Qin et al. *Boundary Value Problems* (2019) 2019:145
https://doi.org/10.1186/s13661-019-1254-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Monotone iterative method for a p-Laplacian boundary value problem with fractional conformable derivatives

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Abstract

By using monotone iterative method, the extremal solutions and the unique solution are obtained for a nonlinear fractional p-Laplacian boundary value problem involving fractional conformable derivatives and nonlocal integral boundary conditions. Comparison theorems related to the proposed study are also proved. The paper concludes with an illustrative example for the main result.

Keywords: Fractional conformable derivative; p-Laplacian operator; Nonlocal integral boundary condition; Extremal solution; Monotone iterative method

1 Introduction

Fractional calculus provides powerful tools to deal with complex phenomena occurring in various areas of applied and technical sciences such as control theory, optical and thermal systems, rheology, materials and mechanical systems, robotics, etc. Numerous researchers have investigated different aspects (existence, uniqueness, stability, etc.) of fractional differential equations involving Caputo, Riemann–Liouville, Hadamard type derivatives, for instance, see [1–10]. For some recent results on Riemann–Liouville fractional differential equations, we refer the reader to the articles [11–15] and the references cited therein. Fractional p-Laplacian boundary value problems also received considerable attention, for example, see [16–26]. The literature on fractional differential equations equipped with integral boundary conditions also contains a variety of interesting results [27–32].

Monotone iterative method is found to be an important and efficient method to obtain sequences of monotone solutions for initial and boundary value problems. For some applications of this technique to nonlinear fractional differential equations, see [15, 33–43]. In 2017, Jarad et. al. [44] proposed a new fractional derivative, which is known as fractional conformable derivative (see definition (2.4)). To the best of the authors' knowledge, the fractional p-Laplacian problem involving fractional conformable derivatives is yet to be investigated. In this paper, we apply monotone iterative method to prove the existence of extremal and uniqueness of solutions for the following nonlinear fractional p-Laplacian problem involving fractional conformable derivatives and nonlocal integral boundary condition:

$$\begin{cases} {}_0^{\beta}D^{\alpha}(\phi_p({}_0^{\gamma}D^{\alpha}h(t))) = f(t,h(t),{}_0^{\gamma}D^{\alpha}h(t)), & t \in (0,d], d > 0, \\ t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha}h(t)|_{t=0} = \int_0^{\tau} a(s)h(s)\,ds, & g(\tilde{h}(0),\tilde{h}(d)) = 0, \tau \in (0,d), \end{cases} \tag{1.1}$$

where $0 < \alpha,\gamma,\beta \le 1$, $\phi_p(t) = |t|^{p-2}t$, $a \in C([0,d],[0,\infty))$, $f \in C([0,d] \times \mathbb{R}^2,\mathbb{R})$, $g \in C(\times\mathbb{R}^2,\mathbb{R})$, ϕ_p, $p > 1$, denotes the p-Laplacian operator and $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\tilde{h}(0) = t^{\alpha(1-\gamma)}h(t)|_{t=0}$, $\tilde{h}(d) = t^{\alpha(1-\gamma)}h(t)|_{t=d}$, and ${}_0^{\gamma}D^{\alpha}$ is the fractional conformable derivative of order γ.

We emphasize that the results obtained for problem (1.1) are new and significantly contribute to the existing literature on p-Laplacian problems with fractional conformable derivatives. In order to establish the desired results, we prove two comparison theorems related to the problem at hand, which are presented in Sect. 2. The main results are presented in Sect. 3.

2 Preliminaries and lemmas

For $\alpha,\gamma \in (0,1)$, we denote by $C_{\alpha(1-\gamma)}([0,d],\mathbb{R})$ a Banach space

$$\{h \in C((0,d],\mathbb{R}) : t^{\alpha(1-\gamma)}h \in C([0,d],\mathbb{R})\}, \tag{2.1}$$

endowed with the norm $\|h\|_{C_{\alpha(1-\gamma)}} = \sup_{t\in[0,d]} t^{\alpha(1-\gamma)}|h(t)|$.

Let

$$Y = \{h(t) \in C_{\alpha(1-\gamma)}([0,d],\mathbb{R}) : {}_0^{\gamma}D^{\alpha}h(t) \in C_k([0,d],\mathbb{R}) \text{ and } t^k\,{}_0^{\gamma}D^{\alpha}h(t)|_{t=0} = \varepsilon\}, \tag{2.2}$$

where $0 < \alpha,\gamma < 1$, $k = \frac{\alpha(1-\beta)}{p-1}$, $\varepsilon = \int_0^{\tau} a(s)h(s)\,ds$, be a Banach space equipped with the norm $\|h\|_Y = \max\{\sup_{t\in[0,d]} t^{\alpha(1-\gamma)}|h(t)|, \sup_{t\in[0,d]} |{}_0^{\gamma}D^{\alpha}h(t)|\}$.

Definition 2.1 ([44]) The Riemann–Liouville type fractional conformable integral of order $\gamma \in \mathbb{C}$, $\mathrm{Re}(\gamma) \ge 0$ is defined by

$${}_a^{\gamma}I^{\alpha}h(t) = \frac{1}{\Gamma(\gamma)}\int_a^t \left(\frac{(t-a)^{\alpha} - (s-a)^{\alpha}}{\alpha}\right)^{\gamma-1} h(s)\frac{ds}{(s-a)^{1-\alpha}}. \tag{2.3}$$

Definition 2.2 ([44]) The fractional conformable derivative of Riemann–Liouville type of order $\gamma \in \mathbb{C}$, $\mathrm{Re}(\gamma) \ge 0$ is defined by

$$\begin{aligned} {}_a^{\gamma}D^{\alpha}h(t) &= {}_a^{n}\mathcal{T}^{\alpha}\left({}_a^{n-\gamma}I^{\alpha}\right)h(t) \\ &= \frac{{}_a^{n}\mathcal{T}^{\alpha}}{\Gamma(n-\gamma)}\int_a^t \left(\frac{(t-a)^{\alpha} - (s-a)^{\alpha}}{\alpha}\right)^{n-\gamma-1} h(s)\frac{ds}{(s-a)^{1-\alpha}}, \end{aligned} \tag{2.4}$$

where

$$n = [\mathrm{Re}(\gamma)] + 1, \qquad {}_a^{n}\mathcal{T}^{\alpha} = \underbrace{{}_a\mathcal{T}^{\alpha}\,{}_a\mathcal{T}^{\alpha}\cdots{}_a\mathcal{T}^{\alpha}}_{n\text{ times}}, \tag{2.5}$$

and ${}_a\mathcal{T}^\alpha$ is the conformable differential operator [45]

$$ {}_a\mathcal{T}^\alpha h(t) = (t-a)^{1-\alpha} h'(t). \tag{2.6}$$

Lemma 2.1 ([44]) *Let* $0 < \operatorname{Re}(\gamma) < 1$, $n = -[-\operatorname{Re}(\gamma)]$, $f \in L((0,d),\mathbb{R})$. *Then*

$$ {}_a^\gamma I^\alpha \big({}_a^\gamma D^\alpha h(t)\big) = h(t) - \frac{{}_a^{\gamma-1} D^\alpha h(a)}{\alpha^{\gamma-1}\Gamma(\gamma)}(t-a)^{\alpha\gamma-\alpha}.$$

Let us first consider the problem

$$\begin{cases} {}_0^\beta D^\alpha(\phi_p({}_0^\gamma D^\alpha h(t))) = f(t,h(t),{}_0^\gamma D^\alpha h(t)), & t \in (0,d],\\ t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^\gamma D^\alpha h(t)|_{t=0} = \int_0^\tau a(s)h(s)\,ds, & \tilde{h}(0) = r, \tau \in (0,d). \end{cases} \tag{2.7}$$

Applying Lemma 2.1 to problem (2.7) with $l(t) = \phi_p({}_0^\gamma D^\alpha h(t))$ and $\tilde{h}(0) = r$, we obtain

$$ h(t) = rt^{\alpha(\gamma-1)} + \frac{1}{\Gamma(\gamma)}\int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \phi_q\big(l(s)\big)\frac{ds}{s^{1-\alpha}} =: Bl(t) \tag{2.8}$$

and

$$ \phi_p\big(t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^\gamma D^\alpha h(t)\big) = t^{\alpha(1-\beta)}\phi_p\big({}_0^\gamma D^\alpha h(t)\big) = t^{\alpha(1-\beta)} l(t). \tag{2.9}$$

Thus problem (2.7) takes the form

$$\begin{cases} {}_0^\beta D^\alpha l(t) = f(t, Bl(t), \phi_q(l(t))), & t \in (0,d],\\ t^{\alpha(1-\beta)} l(t)|_{t=0} = \phi_p[\int_0^\tau a(s)h(s)\,ds], & \tau \in (0,d). \end{cases} \tag{2.10}$$

If (2.10) has a solution $l(t)$, then we get a solution $h(t)$ of Eq. (2.7) after inserting $l(t)$ in Eq.(2.8). This shows the existence of a solution for problem (2.10).

In the following lemma, we use $\|h\|_* = \sup_{t\in[0,d]} |h(t)|$.

Lemma 2.2 *Suppose that* $f \in C([0,d]\times\mathbb{R}^2,\mathbb{R})$, $0 < \alpha, \beta < 1$, *and there exists a nonnegative bounded integrable function* M *on* $[0,d]$ *such that*

$$ \big|f(t,h_1,h_2) - f(t,l_1,l_2)\big| \le M(t)\big|\phi_p(l_2) - \phi_p(h_2)\big|, \quad t \in (0,d].$$

Then problem (2.10) *has a unique solution* $l(t) \in C_{\alpha(1-\beta)}([0,d],\mathbb{R})$, *if*

$$ \frac{d^{\alpha(\beta-1)}\xi^{p-2}\eta^{q-2}}{\Gamma(\gamma+1)}\int_0^\tau a(s)\left(\frac{s^\alpha}{\alpha}\right)^{\gamma-1} ds + \frac{Md^{\alpha\beta}}{\Gamma(\beta+1)\alpha^\beta} < 1, \tag{2.11}$$

where ξ *takes the values between* $\int_0^\tau a(s){}_0^\gamma I^\alpha\phi_q(h(s))\,ds$ *and* $\int_0^\tau a(s){}_0^\gamma I^\alpha \phi_q(l(s))\,ds$, *the values of* η *remain between* $h(u)$ *and* $l(u)$, *and* $M = \sup_{t\in[0,d]} |M(t)|$.

Proof According to Lemma 2.1 and $t^{\alpha(1-\beta)}l(t)|_{t=0} = \phi_p(\int_0^\tau a(s)h(s)\,ds)$, problem (2.10) is equivalent to the following integral equation:

$$\begin{aligned} l(t) &= \phi_p\left[\int_0^\tau a(s)Bl(s)\,ds\right]t^{\alpha(\beta-1)} \\ &\quad + \frac{1}{\Gamma(\beta)}\int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\beta-1} f\bigl(s, Bl(s), \phi_q\bigl(l(s)\bigr)\bigr)\frac{ds}{s^{1-\alpha}} \\ &:= Al(t). \end{aligned} \tag{2.12}$$

For any $h, l \in C_{\alpha(1-\beta)}([0,d], \mathbb{R})$, we have

$$\begin{aligned} &\|Ah - Al\|_* \\ &\quad \le \sup_{t\in[0,d]} t^{\alpha(\beta-1)}\left|\phi_p\left[\int_0^\tau a(s)Bh(s)\,ds\right] - \phi_p\left[\int_0^\tau a(s)Bl(s)\,ds\right]\right| \\ &\qquad + \sup_{t\in[0,d]} \frac{1}{\Gamma(\beta)}\int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\beta-1} \bigl|f\bigl(s, Bh(s), \phi_q\bigl(h(s)\bigr)\bigr) - f\bigl(s, Bl(s), \phi_q\bigl(l(s)\bigr)\bigr)\bigr|\frac{ds}{s^{1-\alpha}} \\ &\quad \le \sup_{t\in[0,d]} t^{\alpha(\beta-1)}\left|\phi_p\left[\int_0^\tau a(s)\left(rs^{\alpha(\gamma-1)} + \frac{1}{\Gamma(\gamma)}\int_0^s \left(\frac{s^\alpha - u^\alpha}{\alpha}\right)^{\gamma-1}\phi_q\bigl(h(u)\bigr)\frac{du}{u^{1-\alpha}}\right)ds\right]\right. \\ &\qquad \left. - \phi_p\left[\int_0^\tau a(s)\left(rs^{\alpha(\gamma-1)} + \frac{1}{\Gamma(\gamma)}\int_0^s \left(\frac{s^\alpha - u^\alpha}{\alpha}\right)^{\gamma-1}\phi_q\bigl(l(u)\bigr)\frac{du}{u^{1-\alpha}}\right)ds\right]\right| \\ &\qquad + \sup_{t\in[0,d]} \frac{1}{\Gamma(\beta)}\int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\beta-1} M(s)\bigl|h(s) - l(s)\bigr|\frac{ds}{s^{1-\alpha}} \\ &\quad \le \sup_{t\in[0,d]} t^{\alpha(\beta-1)}\left|\left[\int_0^\tau a(s)\bigl(rs^{\alpha(\gamma-1)}\bigr)\,ds\right.\right. \\ &\qquad \left. + \int_0^\tau a(s)\frac{1}{\Gamma(\gamma)}\int_0^s \left(\frac{s^\alpha - u^\alpha}{\alpha}\right)^{\gamma-1}\bigl|\phi_q\bigl(h(u)\bigr)\bigr|\frac{du}{u^{1-\alpha}})\,ds\right]^{p-1} \\ &\qquad \left. - \left[\int_0^\tau a(s)\bigl(rs^{\alpha(\gamma-1)}\bigr)\,ds + \int_0^\tau a(s)\frac{1}{\Gamma(\gamma)}\int_0^s \left(\frac{s^\alpha - u^\alpha}{\alpha}\right)^{\gamma-1}\bigl|\phi_q\bigl(l(u)\bigr)\bigr|\frac{du}{u^{1-\alpha}})\,ds\right]^{p-1}\right| \\ &\qquad + \sup_{t\in[0,d]} \frac{1}{\Gamma(\beta)}\int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\beta-1} M(s)\frac{ds}{s^{1-\alpha}}\|h - l\|_* \\ &\quad \le \sup_{t\in[0,d]} t^{\alpha(\beta-1)}\left|(p-1)\xi^{p-2}\left[\int_0^\tau a(s)\bigl(rs^{\alpha(\gamma-1)}\bigr)\,ds + \int_0^\tau a(s)\frac{1}{\Gamma(\gamma)}\int_0^s \left(\frac{s^\alpha - u^\alpha}{\alpha}\right)^{\gamma-1}\right.\right. \\ &\qquad \times \phi_q\bigl(h(u)\bigr)\frac{du}{u^{1-\alpha}}\,ds - \int_0^\tau a(s)\bigl(rs^{\alpha(\gamma-1)}\bigr)\,ds \\ &\qquad \left.\left. - \int_0^\tau a(s)\frac{1}{\Gamma(\gamma)}\int_0^s \left(\frac{s^\alpha - u^\alpha}{\alpha}\right)^{\gamma-1}\phi_q\bigl(l(u)\bigr)\frac{du}{u^{1-\alpha}}\,ds\right]\right| \\ &\qquad + \sup_{t\in[0,d]} \frac{M}{\Gamma(\beta+1)}\left(\frac{t^\alpha}{\alpha}\right)^\beta \|h - l\|_* \\ &\quad \le d^{\alpha(\beta-1)}(p-1)\xi^{p-2}\left|\int_0^\tau a(s)\frac{1}{\Gamma(\gamma)}\int_0^s \left(\frac{s^\alpha - u^\alpha}{\alpha}\right)^{\gamma-1}\bigl(\phi_q\bigl(h(u)\bigr) - \phi_q\bigl(l(u)\bigr)\bigr)\frac{du}{u^{1-\alpha}}\,ds\right| \end{aligned}$$

$$
\begin{aligned}
&+\frac{Md^{\alpha\beta}}{\Gamma(\beta+1)\alpha^{\beta}}\|h-l\|_{*}\\
&\le d^{\alpha(\beta-1)}(p-1)\xi^{p-2}(q-1)\eta^{q-2}\int_0^{\tau}a(s)\frac{1}{\Gamma(\gamma)}\int_0^{s}\left(\frac{s^{\alpha}-u^{\alpha}}{\alpha}\right)^{\gamma-1}\frac{du}{u^{1-\alpha}}\,ds\|h-l\|_{*}\\
&+\frac{Md^{\alpha\beta}}{\Gamma(\beta+1)\alpha^{\beta}}\|h-l\|_{*}\\
&=\left(\frac{d^{\alpha(\beta-1)}\xi^{p-2}\eta^{q-2}}{\Gamma(\gamma+1)}\int_0^{\tau}a(s)\left(\frac{s^{\alpha}}{\alpha}\right)^{\gamma-1}ds+\frac{Md^{\alpha\beta}}{\Gamma(\beta+1)\alpha^{\beta}}\right)\|h-l\|_{*},
\end{aligned}
$$

which, in view of (2.11), implies that the operator A has a unique fixed point by the Banach fixed point theorem. In consequence, problem (2.10) has a unique solution. □

Lemma 2.3 *If* $0<\alpha,\gamma,\beta<1$, $\psi\in C_{\alpha(1-\beta)}([0,d],\mathbb{R})$, *and* M *is a nonnegative bounded integrable function on* $[0,d]$, *then the following problem*

$$
\begin{cases}
{}_0^{\beta}D^{\alpha}(\phi_p({}_0^{\gamma}D^{\alpha}h(t)))+M(t)\phi_p({}_0^{\gamma}D^{\alpha}h(t))=\psi(t), & t\in(0,d],\\
t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha}h(t)|_{t=0}=c, \quad \tilde{h}(0)=r,
\end{cases}
\tag{2.13}
$$

has a unique solution $h\in Y$, *provided that* $Md^{\alpha\beta}<\Gamma(\beta+1)\alpha^{\beta}$.

Proof Letting $l(t)=\phi_p({}_0^{\gamma}D^{\alpha}h(t))$, we have

$$
\begin{cases}
{}_0^{\gamma}D^{\alpha}h(t)=\phi_q(l(t)), & t\in(0,d],\\
\tilde{h}(0)=r,
\end{cases}
\tag{2.14}
$$

and

$$
\begin{cases}
{}_0^{\beta}D^{\alpha}l(t)+M(t)l(t)=\psi(t), & t\in(0,d],\\
t^{\alpha(1-\beta)}l(t)|_{t=0}=\phi_p(c).
\end{cases}
\tag{2.15}
$$

Let $f(t,Bl(t),\phi_q(l(t)))=\psi(t)-M(t)l(t)$. For $l_1,l_2\in C_{\alpha(1-\beta)}([0,d],\mathbb{R})$, we have

$$
\left|f\left(t,Bl_1,\phi_q(l_1)\right)-f\left(t,Bl_2,\phi_q(l_2)\right)\right|=\left|M(t)\right||l_2-l_1|\le M|l_2-l_1|.
$$

Thus, problem (2.15) has a unique solution $l\in C_{\alpha(1-\beta)}([0,d],\mathbb{R})$ by Lemma 2.2, and ${}_0^{\gamma}D^{\alpha}h\in C_{\frac{\alpha(1-\beta)}{p-1}}([0,d],\mathbb{R})$. Moreover, problem (2.14) has a solution $h\in C_{\alpha(1-\gamma)}([0,d],\mathbb{R})$ by Lemma 2.1. By inserting l in h, we get a unique solution $h\in Y$ of problem (2.13). □

Definition 2.3 If $h\in Y$ is a lower solution of (1.1), then

$$
\begin{cases}
{}_0^{\beta}D^{\alpha}(\phi_p({}_0^{\gamma}D^{\alpha}h(t)))\le f(t,h(t),{}_0^{\gamma}D^{\alpha}h(t)), & t\in(0,d], d>0,\\
t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha}h(t)|_{t=0}\le\int_0^{\tau}a(s)h(s)\,ds, \quad g(\tilde{h}(0),\tilde{h}(d))\le 0, \tau\in(0,d).
\end{cases}
\tag{2.16}
$$

If $l\in Y$ is an upper solution of (1.1), then

$$
\begin{cases}
{}_0^{\beta}D^{\alpha}(\phi_p({}_0^{\gamma}D^{\alpha}l(t)))\ge f(t,l(t),{}_0^{\gamma}D^{\alpha}l(t)), & t\in(0,d], d>0,\\
t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha}l(t)|_{t=0}\ge\int_0^{\tau}a(s)h(s)\,ds, \quad g(\tilde{l}(0),\tilde{l}(d))\ge 0, \tau\in(0,d).
\end{cases}
\tag{2.17}
$$

Lemma 2.4 (Comparison theorem)

(C_1) *Let M be a nonnegative bounded integrable function on* $[0,d]$. *If* $m \in C_{\alpha(1-\beta)}([0,d],\mathbb{R})$ *satisfies*

$$\begin{cases} {}_0^{\beta}D^{\alpha}m(t) + M(t)m(t) \geq 0, & t \in (0,d], \\ t^{\alpha(1-\beta)}m(t)|_{t=0} \geq 0, \end{cases}$$

then $m(t) \geq 0$, $t \in (0,d]$.

(C_2) *Assume that* $n \in C_{\alpha(1-\gamma)}([0,d],\mathbb{R})$ *satisfies*

$$\begin{cases} {}_0^{\gamma}D^{\alpha}n(t) \geq 0, & t \in (0,d], \\ t^{\alpha(1-\gamma)}n(t)|_{t=0} \geq 0. \end{cases}$$

Then $n(t) \geq 0$, $t \in (0,d]$.

Proof Assume that $m(t) \geq 0$ is not true. Then there exist $t_1, t_2 \in (0,d]$ such that $m(t_2) < 0$, $m(t_1) = 0$ and $m(t) \geq 0$ for $t \in (0,t_1)$ and $m(t) < 0$ for $t \in (t_1,t_2)$. Since $M(t) \geq 0$, $\forall t \in [0,d]$, we have ${}_0^{\beta}D^{\alpha}m(t) \geq 0$, $\forall t \in (t_1,t_2)$.

According to

$${}_0^{\beta}D^{\alpha}m(t) = t^{1-\alpha}\frac{d}{dt}{}_0^{1-\beta}I^{\alpha}m(t),$$

we obtain that ${}_0^{1-\beta}I^{\alpha}m(t)$ is nondecreasing on (t_1,t_2). Hence ${}_0^{1-\beta}I^{\alpha}m(t) - {}_0^{1-\beta}I^{\alpha}m(t_1) \geq 0$, $t \in (t_1,t_2)$. On the other hand, we have

$$\begin{aligned} &{}_0^{1-\beta}I^{\alpha}m(t) - {}_0^{1-\beta}I^{\alpha}m(t_1) \\ &\quad= \frac{1}{\Gamma(1-\beta)}\int_0^t \left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)^{-\beta} m(s)\frac{ds}{s^{1-\alpha}} - \frac{1}{\Gamma(1-\beta)}\int_0^{t_1}\left(\frac{{t_1}^{\alpha}-s^{\alpha}}{\alpha}\right)^{-\beta} m(s)\frac{ds}{s^{1-\alpha}} \\ &\quad= \frac{1}{\Gamma(1-\beta)}\int_0^{t_1}\left[\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)^{-\beta} - \left(\frac{{t_1}^{\alpha}-s^{\alpha}}{\alpha}\right)^{-\beta}\right] m(s)\frac{ds}{s^{1-\alpha}} \\ &\qquad+ \frac{1}{\Gamma(1-\beta)}\int_{t_1}^{t}\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)^{-\beta} m(s)\frac{ds}{s^{1-\alpha}} \\ &\quad< 0, \quad \forall t \in (t_1,t_2), \end{aligned}$$

which is a contradiction. Therefore, $m(t) \geq 0$, $\forall t \in (0,d]$.

Obviously, the conclusion of (C_2) holds. It follows from (2.8) that $n(t) \geq 0$, $\forall t \in (0,d]$. □

3 Main results

Theorem 3.1 *Assume that*

(L_1) $h_0, l_0 \in Y$ *are lower and upper solutions of* (1.1), *respectively with* $h_0(t) \leq l_0(t)$, $t \in (0,d]$;

(L_2) *there exists a function* $M \in C([0,d],\mathbb{R})$, $t \in [0,d]$ *such that*

$$f\big(t,l(t),{}_0^{\gamma}D^{\alpha}l(t)\big) - f\big(t,h(t),{}_0^{\gamma}D^{\alpha}h(t)\big) \geq -M(t)\big[\phi_p\big({}_0^{\gamma}D^{\alpha}l(t)\big) - \phi_p\big({}_0^{\gamma}D^{\alpha}h(t)\big)\big]$$

for $h_0(t) \leq h(t) \leq l(t) \leq l_0(t)$, $t \in (0,d]$;

(L_3) *the function g satisfies*

$$g(m_2,n_2) - g(m_1,n_1) \geq m_2 - m_1$$

for $\tilde{h}_0(0) \leq m_2 \leq m_1 \leq \tilde{l}_0(0)$, $\tilde{h}_0(d) \leq n_2 \leq n_1 \leq \tilde{l}_0(d)$, *if* $M(t)d^{\alpha\beta} < \Gamma(\beta+1)\alpha^{\beta}$.

Then there exist sequences $\{h_n\},\{l_n\} \in Y$ *such that* (1.1) *has extremal solutions* $m(t)$, $n(t)$ *in* $[h_0,l_0] = \{h \in Y : h_0(t) \leq h(t) \leq l_0(t), t \in (0,d]\}$ *satisfying*

$$\begin{cases} h_0(t) \leq h_1(t) \leq \cdots \leq h_n(t) \leq \cdots \leq m(t) \leq n(t) \leq \cdots \leq l_n(t) \leq \cdots \leq l_1(t) \leq l_0(t), \\ {}_0^{\gamma}D^{\alpha}h_0 \leq {}_0^{\gamma}D^{\alpha}h_1 \leq \cdots \leq {}_0^{\gamma}D^{\alpha}h_n \leq \cdots \leq {}_0^{\gamma}D^{\alpha}m \leq {}_0^{\gamma}D^{\alpha}n \leq \cdots \\ \qquad \leq {}_0^{\gamma}D^{\alpha}l_n \leq \cdots \leq {}_0^{\gamma}D^{\alpha}l_1 \leq {}_0^{\gamma}D^{\alpha}l_0, \\ \phi_p({}_0^{\gamma}D^{\alpha}h_0) \leq \phi_p({}_0^{\gamma}D^{\alpha}h_1) \leq \cdots \leq \phi_p({}_0^{\gamma}D^{\alpha}h_n) \leq \cdots \leq \phi_p({}_0^{\gamma}D^{\alpha}m) \leq \phi_p({}_0^{\gamma}D^{\alpha}n) \leq \cdots \\ \qquad \leq \phi_p({}_0^{\gamma}D^{\alpha}l_n) \leq \cdots \leq \phi_p({}_0^{\gamma}D^{\alpha}l_1) \leq \phi_p({}_0^{\gamma}D^{\alpha}l_0), \end{cases}$$

for $t \in (0,d]$, $n = 1,2,3,\ldots$.

Proof Let $F(h(t)) = f(t,h(t),{}_0^{\gamma}D^{\alpha}h(t))$. For $n = 1,2,\ldots$, we define

$$\begin{cases} {}_0^{\beta}D^{\alpha}(\phi_p({}_0^{\gamma}D^{\alpha}h_n(t))) + M(t)\phi_p({}_0^{\gamma}D^{\alpha}h_n(t)) \\ \quad = F(h_{n-1}(t)) + M(t)\phi_p({}_0^{\gamma}D^{\alpha}h_{n-1}(t)), \quad t \in (0,d], \\ t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha}h_n(t)|_{t=0} = \int_0^{\tau} a(s)h_{n-1}(s)\,ds, \\ \tilde{h}_n(0) = \tilde{h}_{n-1}(0) - g(\tilde{h}_{n-1}(0),\tilde{h}_{n-1}(d)), \quad \tau \in (0,d), \end{cases} \tag{3.1}$$

and

$$\begin{cases} {}_0^{\beta}D^{\alpha}(\phi_p({}_0^{\gamma}D^{\alpha}l_n(t))) + M(t)\phi_p({}_0^{\gamma}D^{\alpha}l_n(t)) \\ \quad = F(l_{n-1}(t)) + M(t)\phi_p({}_0^{\gamma}D^{\alpha}l_{n-1}(t)), \quad t \in (0,d], \\ t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha}l_n(t)|_{t=0} = \int_0^{\tau} a(s)l_{n-1}(s)\,ds, \\ \tilde{l}_n(0) = \tilde{l}_{n-1}(0) - g(\tilde{l}_{n-1}(0),\tilde{l}_{n-1}(d)), \quad \tau \in (0,d). \end{cases} \tag{3.2}$$

Notice that the functions h_1, l_1 are well defined in Y by Lemma 2.3.

Now, we prove that $h_0(t) \leq h_1(t) \leq l_1(t) \leq l_0(t)$, ${}_0^{\gamma}D^{\alpha}h_0(t) \leq {}_0^{\gamma}D^{\alpha}h_1(t) \leq {}_0^{\gamma}D^{\alpha}l_1(t) \leq {}_0^{\gamma}D^{\alpha}l_0(t)$, $t \in (0,d]$, and $\tilde{h}_0(0) \leq \tilde{h}_1(0) \leq \tilde{l}_1(0) \leq \tilde{l}_0(0)$. Let $\lambda(t) = \phi_p({}_0^{\gamma}D^{\alpha}h_1(t)) - \phi_p({}_0^{\gamma}D^{\alpha}h_0(t))$. From (2.9), (3.1), and (L_1), we have

$$\begin{cases} {}_0^{\beta}D^{\alpha}\lambda(t) + M(t)\lambda(t) = F(h_0(t)) - {}_0^{\beta}D^{\alpha}(\phi_p({}_0^{\gamma}D^{\alpha}h_0(t))) \geq 0, \\ t^{\alpha(1-\beta)}\lambda(t)|_{t=0} = \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha}h_1(t))|_{t=0} - \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha}h_0(t))|_{t=0} \\ \qquad \geq \int_0^{\tau} a(s)h_0(s)\,ds - \int_0^{\tau} a(s)h_0(s)\,ds = 0. \end{cases}$$

By (C_1) of Lemma 2.4, we obtain $\lambda(t) \geq 0$, $t \in (0,d]$, which means $\phi_p({}_0^{\gamma}D^{\alpha}h_1(t)) \geq \phi_p({}_0^{\gamma}D^{\alpha}h_0(t))$. The monotone increasing property of $\phi_p(t)$ ensures that ${}_0^{\gamma}D^{\alpha}h_1(t) \geq {}_0^{\gamma}D^{\alpha}h_0(t)$. Thus, ${}_0^{\gamma}D^{\alpha}(h_1(t) - h_0(t)) \geq 0$. According to $\tilde{h}_1(0) - \tilde{h}_0(0) = -g(\tilde{h}_0(0),\tilde{h}_0(d)) \geq 0$, we have $h_1(t) \geq h_0(t)$, $t \in (0,d]$ by (C_2) of Lemma 2.4. In a similar manner, we can obtain that $l_1(t) \leq l_0(t)$, ${}_0^{\gamma}D^{\alpha}h_1(t) \leq {}_0^{\gamma}D^{\alpha}h_0(t)$, $t \in (0,d]$, and $\tilde{l}_1(0) \leq \tilde{l}_0(0)$.

Setting $\eta(t) = \phi_p({}_0^\gamma D^\alpha l_1(t)) - \phi_p({}_0^\gamma D^\alpha h_1(t))$ and using (L_2), we have

$$\begin{cases} {}_0^\beta D^\alpha \eta(t) + M(t)\eta(t) = F(l_0(t)) - F(h_0(t)) + M(t)[\phi_p({}_0^\gamma D^\alpha l_0(t)) - \phi_p({}_0^\gamma D^\alpha h_0(t))] \geq 0, \\ t^{\alpha(1-\beta)}\eta(t)|_{t=0} = \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}} {}_0^\gamma D^\alpha l_1(t))|_{t=0} - \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}} {}_0^\gamma D^\alpha h_1(t))|_{t=0} \geq 0. \end{cases}$$

By (C_1) of Lemma 2.4, we obtain $\eta(t) \geq 0$, $t \in (0,d]$. Then $\phi_p({}_0^\gamma D^\alpha l_1(t)) \geq \phi_p({}_0^\gamma D^\alpha h_1(t))$, and ${}_0^\gamma D^\alpha l_1(t) \geq {}_0^\gamma D^\alpha h_1(t)$. By (L_3), we have

$$\begin{aligned} \tilde{l}_1(0) - \tilde{h}_1(0) &= \tilde{l}_0(0) - g(\tilde{l}_0(0), \tilde{l}_0(d)) - \tilde{h}_0(0) + g(\tilde{h}_0(0), \tilde{h}_0(d)) \\ &= \tilde{l}_0(0) - \tilde{h}_0(0) + g(\tilde{h}_0(0), \tilde{h}_0(d)) - g(\tilde{l}_0(0), \tilde{l}_0(d)) \\ &\geq \tilde{l}_0(0) - \tilde{h}_0(0) + \tilde{h}_0(0) - \tilde{l}_0(0) = 0. \end{aligned}$$

Thus, $l_1(t) \geq h_1(t)$, $t \in (0,d]$ by (C_2) of Lemma 2.4.

Next, we show that h_1, l_1 are lower and upper solutions of (1.1), respectively. By (3.1) and (L_2), we obtain

$$\begin{aligned} &{}_0^\beta D^\alpha \big(\phi_p\big({}_0^\gamma D^\alpha h_1(t)\big)\big) \\ &\quad = F\big(h_0(t)\big) - M(t)\big[\phi_p\big({}_0^\gamma D^\alpha h_1(t)\big) - \phi_p\big({}_0^\gamma D^\alpha h_0(t)\big)\big] - F\big(h_1(t)\big) + F\big(h_1(t)\big) \\ &\quad \leq M(t)\big[\phi_p\big({}_0^\gamma D^\alpha h_1(t)\big) - \phi_p\big({}_0^\gamma D^\alpha h_0(t)\big)\big] - M(t)\big[\phi_p\big({}_0^\gamma D^\alpha h_1(t)\big) \\ &\qquad - \phi_p\big({}_0^\gamma D^\alpha h_0(t)\big)\big] + F\big(h_1(t)\big) \\ &\quad = F\big(h_1(t)\big). \end{aligned}$$

By (L_3), we have

$$\begin{aligned} 0 &= g(\tilde{h}_0(0), \tilde{h}_0(d)) - g(\tilde{h}_1(0), \tilde{h}_1(d)) + g(\tilde{h}_1(0), \tilde{h}_1(d)) + \tilde{h}_1(0) - \tilde{h}_0(0) \\ &\geq \tilde{h}_0(0) - \tilde{h}_1(0) + g(\tilde{h}_1(0), \tilde{h}_1(d)) + \tilde{h}_1(0) - \tilde{h}_0(0) \\ &= g(\tilde{h}_1(0), \tilde{h}_1(d)), \end{aligned}$$

and

$$t^{\frac{\alpha(1-\beta)}{p-1}} {}_0^\gamma D^\alpha h_1(t)|_{t=0} = \int_0^\tau a(s)h_0(s)\,ds \leq \int_0^\tau a(s)h_1(s)\,ds, \tag{3.3}$$

which imply that h_1 is a lower solution of (1.1). Analogously, we can verify that l_1 is an upper solution of (1.1).

Using the mathematical induction, we have

$$\begin{aligned} &h_0(t) \leq h_1(t) \leq \cdots \leq h_n(t) \leq h_{n+1}(t) \leq l_{n+1}(t) \leq l_n(t) \leq \cdots \leq l_1(t) \leq l_0(t), \\ &{}_0^\gamma D^\alpha h_0 \leq {}_0^\gamma D^\alpha h_1 \leq \cdots \leq {}_0^\gamma D^\alpha h_n \leq {}_0^\gamma D^\alpha h_{n+1} \leq {}_0^\gamma D^\alpha l_{n+1} \leq {}_0^\gamma D^\alpha l_n \leq \cdots \\ &\qquad \leq {}_0^\gamma D^\alpha l_1 \leq {}_0^\gamma D^\alpha l_0, \\ &\tilde{h}_0(0) \leq \tilde{h}_1(0) \leq \cdots \leq \tilde{h}_n(0) \leq \tilde{h}_{n+1}(0) \leq \tilde{l}_{n+1}(0) \leq \tilde{l}_n(0) \leq \cdots \leq \tilde{l}_1(0) \leq \tilde{l}_0(0) \end{aligned} \tag{3.4}$$

for $t \in (0,d]$, $n = 1,2,3,\ldots$.

By the standard analysis, we can get that the sequences $\{t^{\alpha(1-\gamma)}h_n\}$ and $\{t^{\alpha(1-\gamma)}l_n\}$ are uniformly bounded and equicontinuous. Thus, in view of Arzela–Ascoli theorem, we obtain

$$\begin{aligned}&\lim_{n\to\infty} h_n(t) = m(t), \qquad \lim_{n\to\infty} l_n(t) = n(t), \quad t\in(0,d],\\ &\lim_{n\to\infty} {}_0^{\gamma}D^{\alpha} h_n(t) = {}_0^{\gamma}D^{\alpha} m(t), \qquad \lim_{n\to\infty} {}_0^{\gamma}D^{\alpha} l_n(t) = {}_0^{\gamma}D^{\alpha} n(t), \quad t\in(0,d].\end{aligned}$$

Hence, $h_0(t)\le m(t)\le n(t)\le l_0(t)$ on $(0,d]$ and $m(t)$, $n(t)$ are solutions of (1.1).

Moreover, we show that $m(t)$, $n(t)$ are extremal solutions of (1.1). Let $h\in[h_0,l_0]$ be any solution of (1.1). Let $h_n(t)\le h(t)\le l_n(t)$, $t\in(0,d]$ and that

$$j(t) = \phi_p\bigl({}_0^{\gamma}D^{\alpha} h(t)\bigr) - \phi_p\bigl({}_0^{\gamma}D^{\alpha} h_{n+1}(t)\bigr), \qquad k(t) = \phi_p\bigl({}_0^{\gamma}D^{\alpha} l_{n+1}(t)\bigr) - \phi_p\bigl({}_0^{\gamma}D^{\alpha} h(t)\bigr).$$

By (L_2), we obtain

$$\begin{cases} {}_0^{\beta}D^{\alpha} j(t) + M(t)j(t) = F(h(t)) - F(h_n(t)) + M(t)[\phi_p({}_0^{\gamma}D^{\alpha} h(t)) - \phi_p({}_0^{\gamma}D^{\alpha} h_n(t))] \ge 0,\\ t^{\alpha(1-\beta)}j(t)|_{t=0} = \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha} h(t))|_{t=0} - \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha} h_{n+1}(t))|_{t=0} \ge 0,\end{cases}$$

and

$$\begin{cases} {}_0^{\beta}D^{\alpha} k(t) + M(t)k(t) = F(l_n(t)) - F(h(t)) + M(t)[\phi_p({}_0^{\gamma}D^{\alpha} l_n(t)) - \phi_p({}_0^{\gamma}D^{\alpha} h(t))] \ge 0,\\ t^{\alpha(1-\beta)}k(t)|_{t=0} = \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha} l_{n+1}(t))|_{t=0} - \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}}\,{}_0^{\gamma}D^{\alpha} h(t))|_{t=0} \ge 0.\end{cases}$$

Thus, by (C_1) of Lemma 2.4, we have $j(t)\ge 0$, $k(t)\ge 0$. Then $\phi_p({}_0^{\gamma}D^{\alpha} h(t))\ge \phi_p({}_0^{\gamma}D^{\alpha} h_{n+1}(t))$, $\phi_p({}_0^{\gamma}D^{\alpha} l_{n+1}(t))\ge \phi_p({}_0^{\gamma}D^{\alpha} h(t))$. Hence, ${}_0^{\gamma}D^{\alpha}(h(t)-h_{n+1}(t))\ge 0$, ${}_0^{\gamma}D^{\alpha}(l_{n+1}(t)-h(t))\ge 0$.

By (L_3), we have

$$\begin{aligned}\tilde h(0) - \tilde h_{n+1}(0) &= \tilde h(0) - \tilde h_n(0) + g\bigl(\tilde h_n(0),\tilde h_n(d)\bigr) - g\bigl(\tilde h(0),\tilde h(d)\bigr)\\ &\ge \tilde h(0) - \tilde h_n(0) + \tilde h_n(0) - \tilde h(0)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\tilde l_{n+1}(0) - \tilde h(0) &= \tilde l_n(0) - \tilde h(0) - g\bigl(\tilde l_n(0),\tilde l_n(d)\bigr) + g\bigl(\tilde h(0),\tilde h(d)\bigr)\\ &\ge \tilde l_n(0) - \tilde h(0) + \tilde h(0) - \tilde l_n(0)\\ &= 0.\end{aligned}$$

Hence, $h_{n+1}(t)\le h(t)\le l_{n+1}(t)$, $t\in(0,d]$ by (C_2) of Lemma 2.4, which, on taking the limit $n\to\infty$, yields $m(t)\le h(t)\le n(t)$. Therefore, $m(t)$, $n(t)$ are extremal solutions of (1.1). □

Theorem 3.2 *If the hypotheses of Theorem* 3.1 *hold*, $a(t)=0$, *and there exists a function* $L(t)\ge 0$ *such that*

$$L(t)\bigl[\phi_p\bigl({}_0^{\gamma}D^{\alpha} l(t)\bigr) - \phi_p\bigl({}_0^{\gamma}D^{\alpha} h(t)\bigr)\bigr] \le f\bigl(t,h(t),{}_0^{\gamma}D^{\alpha} h(t)\bigr) - f\bigl(t,l(t),{}_0^{\gamma}D^{\alpha} l(t)\bigr) \tag{3.5}$$

for $h_0(t) \le h(t) \le l(t) \le l_0(t)$, $t \in (0,d]$ *and* $\tilde{h}_0(0) = \tilde{l}_0(0)$, *then* (1.1) *has a unique solution in* $[h_0, l_0]$.

Proof It follows by Theorem 3.1 that $m(t)$ and $n(t)$ are extremal solutions such that $m(t) \le n(t)$, $t \in (0,d]$. Then we just need to prove $m(t) \ge n(t)$, $t \in (0,d]$. Letting $\lambda(t) = \phi_p({}_0^\gamma D^\alpha m(t)) - \phi_p({}_0^\gamma D^\alpha n(t))$, $t \in (0,d]$ and using (3.5), we obtain

$$\begin{cases} {}_0^\beta D^\alpha \lambda(t) = F(m(t)) - F(n(t)) \ge L(t)[\phi_p({}_0^\gamma D^\alpha n(t)) - \phi_p({}_0^\gamma D^\alpha m(t))] = -L(t)\lambda(t), \\ t^{\alpha(1-\beta)}\lambda(t)|_{t=0} = \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}} {}_0^\gamma D^\alpha m(t))|_{t=0} - \phi_p(t^{\frac{\alpha(1-\beta)}{p-1}} {}_0^\gamma D^\alpha n(t))|_{t=0} = 0. \end{cases} \tag{3.6}$$

Then, by (C_1) of Lemma 2.4, we have $\lambda(t) \ge 0$. Thus, $\phi_p({}_0^\gamma D^\alpha m(t)) \ge \phi_p({}_0^\gamma D^\alpha n(t))$. Since $\phi_p(t)$ is nondecreasing, we have ${}_0^\gamma D^\alpha m(t) \ge {}_0^\gamma D^\alpha n(t)$, $t \in (0,d]$. Then, by (C_2) of Lemma 2.4, we obtain $m(t) \ge n(t)$. Furthermore, we have $\tilde{m}(0) = \tilde{n}(0)$ by $\tilde{h}_0(0) = \tilde{l}_0(0)$ and (3.4). Therefore, we have $m = n$. The proof is completed. □

4 Example

Consider the following problem:

$$\begin{cases} {}_0^{\frac{2}{3}} D^{\frac{1}{2}}(\phi_3({}_0^{\frac{1}{2}} D^{\frac{1}{2}} h(t))) = f(t, h(t), {}_0^{\frac{1}{2}} D^{\frac{1}{2}} h(t)), \quad t \in (0,1], \\ t^{\frac{1}{12}} {}_0^{\frac{1}{2}} D^{\frac{1}{2}} h(t)|_{t=0} = \int_0^\tau a(s)h(s)\,ds, \qquad \frac{1}{2}\tilde{h}(0) - 3\tilde{h}(0)\tilde{h}(1) = 0, \end{cases} \tag{4.1}$$

where $\alpha = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $\beta = \frac{2}{3}$, $d = 1$, $p = 3$, $a(t) = 0$, $\tau = 1$, and $f(t, h(t), {}_0^{\frac{1}{2}} D^{\frac{1}{2}} h(t)) = \frac{1}{2}t + h(t) - 2{}_0^{\frac{1}{2}} D^{\frac{1}{2}} h(t)$, $g(m,n) = \frac{1}{2}m - 3mn$. Let $h_0(t) = 0$, $l_0(t) = \Gamma(\frac{1}{2})t^{\frac{1}{2}}$. Then we have ${}_0^{\frac{1}{2}} D^{\frac{1}{2}} h_0(t) = 0$, ${}_0^{\frac{1}{2}} D^{\frac{1}{2}} l_0(t) = 2^{\frac{1}{2}} t^{\frac{1}{4}}$, and

$$\begin{aligned} &{}_0^{\frac{2}{3}} D^{\frac{1}{2}}\big(\phi_3\big({}_0^{\frac{1}{2}} D^{\frac{1}{2}} h_0(t)\big)\big) = 0 \le \frac{1}{2}t = f\big(t, h_0(t), {}_0^{\frac{1}{2}} D^{\frac{1}{2}} h_0(t)\big), \quad t \in (0,1], \\ &t^{\frac{1}{12}} {}_0^{\frac{1}{2}} D^{\frac{1}{2}} h_0(t)|_{t=0} = 0, \qquad g\big(\tilde{h}_0(0), \tilde{h}_0(1)\big) = 0, \\ &{}_0^{\frac{2}{3}} D^{\frac{1}{2}}\big(\phi_3\big({}_0^{\frac{1}{2}} D^{\frac{1}{2}} l_0(t)\big)\big) = {}_0^{\frac{2}{3}} D^{\frac{1}{2}}\big(2t^{\frac{1}{2}}\big) = \frac{3 \cdot 2^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} t^{\frac{1}{6}} \ge \frac{1}{2}t + \Gamma\left(\frac{1}{2}\right)t^{\frac{1}{2}} - 2^{\frac{3}{2}} t^{\frac{1}{4}} \\ &\qquad = f\big(t, l_0(t), {}_0^{\frac{1}{2}} D^{\frac{1}{2}} l_0(t)\big), \\ &t^{\frac{1}{12}} {}_0^{\frac{1}{2}} D^{\frac{1}{2}} l_0(t)|_{t=0} = 0, \qquad g\big(\tilde{l}_0(0), \tilde{l}_0(1)\big) = 0. \end{aligned}$$

Thus, h_0 and l_0 are lower and upper solutions of (4.1), respectively, and $h_0 \le l_0$ on $[0,1]$.

In addition, for $h_0 \le h \le l \le l_0$, we have

$$\begin{aligned} &f\big(t, h(t), {}_0^{\frac{1}{2}} D^{\frac{1}{2}} h(t)\big) - f\big(t, l(t), {}_0^{\frac{1}{2}} D^{\frac{1}{2}} l(t)\big) \\ &\quad = h(t) - l(t) - 2{}_0^{\frac{1}{2}} D^{\frac{1}{2}} h(t) + 2{}_0^{\frac{1}{2}} D^{\frac{1}{2}} l(t) \\ &\quad \le 2\big[{}_0^{\frac{1}{2}} D^{\frac{1}{2}} l(t) - {}_0^{\frac{1}{2}} D^{\frac{1}{2}} h(t)\big] \\ &\quad \le M(t)\big[\phi_3\big({}_0^{\frac{1}{2}} D^{\frac{1}{2}} l(t)\big) - \phi_3\big({}_0^{\frac{1}{2}} D^{\frac{1}{2}} h(t)\big)\big], \end{aligned}$$

where $M(t) = 2$.

For $\tilde{h}_0(0) \leq m_2 \leq m_1 \leq \tilde{l}_0(0)$, $\tilde{h}_0(1) \leq n_2 \leq n_1 \leq \tilde{l}_0(1)$, we have

$$\begin{aligned} g(m_1,n_1) - g(m_2,n_2) &= \frac{1}{2}m_1 - 3m_1n_1 - \frac{1}{2}m_2 + 3m_2n_2 \\ &\leq \frac{1}{2}(m_1 - m_2) \leq m_1 - m_2. \end{aligned}$$

Hence, assumptions (L_1), (L_2), and (L_3) hold. According to Theorem 3.1, there exist monotone iterative sequences $\{h_n\}$, $\{l_n\}$ such that $\lim_{n\to\infty} h_n = m$, $\lim_{n\to\infty} l_n = n$ on $(0,1]$ and m, n are the extremal solutions on $[h_0, l_0]$ of (4.1).

Acknowledgements
The authors would like to express all gratitude to the anonymous referees for their hard work, helpful comments, and suggestions.

Funding
The work is supported by NSFC (No. 11501342), NSF of Shanxi, China (No. 201701D221007), and STIP (Nos. 201802068 and 201802069).

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors equally contributed this manuscript and approved the final version.

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Publisher's Note

Received: 5 April 2019 Accepted: 13 August 2019 Published online: 02 September 2019

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