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New general decay results in an infinite memory viscoelastic problem with nonlinear damping

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Abstract

This work is concerned with a viscoelastic equation with a nonlinear frictional damping and a relaxation function satisfying $g'(t) \le -\xi(t)g^p(t), t \ge 0, 1 \le p < \frac{3}{2}$. We establish general decay rate results using the multiplier method and some properties of non-homogeneous ordinary differential inequalities. These results extend and improve many results in the literature.

Keywords: General decay; Infinite memory; Viscoelastic problems

1 Introduction

In this paper, we consider the following viscoelastic problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)\,ds + |u_t|^{m-2}u_t = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,-t) = u_0(x,t), u_t(x,0) = u_1(x) & \text{in } \Omega \times (0,+\infty), \end{cases} \tag{1}$$

where u denotes the transverse displacement of waves and Ω is a bounded domain of $\mathbb{R}^N (N \ge 1)$ with a smooth boundary $\partial\Omega$, g is a positive and decreasing function and $m > 1$.

The study of viscoelastic problems has attracted the attention of many authors, and several decay and blow-up results have been established. In [1], Cavalcanti et al. considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds + a(x)u_t + |u|^{p-1}u = 0, \quad \text{in } \Omega \times (0,\infty), \tag{2}$$

where $a : \Omega \to \mathbb{R}^+$ is a function which may vanish on a part of the domain Ω but satisfies $a(x) \ge a_0$ on $\omega \subset \Omega$ and g satisfies, for two positive constants ξ_1 and ξ_2,

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t), \quad t \ge 0.$$

They established an exponential decay result under some restrictions on ω. Berrimi and Messaoudi [2] improved the result of [1], under weaker conditions on both a and g, to a problem where a source term is competing with the damping term. Fabrizio and Polidoro [3] studied the following system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

and showed that the exponential decay of the relaxation function is a necessary condition for the exponential decay of the solution energy. Cavalcanti and Oquendo [4] considered the following problem:

$$u_{tt} - k_0 \Delta u + \int_0^t \operatorname{div}\big[a(x)g(t-s)\Delta u(x,s)\big]\,ds + b(x)h(u_t) + f(u) = 0 \tag{3}$$

and established, for $a(x) + b(x) \geq \rho > 0$, an exponential stability result for g decaying exponentially and h linear and a polynomial stability result for g decaying polynomially and h nonlinear. Rivera [5] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy a bounded domain or the whole space $\mathbb{R}^n$, with zero boundary and history data and in the absence of external body forces. In the bounded domain case, an exponential decay result was proved for exponentially decaying memory kernels, and for the whole space case, a polynomial decay result was established and the rate of the decay was given. This latter result was later pushed to a situation where the kernel is decaying algebraically but not exponentially by Cabanillas and Rivera [6]. In their paper, the authors showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of the decay of the relaxation function and may be improved by the regularity of the initial data. The authors considered both cases, the bounded domains and that of a material occupying the entire space. This result was later improved by Baretto et al. [7], where equations related to linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same decay rate of the relaxation function. For partially viscoelastic materials, Rivera et al. [8, 9] showed that solutions decay exponentially to zero, provided the relaxation function decays in a similar fashion, regardless of the size of the viscoelastic part of the material. Pazoto et al. [10] investigated a class of abstract viscoelastic equations of the form

$$u_{tt} + Au(t) + \beta u(t) - \big(g * A^{\alpha} u\big)(t) = 0 \tag{4}$$

for $0 \leq \alpha \leq \beta, \beta \geq 0$. The main focus was on the case when $0 < \alpha < 1$, and the main result was that solutions for (4) decay polynomially even if the kernel g decays exponentially. This result is sharp (see Theorem 12 [10]). See also Rivera et al. [11], where the authors studied a more general abstract problem than (4) and established a necessary and sufficient condition to obtain an exponential decay. The work of [10] and [11] has been improved by Jamilu and Messaoudi [12].

For infinite history problems, Giorgi et al. [13] considered the following semilinear hyperbolic equation with linear memory in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)\,ds + g(u) = f, \quad \text{in } \Omega \times \mathbb{R}^+$$

with $K(0), K(\infty) > 0$, and $K' \leq 0$ and proved the existence of global attractors for the solutions. Conti and Pata [14] considered the following semilinear hyperbolic equation:

$$u_{tt} + \alpha u_t - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)\,ds + g(u) = f, \quad \text{in } \Omega \times \mathbb{R}^+, \tag{5}$$

where the memory kernel is a convex decreasing smooth function such that $K(0) > K(\infty) > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear term of at most cubic growth satisfying some conditions. They proved the existence of a regular global attractor. In [15], Appleby et al. studied the linear integro-differential equation

$$u_{tt} + Au(t) + \int_{-\infty}^{t} K(t-s)Au(s)\,ds = 0, \quad t > 0$$

and established an exponential decay result for strong solutions in a Hilbert space. Pata [16] discussed the decay properties of the semigroup generated by the following equation:

$$u_{tt} + \alpha Au(t) + \beta u_t(t) - \int_0^{+\infty} \mu(s)Au(t-s)\,ds = 0,$$

where A is a strictly positive self-adjoint linear operator and $\alpha > 0, \beta \geq 0$ and the memory kernel μ is a decreasing function satisfying specific conditions. Subsequently, they established necessary as well as sufficient conditions for the exponential stability. In [17], Guesmia considered

$$u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)\,ds = 0$$

and introduced a new ingenuous approach for proving a more general decay result based on the properties of convex functions and the use of the generalized Young inequality. He used a larger class of infinite history kernels satisfying the following condition:

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))}\,ds + \sup_{s\in\mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty \tag{6}$$

such that

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} G'(t) = +\infty, \tag{7}$$

where $G : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing strictly convex function. Using this approach, Guesmia and Messaoudi [18] later looked into

$$u_{tt} - \Delta u + \int_0^t g_1(t-s)\operatorname{div}\bigl(a_1(x)\nabla u(s)\bigr)\,ds + \int_0^{+\infty} g_2(s)\operatorname{div}\bigl(a_2(x)\nabla u(t-s)\bigr)\,ds = 0,$$

in a bounded domain and under suitable conditions on a_1 and a_2 and for a wide class of relaxation functions g_1 and g_2 that are not necessarily decaying polynomially or exponentially, and established a general decay result from which the usual exponential and polynomial decay rates are only special cases. Messaoudi and Al-Gharabli [19] considered the following nonlinear wave equation:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^{+\infty} g(s)\Delta u(t-s)\,ds = 0, \quad \text{in } \Omega \times (0,+\infty)$$

and proved a general decay result of the solution energy using an approach different from that introduced by Guesmia [17]. For more results in this direction, we refer to [20] and [21].

In the present work, we study the asymptotic behavior of solutions of (1), under assumption (9) (below) instead of (6) considered in Guesmia [17] and Al-Gharabli [22]. This work will extend the result of Belhannache et al. [23] for the finite history case to the infinite history case. The proof of the current result is easier than the one in [17] and [22] since we need no convex function properties or the generalized Young inequality. Moreover, this result gives a better rate of decay in some cases (see Remark 4.3 below).

The rest of this paper is organized as follows. In Sect. 2, we present some assumptions and material needed for our work. Some technical lemmas are presented and proved in Sect. 3. Finally, we state and prove our main decay results and provide some examples in Sect. 4.

2 Preliminaries

In this section, we present some materials needed for the proof of our results and state a well-posedness result of the problem. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms and assume the following hypotheses.

$(A1)$ $g:\mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$g(0) > 0, \quad 1 - \int_0^{+\infty} g(s)\,ds = \ell > 0. \tag{8}$$

$(A2)$ There exist a nonincreasing differentiable function $\xi:\mathbb{R}^+ \to \mathbb{R}^+$ and $1 \le p < \frac{3}{2}$ such that

$$g'(t) \le -\xi(t)g^p(t), \quad \forall t \in \mathbb{R}^+. \tag{9}$$

$(A3)$ For the nonlinearity in the damping, we assume that

$$1 < m \le \frac{2n}{n-2}, \quad \text{if } n > 2 \quad \text{and} \quad m > 1, \quad \text{if } n = 1,2. \tag{10}$$

$(A4)$ There exists $m_0 \ge 0$ such that

$$\|\nabla u_0(\cdot,s)\|_2 \le m_0, \quad \forall s > 0. \tag{11}$$

We introduce the "modified" energy associated to problem (1) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1-\ell}{2}\|\nabla u\|_2^2 + \frac{1}{2}(go\nabla u)(t), \tag{12}$$

where

$$(go\nabla u)(t) = \int_0^{+\infty} g(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2\,ds.$$

Direct differentiation, using (1), leads to

$$E'(t) = \frac{1}{2}\big(g'o\nabla u\big)(t) - \int_\Omega |u_t|^m\,dx \le 0. \tag{13}$$

Now, we state without proof the existence result to problem (1).

Proposition 1 ([22, 23]) *Let $(u_0(\cdot,0), u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that $(A1)$–$(A4)$ hold and $m > 1$. Then problem (1) has a unique weak global solution.*

3 Technical lemmas

In this section, we state and establish several lemmas needed for the proof of our main result.

Lemma 3.1 ([4, 24]) *Assume that g satisfies $(A1)$ and $(A2)$, then*

$$\int_0^{+\infty} \xi(t)g^{1-\sigma}(t)\,dt < +\infty, \quad \forall\sigma < 2-p. \tag{14}$$

Lemma 3.2 ([4, 24]) *Assume that $(A1)$ and $(A2)$ hold and u is the solution of (1) then, for $0 < \sigma < 1$, we have*

$$\int_0^t g(s)\|\nabla u(t-s) - \nabla u(t)\|_2^2\,ds \le c\left[\left(\int_0^t g^{1-\sigma}(t)\,dt\right)E(0)\right]^{\frac{p-1}{p-1+\sigma}} \big(g^p o\nabla u\big)^{\frac{\sigma}{p-1+\sigma}}.$$

By taking $\sigma = \frac{1}{2}$, we get

$$\int_0^t g(s)\|\nabla u(t-s) - \nabla u(t)\|_2^2\,ds \le c\left[\int_0^t g^{\frac{1}{2}}(s)\,ds\right]^{\frac{2p-2}{2p-1}} \big(g^p o\nabla u\big)^{\frac{1}{2p-1}}(t). \tag{15}$$

Remark 3.1 Using (12), $(A4)$, and the fact E is nonincreasing, we obtain

$$\begin{aligned}
&\|\nabla u(t) - \nabla u(t-s)\|_2^2 \\
&\quad\le 2\|\nabla u(t)\|_2^2 + 2\|\nabla u(t-s)\|_2^2 \\
&\quad\le 4\sup_{s>0}\|\nabla u(s)\|_2^2 + 2\sup_{\tau<0}\|\nabla u(\tau)\|_2^2 \\
&\quad\le 4\sup_{s>0}\|\nabla u(s)\|_2^2 + 2\sup_{\tau>0}\|\nabla u_0(\tau)\|_2^2 \\
&\quad\le \frac{8}{1-\ell}E(0) + 2m_0^2 := N_1.
\end{aligned} \tag{16}$$

Corollary 1 *Assume that $(A1)$–$(A4)$ hold and u is a solution of (1), then*

$$\xi(t)\int_0^{+\infty} g(s)\|\nabla u(t-s) - \nabla u(t)\|_2^2\,ds \le C\big[-E'(t)\big]^{\frac{1}{2p-1}} + N_1\xi(t)\int_t^{+\infty} g(s)\,ds. \tag{17}$$

Proof Multiply both sides of (15) by $\xi(t)$ and use (13), (14), and (16) to obtain

$$\begin{aligned}
&\xi(t)\int_0^{+\infty} g(s)\big\|\nabla u(t-s)-\nabla u(t)\big\|_2^2\,ds \\
&\quad= \xi(t)\int_0^{t} g(s)\big\|\nabla u(t-s)-\nabla u(t)\big\|_2^2\,ds + \xi(t)\int_t^{+\infty} g(s)\big\|\nabla u(t-s)-\nabla u(t)\big\|_2^2\,ds \\
&\quad\le C\xi^{\frac{2p-2}{2p-1}}(t)\bigg[\int_0^t g^{\frac{1}{2}}(s)\,ds\bigg]^{\frac{2p-2}{2p-1}}\xi^{\frac{1}{2p-1}}(t)\big(g^p o\nabla u\big)^{\frac{1}{2p-1}}(t) + N_1\xi(t)\int_t^{+\infty} g(s)\,ds \\
&\quad\le C\bigg[\int_0^t \xi(s)g^{\frac{1}{2}}(s)\,ds\bigg]^{\frac{2p-2}{2p-1}}\big(\xi g^p o\nabla u\big)^{\frac{1}{2p-1}}(t) + N_1\xi(t)\int_t^{+\infty} g(s)\,ds \\
&\quad\le C\bigg[\int_0^{+\infty} \xi(s)g^{\frac{1}{2}}(s)\,ds\bigg]^{\frac{2p-2}{2p-1}}\big(-g' o\nabla u\big)^{\frac{1}{2p-1}}(t) + N_1\xi(t)\int_t^{+\infty} g(s)\,ds \\
&\quad\le C\big[-E'(t)\big]^{\frac{1}{2p-1}} + N_1\xi(t)\int_t^{+\infty} g(s)\,ds. \qquad (18)
\end{aligned}$$

∎

Lemma 3.3 ([22]) *Under assumptions* $(A1)$–$(A4)$, *the functional*

$$\psi(t) := \int_\Omega u u_t\,dx$$

satisfies, along the solution, the estimate

$$\begin{aligned}
\psi'(t) \le{}& -\frac{\ell}{4}\|\nabla u\|_2^2 + \|u_t\|_2^2 + \frac{1-\ell}{2\ell}(go\nabla u)(t) \\
&+ c\int_\Omega |u_t|^m\,dx, \quad \text{if } m \ge 2 \qquad (19)
\end{aligned}$$

and

$$\begin{aligned}
\psi'(t) \le{}& -\frac{\ell}{4}\|\nabla u\|_2^2 + \|u_t\|_2^2 + \frac{1-\ell}{2\ell}(go\nabla u)(t) \\
&+ c(\Omega)\bigg(\int_\Omega |u_t|^m\,dx\bigg)^{\frac{2m-2}{m}}, \quad \text{if } m < 2. \qquad (20)
\end{aligned}$$

Lemma 3.4 ([22]) *Under assumptions* $(A1)$–$(A4)$, *the functional*

$$\chi(t) := -\int_\Omega u_t\int_0^{+\infty} g(s)\big(u(t)-u(t-s)\big)\,ds\,dx$$

satisfies, for all $\delta > 0$ *and along the solution, the estimate*

$$\begin{aligned}
\chi'(t) \le{}& -\delta\big[1+2(1-\ell)^2\big]\|\nabla u\|_2^2 - \big((1-\ell)-\delta\big)\|u_t\|_2^2 + C(\delta)(go\nabla u)(t) \\
&+ \frac{g(0)}{4\delta}\big(-\big(g' o\nabla u\big)\big)(t) + C(\delta)\int_\Omega |u_t|^m\,dx, \quad \text{if } m \ge 2 \qquad (21)
\end{aligned}$$

and

$$\chi'(t) \le -\delta\big[1+2(1-\ell)^2\big]\|\nabla u\|_2^2 - \big((1-\ell)-\delta\big)\|u_t\|_2^2 + C(\delta)(go\nabla u)(t)$$

$$+\frac{g(0)}{4\delta}\big(-\big(g'o\nabla u\big)\big)(t)+c(\delta,\Omega)\left(\int_\Omega |u_t|^m\,dx\right)^{\frac{2m-2}{m}}, \quad \text{if } m<2. \tag{22}$$

Lemma 3.5 ([22]) *Assume that* (A1)–(A4) *hold. Then there exist strictly positive constants* $\varepsilon_1,\varepsilon_2,\alpha_1,c$ *such that the functional*

$$L=E(t)+\varepsilon_1\psi(t)+\varepsilon_2\chi(t)$$

satisfies, for all $t\in\mathbb{R}^+$,

$$F\sim E, \tag{23}$$

$$L'(t)\le -\alpha_1E(t)+c(go\nabla u)(t) \quad \text{if } m\ge 2 \tag{24}$$

and

$$L'(t)\le -\alpha_1E(t)+c(go\nabla u)(t)+c\left(\int_\Omega |u_t|^m\,dx\right)^{\frac{2m-2}{m}} \quad \text{if } m<2. \tag{25}$$

4 The main result

In this section we state and prove our decay result. We start with two remarks.

Remark 4.1 If $1+\frac{1}{4p-3}<m<2$, we have

$$\frac{2m-2}{m}>\frac{1}{2p-1}, \tag{26}$$

and if $1<m<1+\frac{1}{4p-3}<2$, we have

$$\frac{2m-2}{m}<\frac{1}{2p-1}. \tag{27}$$

Remark 4.2 Using (13) and (16), we have

$$\big|E'(t)\big|\le\frac{1}{2}\big|g'o\nabla u(t)\big|\le\frac{N_1}{2}g(0)=c. \tag{28}$$

Theorem 4.1 *Let* $(u_0(\cdot,0),u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *be given. Assume that* (A1)–(A4) *hold. Then, for* $m\ge 2$, *we have*

$$E(t)\le\delta_1\left(1+\int_0^t\big(g(s)\big)^{1-\delta_0}\,ds\right)e^{-\delta_0\int_0^t\xi(s)\,ds}+\delta_1\int_t^{+\infty}g(s)\,ds, \quad p=1, \tag{29}$$

and

$$E(t)\le C(1+t)^{\frac{-1}{2p-2}}\xi^{-\frac{2p-1}{2p-2}}(t)\left(1+\int_0^t(1+s)^{\frac{1}{2p-2}}\xi^{\frac{2p-1}{2p-2}}(s)h^{2p-1}\,ds\right), \quad 1<p<\frac{3}{2}. \tag{30}$$

Moreover, for any $1<p<\frac{3}{2}$, *if*

$$\int_0^{+\infty}\left[(1+t)^{\frac{-1}{2p-2}}\xi^{-\frac{2p-1}{2p-2}}(t)\left(1+\int_0^t(1+s)^{\frac{1}{2p-2}}\xi^{\frac{2p-1}{2p-2}}(s)h^{2p-1}(s)\,ds\right)\right]dt<+\infty, \tag{31}$$

then we have

$$E(t) \leq C(1+t)^{\frac{-1}{p-1}} \xi^{-\frac{2p}{p-1}}(t)\left(1+\int_0^t (1+s)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s) h^p(s)\, ds\right), \quad 1 < p < \frac{3}{2}, \tag{32}$$

where $Ch(t) := N_1 \xi(t) \int_t^{+\infty} g(s)\, ds)$, δ_1, C *are strictly positive numbers and* $\delta_0 \in (0, \gamma_0]$, $\gamma_0 \in (0,1)$.

Remark 4.3 Let us compare our estimates (30) and (32) with the one of [17] and [22] obtained for (1). Our estimate (32) improves the decay rate given in [17]. Indeed, let $g(t) = \frac{a}{(1+t)^q}$, $q > 2$, where a is chosen so that hypothesis $(A1)$ remains valid. Then

$$g'(t) = \frac{-aq}{(1+t)^{q+1}} = -b\left(\frac{a}{(1+t)^q}\right)^{\frac{q+1}{q}} = -b g^p(t), \quad p = \frac{q+1}{q} < \frac{3}{2}, b > 0. \tag{33}$$

Let us compute

$$h(t) = \xi(t) \int_t^{+\infty} g(s)\, ds = \frac{ab}{q-1}(1+t)^{1-q}, \quad q = \frac{1}{p-1}. \tag{34}$$

Routine calculations yield, for some positive constant C,

$$\int_0^t (1+s)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s) h^p\, ds = C(1+t)^{p(1-q)+\frac{1}{p-1}+1} - C. \tag{35}$$

Therefore, estimate (32) yields

$$E(t) \leq C(1+t)^{\frac{-q^2+q+1}{q}}, \tag{36}$$

which implies that (36) improves the following decay rate obtained in [17]:

$$E(t) \leq C(1+t)^{-p}, \quad \forall 0 < p < \frac{q-1}{2}. \tag{37}$$

This is because $\frac{q^2-q-1}{2} > \frac{q-1}{2}$ for $q > 2$. As a conclusion our approach improves and has a better decay rate than the one of [17].

Proof of Theorem 4.1 For the proof of (29), see [19]. For (30), we multiply (24) by $\xi^{\alpha+1}(t)E^\alpha(t)$, where $\alpha = 2p-2$, and use (17) to obtain

$$\xi^{\alpha+1}(t)E^\alpha(t)L'(t) \leq -\alpha_1 \xi^{\alpha+1}(t)E^{\alpha+1}(t) + c(\xi E)^\alpha(t)\left[-E'(t)\right]^{\frac{1}{\alpha+1}} + Ch(t)\xi^\alpha(t)E^\alpha(t). \tag{38}$$

Use of Young's inequality, with $q = \alpha+1$ and $q^* = \frac{\alpha+1}{\alpha}$, gives

$$\xi^{\alpha+1}(t)E^\alpha(t)L'(t) \leq -\alpha_1 \xi^{\alpha+1}(t)E^{\alpha+1}(t) + 2\varepsilon\xi^{\alpha+1}(t)E^{\alpha+1} - C_\varepsilon E'(t) + C_\varepsilon h^{\alpha+1}(t). \tag{39}$$

Choosing ε small enough and letting $F := \xi^{\alpha+1}E^\alpha L + C_\varepsilon E \sim E$, we have, for positive constants c_1 and c_2,

$$F'(t) \leq -c_1 \xi^{\alpha+1}(t)F^{\alpha+1}(t) + c_2 h^{\alpha+1}(t). \tag{40}$$

Multiplying both sides of (40) by ξ^β, $\beta > 1$, we get

$$\xi^\beta F'(t) \leq -c_1 \xi^{\alpha+1+\beta}(t) F^{\alpha+1}(t) + c_2 \xi^\beta h^{\alpha+1}(t). \tag{41}$$

Recalling that $\xi > 0$ and nonincreasing, one can see that

$$\%labelL0\big(\xi^\beta F(t)\big)' \leq -c_1 \xi^{\alpha+1+\beta}(t) F^{\alpha+1}(t) + c_2 \xi^\beta h^{\alpha+1}(t). \tag{42}$$

Noting $\varphi = \xi^\beta F$ and taking $\beta = \frac{\alpha+1}{\alpha}$, we obtain

$$\varphi'(t) \leq -c_1 \varphi^{\alpha+1}(t) + c_2 \xi^\beta(t) h^{\alpha+1}(t). \tag{43}$$

Let

$$f(t) := \varphi(t) - \Psi(t); \quad \text{where } \Psi(t) = c_2 (1+t)^{\frac{-1}{\alpha}} \int_0^t \xi^\beta(s) h^{\alpha+1}(s)(1+s)^{\frac{1}{\alpha}}\, ds. \tag{44}$$

From the definition of Ψ, we have

$$c_2 \xi^\beta(t) h^{\alpha+1}(t) = \Psi'(t) + \frac{c_2}{\alpha}(1+t)^{\frac{-1}{\alpha}-1} \int_0^t \xi^\beta(s) h^{\alpha+1}(s)(1+s)^{\frac{1}{\alpha}}\, ds. \tag{45}$$

Since $\xi^\beta(s) h^{\alpha+1}(1+s)^{\frac{1}{\alpha}} > 0$, we have, for all $t \geq t_0 > 0$,

$$\nu := \int_0^{t_0} \xi^\beta(s) h^{\alpha+1}(s)(1+s)^{\frac{1}{\alpha}}\, ds \leq \int_0^t \xi^\beta(s) h^{\alpha+1}(s)(1+s)^{\frac{1}{\alpha}}\, ds,$$

and then

$$\frac{\int_0^t \xi^\beta(s) h^{\alpha+1}(s)(1+s)^{\frac{1}{\alpha}}\, ds}{\nu} \geq 1, \quad \forall t \geq t_0.$$

Thus (45) yields, $\forall t \geq t_0$,

$$c_2 \xi^\beta(t) h^{\alpha+1}(t) \leq \Psi'(t) + \frac{1}{\alpha c_2^\alpha \nu^\alpha} c_2^{\alpha+1} \big[(1+t)^{\frac{-1}{\alpha}}\big]^{\alpha+1} \left[\int_0^t \xi^\beta(s) h^{\alpha+1}(s)(1+s)^{\frac{1}{\alpha}}\, ds\right]^{\alpha+1}. \tag{46}$$

We can choose c_2 large enough so that $\frac{1}{\alpha c_2^\alpha \nu^\alpha} \leq c_1$, and then we get

$$c_2 \xi^\beta(t) h^{\alpha+1}(t) \leq \Psi'(t) + c_1 \Psi^{\alpha+1}, \quad \forall t \geq t_0. \tag{47}$$

Now, using (47) and the definition of f, we get, $\forall t \geq t_0$,

$$\begin{aligned} f'(t) = \varphi'(t) - \Psi'(t) &\leq -c_1 \varphi^{\alpha+1}(t) + c_2 \xi^\beta(t) h^{\alpha+1}(t) - \Psi'(t) \\ &\leq -c_1 \big[(f+\Psi)^{\alpha+1}(t)\big] + c_2 \xi^\beta(t) h^{\alpha+1}(t) - \Psi'(t). \end{aligned} \tag{48}$$

Since $f(0) > 0$, then there exists $t_1 > 0$ such that $f(t) > 0, \forall t \in [0, t_1)$. Hence,

$$f'(t) \leq -c_1 \big[f^{\alpha+1}(t) + \Psi^{\alpha+1}(t)\big] + c_2 \xi^\beta(t) h^{\alpha+1}(t) - \Psi'(t), \quad \forall t \in [t_0, t_1)$$

$$\leq -c_1\left[f^{\alpha+1}(t) + \Psi^{\alpha+1}(t) - \frac{c_2}{c_1}\xi^{\beta}(t)h^{\alpha+1}(t) + \frac{1}{c_1}\Psi'(t)\right]. \tag{49}$$

Thus,

$$f'(t) \leq -c_1 f^{\alpha+1}(t), \quad \forall t \in [t_0, t_1). \tag{50}$$

Integrating over (t_0, t), we have

$$f(t) \leq \frac{c}{(t-t_0)^{\frac{1}{\alpha}}}, \quad \forall t \in [t_0, t_1). \tag{51}$$

If $t_1 = +\infty$, using again the definitions of f and Ψ, we have, for t large enough,

$$\varphi(t) \leq C(1+t)^{\frac{-1}{\alpha}}\left[1 + \int_0^t \xi^{\beta}(s)h^{\alpha+1}(s)(1+s)^{\frac{1}{\alpha}}\,ds\right]. \tag{52}$$

If $t_1 < +\infty$, then there exists $t_2 > t_1$ such that $f(t) \leq 0, \forall t_1 \leq t < t_2$. Hence, (44) yields $\varphi(t) \leq \Psi(t), \forall t_1 \leq t < t_2$; consequently, we get (52). If $t_2 = +\infty$, we are done. Otherwise, there exists $t_3 > t_2$ such that $f(t_2) = 0$ and $f(t) > 0, \forall t_2 < t < t_3$, we then repeat steps (49)–(51) on $[t_2, t_3)$ to obtain (52). Therefore, (52) remains valid for all $t \geq t_0$. Multiply (52) by $\xi^{-\beta}$ and recall the definition of φ, then for $\beta = \frac{\alpha+1}{\alpha}$ we have

$$F(t) \leq C(1+t)^{\frac{-1}{\alpha}}\xi^{-\frac{\alpha+1}{\alpha}}\left[1 + \int_0^t \xi^{\frac{\alpha+1}{\alpha}}(s)h^{\alpha+1}(s)(1+s)^{\frac{1}{\alpha}}\,ds\right]. \tag{53}$$

Using the fact $F \sim E$ and recalling that $\alpha = 2p-2$, we get

$$E(t) \leq C(1+t)^{\frac{-1}{2p-2}}\xi^{-\frac{2p-1}{2p-2}}\left(1 + \int_0^t (1+s)^{\frac{1}{2p-2}}\xi^{\frac{2p-1}{2p-2}}(s)h^{2p-1}\,ds\right). \tag{54}$$

This establishes (30).

To show (32), we note that simple calculations, using (30) and (31), yield

$$\int_{t_0}^{+\infty} E(t)\,dt < +\infty. \tag{55}$$

Use (55) in the following quantity to obtain

$$\begin{aligned}
I(t) &:= \int_0^t \|\nabla u(t-s) - \nabla u(t)\|_2^2\,ds \leq C\int_0^t \left(\|\nabla u(t-s)\|_2^2 + \|\nabla u(t)\|_2^2\right)ds \\
&\leq C\int_0^t \left[E(t-s) + E(t)\right]ds \\
&\leq 2C\int_0^t E(t-s)\,ds \\
&\leq 2C\int_0^t E(s)\,ds < 2C\int_0^\infty E(s)\,ds < \infty.
\end{aligned} \tag{56}$$

Without loss of the generality, we assume that $I(t) > 0$ for all $t \geq t_0$; otherwise (1) yields an exponential decay. Assumption $(A2)$, Jensen's inequality, and the fact that ξ is nonincreasing lead to

$$\begin{aligned}
&\xi(t)\int_0^t g(s)\|\nabla u(t-s)-\nabla u(t)\|_2^2\,ds \\
&\quad\leq \frac{I(t)}{I(t)}\int_0^t \bigl(\xi^p(s)g^p(s)\bigr)^{\frac{1}{p}}\|\nabla u(t-s)-\nabla u(t)\|_2^2\,ds \\
&\quad\leq CI(t)\biggl(\frac{1}{I(t)}\int_0^t \xi^p(s)g^p(s)\|\nabla u(t-s)-\nabla u(t)\|_2^2\,ds\biggr)^{\frac{1}{p}} \\
&\quad\leq CI^{1-\frac{1}{p}}(t)\biggl(\int_0^t \xi^p(s)g^p(s)\|\nabla u(t-s)-\nabla u(t)\|_2^2\,ds\biggr)^{\frac{1}{p}} \\
&\quad\leq CI^{1-\frac{1}{p}}(t)\xi^{p-1}(0)\biggl(\int_0^t \xi(s)g^p(s)\|\nabla u(t-s)-\ \nabla u(t) bigr\|_2^2\,ds\biggr)^{\frac{1}{p}} \\
&\quad\leq C\biggl(\int_0^t -g'(s)\|\nabla u(t-s)-\nabla u(t)\|_2^2\,ds\biggr)^{\frac{1}{p}} \\
&\quad\leq C\bigl(-E'(t)\bigr)^{\frac{1}{p}}.
\end{aligned} \tag{57}$$

Multiply (24) by $\xi(t)$ to get

$$\begin{aligned}
\xi(t)L'(t) \leq{}& -\alpha_1\xi(t)E(t)+\alpha_2\xi(t)\int_0^t g(s)\|\nabla u(t)-\nabla u(t-s)\|_2^2\,ds \\
&+\alpha_2\xi(t)\int_t^{+\infty} g(s)\|\nabla u(t)-\nabla u(t-s)\|_2^2\,ds.
\end{aligned} \tag{58}$$

Then, using (57), (16), and the definition of $Ch(t)$, we have

$$\xi(t)L'(t) \leq -\alpha_1\xi(t)E(t)+C\bigl[-E'(t)\bigr]^{\frac{1}{p}}+Ch(t). \tag{59}$$

Multiply (59) by $\xi^\alpha(t)E^\alpha(t)$, where $\alpha = p-1$, to obtain

$$\begin{aligned}
&\xi^{\alpha+1}(t)E^\alpha(t)L'(t) \\
&\quad\leq -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t)+C(\xi E)^\alpha(t)\bigl[-E'(t)\bigr]^{\frac{1}{\alpha+1}}+Ch(t)\xi^\alpha(t)E^\alpha(t).
\end{aligned} \tag{60}$$

Use of Young's inequality, with $q = \alpha+1$ and $q^* = \frac{\alpha+1}{\alpha}$, gives

$$\xi^{\alpha+1}(t)E^\alpha(t)L'(t) \leq -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t)+2\varepsilon\xi^{\alpha+1}(t)E^{\alpha+1}-C_\varepsilon E'(t)+C_\varepsilon h^{\alpha+1}(t). \tag{61}$$

Choose ε small enough and let $\widetilde{F} := \xi^{\alpha+1}E^\alpha L + C_\varepsilon E \sim E$, then there exist positive constants β_1 and β_2 such that

$$\widetilde{F}'(t) \leq -\beta_1\xi^{\alpha+1}(t)\widetilde{F}^{\alpha+1}(t)+\beta_2 h^{\alpha+1}(t). \tag{62}$$

Repeating the same computations as above, we obtain

$$E(t) \leq C(1+t)^{\frac{-1}{p-1}} \xi^{-\frac{2p}{p-1}} \left(1 + \int_0^t (1+s)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s) h^p \, ds\right). \tag{63}$$

This establishes (32). □

Theorem 4.2 *Let* $(u_0(\cdot,0), u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* $(A1)$–$(A4)$ *hold. Then, for* $1 < m < 2$, $p = 1$, *and positive constants* $c_i, i = 1,2,3$, *we have the following estimate*:

$$E(t) \leq c_1 e^{-\delta_0 \int_0^t \xi(s)\, ds} \left(c_2 + c_3 \int_0^t e^{\delta_0 \int_0^t \xi(s)\, ds} H(s)\, ds\right), \tag{64}$$

where $H(t) = h(t) + \varepsilon \xi^{\frac{m}{2-m}}(t)$.

Proof For (64), we multiply (25) by $\xi(t)$; using (13), (17), and Young's inequality, we have

$$\begin{aligned}
\xi(t) L'(t) &\leq -\alpha_1 \xi(t) E(t) + c\left[-E'(t)\right] + C h(t) + c \xi(t) \left(\int_\Omega |u_t|^m \, dx\right)^{\frac{2m-2}{m}} \\
&\leq -\alpha_1 \xi(t) E(t) + c\left[-E'(t)\right] + C h(t) + c \xi(t) \left(-E'(t)\right)^{\frac{2m-2}{m}} \\
&\leq -\alpha_1 \xi(t) E(t) + C h(t) + \varepsilon \xi^{\frac{m}{2-m}}(t) - \left(c(\varepsilon) + c\right) E'(t).
\end{aligned} \tag{65}$$

By letting $\mathbb{F}(t) := \xi(t) L(t) + (c(\varepsilon) + c) E(t) \sim E(t)$, we arrive at

$$\mathbb{F}'(t) \leq -\alpha_1 \xi(t) \mathbb{F}(t) + C H(t), \tag{66}$$

where $H(t) = h(t) + \varepsilon \xi^{\frac{m}{2-m}}(t)$. Repeating the same steps of [25], then (64) is established. □

Remark 4.4 Estimate (64) gives a decay estimate on $E(t)$ if $\xi(t)$ converges to zero when t goes to infinity. If $\xi(t)$ is a constant, that is, $g'(t) \leq -\xi g(t)$, then $g(t)$ converges to zero exponentially when t goes to infinity. In this case, we have the following estimates:

$$\begin{cases} E(t) \leq c_2 e^{-c_1 t} & m \geq 2; \\ E(t) \leq c(1+t)^{-\frac{2m-2}{2-m}} & 1 < m < 2. \end{cases} \tag{67}$$

For the proof, see Theorem 4.1 in [22].

Theorem 4.3 *Let* $(u_0(\cdot,0), u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* $(A1)$–$(A4)$ *hold. Then we have, for* $1 < m < 2$ *and* $1 < p < \frac{3}{2}$, *the following estimates*:

$$\begin{aligned}
&E(t) \leq C(1+t)^{\frac{-1}{2p-2}} \xi^{-\frac{2p-1}{2p-2}}(t) \left(1 + \int_0^t (1+s)^{\frac{1}{2p-2}} \xi^{\frac{2p-1}{2p-2}}(s) h^{2p-1}(s)\, ds\right), \\
&\quad 1 + \frac{1}{4p-3} < m < 2,
\end{aligned} \tag{68}$$

and

$$E(t) \le C(1+t)^{-\frac{2m-2}{m-2}} \xi^{-\frac{m}{2-m}}(t)\left(1+\int_0^t (1+s)^{\frac{2m-2}{m-2}} \xi^{\frac{m}{2-m}}(s) h^{\frac{m}{2m-2}}(s)\,ds\right),$$
$$1 < m < 1 + \frac{1}{4p-3}. \tag{69}$$

Moreover, if $1 < m < 1 + \frac{1}{4p-3}$ *and*

$$\int_0^{+\infty} \left[(1+t)^{-\frac{2m-2}{m-2}} \xi^{-\frac{m}{2-m}}(t)\left(1+\int_0^t (1+s)^{\frac{2m-2}{m-2}} \xi^{\frac{m}{2-m}}(s) h^{\frac{m}{2m-2}}(s)\,ds\right)\right] < +\infty, \tag{70}$$

then

$$E(t) \le C(1+t)^{-\frac{4m-4}{2-m}} \xi^{-\frac{6m-4}{2-m}}(t)\left(1+\int_0^t (1+s)^{\frac{4m-4}{2-m}} \xi^{\frac{6m-4}{2-m}}(s) h^{\frac{3m-2}{4m-4}}(s)\,ds\right), \tag{71}$$

where $h(t) = \xi(t)\int_t^{\infty} g(s)\,ds$ *and* C *is a positive constant.*

Proof For (68), we multiply (25) by $\xi^{\alpha+1}(t)E^{\alpha}(t)$, where $\alpha = 2p-2$. Recall the definition of $h(t)$ and use (17) to obtain

$$\begin{aligned}\xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \le{}& -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) + c(\xi E)^{\alpha}(t)\big[-E'(t)\big]^{\frac{1}{2p-1}} \\ &+ Ch(t)\xi^{\alpha}(t)E^{\alpha}(t) + c\xi(t)(\xi E)^{\alpha}(t)\left(\int_{\Omega} |u_t|^m\,dx\right)^{\frac{2m-2}{m}}.\end{aligned} \tag{72}$$

Then exploit (13) to get

$$\begin{aligned}\xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \le{}& -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) + c(\xi E)^{\alpha}(t)\big[-E'(t)\big]^{\frac{1}{2p-1}} + Ch(t)\xi^{\alpha}(t)E^{\alpha}(t) \\ &+ c(\xi E)^{\alpha}(t)\big[-E'(t)\big]^{\frac{2m-2}{m}}.\end{aligned} \tag{73}$$

Using (26), then (73) becomes

$$\begin{aligned}\xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \le{}& -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) + c(\xi E)^{\alpha}(t)\big[-E'(t)\big]^{\frac{1}{2p-1}} + Ch(t)\xi^{\alpha}(t)E^{\alpha}(t) \\ &+ c(\xi E)^{\alpha}(t)\big[-E'(t)\big]^{\frac{2m-2}{m}-\frac{1}{2p-1}}\big[-E'(t)\big]^{\frac{1}{2p-1}}.\end{aligned} \tag{74}$$

Recalling Remark (4.2), we get

$$\xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \le -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) + c(\xi E)^{\alpha}(t)\big[-E'(t)\big]^{\frac{1}{2p-1}} + Ch(t)\xi^{\alpha}(t)E^{\alpha}(t). \tag{75}$$

Since $\alpha = 2p-2$, then we have

$$\xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \le -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) + c(\xi E)^{\alpha}(t)\big[-E'(t)\big]^{\frac{1}{\alpha+1}} + Ch(t)\xi^{\alpha}(t)E^{\alpha}(t). \tag{76}$$

Now, repeating the same calculation as that in the proof of Theorem (4.1), we obtain (68).

For the proof of (69), we multiply (25) by $\xi^{\alpha+1}(t)E^{\alpha}(t)$, use (17), recall the definition of $h(t)$ and Remark 4.1, then (73) becomes

$$\begin{aligned} \xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \leq{}& -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) + c(\xi E)^{\alpha}(t)\bigl[-E'(t)\bigr]^{\frac{1}{2p-1}-\frac{2m-2}{m}}\bigl[-E'(t)\bigr]^{\frac{2m-2}{m}} \\ & + Ch(t)\xi^{\alpha}(t)E^{\alpha}(t) + c(\xi E)^{\alpha}(t)\bigl[-E'(t)\bigr]^{\frac{2m-2}{m}}. \end{aligned} \tag{77}$$

Using Remark (4.2), we get

$$\xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \leq -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) + c(\xi E)^{\alpha}(t)\bigl[-E'(t)\bigr]^{\frac{2m-2}{m}} + Ch(t)\xi^{\alpha}(t)E^{\alpha}(t). \tag{78}$$

Since $\alpha = \frac{2-m}{2m-2}$, then we have

$$\xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \leq -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) + c(\xi E)^{\alpha}(t)\bigl[-E'(t)\bigr]^{\frac{1}{\alpha+1}} + Ch(t)\xi^{\alpha}(t)E^{\alpha}(t). \tag{79}$$

Now, repeating the same calculation as that in the proof of Theorem (4.1), we can establish (69) and (71). □

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