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Multiple positive solutions for mixed fractional differential system with p-Laplacian operators

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Abstract

This paper is focused on researching a class of mixed fractional differential system with p-Laplacian operators. Based on the properties of the corresponding Green's function, different combinations of superlinearity or sublinearity for the nonlinearities and other appropriate conditions, the existence of multiple positive solutions are derived via the Guo–Krasnosel'skii fixed point theorem. An example is then given to illustrate the usability of the main results.

MSC: 26A33; 34B18

Keywords: Multiple positive solutions; Mixed fractional differential system; p-Laplacian operators; Coupled integral boundary conditions

1 Introduction

In this paper, we investigate the following mixed fractional differential system:

$$\begin{cases} D^{\beta_1}(\varphi_{p_1}({}^cD^{\alpha_1}u(t))) + f_1(t,u(t),v(t)) = 0, \\ D^{\beta_2}(\varphi_{p_2}({}^cD^{\alpha_2}v(t))) + f_2(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0, \qquad u(1) = \mu_1 \int_0^1 a(s)v(s)\,dA_1(s), \\ v'(0) = v''(0) = \cdots = v^{(m-1)}(0) = 0, \qquad v(1) = \mu_2 \int_0^1 b(s)u(s)\,dA_2(s), \\ {}^cD^{\alpha_1}u(0) = 0, \qquad {}^cD^{\alpha_1}u(1) = \varepsilon_1 {}^cD^{\alpha_1}u(\eta_1), \\ {}^cD^{\alpha_2}v(0) = 0, \qquad {}^cD^{\alpha_2}v(1) = \varepsilon_2 {}^cD^{\alpha_2}v(\eta_2), \end{cases} \tag{1.1}$$

where $1 < \beta_i \le 2$, $1 \le n-1 < \alpha_1 \le n$, $1 \le m-1 < \alpha_2 \le m$, $n,m \ge 2$, D^{β_i} is the Riemann–Liouville derivative operator, ${}^cD^{\alpha_i}$ is the Caputo derivative. $\mu_i > 0$ is a constant, $\eta_i \in (0,1)$, $\varepsilon_i > 0$ and satisfies $1 - \varepsilon_i^{p_i-1}\eta^{\beta_i-1} > 0$, φ_{p_i} is the Laplacian operator defined by $\varphi_{p_i}(s) = |s|^{p_i-2}s$, $(\varphi_{p_i})^{-1} = \varphi_{q_i}$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $p_i > 1$, $\int_0^1 a(s)v(s)\,dA_1(s)$, $\int_0^1 b(s)u(s)\,dA_2(s)$ denote the Riemann–Stieltjes integrals with a signed measure, that is $A_i : [0,1] \to [0,+\infty)$ is the function of bounded variation. $a,b : [0,1] \to [0,+\infty)$ are continuous, $f_i : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function, $i = 1,2$.

Compared with the integer order systems, fractional differential systems are regarded as a better tool in the description of some problems in science and engineering. Arafal et al. [1] presented a fractional order model for infection of $CD4^{+}T$ cells:

$$\begin{cases} D^{\alpha_1}(T) = s - KVT - dT + bI, \\ D^{\alpha_2}(I) = KVT - (b+\delta)I, \\ D^{\alpha_3}(V) = N\delta I - cV, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$. In the mathematical context, many mathematicians and applied scholars have studied the fractional differential equation or system in recent years [2–15]. In addition, by applying the functional analysis methods such as the lower and upper solutions, monotone iterative techniques, fractional integro-differential equations or singular equations are researched by Dumitru et al. [16], Denton et al. [17], Lyons and Neugebauer [18], Ambrosio [19], Zhou and Qiao [20]. There are also related books [21, 22].

Cabada and Wang in [23] studied the following factional differential equation:

$$\begin{cases} D^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(1) = u'(0) = 0, \qquad u(1) = \lambda \int_0^1 u(s)\,dA(s), \end{cases} \tag{1.2}$$

where $2 < \alpha \le 3$, $0 < \lambda$, $\lambda \ne \alpha$, D^{α} is the Caputo fractional derivative, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function. By the use of Guo–Krasnosel'skii fixed point theorem, the authors in [23] obtained the positive solution to Eq. (1.2). Cabada and Wang also discussed the solution of Eq. (1.2) when D^{α} is the Riemann–Liouville fractional derivative [24].

The p-Laplacian equation is the second order quasilinear differential operator, it arises in the modeling of various physical and natural phenomena. Fractional differential equation with p-Laplacian operator can describe the nonlinear phenomena in non-Newtonian fluids and establishes complex process models; for some related articles, see [25–31]. Via variational methods, Li and Wei [32] dealt with fractional p-Laplacian equations, the existence and multiplicity of nontrivial solutions were obtained. Wu et al. [33] researched the following fractional differential turbulent flow model and obtained the iterative solutions of the equation:

$$\begin{cases} -D^{\alpha}(\varphi_p(-D^{\gamma}u(t))) = g(t)h(u), \quad 0 < t < 1, \\ u(0) = 0, \qquad D^{\gamma}u(0) = D^{\gamma}u(1) = 0, \qquad u(1) = \int_0^1 u(s)\,dA(s), \end{cases} \tag{1.3}$$

where $1 < \alpha, \gamma \le 2$, D^{α}, D^{γ} are the Riemann–Liouville fractional derivatives, $h : [0,+\infty) \to [0,+\infty)$ is a continuous and increasing function.

Fractional differential systems with p-Laplacian operators have also attracted tremendous attention [34–40]. Among them, applying the monotone iterative approach, the authors in [34] got the extremal solutions of the following system:

$$\begin{cases} D_{0^+}^{\alpha_1}(\varphi_{p_1}(D_{0^+}^{\beta_1}u(t))) = f_1(t,v(t)), \\ D_{0^+}^{\alpha_2}(\varphi_{p_2}(D_{0^+}^{\beta_2}v(t))) = f_2(t,u(t)), \quad 0 < t < 1, \\ u(0) = D_{0^+}^{\beta_1}u(0) = 0, \qquad D_{0^+}^{\gamma_1}u(1) = \sum_{j=1}^{m-2} a_{1j}D_{0^+}^{\gamma_1}u(\eta_j) = 0, \\ v(0) = D_{0^+}^{\beta_2}v(0) = 0, \qquad D_{0^+}^{\gamma_2}v(1) = \sum_{j=1}^{m-2} a_{2j}D_{0^+}^{\gamma_2}v(\eta_j) = 0, \end{cases} \tag{1.4}$$

where $0 < \alpha_i, \gamma_i \le 1$, $1 < \beta_i \le 2$, $D_{0^+}^{\alpha_i}$, $D_{0^+}^{\beta_i} D_{0^+}^{\gamma_i}$ are the Riemann–Liouville fractional derivatives, $i = 1, 2$.

Inspired by the above articles, in this article we discuss the mixed fractional differential system with p-Laplacian operators under integral boundary value conditions. To the best of our knowledge, there is very little research on mixed fractional differential systems, especially if the system has p-Laplacian operators. Through the application of the Guo–Krasnosel'skii fixed point theorem, the existence of multiple positive solutions of the system is achieved.

2 Preliminaries and lemmas

Definition 2.1 ([41, 42]) The Caputo fractional order derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$^{c}D^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds,$$

where $u \in C^n(J, \mathbb{R})$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ denotes the natural number set, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of α.

Definition 2.2 ([41, 42]) Let $\alpha > 0$ and let u be piecewise continuous on $(0, +\infty)$ and integrable on any finite subinterval of $[0, +\infty)$. Then, for $t > 0$, we call

$$I^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds$$

the Riemann–Liouville fractional integral of u of order α.

Lemma 2.1 ([41, 42]) *Let $n - 1 < \alpha \le n$, $u \in C^n[0,1]$. Then*

$$I^{\alpha}\left({}^{c}D^{\alpha} u\right)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n-1)$, n is the smallest integer greater than or equal to α.

Let $\varphi_{p_1}({}^{c}D_{0^+}^{\alpha_1} u(t)) = \overline{u}(t)$, $\varphi_{p_2}({}^{c}D_{0^+}^{\alpha_2} v(t)) = \overline{v}(t)$, then $\overline{u}(0) = 0$, $\overline{u}(1) = \varepsilon_1^{p_1-1} \overline{u}(\eta_1)$, $\overline{v}(0) = 0$, $\overline{v}(1) = \varepsilon_2^{p_2-1} \overline{v}(\eta_2)$, we now consider the following system:

$$\begin{cases} D^{\beta_1} \overline{u}(t) + y_1(t) = 0, \qquad D^{\beta_2} \overline{v}(t) + y_2(t) = 0, \quad 0 < t < 1, \\ \overline{u}(0) = \overline{v}(0) = 0, \qquad \overline{u}(1) = \varepsilon_1^{p_1-1} u(\eta_1), \\ \overline{v}(1) = \varepsilon_2^{p_2-1} v(\eta_2). \end{cases} \tag{2.1}$$

Similar to [43], if $y_i \in C[0,1]$, then the system (2.1) has a unique solution,

$$\begin{cases} \overline{u}(t) = \int_0^1 \overline{H}_1(t,s) y_1(s)\, ds, \\ \overline{v}(t) = \int_0^1 \overline{H}_2(t,s) y_2(s)\, ds, \end{cases}$$

where

$$\begin{aligned} &\overline{H}_i(t,s) = \overline{h}_i(t,s) + \frac{\varepsilon_i^{p_i-1} t^{\beta_i-1}}{1-\varepsilon_i^{p_i-1}\eta_i{}^{\beta_i-1}}, \\ &\overline{h}_i(t,s) = \begin{cases} \frac{(t(1-s))^{\beta_i-1}-(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}, & 0 \le s \le t \le 1, \\ \frac{(t(1-s))^{\beta_i-1}}{\Gamma(\beta_i)}, & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{2.2}$$

For $y_i \in C[0,1]$, consider the system

$$\begin{cases} D^{\beta_1}(\varphi_{p_1}({}^cD^{\alpha_1}u(t))) + y_1(t) = 0, \quad D^{\beta_2}(\varphi_{p_2}({}^cD^{\alpha_2}v(t))) + y_2(t) = 0, \quad 0 < t < 1, \\ u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0, \quad u(1) = \mu_1 \int_0^1 a(s)v(s)\,dA_1(s), \\ v'(0) = v''(0) = \cdots = v^{(m-1)}(0) = 0, \quad v(1) = \mu_2 \int_0^1 b(s)u(s)\,dA_2(s), \\ {}^cD^{\alpha_1}u(0) = 0, \quad {}^cD^{\alpha_1}u(1) = \varepsilon_1 {}^cD^{\alpha_1}u(\eta_1), \\ {}^cD^{\alpha_2}v(0) = 0, \quad {}^cD^{\alpha_2}v(1) = \varepsilon_2 {}^cD^{\alpha_2}v(\eta_2). \end{cases} \tag{2.3}$$

Through calculation, we conclude that system (2.3) is equal to

$$\begin{cases} {}^cD^{\alpha_1}u(t) + \varphi_{q_1}(\int_0^1 \overline{H}_1(t,s)y_1(s)\,ds) = 0, \\ {}^cD^{\alpha_2}v(t) + \varphi_{q_2}(\int_0^1 \overline{H}_2(t,s)y_2(s)\,ds) = 0, \quad 0 < t < 1, \\ u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0, \quad u(1) = \mu_1 \int_0^1 a(s)v(s)\,dA_1(s), \\ v'(0) = v''(0) = \cdots = v^{(m-1)}(0) = 0, \quad v(1) = \mu_2 \int_0^1 b(s)u(s)\,dA_2(s). \end{cases}$$

Lemma 2.2 was obtained by the author herself and her collaborator in [44]

Lemma 2.2 *Assume the following condition* ($\mathbf{H}_0$) *holds.*
($\mathbf{H}_0$)

$$k_1 = \int_0^1 a(s)\,dA_1(s) > 0, \qquad k_2 = \int_0^1 b(s)\,dA_2(s) > 0, \quad 1 - \mu_1\mu_2k_1k_2 > 0.$$

Let $h_i \in C(0,1) \cap L(0,1)$ $(i = 1,2)$, *then the system with the coupled boundary conditions*

$$\begin{cases} {}^cD^{\alpha_1}u(t) + h_1(t) = 0, \quad {}^cD^{\alpha_2}v(t) + h_2(t) = 0, \quad 0 < t < 1, \\ u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0, \quad u(1) = \mu_1 \int_0^1 a(s)v(s)\,dA_1(s), \\ v'(0) = v''(0) = \cdots = v^{(m-1)}(0) = 0, \quad v(1) = \mu_2 \int_0^1 b(s)u(s)\,dA_2(s), \end{cases} \tag{2.4}$$

has a unique integral representation,

$$\begin{cases} u(t) = \int_0^1 K_1(t,s)h_1(s)\,ds + \int_0^1 H_1(t,s)h_2(s)\,ds, \\ v(t) = \int_0^1 K_2(t,s)h_2(s)\,ds + \int_0^1 H_2(t,s)h_1(s)\,ds, \end{cases} \tag{2.5}$$

where

$$\begin{aligned} K_1(t,s) &= \frac{\mu_1\mu_2 k_1}{1-\mu_1\mu_2 k_1 k_2}\int_0^1 G_1(t,s)b(t)\,dA_2(t) + G_1(t,s), \\ H_1(t,s) &= \frac{\mu_1}{1-\mu_1\mu_2 k_1 k_2}\int_0^1 G_2(t,s)a(t)\,dA_1(t), \\ K_2(t,s) &= \frac{\mu_2\mu_1 k_2}{1-\mu_1\mu_2 k_1 k_2}\int_0^1 G_2(t,s)a(t)\,dA_1(t) + G_2(t,s), \\ H_2(t,s) &= \frac{\mu_2}{1-\mu_1\mu_2 k_1 k_2}\int_0^1 G_1(t,s)b(t)\,dA_2(t), \end{aligned} \tag{2.6}$$

and

$$G_i(t,s) = \begin{cases} \frac{(1-s)^{\alpha_i-1}-(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & 0 \le t \le s \le 1, \end{cases} \quad i = 1,2. \tag{2.7}$$

Lemma 2.3 *The Green function $\overline{H}_i(t,s)$, $G_i(t,s)$ $(i = 1,2)$ defined separately by* (2.2), (2.7) *has the following properties*:

(i) $\overline{H}_i(t,s), G_i(t,s) : [0,1] \times [0,1] \to [0,+\infty)$ *are continuous*,

(ii)

$$\frac{(1-s)^{\alpha_i-1}(1-t^{\alpha_i-1})}{\Gamma(\alpha_i)} \le G_i(t,s) \le \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, \quad t,s \in [0,1].$$

Proof Obviously, (i) holds, we only prove (ii). From the definition of $G_i(t,s)$, for $0 \le t \le s \le 1$, it is obvious that (ii) holds.

For $0 \le s \le t \le 1$, we have $t - ts \ge t - s$, then

$$\begin{aligned} (1-s)^{\alpha_i-1} - (t-s)^{\alpha_i-1} &\ge (1-s)^{\alpha_i-1} - (t-ts)^{\alpha_i-1} \\ &\ge (1-s)^{\alpha_i-1} - t^{\alpha_i-1}(1-s)^{\alpha_i-1} \\ &= (1-s)^{\alpha_i-1}\bigl(1-t^{\alpha_i-1}\bigr), \end{aligned}$$

so, we know $\frac{(1-s)^{\alpha_i-1}(1-t^{\alpha_i-1})}{\Gamma(\alpha_i)} \le G_i(t,s)$. It is also defined by $G_i(t,s)$, and we obtain $G_i(t,s) \le \frac{(1-s)^{\alpha_i}}{\Gamma(\alpha_i)}$. Thus, (ii) holds. The proof is completed. □

Similar to the proof in [35], Lemma 2.4 was obtained.

Lemma 2.4 *For $t,s \in [0,1]$, the functions $K_i(t,s)$ and $H_i(t,s)$ $(i = 1,2)$ defined as* (2.3) *satisfy*

$$K_1(t,s), H_2(t,s) \le \rho(1-s)^{\alpha_1-1}, \qquad K_2(t,s), H_1(t,s) \le \rho(1-s)^{\alpha_2-1}, \tag{2.8}$$

$$K_1(t,s), H_2(t,s) \ge \varrho(1-s)^{\alpha_1-1}, \qquad K_2(t,s), H_1(t,s) \ge \varrho(1-s)^{\alpha_2-1}, \tag{2.9}$$

where

$$\rho = \max\left\{\begin{matrix} \frac{\mu_1\mu_2k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\,dA_2(t) + \frac{1}{\Gamma(\alpha_1)}, \frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\,dA_2(t), \\ \frac{\mu_1\mu_2k_2}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1 a(t)\,dA_1(t) + \frac{1}{\Gamma(\alpha_2)}, \frac{\mu_1}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1 a(t)\,dA_1(t), \end{matrix}\right\}$$

$$\varrho = \max\left\{\begin{matrix} \frac{\mu_1\mu_2k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)(1-t^{\alpha_1-1})\,dA_2(t), \\ \frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)(1-t^{\alpha_1-1})\,dA_2(t), \\ \frac{\mu_1\mu_2k_2}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1 a(t)(1-t^{\alpha_2-1})\,dA_1(t), \\ \frac{\mu_1}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1 a(t)(1-t^{\alpha_2-1})\,dA_1(t). \end{matrix}\right\}$$

Remark 2.1 From Lemma 2.4, for $t,\tau,s \in [0,1]$, we have

$$\begin{aligned} &K_i(t,s) \geq \omega K_i(\tau,s), \qquad H_i(t,s) \geq \omega H_i(\tau,s), \quad i = 1,2, \\ &K_1(t,s) \geq \omega H_2(\tau,s), \qquad H_2(t,s) \geq \omega K_1(\tau,s), \\ &K_2(t,s) \geq \omega H_1(\tau,s), \qquad H_1(t,s) \geq \omega K_2(\tau,s), \end{aligned}$$

where $\omega = \frac{\varrho}{\rho}$, ϱ, ρ are defined as Lemma 2.4, $0 < \omega < 1$.

Let $X = C[0,1] \times C[0,1]$, then X is a Banach space with the norm

$$\|(u,v)\| = \max\{\|u\|, \|v\|\}, \qquad \|u\| = \max_{t\in[0,1]}|u(t)|, \qquad \|v\| = \max_{t\in[0,1]}|v(t)|.$$

Let

$$K = \{(u,v) \in X : u(t) \geq \omega\|(u,v)\|, v(t) \geq \omega\|(u,v)\|, t \in [0,1]\},$$

where ω is defined as Remark 2.1. It is easy to see that K is a positive cone in X. For any $(u,v) \in K$, we can define an integral operator $T : K \to X$ by

$$\begin{aligned} T(u,v)(t) &= (T_1(u,v)(t), T_2(u,v)(t)), \quad 0 \leq t \leq 1, \\ T_1(u,v)(t) &= \int_0^1 K_1(t,s)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1(\tau,u(\tau),v(\tau))\,d\tau\right)ds \\ &\quad + \int_0^1 H_1(t,s)\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau\right)ds, \quad 0 \leq t \leq 1, \\ T_2(u,v)(t) &= \int_0^1 K_2(t,s)\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau\right)ds \\ &\quad + \int_0^1 H_2(t,s)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1(\tau,u(\tau),v(\tau))\,d\tau\right)ds, \quad 0 \leq t \leq 1. \end{aligned} \tag{2.10}$$

We know that (u,v) is a positive solutions of system (1.1) if and only if (u,v) is a fixed point of T in K.

Lemma 2.5 *$T : X \to X$ is a completely continuous operator and $T(K) \subseteq K$.*

Proof By a routine discussion, we see that $T : X \to X$ is well defined, so we only prove $T(K) \subseteq K$. For any $(u,v) \in K$, $0 \le t, t' \le 1$, by Remark 2.1, we have

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,s)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\quad + \int_0^1 H_1(t,s)\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\ge \int_0^1 \omega K_1\big(t',s\big)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\quad + \int_0^1 \omega H_1\big(t',s\big)\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\ge \omega\left(\int_0^1 K_1\big(t',s\big)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \right.\\ &\quad \left. + \int_0^1 H_1\big(t',s\big)\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds\right) \\ &\ge \omega T_1(u,v)\big(t'\big), \end{aligned} \tag{2.11}$$

$$\begin{aligned} T_1(u,v)(t) &\ge \int_0^1 \omega H_2\big(t',s\big)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\quad + \int_0^1 \omega K_2\big(t',s\big)\left(\int_0^1 \overline{H}_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\ge \omega\left(\int_0^1 H_2\big(t',s\big)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \right.\\ &\quad \left. + \int_0^1 K_2\big(t',s\big)\left(\int_0^1 \overline{H}_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)\right) \\ &\ge \omega T_2(u,v)\big(t'\big). \end{aligned} \tag{2.12}$$

So we have

$$T_1(u,v)(t) \ge \omega\big\|T_1(u,v)\big\|, \qquad T_1(u,v)(t) \ge \omega\big\|T_2(u,v)\big\|,$$

i.e.,

$$T_1(u,v)(t) \ge \omega\big\|\big(T_1(u,v),T_2(u,v)\big)\big\|.$$

In the same way as (2.11) and (2.12), we can prove that

$$T_2(u,v)(t) \ge \omega\big\|\big(T_1(u,v),T_2(u,v)\big)\big\|.$$

Therefore, we have $T(K) \subseteq K$.

According to the Ascoli–Arzela theorem, we see that $T : K \to K$ is completely continuous. The proof is completed. □

Lemma 2.6 ([45]) *Let K be a positive cone in a Banach space E, Ω_1 and Ω_2 are bounded open sets in E, $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $T : K \cap \overline{\Omega}_2 \backslash \Omega_1 \to K$ is a completely continuous operator.*

If the following conditions are satisfied:

$$\|Tx\| \le \|x\|, \quad \forall x \in K \cap \partial\Omega_1, \qquad \|Tx\| \ge \|x\|, \quad \forall x \in K \cap \partial\Omega_2,$$

or

$$\|Tx\| \ge \|x\|, \quad \forall x \in K \cap \partial\Omega_1, \qquad \|Tx\| \le \|x\|, \quad \forall x \in K \cap \partial\Omega_2,$$

then T *has at least one fixed point in* $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Main results

Denote

$$f_{10} = \liminf_{x\to 0^+} \inf_{\substack{t\in[a,b]\subset(0,1)\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{\varphi_{p_1}(x)}, \qquad f_1^0 = \limsup_{x\to 0^+} \sup_{\substack{t\in[0,1]\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{\varphi_{p_1}(x)},$$

$$f_{20} = \liminf_{y\to 0^+} \inf_{\substack{t\in[a,b]\subset(0,1)\\ x\in[0,+\infty)}} \frac{f_2(t,x,y)}{\varphi_{p_2}(y)}, \qquad f_2^0 = \limsup_{y\to 0^+} \sup_{\substack{t\in[0,1]\\ x\in[0,+\infty)}} \frac{f_2(t,x,y)}{\varphi_{p_2}(y)},$$

$$f_{1\infty} = \liminf_{x\to +\infty} \inf_{\substack{t\in[a,b]\subset(0,1)\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{\varphi_{p_1}(x)}, \qquad f_1^\infty = \limsup_{x\to +\infty} \sup_{\substack{t\in[0,1]\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{\varphi_{p_1}(x)},$$

$$f_{2\infty} = \liminf_{y\to +\infty} \inf_{\substack{t\in[a,b]\subset(0,1)\\ x\in[0,+\infty)}} \frac{f_2(t,x,y)}{\varphi_{p_2}(y)}, \qquad f_2^\infty = \limsup_{y\to +\infty} \sup_{\substack{t\in[0,1]\\ x\in[0,+\infty)}} \frac{f_2(t,x,y)}{\varphi_{p_2}(y)},$$

$$L_i = \left(\frac{1}{2}\int_0^1 \rho(1-s)^{\alpha_i-1}\varphi_{q_i}\left(\int_0^1 \overline{H}_i(s,\tau)\,d\tau\right)ds\right)^{-1},$$

$$l_i = \left(\frac{1}{2}\int_0^1 \varrho(1-s)^{\alpha_i-1}\varphi_{q_i}\left(\int_a^b \overline{H}_i(s,\tau)\,d\tau\right)ds\right)^{-1}, \quad i = 1,2.$$

In what follows, we list the conditions to be used later:

($\mathbf{H}_1$) $f_{i0} \in (\varphi_{p_i}(\frac{l_i}{\omega}), +\infty], f_{i\infty} \in (\varphi_{p_i}(\frac{l_i}{\omega}), +\infty]$.

($\mathbf{H}_2$) $f_i^0 \in [0, \varphi_{p_i}(L_i)), f_i^\infty \in [0, \varphi_{p_i}(L_i))$.

($\mathbf{H}_3$) There exist constants $d_i \in (0, L_i)$ and $r_1 > 0$, such that

$$f_i(t,x,y) \le \varphi_{p_i}(d_i r_1), \quad 0 \le t \le 1, 0 \le x,y \le r_1.$$

($\mathbf{H}_4$) There exist constants $d_i^* \in (l_i, +\infty)$ and $R_1 > 0$, $[a,b] \subset (0,1)$, such that

$$f_i(t,x,y) \ge \varphi_{p_i}(d_i^* R_1), \quad a \le t \le b, \omega R_1 \le x,y \le R_1.$$

Theorem 3.1 *Assume that* ($\mathbf{H}_0$), ($\mathbf{H}_1$), ($\mathbf{H}_3$) *hold, then system* (1.1) *has at least two positive solutions* (u_1,v_1) *and* (u_2,v_2) *such that* $0 < \|(u_1,v_1)\| < r_1 < \|(u_2,v_2)\|$.

Proof (I) By ($\mathbf{H}_3$), there exist constants $d_i \in (0, L_i)$ and $r_1 > 0$, such that

$$f_i(t,x,y) \le \varphi_{p_i}(d_i r_1), \quad 0 \le t \le 1, 0 \le x,y \le r_1. \tag{3.1}$$

Let $K_{r_1} = \{(u,v) \in K : \|(u,v)\| < r_1\}$. For any $(u,v) \in \partial K_{r_1}$, by the definition of $\|\cdot\|$, we know that

$$\begin{aligned} &u(t) \le |u(t)| \le \|u\| \le \|(u,v)\| \le r_1, \\ &v(t) \le |v(t)| \le \|v\| \le \|(u,v)\| \le r_1, \quad 0 \le t \le 1. \end{aligned} \tag{3.2}$$

Thus, for any $(u,v) \in \partial K_{r_1}$, by (3.1) and (3.2), we can obtain

$$f_i(t,u(t),v(t)) \le \varphi_{p_i}(d_i r_1), \quad 0 \le t \le 1. \tag{3.3}$$

Hence, for any $(u,v) \in \partial K_{r_1}$, by Lemmas 2.3, 2.4 and (3.3), we have

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,s)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1(\tau,u(\tau),v(\tau))\,d\tau\right)ds \\ &\quad + \int_0^1 H_1(t,s)\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau\right)ds \\ &\le \int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1(\tau,u(\tau),v(\tau))\,d\tau\right)ds \\ &\quad + \int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2(\tau,u(\tau),v(\tau))\,d\tau\right)ds \\ &\le \int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)\varphi_{p_1}(d_1r_1)\,d\tau\right)ds \\ &\quad + \int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)\varphi_{p_2}(d_2r_1)\,d\tau\right)ds \\ &\le r_1\left(L_1\int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)\,d\tau\right)ds \right. \\ &\quad \left. + L_2\int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)\,d\tau\right)ds\right) \\ &= r_1 = \|(u,v)\|. \end{aligned} \tag{3.4}$$

Similar to (3.4), for any $(u,v) \in \partial K_{r_1}$, we also have

$$\|T_2(u,v)\| \le r_1 = \|(u,v)\|.$$

Consequently

$$\|T(u,v)\| = \max\{\|T_1(u,v)\|,\|T_2(u,v)\|\} \le r_1 = \|(u,v)\|, \quad (u,v) \in \partial K_{r_1}. \tag{3.5}$$

(II) With the first inequality of $(\mathbf{H}_1)$, $f_{i0} \in (\varphi_{p_i}(\frac{l_i}{\omega}),+\infty]$, there exists a real number $r \in (0,r_1)$, such that

$$\begin{aligned} &f_1(t,x,y) \le \varphi_{p_1}(x)\varphi_{p_1}\left(\frac{l_1}{\omega}\right), \quad a \le t \le b, 0 \le x \le r, y \ge 0, \\ &f_2(t,x,y) \le \varphi_{p_2}(y)\varphi_{p_2}\left(\frac{l_2}{\omega}\right), \quad a \le t \le b, 0 \le y \le r, x \ge 0. \end{aligned} \tag{3.6}$$

Let $K_r = \{(u,v) \in K : \|(u,v)\| < r\}$. For any $(u,v) \in \partial K_r$,

$$\begin{aligned} r &= \|(u,v)\| \geq u(t) \geq \omega\|(u,v)\| \geq \omega r, \\ r &= \|(u,v)\| \geq v(t) \geq \omega\|(u,v)\| \geq \omega r, \quad 0 \leq t \leq 1. \end{aligned} \tag{3.7}$$

By Lemmas 2.3, 2.4 and (3.6), (3.7), we have

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,s)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\ &\quad + \int_0^1 H_1(t,s)\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\ &\geq \int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\ &\quad + \int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\ &\geq \int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_a^b \overline{H}_1(s,\tau)\varphi_{p_1}\bigl(u(\tau)\bigr)\varphi_{p_1}\left(\frac{l_1}{\omega}\right)d\tau\right)ds \\ &\quad + \int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_a^b \overline{H}_2(s,\tau)\varphi_{p_2}\bigl(v(\tau)\bigr)\varphi_{p_2}\left(\frac{l_2}{\omega}\right)d\tau\right)ds \\ &\geq r\left(l_1\int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_a^b \overline{H}_1(s,\tau)\,d\tau\right)ds \right. \\ &\quad \left. + l_2\int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_a^b \overline{H}_2(s,\tau)\,d\tau\right)ds\right) \\ &= r = \|(u,v)\|. \end{aligned} \tag{3.8}$$

Therefore, we obtain

$$\|T(u,v)\| = \max\bigl\{\|T_1(u,v)\|, \|T_2(u,v)\|\bigr\} \geq r = \|(u,v)\|, \quad \text{for any } (u,v) \in \partial K_r. \tag{3.9}$$

(III) With the second inequality of $(\mathbf{H}_1)$, $f_{i\infty} \in (\varphi_{p_i}(\frac{l_i}{\omega}), +\infty]$, there exist real numbers r_2^*, r_2^{**}, such that

$$\begin{aligned} f_1(t,x,y) &\geq \varphi_{p_1}(x)\varphi_{p_1}\left(\frac{l_1}{\omega}\right), \quad a \leq t \leq b, x \geq r_2^*, y \geq 0, \\ f_2(t,x,y) &\geq \varphi_{p_2}(y)\varphi_{p_2}\left(\frac{l_2}{\omega}\right), \quad a \leq t \leq b, y \geq r_2^{**}, x \geq 0. \end{aligned} \tag{3.10}$$

Choose $r_2 = \max\{2r_1, \frac{r^*}{\omega\theta}, \frac{r_2^{**}}{\omega\theta}\}$. Let $K_{r_2} = \{(u,v) \in K : \|(u,v)\| < r_2\}$. For any $(u,v) \in \partial K_{r_2}$, by the definition of $\|\cdot\|$, we have

$$\begin{aligned} r_2 &= \|(u,v)\| \geq u(t) \geq \omega\|(u,v)\| \geq \omega r_2 \geq r_2^*, \quad 0 \leq t \leq 1, \\ r_2 &= \|(u,v)\| \geq v(t) \geq \omega\|(u,v)\| \geq \omega r_2 \geq r_2^{**}, \quad 0 \leq t \leq 1. \end{aligned} \tag{3.11}$$

Thus, for any $(u,v) \in \partial K_{r_2}$, by (3.10), (3.11), we have

$$\begin{aligned} f_1\big(t,u(t),v(t)\big) &\geq \varphi_{p_1}\big(u(t)\big)\varphi_{p_1}\left(\frac{l_1}{\omega}\right) \geq \varphi_{p_1}(\omega r_2)\varphi_{p_1}\left(\frac{l_1}{\omega}\right), \quad a \leq t \leq b, \\ f_2\big(t,u(t),v(t)\big) &\geq \varphi_{p_2}\big(v(t)\big)\varphi_{p_2}\left(\frac{l_2}{\omega}\right) \geq \varphi_{p_2}(\omega r_2)\varphi_{p_2}\left(\frac{l_2}{\omega}\right), \quad a \leq t \leq b. \end{aligned} \tag{3.12}$$

So, for any $(u,v) \in \partial K_{r_2}$, by Lemmas 2.3, 2.4 and (3.12), we know

$$\begin{aligned} T_1(u,v)(t) &\geq \int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\quad + \int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\geq \int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_a^b \overline{H}_1(s,\tau)\varphi_{p_1}\big(u(\tau)\big)\varphi_{p_1}\left(\frac{l_1}{\omega}\right)d\tau\right)ds \\ &\quad + \int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_a^b \overline{H}_2(s,\tau)\varphi_{p_2}\big(v(\tau)\big)\varphi_{p_2}\left(\frac{l_2}{\omega}\right)d\tau\right)ds \\ &\geq \int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_a^b \overline{H}_1(s,\tau)\varphi_{p_1}(\omega r_2)\varphi_{p_1}\left(\frac{l_1}{\omega}\right)d\tau\right)ds \\ &\quad + \int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_a^b \overline{H}_2(s,\tau)\varphi_{p_2}(\omega r_2)\varphi_{p_2}\left(\frac{l_2}{\omega}\right)d\tau\right)ds \\ &\geq r_2\left(l_1\int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_a^b \overline{H}_1(s,\tau)\,d\tau\right)ds\right. \\ &\quad \left. + l_2\int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_a^b \overline{H}_2(s,\tau)\,d\tau\right)ds\right) \\ &= r_2 = \big\|(u,v)\big\|. \end{aligned} \tag{3.13}$$

Hence, we obtain

$$\big\|T(u,v)\big\| = \max\big\{\big\|T_1(u,v)\big\|,\big\|T_2(u,v)\big\|\big\} \geq r_2 = \big\|(u,v)\big\|, \quad \text{for any } (u,v) \in \partial K_{r_2}. \tag{3.14}$$

It follows from the above discussion, (3.5), (3.9), (3.14), Lemmas 2.5, 2.6, that T has fixed points $(u_1,v_1) \in \overline{K}_{r_2}\backslash K_r$, $(u_2,v_2) \in \overline{K}_r\backslash K_{r_1}$, that is to say, system (1.1) has at least two positive solutions (u_1,v_1), (u_2,v_2), satisfying $0 < \|(u_1,v_1)\| < r_1 < \|(u_2,v_2)\|$. The proof is completed. □

Theorem 3.2 *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_2)$, $(\mathbf{H}_4)$ *hold, then system* (1.1) *has at least two positive solutions* (u_1,v_1) *and* (u_2,v_2) *such that* $0 < \|(u_1,v_1)\| < R_1 < \|(u_2,v_2)\|$.

Proof (I) By $(\mathbf{H}_4)$, there exist constants $d_i^* \in (l_i,+\infty)$ and $R_1 > 0$, such that

$$f_i(t,x,y) \geq \varphi_{p_i}\big(d_i^* R_1\big), \quad a \leq t \leq b, \omega R_0 \leq x,y \leq R_1. \tag{3.15}$$

Let $K_{R_1} = \{(u,v) \in K : \|(u,v)\| < R_1\}$. For any $(u,v) \in \partial K_{R_1}$,

$$\begin{aligned} &R_1 = \|(u,v)\| \geq u(t) \geq \omega\|(u,v)\| \geq \omega R_1, \\ &R_1 = \|(u,v)\| \geq v(t) \geq \omega\|(u,v)\| \geq \omega R_1, \quad 0 \leq t \leq 1. \end{aligned} \tag{3.16}$$

Thus, for any $(u,v) \in \partial K_{R_1}$, by Lemmas 2.3, 2.4 and (3.15), (3.16), we get

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,s)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\quad + \int_0^1 H_1(t,s)\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\geq \int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\quad + \int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\geq \int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_a^b \overline{H}_1(s,\tau)\varphi_{p_1}\big(d_1^* R_0\big)\,d\tau\right)ds \\ &\quad + \int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_a^b \overline{H}_2(s,\tau)\varphi_{p_2}\big(d_2^* R_0\big)\,d\tau\right)ds \\ &\geq R_1\left(l_1\int_0^1 \varrho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_a^b \overline{H}_1(s,\tau)\,d\tau\right)ds\right. \\ &\quad \left. + l_2\int_0^1 \varrho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_a^b \overline{H}_2(s,\tau)\,d\tau\right)ds\right) \\ &= R_1 = \|(u,v)\|. \end{aligned} \tag{3.17}$$

So, we have

$$\begin{aligned} \|T(u,v)\| &= \max\{\|T_1(u,v)\|, \|T_2(u,v)\|\} \\ &\geq R_1 = \|(u,v)\|, \quad \text{for any } (u,v) \in \partial K_{R_1}. \end{aligned} \tag{3.18}$$

(II) With the first inequality of $(\mathbf{H}_2)$, $f_i^0 \in [0, \varphi_{p_i}(L_i))$, there exists a real number $R_2 \in (0, R_1)$, such that

$$\begin{aligned} &f_1(t,x,y) \leq \varphi_{p_1}(xL_1) \leq \varphi_{p_1}(R_2L_1), \quad 0 \leq t \leq 1, 0 \leq x \leq R_2, y \geq 0, \\ &f_2(t,x,y) \leq \varphi_{p_2}(yL_2) \leq \varphi_{p_2}(R_2L_2), \quad 0 \leq t \leq 1, 0 \leq y \leq R_2, x \geq 0. \end{aligned} \tag{3.19}$$

Let $K_{R_2} = \{(u,v) \in K : \|(u,v)\| < R_2\}$. For any $(u,v) \in \partial K_{R_2}$,

$$\begin{aligned} &u(t) \leq |u(t)| \leq \|u\| \leq \|(u,v)\| \leq R_2, \\ &v(t) \leq |v(t)| \leq \|v\| \leq \|(u,v)\| \leq R_2, \quad 0 \leq t \leq 1. \end{aligned} \tag{3.20}$$

Therefore, for any $(u,v) \in \partial K_{R_2}$, by Lemmas 2.3, 2.4 and (3.19), (3.20), we have

$$\begin{aligned}
T_1(u,v)(t) &= \int_0^1 K_1(t,s)\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + \int_0^1 H_1(t,s)\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\leq \int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + \int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\leq \int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)\varphi_{p_1}(R_2L_1)\,d\tau\right)ds \\
&\quad + \int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)\varphi_{p_2}(R_2L_2)\,d\tau\right)ds \\
&\leq R_2\left(L_1\int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)\,d\tau\right)ds\right. \\
&\quad \left. + L_2\int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)\,d\tau\right)ds\right) \\
&= R_2 = \bigl\|(u,v)\bigr\|.
\end{aligned} \tag{3.21}$$

By a similar proof to (3.21), for any $(u,v) \in \partial K_{R_2}$, we also have

$$\bigl\|T_2(u,v)\bigr\| \leq R_2 = \bigl\|(u,v)\bigr\|.$$

Thus,

$$\bigl\|T(u,v)\bigr\| = \max\bigl\{\bigl\|T_1(u,v)\bigr\|,\bigl\|T_2(u,v)\bigr\|\bigr\} \leq R_2 = \bigl\|(u,v)\bigr\|, \quad (u,v) \in \partial K_{R_2}. \tag{3.22}$$

(III) With the second inequality of $(\mathbf{H}_2)$, $f_i^\infty \in [0,\varphi_{p_i}(L_i))$, there exists $R^* > 0$, such that

$$\begin{aligned}
&f_1(t,x,y) \leq \varphi_{p_1}(xL_1), \quad 0 \leq t \leq 1, x \geq R^*, y \geq 0, \\
&f_2(t,x,y) \leq \varphi_{p_2}(yL_2), \quad 0 \leq t \leq 1, y \geq R^*, x \geq 0.
\end{aligned} \tag{3.23}$$

Now there are two situations.

Case 1. f_i is bounded on $[0,+\infty)$, then we choose $\overline{R} > 0$, such that

$$f_i(t,x,y) \leq \varphi_{p_i}(\overline{R}L_i), \quad 0 \leq t \leq 1, x,y \geq 0, i = 1,2. \tag{3.24}$$

Let $R_3 = \max\{2R_1,\overline{R}\}$, $K_{R_3} = \{(u,v) \in K : \|(u,v)\| < R_3\}$. For any $(u,v) \in \partial K_{R_3}$, we know

$$\begin{aligned}
T_1(u,v)(t) &\leq \int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + \int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)\varphi_{p_1}(R_3L_1)\,d\tau\right)ds \\
&\quad + \int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)\varphi_{p_2}(R_3L_2)\,d\tau\right)ds \\
&\le R_3\left(L_1\int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)\,d\tau\right)ds \right.\\
&\quad \left. + L_2\int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)\,d\tau\right)ds\right) \\
&= R_3 = \big\|(u,v)\big\|.
\end{aligned}
\tag{3.25}
$$

Similar to (3.25), for any $(u,v)\in\partial K_{R_3}$, we have

$$
\big\|T_2(u,v)\big\| \le R_3 = \big\|(u,v)\big\|.
$$

Thus,

$$
\big\|T(u,v)\big\| = \max\bigl\{\big\|T_1(u,v)\big\|,\big\|T_2(u,v)\big\|\bigr\} \le R_3 = \big\|(u,v)\big\|, \quad (u,v)\in\partial K_{R_3}. \tag{3.26}
$$

Case 2. f_1 and f_2 have at least one unbounded function, assume both f_1 and f_2 are unbounded. (If f_1 or f_2 is unbounded, the proof is similar.) Choose $R_3 = \max\{2R_1, \frac{R^*}{\omega}\}$, such that

$$
f_i(t,x,y) \le f_i(t,R_3,R_3), \quad 0\le t\le 1, 0\le x,y\le R_3, i=1,2. \tag{3.27}
$$

Let $K_{R_3} = \{(u,v)\in K : \|(u,v)\| < R_3\}$. For any $(u,v)\in\partial K_{R_3}$, by (3.24), (3.27), we have

$$
\begin{aligned}
T_1(u,v)(t) &\le \int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + \int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\le \int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)f_1(\tau,R_3,R_3)\,d\tau\right)ds \\
&\quad + \int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)f_2(\tau,R_3,R_3)\,d\tau\right)ds \\
&\le \int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)\varphi_{p_1}(R_3L_1)\,d\tau\right)ds \\
&\quad + \int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)\varphi_{p_2}(R_3L_2)\,d\tau\right)ds \\
&\le R_3\left(L_1\int_0^1 \rho(1-s)^{\alpha_1-1}\varphi_{q_1}\left(\int_0^1 \overline{H}_1(s,\tau)\,d\tau\right)ds \right.\\
&\quad \left. + L_2\int_0^1 \rho(1-s)^{\alpha_2-1}\varphi_{q_2}\left(\int_0^1 \overline{H}_2(s,\tau)\,d\tau\right)ds\right) \\
&= R_3 = \big\|(u,v)\big\|.
\end{aligned}
\tag{3.28}
$$

Similar to (3.28), for any $(u,v) \in \partial K_{R_3}$, we have

$$\|T_2(u,v)\| \le R_3 = \|(u,v)\|.$$

Thus,

$$\|T(u,v)\| = \max\{\|T_1(u,v)\|, \|T_2(u,v)\|\} \le R_3 = \|(u,v)\|, \quad (u,v) \in \partial K_{R_3}. \tag{3.29}$$

Through the above discussion, (3.18), (3.22), (3.26) (or (3.29)), Lemmas 2.5, 2.6, T has fixed points $(u_1,v_1) \in \overline{K}_{R_1} \backslash K_{R_2}$, $(u_2,v_2) \in \overline{K}_{R_3} \backslash K_{R_1}$, that is to say, system (1.1) has at least two positive solutions (u_1,v_1), (u_2,v_2), satisfying $0 < \|(u_1,v_1)\| < R_1 < \|(u_2,v_2)\|$. The proof is completed. □

4 An example

Consider the following fractional differential system:

$$\begin{cases} D^{\frac{3}{2}}({}^cD^{\frac{5}{2}}u(t)) + f_1(t,u(t),v(t)) = 0, \\ D^{\frac{3}{2}}({}^cD^{\frac{5}{2}}v(t)) + f_2(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ u'(0) = u''(0) = 0, \qquad v'(0) = v''(0) = 0, \\ u(1) = \frac{1}{2}\int_0^1 s^2 v(s)\,ds^{\frac{1}{3}}, \qquad v(1) = \int_0^1 su(s)\,ds, \\ {}^cD^{\frac{5}{2}}u(0) = 0, \qquad {}^cD^{\frac{5}{2}}u(1) = \frac{1}{4}{}^cD^{\alpha_1}u(\frac{1}{2}), \\ {}^cD^{\frac{5}{2}}v(0) = 0, \qquad {}^cD^{\frac{5}{2}}v(1) = \frac{1}{4}{}^cD^{\alpha_2}v(\frac{1}{2}), \end{cases} \tag{4.1}$$

where $\beta_1 = \beta_2 = \frac{3}{2}$, $\alpha_1 = \alpha_2 = \frac{5}{2}$, $\mu_1 = \frac{1}{2}$, $\mu_2 = 1$, $A_1(t) = t^{\frac{1}{3}}$, $A_2(t) = t$, $\varepsilon_1 = \varepsilon_2 = \frac{1}{4}$, $\eta_1 = \eta_2 = \frac{1}{2}$, $a(s) = s^2$, $b(s) = s$, $p_1 = p_2 = 2$. Then we have

$$\begin{aligned} k_1 &= \int_0^1 a(s)\,dA_1(s) = \int_0^1 s^2\,ds^{\frac{1}{3}} = \frac{1}{7} > 0, \\ k_2 &= \int_0^1 b(s)\,dA_2(s) = \int_0^1 s\,ds = \frac{1}{2} > 0, \\ 1 - \mu_1\mu_2k_1k_2 &= \frac{27}{28} > 0. \end{aligned}$$

Condition $(\mathbf{H}_0)$ holds. Through calculation, $L_1 = L_2 = 2.43299$, $l_1 = l_2 = 6.80274$, $\omega = 0.01953$. Choose

$$\begin{aligned} f_1(t,x,y) &= 10^{-5}(x^2+y^2)\cos t + 350\sin x, \\ f_2(t,x,y) &= 10^{-4}t(x^2+y^2) + 350\sin y, \\ f_{10} &= 350 > 348.32258 = \varphi_{p_1}\left(\frac{l_1}{\omega}\right), \\ f_{20} &= 350 > 348.32258 = \varphi_{p_2}\left(\frac{l_2}{\omega}\right), \\ f_{1\infty} &= +\infty > 348.32258 = \varphi_{p_1}\left(\frac{l_1}{\omega}\right), \\ f_{2\infty} &= +\infty > 348.32258 = \varphi_{p_2}\left(\frac{l_2}{\omega}\right). \end{aligned}$$

Take $d_1 = d_2 = 2$, $r_1 = 180$, we have

$$f_1(t,x,y) \leq 350.648 < 360 = d_1 r_1,$$
$$f_2(t,x,y) \leq 356.48 < 360 = d_2 r_1, \quad 0 \leq t \leq 1, 0 \leq x,y \leq 180.$$

Then, by Theorem 3.1, system (4.1) has at least two positive solutions (u_1, v_1) and (u_2, v_2) such that $0 < \|(u_1, v_1)\| < 180 < \|(u_2, v_2)\|$.

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