point of J_{q,λ_n}^T at level c_{λ_n} . Then there exist $T_0 > 1$ and $q_0 > 0$ with $16q_0^2 T_0^2 \tilde{T} < a$ such that, for any $q \in [0, q_0)$ and $b \leq 1/4\gamma_2^2$, $\|u_n\| \leq T_0/\sqrt{2}$ for all $n \in \mathbb{N}$.*

Proof We will argue by contradiction. Assume that for every $T > 1$ there exists q_T satisfying $16q_T^2 T^2 \tilde{T} < a$ such that

$$\limsup_{n \rightarrow \infty} \|u_n\| \geq T/\sqrt{2}. \quad (4.8)$$

Firstly, since $(J_{q,\lambda_n}^T)'(u_n) = 0$, by (4.7), u_n satisfies the following Pohozaev identity:

$$\begin{aligned} & [a + \|u_n\|^2 + a_q^T(u_n)] \left(\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3}{2} b \int_{\mathbb{R}^3} u_n^2 dx \right) \\ & + \frac{5}{2} qh_T(u_n) \int_{\mathbb{R}^3} F(u_n)\phi_{u_n} dx \\ & = 3\lambda_n \int_{\mathbb{R}^3} G(u_n) dx + \int_{\mathbb{R}^3} (3h(x) + \langle \nabla h(x), x \rangle) u_n dx. \end{aligned} \quad (4.9)$$

By using $J_{q,\lambda_n}^T(u_n) = c_{\lambda_n}$, we have that

$$\begin{aligned} & \frac{3a}{2} \|u_n\|^2 + \frac{3}{4} \|u_n\|^4 + \frac{3q}{2} h_T(u_n) \int_{\mathbb{R}^3} F(u_n)\phi_{u_n} dx - 3\lambda_n \int_{\mathbb{R}^3} G(u_n) dx - 3 \int_{\mathbb{R}^3} hu_n dx \\ & = 3c_{\lambda_n}. \end{aligned} \quad (4.10)$$

Hence, by (4.5), (4.9), (4.10), and (h_3) , we can obtain that

$$\begin{aligned}
 & \left(\|u_n\|^2 + \frac{a}{2} \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \\
 & \leq \left[a + \|u_n\|^2 + a_q^T(u_n) \right] \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \\
 & = 3c_{\lambda_n} + \frac{3}{2} a_q^T(u_n) \|u_n\|^2 + \frac{3}{4} \|u_n\|^4 + qh_T(u_n) \int_{\mathbb{R}^3} F(u_n) \phi_{u_n} \, dx \\
 & \quad - \int_{\mathbb{R}^3} \langle \nabla h(x), x \rangle u_n \, dx \\
 & \leq 3c_{\lambda_n} + \frac{3}{2} a_q^T(u_n) \|u_n\|^2 + \frac{3}{4} \|u_n\|^4 + qh_T(u_n) \int_{\mathbb{R}^3} F(u_n) \phi_{u_n} \, dx.
 \end{aligned} \tag{4.11}$$

Now we estimate the right-hand side of (4.11), by Lemma 4.1 and (4.3), we have

$$\begin{aligned}
 c_{\lambda_n} & \leq \max_t J_{q, \lambda_n}^T(t\psi) \\
 & \leq \max_t \left\{ \frac{at^2}{2} + \frac{t^4}{4} - \lambda_n \int_{\mathbb{R}^3} G(t\psi) \, dx \right\} + \max_t \frac{q}{2} \chi \left(\frac{t^2}{T^2} \right) \int_{\mathbb{R}^3} F(t\psi) \phi_{t\psi} \, dx \\
 & \leq \max_t \left\{ \frac{at^2}{2} + \frac{t^4}{4} - \frac{1}{2} C_2 t^4 \int_{\mathbb{R}^3} \psi^4 \, dx + C_4 \right\} + A_q(T) \\
 & = C_5 + A_q(T).
 \end{aligned}$$

If $t \geq T$, then $\chi(\frac{t^2}{T^2}) = 0$. Thus, by (4.4), we have that

$$A_q(T) \leq \frac{q}{2} \max_{t \in [0, T]} \left| \int_{\mathbb{R}^3} F(t\psi) \phi_{t\psi} \, dx \right| \leq \frac{1}{2} q^2 T^2 \tilde{T}.$$

By (4.4) and (4.5), we also have that

$$qh_T(u_n) \int_{\mathbb{R}^3} F(u_n) \phi_{u_n} \, dx \leq q^2 T^2 \tilde{T}$$

and

$$|a_q^T(u_n)| \|u_n\|^2 \leq 4q^2 T^2 \tilde{T}.$$

Then, by (4.11), we can obtain that

$$\left(\|u_n\|^2 + \frac{3a}{4} \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \leq \frac{3}{4} \|u_n\|^4 + 3 \left(C_5 + \frac{1}{2} q^2 T^2 \tilde{T} \right) + 6q^2 T^2 \tilde{T} + q^2 T^2 \tilde{T},$$

which implies that

$$\left(\frac{1}{4} \|u_n\|^2 + \frac{3a}{4} \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \leq \frac{3b}{4} \|u_n\|^2 \int_{\mathbb{R}^3} u_n^2 \, dx + 3C_5 + \frac{17}{2} q^2 T^2 \tilde{T}. \tag{4.12}$$

Since $b \leq 1/4\gamma_2^2$, (4.12) implies that

$$\frac{3a}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \leq 3C_5 + \frac{17}{2} q^2 T^2 \tilde{T}.$$

On the other hand, since $\langle (J_{q,\lambda_n}^T)'(u_n), u_n \rangle = 0$, by (4.1), (g_1) , and (g_2) , we have that

$$\begin{aligned} & (a + \|u_n\|^2 + a_q^T(u_n))\|u_n\|^2 + qh_T(u_n) \int_{\mathbb{R}^3} f(u_n)\phi_{u_n}n_n \, dx \\ &= \lambda_n \int_{\mathbb{R}^3} g(u_n)n_n \, dx + \int_{\mathbb{R}^3} hu_n \, dx \\ &\leq \frac{a}{2\gamma_2^2} \|u_n\|_2^2 + C\|u_n\|_6^6 + \gamma_2 \|h\|_2 \|u_n\|. \end{aligned} \quad (4.13)$$

Thus, by (4.12) and (4.13), we obtain that

$$\frac{3a}{4} \|u_n\|^2 - \gamma_2 \|h\|_2 \|u_n\| \leq C\|u_n\|_6^6 \leq S^{-3} C \left(4C_5/a + \frac{34}{3a} q^2 T^2 \tilde{T} \right)^3.$$

Since $\gamma_2 \|h\|_2 < \eta(\rho)$ in Lemma 3.1 and $16q_T^2 T^2 \tilde{T} < a$, then we have

$$\frac{3a}{4} \|u_n\|^2 - \eta(\rho) \|u_n\| < S^{-3} C (4C_5/a + 17/24)^3. \quad (4.14)$$

By (4.8) and (4.14), it is impossible for $T > 1$ large enough. Thus, we obtain the conclusion. \square

Proof of Theorem 1.1 By virtue of the result of Theorem 3.4, in order to prove Theorem 1.1, here we just need to prove that problem (1.1) has a positive radial solution v_0 with positive energy. Let T_0, q_0 be defined as in Lemma 4.5 and u_n be a critical point of $J_{q,\lambda_n}^{T_0}$ at level c_{λ_n} . From Lemma 4.5 we have that $\|u_n\| \leq T_0/\sqrt{2}$ and $\{c_{\lambda_n}\}$ is bounded. So

$$\begin{aligned} J_{q,\lambda_n}^{T_0}(u_n) &= \frac{a}{2} \|u_n\|^2 + \frac{1}{4} \|u_n\|^4 + \frac{q}{2} h_T(u_n) \int_{\mathbb{R}^3} F(u_n)\phi_{u_n} \, dx \\ &\quad - \lambda_n \int_{\mathbb{R}^3} G(u_n) \, dx - \int_{\mathbb{R}^3} hu_n \, dx. \end{aligned}$$

Now we claim that $\{u_n\}$ is a (PS) sequence of J_q . Indeed,

$$J_q(u_n) = J_{q,\lambda_n}^{T_0}(u_n) + (\lambda_n - 1) \int_{\mathbb{R}^3} G(u_n) \, dx,$$

and

$$\langle J_q'(u_n), v \rangle = \langle (J_{q,\lambda_n}^{T_0})'(u_n), v \rangle + (\lambda_n - 1) \int_{\mathbb{R}^3} g(u_n)v \, dx.$$

The fact $\{u_n\}$ is bounded implies that $\int_{\mathbb{R}^3} G(u_n) \, dx$ is bounded and $\{\int_{\mathbb{R}^3} g(u_n)v \, dx\} \leq C\|v\|$. Thus, when $\lambda_n \rightarrow 1$, we have that $\{u_n\}$ is a bounded (PS) sequence of J_q . By Lemma 3.3, $\{u_n\}$ has a convergent subsequence, we may assume that $u_n \rightarrow v_0$. Consequently, $J_q'(v_0) = 0$. According to Lemma 4.1, we have that

$$J_q(v_0) = \lim_{n \rightarrow \infty} J_q(u_n) = \lim_{n \rightarrow \infty} J_{q,\lambda_n}^{T_0}(u_n) \geq c > 0,$$

and v_0 is a positive solution by the condition of $f(t) = g(t) = 0$ for $t < 0$. Thus, the proof is finished. \square

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