Lei and Li *Boundary Value Problems* (2019) 2019:138
https://doi.org/10.1186/s13661-019-1255-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Boundedness in a quasilinear chemotaxis–haptotaxis model of parabolic–parabolic–ODE type

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Abstract

This paper deals with the boundedness of solutions to the following quasilinear chemotaxis–haptotaxis model of parabolic–parabolic–ODE type:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u\nabla v) - \xi \nabla \cdot (u\nabla w) + \mu u(1 - u^{r-1} - w), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u^{\eta}, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0, \end{cases}$$

under zero-flux boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$, with parameters $r \geq 2$, $\eta \in (0, 1]$ and the parameters $\chi > 0$, $\xi > 0$, $\mu > 0$. The diffusivity $D(u)$ is assumed to satisfy $D(u) \geq \delta u^{-\alpha}$, $D(0) > 0$ for all $u > 0$ with some $\alpha \in \mathbb{R}$ and $\delta > 0$. It is proved that if $\alpha < \frac{n+2-2n\eta}{2+n}$, then, for sufficiently smooth initial data (u_0, v_0, w_0), the corresponding initial-boundary problem possesses a unique global-in-time classical solution which is uniformly bounded.

MSC: 35B65; 35K55; 35Q92; 92C17

Keywords: Chemotaxis; Haptotaxis; Nonlinear diffusion; Boundedness; Logistic source; Nonlinear production

1 Introduction

Chemotaxis is the motion of cells moving towards the higher concentration of a chemical signal. A classical mathematical model for chemotaxis was proposed by Keller and Segel [9]. In the recent 40 years, a large quantity of the Keller–Segel system were proposed and have been extensively studied; see Hillen and Painter [15] for example.

Another important extension of the classical Keller–Segel model to a more complex cell migration mechanism was proposed by Chaplain and Lolas [4, 5] in order to describe processes of cancer cell invasion of surrounding healthy tissue. In addition to random motion, cancer cells bias their movement toward increasing concentrations of a diffusible enzyme as well as according to gradients of non-diffusible tissue by detecting matrix molecules such as vitronectin adhered therein. The latter type of directed migration toward immovable cues is commonly referred to as haptotaxis. Apart from that, in this modeling context the cancer cells are usually also assumed to follow a logistic growth competing for space with healthy tissue. The enzyme is produced by cancer cells and it is supposed to be influenced by diffusion and degradation. The tissue, also named extracellular matrix, can be degraded by enzyme upon contact; on the other hand, the tissue might possess the ability to remodel the healthy level. In [10, 21, 28, 29, 46], authors studied the following parabolic–parabolic–ODE chemotaxis–haptotaxis model:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u\nabla v) & \\ \qquad - \xi \nabla \cdot (u\nabla w) + \mu u(1-u-w), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0, \\ D(u)\frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

in smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where $\chi > 0$, $\xi > 0$, $\mu > 0$ are parameters, the variables u, v and w represent the density of cancer cells, the enzyme concentration and the density of the extracellular matrix, $D(u)$ describes the density-dependent motility of cancer cells through the extracellular matrix, χ and ξ represent the chemotactic and haptotactic sensitivities, μ is the proliferation rate of cells.

For the special case $D(u) = 1$ in (1.1), Tao and Wang [19] proved that model (1.1) possesses a unique global-in-time classical solution for any $\chi > 0$ in one space dimension, or for small $\frac{\chi}{\mu} > 0$ in two and three space dimensions. Later, Tao [17] improved the result of [19] for any $\mu > 0$ in two space dimensions. Hillen, Painter and Winkler [6] studied the global boundedness and asymptotic behavior of the solution to (1.1) in one space dimension. Tao [18] proved that the model has a unique classical solution which is global-in-time and bounded in two space dimensions. Cao [3] proved that the model has a unique classical solution which is global-in-time and bounded in three space dimensions. Tao and Winkler [26] claimed that if $n \leq 3$ and (u, v, w) is a bounded global classical solution, then under the fully explicit condition $\mu > \frac{\chi 2}{8}$ the solution (u, v, w) approaches the spatially uniform state $(1, 1, 0)$ as time goes to infinity. Then Wang and Ke [30] proved that the model possesses a unique global-in-time classical solution that is bounded in the case $3 \leq n \leq 8$ and μ is appropriately large.

When $D \in C^2([0,\infty))$, $D(0) > 0$ and $D(u) \geq \delta u^{-\alpha}$ for all $u \geq 0$ with some $\delta > 0$, the global existence of a unique classical solution to (1.1) was proved by Tao and Winkler in [21] under the assumption that either $n \leq 8$ and $\alpha < \frac{4-n^2}{n^2+4n}$ or $n \geq 9$ and $\alpha < (\sqrt{8n(n+1)} - n^2 - n - 2)/(n^2 + 2n)$. When $D \in C^2([0,\infty))$, $D(u) \geq \delta u^{-\alpha}$ for all $u \geq 0$ with some $\delta > 0$ and $\alpha < 0$, Zheng et al. [46] studied model (1.1) and found that (1.1) possesses a unique global classical solution which is uniformly bounded in the case of non-degenerate diffusion (i.e. $D(0) > 0$) and possesses at least one nonnegative global weak solution in the case of degenerate diffusion ($D(0) \geq 0$) in two space dimensions. Li and Lankeit [10] proved that for sufficiently regular initial date global bounded solutions exist whenever $\alpha < \frac{2}{n} - 1$ in two, three and four space dimensions. When $D \in C^2([0,\infty))$, $D(u) \geq \delta(u+1)^{-\alpha}$ for all $u \geq 0$ with some $\delta > 0$, Wang clarified the issue of the global boundedness to solutions of (1.1) without any restriction on the space dimension with $\alpha < \frac{2-n}{n+2}$ in [28, 29].

When the second PDE in (1.1) is replaced by $0 = \Delta v - v + u$ and $D(u) = 1$ in (1.1), Tao and Wang [20] proved that model (1.1) possesses a unique global bounded classical solution for any $\mu > 0$ in two space dimension, and for large $\mu > 0$ in three space dimensions. Tao and Winkler [23] proved that model (1.1) possesses a unique global smooth solution for first-order compatibility conditions in two space dimension. For all $n \geq 1$, Tao and Winkler [24] proved that model (1.1) possesses a unique global bounded classical solution for $\mu > \chi$. In particular, the global solution (u, v, w) approaches the spatially uniform state $(1, 1, 0)$ as time goes to infinity under an additional assumption on the size of μ and the initial data u_0 and w_0. Later, Tao and Winkler [25] studied global boundedness for model (1.1) under the condition $\mu > \frac{(n+2)_+}{n}\chi$. Furthermore, in addition to the explicit smallness on $w \equiv 0$, they gave the exponential decay of w in the large time limit.

Zheng [41] considered the following chemotaxis–haptotaxis model with generalized logistic source:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u\nabla v) & \\ \quad - \xi \nabla \cdot (u\nabla w) + u(1 - u^{r-1} - w), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

in smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, where $\chi > 0$, $\xi > 0$, $r > 1$ are parameters. Zheng [41] proved that model (1.2) possesses a unique global classical solution which is uniformly bounded in $\Omega \times (0, \infty)$ in the case of $D(u) \geq \delta(u + 1)^{-\alpha}$ for all $u > 0$ with some $\delta > 0$ and some

$$\alpha \begin{cases} < \frac{2}{n} - 1, & \text{if } 1 < r < \frac{n+2}{n}, \\ < -\frac{(n+2-2r)_+}{n+2}, & \text{if } \frac{n+2}{2} \geq r \geq \frac{n+2}{n}, \\ \leq 0, & \text{if } r > \frac{n+2}{2}. \end{cases}$$

For the special case $D(u) = 1$ in (1.2) and the logistic source replaced by $u(a - \mu u^{r-1} - w)$, $a \in \mathbb{R}$, $n \geq 1$, $\mu > 0$, Zheng [43] has shown that, when $r > 2$, or

$$\mu > \mu^* = \begin{cases} \frac{(n-2)_+}{n} \chi C_{\frac{n}{2}+1}^{\frac{1}{\frac{n}{2}+1}}, & \text{if } r = 2 \text{ and } n \leq 4, \\ \text{is appropriately large}, & \text{if } r = 2 \text{ and } n > 5, \end{cases}$$

the problem (1.2) possesses a global classical solution which is bounded, where $C_{\frac{n}{2}+1}^{\frac{1}{\frac{n}{2}+1}}$ is a positive constant which corresponds to the maximal Sobolev regularity. When $D(u) = 1$ in (1.2) and the logistic source is replaced by $u(a - \mu u^{r-1} - \lambda w)$, $a \in \mathbb{R}$, $n \geq 1$, $\mu > 0$, $\lambda > 0$, Zheng [44] has shown that when $r > 2$, or $r = 2$, with $\mu > \mu^\star = \frac{(n-2)_+}{n}(\chi + C_\beta)C_{\frac{n}{2}+1}^{\frac{1}{\frac{n}{2}+1}}$ the problem (1.2) possesses a global classical solution which is bounded, where C_β and $C_{\frac{n}{2}+1}$ are positive constants.

Many authors considered the following Keller–Segel system:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\nabla v) + f(u), & x\in\Omega, t>0,\\ v_t = \Delta v - v + g(u), & x\in\Omega, t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), \qquad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

in a smoothly bounded domain $\Omega\subset\mathbb{R}^n$, $n\geq 2$. Here the positive function $D(u)$ represents the diffusivity of the cells, and the nonnegative function $S(u)$ measures the chemotactic sensitivity. The functions $f(u)$ and $g(u)$ are the growth of u and the production of v, respectively.

For $D(u)=(1+u)^{-\alpha}(\alpha\in\mathbb{R})$, $S(u)=u(1+u)^{\beta-1}(\beta\in\mathbb{R})$ and $g(u)=u^{\eta}(\eta>0)$, Tao et al. [16] proved that model (1.3) possesses a uniform-in-time boundedness of solutions in the case of $f\equiv 0$, $\eta\in(0,1]$ and $\alpha+\beta+\eta<1+\frac{2}{n}$ or in the case of $f(u)=\gamma u-\mu u^r$, $\gamma\in\mathbb{R}$, $r>1$ and $\beta+\eta<r$ or $\beta+\eta=r$, $\mu\geq\mu_0$ for some $\mu_0>0$; Wang et al. [27] found that model (1.3) possesses a unique global-in-time classical solution for $0<\alpha+\beta<\frac{2}{n}$ when $f(u)=\gamma u-\mu u^r$, $g(u)=u$, $\gamma\in\mathbb{R}$, $r>1$; Zheng [39] proved that model (1.3) possesses a unique global-in-time classical solution that is bounded in the case $0<\alpha+\beta<\max\{r-1+\alpha,\frac{2}{n}\}$, $n\geq 1$, or $\beta=r-1$ and μ is large enough. Afterwards,Wang and Liu [31] improved the previous results on the boundedness of solutions to (1.3). When $f(u)=u(1-u)$, $g(u)=u$, $n\geq 3$, Zheng [42] shown that model (1.3) possesses a unique global-in-time classical solution that is bounded in the case $0<\alpha+\beta<\frac{4}{n+2}$. When $f(u)=\gamma u-\mu u^2$, $g(u)=u$, $D(u)\geq\delta u^{-\alpha}$, $\delta u^{\beta}\leq S(u)\leq\delta_1 u^{\beta}$ $\alpha,\beta\in\mathbb{R}$, $\delta_1>\delta>0$, $u\geq u_0$ with some $u_0>1$, Cao [2] proved that model (1.3) possesses a unique global-in-time classical solution that is bounded in the case $\beta<1$. For research on the corresponding quasilinear parabolic-elliptic problems, we refer to [37, 40] and the references therein.

For the special case $f\equiv 0$, $g(u)=u$ in (1.3), Winkler [32] found that if $\frac{S(u)}{D(u)}$ grows faster than $u^{\frac{2}{n}}$ as $u\to\infty$ and some further technical conditions are fulfilled, then there exist solutions that blow up in either finite or infinite time. Afterwards, Tao and Winkler [22] proved that solutions (1.3) remain bounded under the condition that $\frac{S(u)}{D(u)}\leq cu^{\alpha}$ with $\alpha<\frac{2}{n}$ and $c>0$ for all $u>1$, provided that Ω is a convex domain and $D(u)$ satisfies some other technical conditions. Then Ishida et al. [8] generalized the result obtained in [22] to non-convex domains.

For the special case $D(u)=1$, $S(u)=u$, $f(u)=0$ and $g(u)=u^{\eta}$ in (1.3), Liu and Tao [11] shown the global boundedness of solutions when $0<\eta<\frac{2}{n}$. As to the case $D(u)=1$, $S(u)=\chi u$, $f(u)=u-\mu u^r$ and $g(u)=u(u+1)^{\eta-1}$ in (1.3) for all $u\geq 0$ with some $\chi>0$, $r>1$, $\eta>0$, Zhuang et al. [47] proved that model (1.3) possesses a globally bounded classical solution if $r>\eta+1$, or $r=\eta+1$ and μ is large enough. When $D(u)=1$, $S(u)=\chi u$, $f(u)=\gamma u-\mu u^2$, $g(u)=u$, Osaki et al. [14] proved the solutions of (1.3) are globally bounded in two space dimension regardless of the size of $\mu>0$. Later, Winkler [33] found that model (1.3) possesses a global solution that is bounded under the condition $n\leq 3$, Ω convex, and $\mu>0$ sufficiently large.

When the second PDE in (1.3) is replaced by $0=\Delta v-\frac{1}{|\Omega|}\int_{\Omega}u+u$ and $D(u)=1$, $S(u)=\chi u$, $f(u)=\gamma u-\mu u^r$, in (1.3) for all $u\geq 0$ with some $\chi>0$, $\gamma\in\mathbb{R}$, $\mu\geq 0$, $r\geq 1$, Winkler [34] found that model (1.3) possesses a local-in-time solution of (1.3) that blows up in finite time for $r < \frac{3}{2} + \frac{1}{2n-2}$ in dimension $n \geq 5$. When the second PDE in (1.3) is replaced by $0 = \Delta v - v + u$ and $D(u) = 1$, $S(u) = u, f(u) = \gamma u - \mu u^r, g(u) = u$ in (1.3) for all $u \geq 0$ with some $\gamma \in \mathbb{R}$, $\mu \geq 0$, $r \geq 1$, Winkler [36] shown that model (1.3) possesses a corresponding solution of (1.3) blows up in finite time for $r < \frac{7}{6}$ in dimension $n = 3, 4$ or for $r < 1 + \frac{1}{2n-2}$ in dimension $n \geq 5$. When the second PDE in (1.3) is replaced by $0 = \Delta v - \frac{1}{|\Omega|} \int_\Omega g(u) + g(u)$ and $D(u) = 1$, $S(u) = u, f(u) = 0, g(u) = u^\eta$ in (1.3) for all $u \geq 0$ with some $\eta > 0$, Winkler [35] proved the global boundedness of solutions when $0 < \eta < \frac{2}{n}$. Moreover, it is presented in [35] that if Ω is a ball and then there exists initial data such that the corresponding radially symmetric solution blows up in finite time if $\eta > \frac{2}{n}$, hence $\eta = \frac{2}{n}$ is critical. In addition, for the studies on the parabolic-elliptic version, we suggest the reader to read the recent papers [7, 12, 38, 45].

Motivated the above papers, we consider the boundedness of solutions to the following quasilinear chemotaxis–haptotaxis model of parabolic–parabolic–ODE type:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u \nabla v) & \\ \qquad - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u^{r-1} - w), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u^\eta, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0, \\ D(u)\frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

under zero-flux boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$, with parameters $r \geq 2$, $\eta \in (0,1]$ and the parameters $\chi > 0$, $\xi > 0$, $\mu > 0$. This paper mainly aims to understand the competition among the nonlinear diffusion, the haptotaxis, the nonlinear logistic source and the nonlinear production.

The functions u_0, v_0, w_0 are supposed to satisfy the smoothness assumptions

$$\begin{cases} u_0 \in C(\bar{\Omega}) & \text{with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) & \text{with } v_0 \geq 0 \text{ in } \Omega, \\ w_0 \in C^{2+\vartheta}(\bar{\Omega}) & \text{for some } \vartheta \in (0,1) \text{ with } w_0 \geq 0 \text{ in } \bar{\Omega} \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

We furthermore assume that

$$D \in C^2([0,\infty)), \quad D(0) > 0 \tag{1.6}$$

and

$$D(u) \geq \delta u^{-\alpha} \quad \text{for all } u \geq 0 \tag{1.7}$$

with some $\alpha \in \mathbb{R}$ and $\delta > 0$.

The main result of this paper reads as follows.

Theorem 1.1 *Let $n \geq 2$, $\chi > 0$, $\xi > 0$, $\mu > 0$, $r \geq 2$ and $\eta \in (0,1]$, and let D be a function satisfying* (1.6) *and* (1.7) *with $\alpha < \frac{n+2-2n\eta}{2+n}$. Then, for any initial data fulfilling* (1.5), *the problem* (1.4) *admits a unique classical solution which is global and bounded in $\Omega \times (0,\infty)$.*

Remark 1.1

(i) From our results, it is worth to point out that the nonlinear production affect the nonlinear diffusion to guarantee the global boundedness of the solution to (1.4).

(ii) Obviously, since (1.6) and (1.7) are equivalent to $D(u) \geq \delta(u+1)^{-\alpha}$, for $r = 2$ and $\eta = 1$, Theorem 1.1 agrees with Wang [28, 29], who proved the boundedness of the solutions in the case $n \geq 2$.

This paper is structured as follows. In Sect. 2, we collect basic facts which will be used later. Section 3 is devoted to proving global existence and boundedness by using some L^p-estimate techniques and Moser–Alikakos iteration (see e.g. [1] and Lemma A.1 in [22]).

2 Preliminaries

We first state one result concerning local-in-time existence of a classical solution to model (1.4).

Lemma 2.1 *Let $\chi > 0$, $\xi > 0$ and $\mu > 0$, and assume that u_0, v_0 and w_0 satisfy* (1.5). *Then the problem* (1.4) *admits a unique classical solution*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ v \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ w \in C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \end{cases} \tag{2.1}$$

with $u \geq 0$, $v \geq 0$ and $0 \leq w \leq \|w_0\|_{L^\infty(\Omega)}$ for all $(x,t) \in \Omega \times [0, T_{\max})$, where $T_{\max}$ denotes the maximal existence time. In addition, if $T_{\max} < +\infty$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}. \tag{2.2}$$

Proof The local-in-time existence of classical solution to model (1.4) is well established by a fixed point theorem in the context of chemotaxis–haptotaxis systems. By the maximum principle, it is easy to obtain $u \geq 0$ and $v \geq 0$ for all $(x,t) \in \Omega \times [0, T_{\max})$. Integrating the third equation in (1.4), it follows from (1.5) and $v \geq 0$ that $0 \leq w \leq \|w_0\|_{L^\infty(\Omega)}$ for all $(x,t) \in \Omega \times [0, T_{\max})$. The proof is quite standard, for details, we refer the reader to [46]. □

For reference, we begin with Young's inequality, which states, for any positive numbers p and q with $\frac{1}{p} + \frac{1}{q} = 1$, that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b \geq 0.$$

This immediately yields the so-called Young inequality with ϵ.

Lemma 2.2 (Young's inequality with ϵ) *Let p and q be two given positive numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $\epsilon > 0$,*

$$ab \leq \epsilon a^p + \frac{b^q}{(\epsilon p)^{\frac{q}{p}} q}, \quad \forall a, b \geq 0. \tag{2.3}$$

In the proof of main result, we will frequently use the following version of the Gagliardo–Nirenberg inequality, for detail we refer to the reader to [10].

Lemma 2.3 (Gagliardo–Nirenberg inequality) *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $r \geq 1$, $0 < q \leq p \leq \infty$, $s > 0$ be such that*

$$\frac{1}{r} \leq \frac{1}{n} + \frac{1}{p}.$$

Then there exists $c > 0$ such that

$$\|u\|_{L^p(\Omega)} \leq c\bigl(\|\nabla u\|^a_{L^r(\Omega)}\|u\|^{1-a}_{L^q(\Omega)} + \|u\|_{L^s(\Omega)}\bigr) \quad \text{for all } u \in W^{1,r}(\Omega) \cap L^q(\Omega),$$

where

$$a = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} + \frac{1}{n} - \frac{1}{r}}.$$

Proof This can be found in [10, Lemma 2.3]. □

The following lemma provides the basic estimates of solutions to (1.4).

Lemma 2.4 *Let (u, v, w) be the solution of (1.4). Then there exists $C > 0$ depending on n, $\|v_0\|_{L^1(\Omega)}$ and $\|u_0\|_{L^1(\Omega)}$ such that*

$$\begin{aligned} &\bigl\|u(\cdot,t)\bigr\|_{L^1(\Omega)} \leq C, \qquad \bigl\|v(\cdot,t)\bigr\|_{L^1(\Omega)} \leq C, \\ &\bigl\|\nabla v(\cdot,t)\bigr\|_{L^2(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \tag{2.4}$$

Proof (i) Integrating the first equation in (1.4) with respect to $x \in \Omega$, we have

$$\frac{d}{dt}\int_\Omega u \leq \mu \int_\Omega u - \mu \int_\Omega u^r.$$

Then

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u \leq (\mu + 1)\int_\Omega u - \mu \int_\Omega u^r, \tag{2.5}$$

since $w \geq 0$ by Lemma 2.1. Moreover, by Young's inequality (2.3), we get

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u \leq \widetilde{C}_1,$$

where $\widetilde{C}_1 > 0$, as all subsequently appearing constants $\widetilde{C}_2 > 0$ and $\widetilde{C}_3 > 0$ are depending on n, $\|v_0\|_{L^1(\Omega)}$ and $\|u_0\|_{L^1(\Omega)}$.

Upon ODE comparison, we can prove that $\|u(\cdot,t)\|_{L^1(\Omega)} \leq C$.

(ii) Integrating the second equation in (1.4) with respect to $x \in \Omega$ yields

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v = \int_\Omega u^\eta.$$

Moreover, if $\eta \in (0,1)$, by Young's inequality(2.3), we get

$$\int_\Omega u^\eta \le \int_\Omega u + (1-\eta)|\Omega| \le \widetilde{C}_2.$$

If $\eta = 1$, we get

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v = \int_\Omega u \le C.$$

In summary, upon ODE comparison, we can prove $\|v(\cdot,t)\|_{L^1(\Omega)} \le C$.

(iii) Multiplying the second equation in (1.4) by $-\Delta v$ and integrating over Ω, and using Young's inequality, we find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\Delta v|^2 + \int_\Omega |\nabla v|^2 = -\int_\Omega u^\eta \Delta v \le \int_\Omega |\Delta v|^2 + \frac{1}{4}\int_\Omega u^{2\eta}$$

and thus

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + 2\int_\Omega |\nabla v|^2 \le \frac{1}{2}\int_\Omega u^{2\eta}.$$

Combining this with (2.5), we obtain

$$\frac{d}{dt}\int_\Omega \left(|\nabla v|^2 + u\right) + 2\int_\Omega \left(|\nabla v|^2 + u\right) \le \frac{1}{2}\int_\Omega u^{2\eta} + (\mu+2)\int_\Omega u - \mu\int_\Omega u^r.$$

Moreover, by Young's inequality (2.3), we get

$$\frac{d}{dt}\int_\Omega \left(|\nabla v|^2 + u\right) + 2\int_\Omega \left(|\nabla v|^2 + u\right) \le \widetilde{C}_3.$$

Upon ODE comparison, we can prove $\|\nabla v(\cdot,t)\|_{L^2(\Omega)} \le C$. □

Lemma 2.5 *Let (u,v,w) be the classical solution of* (1.4) *in $\Omega \times (0,T_{\max})$. Then, for any $k > 1$,*

$$-\int_\Omega u^{k-1}\nabla\cdot(u\nabla w) \le c_1\left(\int_\Omega u^k + \int_\Omega u^k v + k\int_\Omega u^{k-1}|\nabla u|\right) \tag{2.6}$$

with constant $c_1 > 0$ independent of k.

Proof Firstly, we follow the well-known precedent in [18] and give the estimate for Δw. Since the third equation in (1.4) is an ODE, we have

$$\begin{aligned} &w(x,t) = w_0(x)e^{-\int_0^t v(x,s)\,ds}, \\ &\nabla w(x,t) = \nabla w_0(x)e^{-\int_0^t v(x,s)\,ds} - w_0(x)e^{-\int_0^t v(x,s)\,ds}\int_0^t \nabla v(x,s)\,ds, \end{aligned} \tag{2.7}$$

as well as

$$\begin{aligned}\Delta w(x,t) &= \Delta w_0(x)e^{-\int_0^t v(x,s)\,ds} - 2e^{-\int_0^t v(x,s)\,ds}\nabla w_0(x)\cdot\int_0^t \nabla v(x,s)\,ds \\ &\quad + w_0(x)e^{-\int_0^t v(x,s)\,ds}\cdot\left|\int_0^t \nabla v(x,s)\,ds\right|^2 - w_0(x)e^{-\int_0^t v(x,s)\,ds}\int_0^t \Delta v(x,s)\,ds\end{aligned}$$

and

$$\begin{aligned}\Delta w(x,t) &\geq \Delta w_0(x)e^{-\int_0^t v(x,s)\,ds} - 2e^{-\int_0^t v(x,s)\,ds}\nabla w_0(x)\cdot\int_0^t \nabla v(x,s)\,ds \\ &\quad - w_0(x)e^{-\int_0^t v(x,s)\,ds}\int_0^t \Delta v(x,s)\,ds.\end{aligned} \tag{2.8}$$

Note that $\frac{\partial w_0}{\partial \nu} = 0$ and $\frac{\partial v}{\partial \nu} = 0$, (2.7) shows that $\frac{\partial w}{\partial \nu} = 0$. Therefore, the zero-flux boundary condition in (1.4) becomes

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0.$$

Hence, for any $k \geq 1$, integrating by parts and using (2.8), we obtain

$$\begin{aligned} &-\int_\Omega u^{k-1}\nabla\cdot(u\nabla w)\,dx \\ &= (k-1)\int_\Omega u^{k-1}\nabla u\cdot\nabla w\,dx \\ &= -\frac{k-1}{k}\int_\Omega u^k\Delta w\,dx \\ &\leq \frac{k-1}{k}\int_\Omega u^k\Big(-\Delta w_0(x)e^{-\int_0^t v(x,s)\,ds} + 2e^{-\int_0^t v(x,s)\,ds}\nabla w_0(x)\cdot\int_0^t \nabla v(x,s)\,ds \\ &\quad + w_0(x)e^{-\int_0^t v(x,s)\,ds}\int_0^t \Delta v(x,s)\,ds\Big)\,dx \\ &=: J_1 + J_2 + J_3, \end{aligned} \tag{2.9}$$

where

$$J_1 = -\frac{k-1}{k}\int_\Omega u^k\Delta w_0(x)e^{-\int_0^t v(x,s)\,ds}\,dx,$$

$$J_2 = \frac{2(k-1)}{k}\int_\Omega u^k e^{-\int_0^t v(x,s)\,ds}\nabla w_0(x)\cdot\int_0^t \nabla v(x,s)\,ds\,dx$$

and

$$J_3 = \frac{k-1}{k}\int_\Omega u^k w_0(x)e^{-\int_0^t v(x,s)\,ds}\int_0^t \Delta v(x,s)\,ds\,dx.$$

Now, since $v \geq 0$ leads to

$$-\Delta w_0(x)e^{-\int_0^t v(x,s)\,ds} \leq \|\Delta w_0\|_{L^\infty(\Omega)} \quad \text{for all } (x,t) \in \Omega\times(0,T_{\max}),$$

we have

$$J_1 \le \|\Delta w_0\|_{L^\infty(\Omega)} \int_\Omega u^k\,dx, \tag{2.10}$$

$$\begin{aligned} J_2 &= -\frac{2(k-1)}{k}\int_\Omega u^k \nabla e^{-\int_0^t v(x,s)\,ds}\cdot\nabla w_0(x)\,dx \\ &= 2(k-1)\int_\Omega u^{k-1}\nabla u\cdot\nabla w_0(x)e^{-\int_0^t v(x,s)\,ds}\,dx \\ &\quad + \frac{2(k-1)}{k}\int_\Omega u^k\Delta w_0(x)e^{-\int_0^t v(x,s)\,ds}\,dx \\ &\le c_1k\int_\Omega u^{k-1}|\nabla u|\,dx + c_1\int_\Omega u^k\,dx, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} J_3 &= \frac{k-1}{k}\int_\Omega u^k w_0(x)e^{-\int_0^t v(x,s)\,ds}\int_0^t \bigl(v_s(x,s)+v(x,s)-u^\eta(x,s)\bigr)\,ds\,dx \\ &\le \frac{k-1}{k}\int_\Omega u^k w_0(x)e^{-\int_0^t v(x,s)\,ds}\left(v(x,s)-v_0(x)+\int_0^t v(x,s)\,ds\right)dx \\ &\le c_1\int_\Omega u^k v\,dx + c_1\int_\Omega u^k\,dx \end{aligned} \tag{2.12}$$

for all $(x,t)\in\Omega\times(0,T_{\max})$, where we have used the facts that $ze^{-z}\le\frac{1}{e}$ for all $z\in\mathbb{R}$ and $0<e^{-\int_0^t v(x,s)\,ds}\le 1$ thanks to $v\ge 0$. Inserting (2.10)–(2.12) into (2.9) yields

$$-\int_\Omega u^{k-1}\nabla\cdot(u\nabla w)\,dx \le c_1\int_\Omega u^k\,dx + c_1\int_\Omega u^k v\,dx + c_1k\int_\Omega u^{k-1}|\nabla u|\,dx. \qquad \square$$

Lemma 2.6 *Let* $n\ge 2$, $\eta\in(0,1]$, $\alpha<\frac{n+2-2n\eta}{2+n}$, $\theta_1=\frac{2(k+1)}{1-\alpha}$, $\theta_2=\frac{2(k+1)(m-1)}{k-2\eta+1}$ *and* $\kappa_i=\frac{\frac{m}{2}-\frac{m}{\theta_i}}{\frac{m}{2}-\frac{1}{2}+\frac{1}{n}}$, $i=1,2$. *Then, for all sufficiently large* $k>1$, *there exists a large* $m>1$ *such that the following inequalities are valid*:

$$\theta_i>2,\qquad m>\frac{n-2}{2n}\theta_i,\qquad 2m>\max\{\theta_i\kappa_i,k+1\}\quad \textit{for } i=1,2. \tag{2.13}$$

Proof Since $2m>\theta_i\kappa_i$ is equivalent to $m>\frac{\theta_i}{2}-\frac{2}{n}$, it is sufficient to show that if $\alpha<\frac{n+2-2n\eta}{2+n}$, then, for all sufficiently large $k>1$, there exists a large $m>1$ satisfying $m>\frac{\theta_i}{2}-\frac{2}{n}(i=1,2)$ and $2m>k+1$, which can be achieved by the fact that $m>\frac{\theta_i}{2}-\frac{2}{n}(i=1,2)$ is equivalent to $\frac{k+1}{1-\alpha}-\frac{2}{n}<m<\frac{(k+1)(n+2)}{2n\eta}-\frac{2}{n}$. $\square$

3 Proof of Theorem 1.1

In this section, we are going to establish an iteration step to develop the main ingredient of our result. The iteration depends on a series of a priori estimates. Firstly, based on the estimates in Lemma 2.3, we use test function arguments to derive the bound of u in $L^k(\Omega)$ and ∇v in $L^{2m}(\Omega)$ for all sufficiently large $k,m>1$, which is the main step towards our proof of Theorem 1.1.

Lemma 3.1 *Assume that D satisfies (1.6) and (1.7) with $\alpha < \frac{n+2-2n\eta}{2+n}$. Then, for all large numbers $m > 2$, $k > 1$ as provided by Lemma 2.6, there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^k(\Omega)} \le C, \qquad \|\nabla v(\cdot,t)\|_{L^{2m}} \le C \quad \textit{for all } t \in (0, T_{\max}). \tag{3.1}$$

Proof Multiplying the first equation in (1.4) by ku^{k-1} and integrating over Ω, we get

$$\begin{aligned}
&\frac{d}{dt}\|u\|_{L^k(\Omega)}^k + k(k-1)\int_\Omega u^{k-2}D(u)|\nabla u|^2 + k\mu\int_\Omega u^{k+r-1} \\
&\quad \le -k\chi\int_\Omega \nabla\cdot(u\nabla v)u^{k-1} - k\xi\int_\Omega \nabla\cdot(u\nabla w)u^{k-1} + k\mu\int_\Omega u^k.
\end{aligned} \tag{3.2}$$

By (1.7), we have

$$\delta k(k-1)\int_\Omega u^{k-2-\alpha}|\nabla u|^2 \le k(k-1)\int_\Omega u^{k-2}D(u)|\nabla u|^2. \tag{3.3}$$

By Young's inequality, the first item on the right side of the inequality (3.2) becomes

$$\begin{aligned}
&-k\chi\int_\Omega \nabla\cdot(u\nabla v)u^{k-1} \\
&\quad = k\chi\int_\Omega u\nabla v\cdot\nabla u^{k-1} \\
&\quad = k(k-1)\chi\int_\Omega u^{k-1}\nabla u\cdot\nabla v \\
&\quad \le \frac{\delta k(k-1)}{2}\int_\Omega u^{k-2-\alpha}|\nabla u|^2 + \frac{\chi^2 k(k-1)}{2\delta}\int_\Omega u^{k+\alpha}|\nabla v|^2.
\end{aligned} \tag{3.4}$$

The second item of the right side of the inequality (3.2), combining with (2.6), yields

$$\begin{aligned}
&-k\xi\int_\Omega u^{k-1}\nabla\cdot(u\nabla w) \le c_1k\xi\int_\Omega u^k + c_1k\xi\int_\Omega u^k v + c_1k^2\xi\int_\Omega u^{k-1}|\nabla u| \\
&\quad \le c_1k\xi\int_\Omega u^k + c_1k\xi\int_\Omega u^k v + \frac{\delta k(k-1)}{4}\int_\Omega u^{k-2-\alpha}|\nabla u|^2 + \frac{c_1^2\xi^2k^3}{\delta(k-1)}\int_\Omega u^{k+\alpha}.
\end{aligned} \tag{3.5}$$

Hence, inserting (3.3)–(3.5) into (3.2) yields

$$\begin{aligned}
&\frac{d}{dt}\|u\|_{L^k(\Omega)}^k + \frac{\delta k(k-1)}{4}\int_\Omega u^{k-2-\alpha}|\nabla u|^2 + k\mu\int_\Omega u^{k+r-1} \\
&\quad \le \frac{\chi^2k(k-1)}{2\delta}\int_\Omega u^{k+\alpha}|\nabla v|^2 + c_1k\xi\int_\Omega u^k + c_1k\xi\int_\Omega u^k v \\
&\qquad + \frac{c_1^2\xi^2k^3}{\delta(k-1)}\int_\Omega u^{k+\alpha} + k\mu\int_\Omega u^k.
\end{aligned} \tag{3.6}$$

Removing the nonnegative number on the left of the inequality (3.6), we have

$$\begin{aligned}
\frac{d}{dt}\|u\|_{L^k(\Omega)}^k + k\mu\int_\Omega u^{k+r-1} &\le \frac{\chi^2k(k-1)}{2\delta}\int_\Omega u^{k+\alpha}|\nabla v|^2 + c_1k\xi\int_\Omega u^k + c_1k\xi\int_\Omega u^k v \\
&\quad + \frac{c_1^2\xi^2k^3}{\delta(k-1)}\int_\Omega u^{k+\alpha} + k\mu\int_\Omega u^k.
\end{aligned}$$

Furthermore, using Young's inequality, we can find

$$\frac{d}{dt}\|u\|_{L^k(\Omega)}^k + c_2\int_\Omega u^{k+1} \le \frac{\chi^2 k(k-1)}{2\delta}\int_\Omega u^{k+\alpha}|\nabla v|^2 + c_2\int_\Omega v^{k+1} + c_2, \tag{3.7}$$

where $c_2 > 0$, as all subsequently appearing constants $c_3, c_4, \dots, c_{16}$ possibly depend on k, m, μ, ξ, r, η, $|\Omega|$ and δ.

Differentiating the second equation in (1.4), we obtain

$$\frac{d}{dt}|\nabla v|^2 = 2\nabla v \cdot \nabla\Delta v - 2|\nabla v|^2 + 2\nabla u^\eta \cdot \nabla v,$$

and hence, according to the identity

$$\Delta|\nabla v|^2 = 2\nabla v \cdot \nabla\Delta v + 2\left|D^2 v\right|^2,$$

we obtain

$$\frac{d}{dt}|\nabla v|^2 = \Delta|\nabla v|^2 - 2\left|D^2 v\right|^2 + 2\nabla u^\eta \cdot \nabla v.$$

Testing this by $m|\nabla v|^{2m-2}$ yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^{2m} + m(m-1)\int_\Omega |\nabla v|^{2m-4}\left|\nabla|\nabla v|^2\right|^2 \\
&\quad + 2m\int_\Omega |\nabla v|^{2m-2}\left|D^2 v\right|^2 + 2m\int_\Omega |\nabla v|^{2m} \\
&\le 2m\int_\Omega |\nabla v|^{2m-2}\nabla u^\eta \cdot \nabla v + m\int_{\partial\Omega} \frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^{2m-2}.
\end{aligned} \tag{3.8}$$

On the other hand, based on the estimate of Mizoguchi–Souplet [13], the Gagliardo–Nirenberg inequality and boundedness of ∇v in $L^2(\Omega)$, we can conclude that

$$m\int_{\partial\Omega} \frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^{2m-2} \le c_3\left(\int_\Omega \left|\nabla|\nabla v|^m\right|^2\right)^b + c_3 \tag{3.9}$$

with some $b \in (0,1)$. Therefore, combining (3.8) with (3.9) and applying Young's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^{2m} + \frac{m(m-1)}{2}\int_\Omega |\nabla v|^{2m-4}\left|\nabla|\nabla v|^2\right|^2 \\
&\quad + 2m\int_\Omega |\nabla v|^{2m-2}\left|D^2 v\right|^2 + 2m\int_\Omega |\nabla v|^{2m} \\
&\le 2m\int_\Omega |\nabla v|^{2m-2}\nabla u^\eta \cdot \nabla v + c_4
\end{aligned} \tag{3.10}$$

due to $\int_\Omega |\nabla v|^{2m-4}|\nabla|\nabla v|^2|^2 = \frac{4}{m}\int_\Omega |\nabla|\nabla v|^m|^2$.

Hence, due to the pointwise identities $\nabla|\nabla v|^{2m-2} = (m-1)|\nabla v|^{2m-4}\nabla|\nabla v|^2$ and $|\Delta v|^2 \le n|D^2 v|^2$, and together with an integration by the right part in (3.10) and using Young's inequality, we have

$$\begin{aligned}
&2m\int_\Omega |\nabla v|^{2m-2}\nabla u^\eta\cdot\nabla v\\
&= -2m(m-1)\int_\Omega u^\eta|\nabla v|^{2m-4}\nabla v\cdot\nabla|\nabla v|^2 - 2m\int_\Omega u^\eta|\nabla v|^{2m-2}\Delta v\\
&\le \frac{m(m-1)}{4}\int_\Omega |\nabla v|^{2m-4}\big|\nabla|\nabla v|^2\big|^2 + 4m(m-1)\int_\Omega u^{2\eta}|\nabla v|^{2m-2}\\
&\quad + \frac{m}{n}\int_\Omega |\nabla v|^{2m-2}|\Delta v|^2 + mn\int_\Omega u^{2\eta}|\nabla v|^{2m-2}\\
&\le \frac{m(m-1)}{4}\int_\Omega |\nabla v|^{2m-4}\big|\nabla|\nabla v|^2\big|^2 + \big(4m(m-1)+mn\big)\int_\Omega u^{2\eta}|\nabla v|^{2m-2}\\
&\quad + m\int_\Omega |\nabla v|^{2m-2}\big|D^2v\big|^2.
\end{aligned}\tag{3.11}$$

Hence, inserting (3.11) into (3.10) yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^{2m} + (m-1)\int_\Omega \big|\nabla|\nabla v|^m\big|^2 + 2m\int_\Omega |\nabla v|^{2m}\\
&\le \big(4m(m-1)+mn\big)\int_\Omega u^{2\eta}|\nabla v|^{2m-2} + c_4.
\end{aligned}\tag{3.12}$$

Hence combining (3.7) with (3.12) and using Young's inequality, we can find

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \big(u^k + |\nabla v|^{2m}\big) + c_5\int_\Omega \big(\big|\nabla|\nabla v|^m\big|^2 + |\nabla v|^{2m}\big) + c_5\int_\Omega u^{k+1}\\
&\le c_6\int_\Omega u^{k+\alpha}|\nabla v|^2 + c_6\int_\Omega u^{2\eta}|\nabla v|^{2m-2} + c_6\int_\Omega v^{k+1} + c_6\\
&\le \frac{c_5}{2}\int_\Omega u^{k+1} + c_7\int_\Omega \big(|\nabla v|^{\theta_1} + |\nabla v|^{\theta_2}\big) + c_6\int_\Omega v^{k+1} + c_6
\end{aligned}\tag{3.13}$$

with θ_i $(i=1,2)$ as shown in Lemma 2.6. According to the Gagliardo–Nirenberg inequality, (2.4) and Lemma 2.6, we have

$$\begin{aligned}
c_7\int_\Omega |\nabla v|^{\theta_i} &= c_7\big\||\nabla v|^m\big\|_{L^{\frac{\theta_i}{m}}}^{\frac{\theta_i}{m}}\\
&\le c_8\Big(\big\|\nabla|\nabla v|^m\big\|_{L^2(\Omega)}^{\kappa_i}\big\||\nabla v|^m\big\|_{L^{\frac{2}{m}}(\Omega)}^{1-\kappa_i} + \big\||\nabla v|^m\big\|_{L^{\frac{2}{m}}(\Omega)}\Big)^{\frac{\theta_i}{m}}\\
&\le c_9\big\|\nabla|\nabla v|^m\big\|_{L^2(\Omega)}^{\frac{\theta_i\kappa_i}{m}} + c_9\\
&\le \frac{c_5}{2}\big\|\nabla|\nabla v|^m\big\|_{L^2(\Omega)}^2 + c_{10}.
\end{aligned}\tag{3.14}$$

Due to the boundedness of $\|v\|_{W^{1,2}(\Omega)}$ (see Lemma 2.4) and Lemma 2.6, and by the Sobolev inequality and Young's inequality, we can find

$$\begin{aligned}
c_6\int_\Omega v^{k+1} &\le c_{11}\|v\|_{L^\infty(\Omega)}^{k+1} \le c_{12}\|v\|_{L^{n+1}(\Omega)}^{k+1} + c_{12}\\
&\le c_{13}\|v\|_{L^{2m}(\Omega)}^{k+1} + c_{12} \le \frac{c_5}{2}\int_\Omega |\nabla v|^{2m} + c_{14}.
\end{aligned}\tag{3.15}$$

Hence substituting (3.14) and (3.15) into (3.13) yields

$$\frac{d}{dt}\int_\Omega\left(u^k+|\nabla v|^{2m}\right)+\frac{c_5}{2}\int_\Omega\left(u^{k+1}+|\nabla v|^{2m}\right)\le c_{15},$$

by Young's inequality, we can find

$$\frac{d}{dt}\int_\Omega\left(u^k+|\nabla v|^{2m}\right)+\frac{c_5}{2}\int_\Omega\left(u^{k}+|\nabla v|^{2m}\right)\le c_{16}$$

for sufficiently large $k>1$, $m>1$. Consequently, $y(t):=\int_\Omega(u^k+|\nabla v|^{2m})$ satisfies $y'(t)+\frac{c_5}{2}y(t)\le c_{16}$.

Upon an ODE comparison argument, we have $y(t)\le\max\{y(0),\frac{2c_{16}}{c_5}\}$ for all $t\in(0,T_{\max})$. The proof of Lemma 3.1 is complete. □

Due to $\|u(\cdot,t)\|_{L^k(\Omega)}\le C$ is bounded for any large k, by the fundamental estimates for Neumann semigroup (see [8, Lemma 2.1]) or the standard regularity theory of parabolic equation, we immediately have the following corollary.

Corollary 3.1 *Let $\chi>0$, $\xi>0$ and $\mu>0$, and assume that (u_0,v_0,w_0) satisfy (1.5). Then there exists $C>0$ such that the solution (u,v,w) of (1.4) satisfies*

$$\left\|v(\cdot,t)\right\|_{W^{1,\infty}(\Omega)}\le C\quad\text{for all } t\in(0,T_{\max}).\tag{3.16}$$

Now we can prove our main result. The derivation of following statement can be obtained by a well-established Moser–Alikakos iteration technique (see e.g. [1] and Lemma A.1 in [22]). We choose (3.6) as a starting point for our proof.

Lemma 3.2 *Under the same assumption of Lemma 3.1, there exists $C>0$ such that the solution (u,v,w) of (1.4) satisfies*

$$\left\|u(\cdot,t)\right\|_{L^{\infty}(\Omega)}\le C\quad\text{for all } t\in(0,T_{\max}).\tag{3.17}$$

Proof We begin with (3.6)

$$\begin{aligned}\frac{d}{dt}\|u\|^k_{L^k(\Omega)}&+\frac{\delta k(k-1)}{4}\int_\Omega u^{k-2-\alpha}|\nabla u|^2+k\mu\int_\Omega u^{k+r-1}\\&\le\frac{\chi^2k(k-1)}{2\delta}\int_\Omega u^{k+\alpha}|\nabla v|^2+c_1k\xi\int_\Omega u^k+c_1k\xi\int_\Omega u^kv\\&\quad+\frac{c_1^2\xi^2k^3}{2\delta(k-1)}\int_\Omega u^{k+\alpha}+k\mu\int_\Omega u^k,\end{aligned}$$

which, along with (3.16), implies that

$$\begin{aligned}\frac{d}{dt}\|u\|^k_{L^k(\Omega)}&+\frac{\delta k(k-1)}{4}\int_\Omega u^{k-2-\alpha}|\nabla u|^2+k\mu\int_\Omega u^{k+r-1}\\&\le c_{17}k(k-1)\int_\Omega u^{k+\alpha}+c_{17}k\int_\Omega u^k+\frac{c_1^2\xi^2k^3}{2\delta(k-1)}\int_\Omega u^{k+\alpha}+k\mu\int_\Omega u^k,\end{aligned}$$

where $c_{17}>0$, as all subsequently appearing constants $c_{18},c_{19},\dots$ are independent of k.

Due to $\alpha < 1$, $r > 2$ and $u \geq 0$ and by Young's inequality, we can find

$$\frac{d}{dt}\int_\Omega u^k + c_{18}\int_\Omega \bigl|\nabla u^{\frac{k-\alpha}{2}}\bigr|^2 + \int_\Omega u^k \leq c_{19}k^2\int_\Omega u^k. \tag{3.18}$$

We now recursively define

$$k := b_i = \frac{2}{s}\cdot b_{i-1} + \alpha, \quad i \geq 1, \tag{3.19}$$

$$\epsilon_i := \frac{2b_i(1-a)}{s(b_i - b_i a - \alpha)}, \quad i \geq 1, \tag{3.20}$$

and

$$M_i := \sup_{t\in(0,T)}\int_\Omega u^{b_i}, \quad i \in \mathbb{N}. \tag{3.21}$$

Note that $(b_i)_{i\in\mathbb{N}}$ increases and

$$c_{23}\cdot\left(\frac{2}{s}\right)^i \leq b_i \leq c_{24}\cdot\left(\frac{2}{s}\right)^i \quad \text{for all } i \in \mathbb{N}, \tag{3.22}$$

where we chose $s \in (0,2)$.

Now invoking the Gagliardo–Nirenberg inequality (Lemma 2.3), we find $c_{20} > 0$ independent of k, such that

$$\int_\Omega u^{b_i} = \bigl\|u^{\frac{b_i-\alpha}{2}}\bigr\|_{L^{\frac{2b_i}{b_i-\alpha}}}^{\frac{2b_i}{b_i-\alpha}} \leq c_{20}\bigl\|\nabla u^{\frac{b_i-\alpha}{2}}\bigr\|_{L^2(\Omega)}^{\frac{2b_i}{b_i-\alpha}a}\cdot\bigl\|u^{\frac{b_i-\alpha}{2}}\bigr\|_{L^s(\Omega)}^{\frac{2b_i}{b_i-\alpha}(1-a)} + c_{20}\bigl\|u^{\frac{b_i-\alpha}{2}}\bigr\|_{L^s(\Omega)}^{\frac{2b_i}{b_i-\alpha}} \tag{3.23}$$

for all $t \in (0, T_{\max})$, with

$$a = \frac{\frac{1}{s} - \frac{1}{\frac{2b_i}{b_i-\alpha}}}{\frac{1}{s} + \frac{1}{n} - \frac{1}{2}} = \frac{\frac{n}{s} - \frac{n}{\frac{2b_i}{b_i-\alpha}}}{\frac{n}{s} + 1 - \frac{n}{2}} \in (0,1). \tag{3.24}$$

Assume $b_i > \max\{\frac{n\alpha}{2}, \frac{\alpha}{1-a}\}$, so we have $\frac{b_i a}{b_i-\alpha} < 1$. Combining (3.18) with (3.23) and using Young's inequality, we obtain

$$\frac{d}{dt}\int_\Omega u^{b_i} + \int_\Omega u^{b_i} \leq c_{21}b_i^2\left(\left(\int_\Omega u^{\frac{s(b_i-\alpha)}{2}}\right)^{\frac{2b_i(1-a)}{s(b_i-\alpha)}}\right)^{\frac{b_i-\alpha}{b_i-\alpha-b_i a}} + c_{21}b_i^2\left(\int_\Omega u^{\frac{s(b_i-\alpha)}{2}}\right)^{\frac{2b_i}{s(b_i-\alpha)}}.$$

To simplify this, we observe that $\frac{2b_i(1-a)}{s(b_i-\alpha)}\cdot\frac{b_i-\alpha}{b_i-\alpha-b_i a} > \frac{2b_i}{s(b_i-\alpha)}$, and thus

$$\frac{d}{dt}\int_\Omega u^{b_i} + \int_\Omega u^{b_i} \leq c_{22}b_i^2\left(\int_\Omega u^{\frac{s(b_i-\alpha)}{2}}\right)^{\frac{2b_i(1-a)}{s(b_i-\alpha-b_i a)}}. \tag{3.25}$$

Inserting (3.19)–(3.22) into (3.25) yields

$$\frac{d}{dt}\int_\Omega u^{b_i} + \int_\Omega u^{b_i} \leq c_{25}\cdot\left(\frac{4}{s^2}\right)^i\cdot M_{i-1}^{\epsilon_i}.$$

Upon invoking an ODE comparison argument, we have

$$M_i \le \max\left\{ \|u_0\|^{b_i}_{L^\infty(\Omega)}, c_{25}\cdot\left(\frac{4}{s^2}\right)^i \cdot M_{i-1}^{\epsilon_i}\right\}.$$

We easily deduce from (3.19), (3.20) and (3.22) that

$$\epsilon_i = \frac{2b_i(1-a)}{s(b_i - b_i a - \alpha)} = \frac{2}{s}\cdot\frac{b_i(1-a)}{b_i - b_i a - \alpha} = \frac{2}{s}\cdot(1+\varepsilon_i), \quad i\ge 1 \tag{3.26}$$

holds with some $\varepsilon_i \ge 0$ satisfying

$$\varepsilon_i = \frac{\alpha}{b_i - b_i a - \alpha} \le \frac{c_{26}}{b_i} \le c_{27}\cdot\left(\frac{s}{2}\right)^i \tag{3.27}$$

for all $i\ge 1$ and appropriately large $c_{26}>0$ and $c_{27}>0$.

Now if $\|u_0\|^{b_i}_{L^\infty(\Omega)} \ge c_{25}\cdot(\frac{2}{s})^i \cdot M_{i-1}^{\epsilon_i}$ for infinitely many $i\ge 1$, we get (3.17) with $C = \|u_0\|_{L^\infty(\Omega)}$ for all $t\in(0,T_{\max})$.

Otherwise

$$M_i \le c_{25}\cdot\left(\frac{4}{s^2}\right)^i \cdot M_{i-1}^{\epsilon_i} \quad \text{for all } i\ge 1.$$

By a straightforward induction, this yields

$$M_i \le c_{25}^{1+\sum_{j=0}^{i-2}\prod_{l=i-j}^{i}\epsilon_i}\cdot\left(\frac{4}{s^2}\right)^{i+\sum_{j=0}^{i-2}(i-j-1)\cdot\prod_{l=i-j}^{i}\epsilon_i}\cdot M_0^{\prod_{l=1}^{i}\epsilon_i}$$

for all $i\ge 2$, and hence in view of (3.22) and (3.26) we obtain

$$M_i^{\frac{1}{b_i}} \le c_{25}^{\frac{1}{c_{23}}(\frac{s}{2})^i + \frac{1}{c_{23}}\cdot\sum_{j=0}^{i-2}(\frac{s}{2})^{i-j-1}\cdot\prod_{l=i-j}^{i}(1+\varepsilon_i)} \times \left(\frac{4}{s^2}\right)^{i\frac{1}{c_{23}}(\frac{s}{2})^i + \frac{1}{c_{23}}\cdot\sum_{j=0}^{i-2}(i-j-1)(\frac{s}{2})^{i-j-1}\cdot\prod_{l=i-j}^{i}(1+\varepsilon_i)}$$
$$\times M_0^{\frac{1}{c_{23}}\cdot\prod_{l=1}^{i}(1+\varepsilon_i)}$$

for all $i\ge 2$. Since $\ln(1+z)\le z$ for $z\ge 0$, from (3.27) and the fact that $s<2$ we get

$$\ln\left(\prod_{l=1}^{i}(1+\varepsilon_i)\right) \le \sum_{l=1}^{i}\varepsilon_i \le \frac{c_{27}}{1-\frac{s}{2}},$$

so that using $\sum_{j=0}^{i-2}(i-j-1)\cdot(\frac{s}{2})^{i-j-1} \le \sum_{h=1}^{\infty} h(\frac{s}{2})^h < \infty$, from this we conclude that also in this case $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded from above by a constant independent of $t\in(0,T_{\max})$. This clearly proves (3.17). □

We are now in a position to pass to the proof of Theorem 1.1.

Proof of Theorem 1.1 First we see that boundedness of u and v follows from Lemma 3.2 and Corollary 3.1, respectively. Therefore the assertion of Theorem 1.1 is immediately obtained from Lemma 2.1. □

Acknowledgements
The paper is supported by the National Science Foundation of China (11301419) and the Meritocracy Research Funds of China West Normal University [17YC382].

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This paper is the result of joint work of all authors who contributed equally to the final version of this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 21 January 2019 Accepted: 14 August 2019 Published online: 27 August 2019

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