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A modified Schrödinger-type identity: uniqueness of solutions for singular boundary value problem for the Schrödinger equation

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Abstract

This paper is devoted to modifying the Schrödinger-type identity related to singular boundary value problem in (Zhang et al. in *Bound. Value Probl.* 2018:135, 2018). We also present some mathematical consequences of the method, including a stability result. The main technical tools used to develop the mathematical analysis are local and global bifurcation, monotonicity techniques, fixed point theory in b -metric spaces in (Liu et al. in *Bull. Aust. Math. Soc.* 94(1):121–130, 2016) and the maximum principle approach with respect to the Schrödinger operator in (Fan et al. in *Math. Appl.* 31(1):42–48, 2018). As an application, the uniqueness of solutions for singular boundary value problem for the Schrödinger equation is proved.

Keywords: Boundary value problem; Schrödinger-type identity; Uniqueness

1 Introduction

In this paper we consider a singular boundary value problem with mixed boundary conditions and spatial heterogeneities given by (see [4–6])

$$\begin{aligned} -\Delta f &= \chi f \quad \text{in } \mathfrak{J}, \\ f &= 0 \quad \text{on } \mathfrak{T}_1, \\ \partial f + V(t)f &= \chi \omega(t)u^q \quad \text{on } \mathfrak{T}_2, q > 1, \end{aligned} \tag{1}$$

where:

- (i) $\mathfrak{J} = (0, W) \times (0, w)$ is a bounded rectangular domain in \mathbb{R}^2 , \mathfrak{J} represents a porous medium, with Lipschitz boundary $\partial\mathfrak{J} = \mathfrak{T}_1 \cup \mathfrak{T}_2$ where

$$\mathfrak{T}_2 = (\{0\} \times [0, w]) \cup ([0, W] \times \{w\}) \cup (\{W\} \times [0, w])$$

is the part in contact with air or covered by fluid and

$$\mathfrak{T}_1 = [0, W] \times \{0\}$$

is the impervious part of $\partial\mathfrak{J}$. Let $P = \mathfrak{J} \times (0, M)$, where $M > 0$;

- (ii) $-\Delta$ stands for the minus Laplacian operator, χ is a function of the variable t satisfying

$$c_1 \leq \chi(t) \leq c_2 \quad \text{a.e. } t \in (0, W) \quad (2)$$

for two positive constants c_1 and c_2 and $\omega(t)$ satisfies

$$0 \leq \omega(t) \leq 1 \quad \text{a.e. } t \in \mathbb{J}; \quad (3)$$

- (iii) the spatial heterogeneities on the boundary come given by the potentials

$V, b \in C(\overline{\mathbb{T}}_2)$, where $b > 0$ on $\overline{\mathbb{T}}_2$ and V possesses arbitrary sign in each point $x \in \overline{\mathbb{T}}_2$;

- (iv) $\partial f(x)$ stands for the outer normal derivative of f at $t \in \overline{\mathbb{T}}_2$.

In 2008, Polidoro and Ragusa in [7] proved a Harnack inequality for the positive solutions of ultraparabolic equations of the type $Lf + Vf = 0$, where L is a linear second order hypoelliptic operator and V belongs to a class of functions of Stummel-Kato type. They also obtained the existence of a Green function and an uniqueness result for the Cauchy–Dirichlet problem. In 2016, Guariglia and Silvestrov in [8] described a wavelet expansion theory for positive definite distributions over the real line and define a fractional derivative operator for complex functions in the distribution sense. In 2017, Goubet and Hamraoui in [9] investigated both numerically and theoretically the influence of a defect on the blow-up of radial solutions to a cubic NLS equation in dimension 2. Colorado in [10] showed the existence of positive bound and ground states for a system of coupled nonlinear Schrödinger–Korteweg–de Vries equations. In 2018, Khader and Adel in [11] introduced a study of the convergence analysis and error estimation of the obtained approximation solution. The FLDE is reduced to a system of algebraic equations with the help of the properties of wavelets polynomials. Rybalko in [12] studied an initial value problem for the one-dimensional non-stationary linear Schrödinger equation with a point singular potential. Zhang and Gu in [13] considered a three components system of nonlinear Schrödinger equations related to the Raman amplification in a plasma. In 2019, Scapellato in [14] showed some regularity properties of solutions to the elliptic equations on Herz spaces with two variable exponents. Meng in [15, 16] discussed the application of the new criteria for minimally thin sets with respect to the Schrödinger operator to an approximate solution of singular Schrödinger-type boundary value problems.

The classical Randon transform (see [17]) is defined by the Cauchy principal value of the singular integral. In 2018, Zhang et al. in [1] established the Schrödinger-type identity and applied it to a Schrödinger integral equation and then gave three examples where the kernel function is a Green's function for a two-order system of Schrödinger integral equations.

In this paper, we shall use this identity to study the problem (1) with a power nonlinearity and an exponential nonlinearity both of which are singular as the solutions approaches 0. The results from previous applications are generalized and extended. In our approach, the problem is considered as a system of coupled Schrödinger boundary value problems on two half-lines. This system can exhibit different qualitative behaviors depending on the value of the vaccination-isolation reproductive number. Following the ideas of this method, we studied the existence and the behavior of periodic solutions of a generalized model with general heterogeneous coefficients, by using a continuation theorem based on

coincidence degree theory. Regarding existence of a solution of the problem (1) with respect to the Schrödinger operator was obtained in [2] by applying the maximum principle approach for a two-order system of Schrödinger operator in [18–22]. The regularity of the solution of the problem (1) with respect to a two-order system of Schrödinger operator was also discussed in [23], where it was proved that $\omega \in C^0([0, T]; L^p(\mathbb{I}))$ for all $p \in [1, +\infty)$ in both classes of free boundary problem of types (see [6])

$$\operatorname{div}(\chi(t)\nabla f + \mathcal{H}(t)\omega) - \omega_s$$

and

$$\operatorname{div}(\chi(t)\nabla f + \mathcal{H}(t)\omega) - (f + \omega)_s,$$

and that $f \in C^0([0, M]; L^p(\mathbb{I}))$ for all $p \in [1, 2]$ in the second class.

2 A modified Schrödinger-type identity

For simplicity we shall denote the solution of (1) as (f, ω) .

Lemma 2.1 *Let*

$$\Sigma_1 = \mathbb{T}_1 \times (0, M), \quad \Sigma_2 = \mathbb{T}_2 \times (0, M), \quad \Sigma_3 = \Sigma_2 \cap \{\phi > 0\}$$

and

$$\Sigma_4 = \Sigma_2 \cap \{\phi = 0\}.$$

(i) *Let $\epsilon > 0, k \geq 0$ and $\zeta \in D(\mathbb{R}^2 \times (0, M))$ such that $\zeta \geq 0, \zeta = 0$ on Σ_3 . Then*

$$\int_P \chi(t_1)(f_{t_2} + \omega) \left(\min \left(\frac{(f-k)^+}{\epsilon}, \zeta \right) \right)_{t_2} dt ds = 0. \tag{4}$$

(ii) *Let $\zeta = 0$ on Σ_2 . Then*

$$\int_P \chi(t_1)(f_{t_2} + \omega) \left(\min \left(\frac{(k-u)^+}{\epsilon}, \zeta \right) - \min \left(\frac{k}{\epsilon}, \zeta \right) \right)_{t_2} dt ds = 0. \tag{5}$$

Proof Let ψ be a measure function satisfying

$$d(\operatorname{supp}(\psi), \Sigma_2) > 0$$

and

$$\operatorname{supp}(\psi) \subset \mathbb{R}^2 \times (0, M).$$

Then we know that

$$(t, s) \mapsto \pm \psi(t, t - \kappa)$$

vanishes on Σ_2 for any $\kappa \in (-\kappa_0, \kappa_0)$, where $\kappa_0 > 0$.

So

$$\int_P [\chi(t_1)(f_{t_2} + \omega)\psi_{t_2}(t, s - \kappa) + (1 - \omega)\psi_s(t, s - \kappa)] dt ds = 0,$$

which yields

$$\int_P \chi(t_1)(f_{t_2} + \omega)\psi_{t_2}(t, s - \kappa) dt ds = \frac{\partial}{\partial \kappa} \left(\int_P (1 - \omega(t, t + \kappa))\psi(t, s) dt ds \right). \tag{6}$$

Equation (6) still holds for any $\psi \in L^2(0, T; H^1(\mathbb{I}))$, where $\psi = 0$ on Σ_2 and $\psi = 0$ on $\mathbb{J} \times ((0, \kappa_0) \cup (M - \kappa_0, M))$.

So

$$\zeta \in D(\mathbb{R}^2 \times (\kappa_0, M - \kappa_0)),$$

which shows that $\zeta \geq 0$, $\zeta = 0$ on Σ_3 .

Set

$$\psi = \min\left(\frac{(f - k)^+}{\epsilon}, \zeta\right).$$

Then (6) also gives

$$\begin{aligned} & \int_P \chi(t_1)(f_{t_2} + \omega) \left(\min\left(\frac{(f - k)^+}{\epsilon}, \zeta\right) \right)_{t_2}(t, s - \kappa) dt ds \\ &= \frac{\partial}{\partial \kappa} \left(\int_P (1 - \omega(t, t + \kappa)) \min\left(\frac{(f - k)^+}{\epsilon}, \zeta\right)(t, s) dt ds \right) := \mathcal{G}'(\kappa) \end{aligned} \tag{7}$$

for any $\kappa \in (-\kappa_0, \kappa_0)$, where

$$\mathcal{G}(\kappa) = \int_P (1 - \omega(t, t + \kappa)) \min\left(\frac{(f - k)^+}{\epsilon}, \zeta\right)(t, s) dt ds.$$

So

$$\mathcal{G}(\kappa) \geq 0 = \mathcal{G}(0)$$

for any $\kappa \in (-\kappa_0, \kappa_0)$.

By applying (7), we know that (4) holds for $\zeta \in D(\mathbb{R}^2 \times (0, M))$ such that $\zeta \geq 0$, $\zeta = 0$ on Σ_3 .

If we put $\zeta = 0$ on Σ_2 and set

$$\psi = \min\left(\frac{(k - u)^+}{\epsilon}, \zeta\right) - \min\left(\frac{k}{\epsilon}, \zeta\right),$$

then (6) also yields

$$\begin{aligned} & \int_P \chi(t_1)(f_{t_2} + \omega) \left(\min\left(\frac{(k-u)^+}{\epsilon}, \varsigma\right) - \min\left(\frac{k}{\epsilon}, \varsigma\right) \right)_{t_2}(t, s - \kappa) dt ds \\ &= \frac{\partial}{\partial \kappa} \left(\int_P (1 - \omega(t, t + \kappa)) \left(\min\left(\frac{(k-u)^+}{\epsilon}, \varsigma\right) - \min\left(\frac{k}{\epsilon}, \varsigma\right) \right) (t, s) dt ds \right) \\ &:= \mathcal{K}'(\kappa) \end{aligned} \tag{8}$$

for any $\kappa \in (-\kappa_0, \kappa_0)$, where

$$\mathcal{K}(\kappa) = \int_P (1 - \omega(t, t + \kappa)) \left(\min\left(\frac{(k-u)^+}{\epsilon}, \varsigma\right) - \min\left(\frac{k}{\epsilon}, \varsigma\right) \right) (t, s) dt ds.$$

So

$$\mathcal{K}(\kappa) \leq 0 = \mathcal{K}(0)$$

for all $\kappa \in (-\kappa_0, \kappa_0)$.

By applying (8), we know that (5) holds for $\varsigma \in D(\mathbb{R}^2 \times (0, M))$ such that $\varsigma \geq 0$, $\varsigma = 0$ on Σ_2 . □

3 Uniqueness of the solution

In this section, we shall state and prove our main result: the solution of problem (1) is unique. Let us assume that

$$\chi \in C^1([0, W]). \tag{9}$$

Now, we can state our uniqueness theorem.

Theorem 3.1 *The solution of the problem (1) associated with the initial data ω_0 is unique.*

Proof Let (f_1, ω_1) and (f_2, ω_2) be two solutions of the problem (1) satisfying

$$\omega_1(t, 0) = \omega_2(t, 0) = \omega_0(t)$$

a.e. in \mathfrak{J} .

Set

$$v = (f_1 - f_2)^+$$

and

$$\gamma = (1 - \omega_2(t, s))\omega_{\{f_1 > f_2\}} + ((1 - \omega_2(t, s)) + (1 - f_{2t_2}(t, s)))\omega_{\{f_1 > 0\}}.$$

It follows that

$$\int_P \chi(t_1)(v_{t_2} + \gamma)_{\varsigma_{t_2}} dt ds \leq 0 \tag{10}$$

from Lemma 2.1.

Let

$$\varepsilon_0 = d(\text{supp}(\zeta), \partial \mathbb{J})$$

and

$$\mathcal{A}_{\varepsilon_0} = \{t \in \mathbb{R}^2 / d(t, \partial \mathbb{J}) > \varepsilon_0\}.$$

Note that the function χ admits an extension to \mathbb{R} from (9), still denoted by χ , such that $\chi \in C^1(\mathbb{R}, \mathbb{R})$ (see [24]).

It follows that

$$\begin{aligned} & \int_{\mathbb{R}^2 \times (0, M)} \iota \chi(t_1)(v_{\varepsilon t_2} + \gamma_{\varepsilon}) \zeta_{t_2} dt ds \\ &= \int_{\mathbb{R}^2 \times (0, M)} \iota \left\{ \int_{\mathbb{R}^2} (v_{t_2}(t-x, s) + \gamma(t-x, s)) \rho_{\varepsilon}(y) dy \right\} \chi(t_1) \zeta_{t_2}(t_1, t_2) dt ds \\ &= \int_{\mathbb{R}^2} \rho_{\varepsilon}(y) \left\{ \int_{\mathbb{R}^2 \times (0, M)} \iota (v_{t_2}(t-x, s) + \gamma(t-x, s)) \chi(t_1) \zeta_{t_2}(t_1, t_2) dt ds \right\} dy \\ &= \int_{B(0, \varepsilon)} \rho_{\varepsilon}(y) \left\{ \int_P \iota (v_{z_2}(z, s) + \gamma(z, s)) (\chi(z_1 + x_1) \zeta(z+y))_{z_2} dz ds \right\} dy \\ &= \int_{B(0, \varepsilon)} \rho_{\varepsilon}(y) \left\{ \int_P \iota \chi(z_1) (v_{z_2}(z, s) + \gamma(z, s)) \left(\frac{\chi(z_1 + x_1) \zeta(z+y)}{\chi(z_1)} \right)_{z_2} dz ds \right\} dy \end{aligned}$$

from Fubini’s theorem, where $\varepsilon \in (0, \frac{\varepsilon_0}{2})$, $\rho_{\varepsilon} \in D(\mathbb{R}^2)$ with

$$\text{supp}(\rho_{\varepsilon}) \subset B(0, \varepsilon)$$

and $f_{\varepsilon} = \rho_{\varepsilon} * f$.

Note that

$$z \mapsto \frac{\chi(z_1 + x_1) \zeta(z+y)}{\chi(z_1)}$$

is nonnegative for all $y \in B(0, \varepsilon)$ and belongs to $C_0^1(\mathbb{J})$.

Since (10) holds for $0 \leq \varphi \in C_0^1(\mathbb{J})$, we have

$$\int_{\mathbb{R}^2 \times (0, M)} \iota \chi(t_1)(v_{\varepsilon t_2} + \gamma_{\varepsilon}) \zeta_{t_2} dt ds \leq 0,$$

which yields

$$\begin{aligned} & - \int_{\mathcal{A}_{\varepsilon_0} \times (0, M)} \iota \chi(t_1) v_{\varepsilon} \zeta_{t_2} dt ds + \int_{\mathcal{A}_{\varepsilon_0} \times (0, M)} \iota \chi(t_1) \gamma_{\varepsilon} \zeta_{t_2} dt ds \leq 0, \\ & \zeta \in D(\mathbb{J}), \zeta \geq 0, \quad d(\text{supp}(\zeta), \partial \mathbb{J}) = \varepsilon_0 > 0, \quad \forall \iota \in D(0, M), \iota \geq 0. \end{aligned} \tag{11}$$

Put

$$\alpha_{\varepsilon}(t) = \chi(t_1) \int_0^M \iota v_{\varepsilon} ds,$$

and suppose that there exists $t_0 \in \mathcal{A}_{\varepsilon_0} \cap \mathbb{J}$ and $\varepsilon_1 \in (0, \frac{\varepsilon_0}{2})$ such that $\alpha_{\varepsilon_1}(t_0) > 0$.

Since α_{ε_1} is continuous and $\mathcal{A}_{\varepsilon_0} \cap \mathbb{J}$ is an open set,

$$\overline{B(t_0, r)} \subset \mathcal{A}_{\varepsilon_0} \cap \mathbb{J}$$

and $\alpha_{\varepsilon_1}(t) > 0$ for all $t \in \overline{B(t_0, r)}$, where $r > 0$.

It follows that

$$\int_0^M \iota v_{\varepsilon_1} ds > 0$$

in $\overline{B(t_0, r)}$ from (2).

Let us consider the following Dirichlet problem associated to a linear second order partial differential equation:

$$\begin{aligned} -\frac{1}{\alpha_{\varepsilon_1}(t)} \varsigma_{t_1 t_1} - \varsigma_{t_2 t_2} + \frac{\int_0^M \iota \gamma_{\varepsilon_1} ds}{\int_0^M \iota v_{\varepsilon_1} ds} \varsigma_{t_2} &= \frac{1}{\alpha_{\varepsilon_1}(t)} \quad \text{in } B(t_0, r), \\ \varsigma &= 0 \quad \text{on } \partial(B(t_0, r)), \end{aligned} \tag{12}$$

which yields

$$\begin{aligned} -\sum_{i,j=1}^2 a_{\varepsilon_1 ij}(t) \varsigma_{t_i t_j} + \beta_{\varepsilon_1}(t) \varsigma_{t_2} &= \frac{1}{\alpha_{\varepsilon_1}(t)} \quad \text{in } B(t_0, r), \\ \varsigma &= 0 \quad \text{on } \partial(B(t_0, r)), \end{aligned}$$

where

$$\begin{aligned} a_{\varepsilon_1 11}(t) &= \frac{1}{\alpha_{\varepsilon_1}(t)}, & a_{\varepsilon_1 12}(t) &= a_{\varepsilon_1 21}(t) = 0, & a_{\varepsilon_1 22}(t) &= 1, \\ \beta_{\varepsilon_1}(t) &= \frac{\int_0^M \iota \gamma_{\varepsilon_1} ds}{\int_0^M \iota v_{\varepsilon_1} ds}. \end{aligned}$$

Note that $(a_{\varepsilon_1 ij}(t))_{ij}$ is strictly elliptic in $B(t_0, r)$ with a positive constant

$$\min\left(\frac{1}{\max_{t \in \overline{B(t_0, r)}} \alpha_{\varepsilon_1}(t)}, 1\right)$$

and the coefficients $\frac{1}{\alpha_{\varepsilon_1}}, \beta_{\varepsilon_1}$ are in $C^1(\overline{B(t_0, r)})$. So, by the regularity theory (see [25] for example), the problem (12) has a unique solution $\hat{\varsigma} \in C^2(\overline{B(t_0, r)})$.

Moreover, since the function in the right side of the first equation of (12) satisfies $\frac{1}{\alpha_{\varepsilon_1}} > 0$ in $B(t_0, r)$, we have $\hat{\varsigma} \geq 0$ in $\overline{B(t_0, r)}$ from the maximum principle (see [26]).

So (11) yields

$$\int_{\mathcal{A}_{\varepsilon_0} \times (0, M)} \{-\iota \chi(t_1) v_{\varepsilon} \hat{\varsigma}_{t_2 t_2} + \iota \chi(t_1) \gamma_{\varepsilon} \hat{\varsigma}_{t_2}\} dt ds \leq 0,$$

where $\varepsilon \in (0, \frac{\varepsilon_0}{2})$.

It follows that

$$-\hat{S}_{t_1 t_1} - \alpha_{\varepsilon_1}(t) \hat{S}_{t_2 t_2} + \chi(t_1) \hat{S}_{t_2} \int_0^M \iota \gamma_{\varepsilon_1} ds = 1$$

and

$$\begin{aligned} & \int_{\overline{B(t_0, r)} \times (0, M)} \{-\iota \chi(t_1) \nu_{\varepsilon_1} \hat{S}_{t_2 t_2} + \iota \chi(t_1) \gamma_{\varepsilon_1} \hat{S}_{t_2}\} dt ds \\ &= \int_{\overline{B(t_0, r)}} \hat{S}_{t_1 t_1} dt + \int_{\overline{B(t_0, r)}} dt \\ &= \int_{\mathfrak{J}} \hat{S}_{t_1 t_1} dt + \int_{\overline{B(t_0, r)}} dt \end{aligned}$$

from (12), which gives that

$$\int_{\mathcal{A}_{\varepsilon_0} \times (0, M)} \{-\iota \chi(t_1) \nu_{\varepsilon_1} \hat{S}_{t_2 t_2} + \iota \chi(t_1) \gamma_{\varepsilon_1} \hat{S}_{t_2}\} dt ds = |\overline{B(t_0, r)}| > 0.$$

So

$$\alpha_{\varepsilon}(t) = \chi(t_1) \int_0^M \iota(t) \nu_{\varepsilon}(t, s) ds \leq 0,$$

where $\varepsilon \in (0, \frac{\varepsilon_0}{2})$ and $t \in \mathcal{A}_{\varepsilon_0} \cap \mathfrak{J}$, which yields

$$\int_{(\mathcal{A}_{\varepsilon_0} \cap \mathfrak{J}) \times (0, M)} \iota(t) \nu_{\varepsilon}(t, s) dt ds \leq 0,$$

for any $\varepsilon \in (0, \frac{\varepsilon_0}{2})$.

By passing to the limit as $\varepsilon \rightarrow 0$, we obtain

$$0 \leq \int_{(\mathcal{A}_{\varepsilon_0} \cap \mathfrak{J}) \times (0, M)} \iota(t) (f_1 - f_2)^+(t, s) dt ds \leq 0$$

and

$$\int_P \iota(t) (f_1 - f_2)^+(t, s) dt ds = 0.$$

Notice that

$$\iota(f_1 - f_2)^+ = 0 \quad \text{a.e. in } P$$

for all $0 \leq \iota \in D(0, M)$, which gives that

$$f_1 = f_2 := u \quad \text{a.e. in } P. \tag{13}$$

Next we prove

$$\omega_1 = \omega_2 \quad \text{a.e. in } P. \tag{14}$$

Put

$$\iota(t) = \begin{cases} 2\left(\frac{t}{\sigma}\right)^2 & \text{if } t \in [0, \frac{\sigma}{2}], \\ 1 - 2\left(1 - \frac{t}{\sigma}\right)^2 & \text{if } t \in (\frac{\sigma}{2}, \sigma], \\ 1 & \text{if } t \in (\sigma, s - \sigma], \\ 1 - 2\left(1 - \frac{s-t}{\sigma}\right)^2 & \text{if } t \in (s - \sigma, s - \frac{\sigma}{2}], \\ 2\left(\frac{s-t}{\sigma}\right)^2 & \text{if } t \in (s - \frac{\sigma}{2}, s], \end{cases}$$

where $\sigma > 0$ and $s \in (0, M]$.

Note that $\iota \in C^1([0, s])$ and

$$\iota'(t) = \begin{cases} \frac{4-t}{\sigma^2} & \text{if } t \in [0, \frac{\sigma}{2}], \\ \frac{4}{\sigma}\left(1 - \frac{t}{\sigma}\right) & \text{if } t \in (\frac{\sigma}{2}, \sigma], \\ 0 & \text{if } t \in (\sigma, s - \sigma], \\ -\frac{4}{\sigma}\left(1 - \frac{s-t}{\sigma}\right) & \text{if } t \in (s - \sigma, s - \frac{\sigma}{2}], \\ -\frac{4}{\sigma}\left(\frac{s-t}{\sigma}\right) & \text{if } t \in (s - \frac{\sigma}{2}, s]. \end{cases}$$

Note that $\zeta \iota^2 \in H^1(P)$, $\zeta \iota^2 = 0$ on $\partial \mathfrak{J} \times (0, M)$ and

$$(\zeta \iota^2)(t, 0) = (\zeta \iota^2)(t, M) = 0 \quad \text{a.e. in } \mathfrak{J}.$$

Choosing $\pm \zeta \iota^2$ as test functions for (1) (see [27] for example), we obtain

$$\int_{\mathfrak{J} \times (0, s)} [\chi(t_1)(f_{t_2} + \omega_1)\zeta \iota^2 - 2\omega_1 u' \zeta] dt ds = 0, \tag{15}$$

$$\int_{\mathfrak{J} \times (0, s)} [\chi(t_1)(f_{t_2} + \omega_2)\zeta \iota^2 - 2\omega_2 u' \zeta] dt ds = 0. \tag{16}$$

It follows from (16) and (15) that

$$\begin{aligned} 0 &= \int_{\mathfrak{J} \times (0, s)} \chi(t_1)(\omega_1 - \omega_2)\zeta \iota^2 dt ds - \int_{\mathfrak{J} \times (0, s)} 2(\omega_1 - \omega_2)u' \zeta dt ds \\ &:= \mathcal{R}_\sigma^1 - \mathcal{R}_\sigma^2. \end{aligned} \tag{17}$$

By using the Lebesgue theorem to \mathcal{R}_σ^1 (see [28]), we have

$$\lim_{\sigma \rightarrow 0} \mathcal{R}_\sigma^1 = \int_{\mathfrak{J} \times (0, s)} \chi(t_1)(\omega_1 - \omega_2)\zeta_{t_2} dt ds. \tag{18}$$

We now estimate \mathcal{R}_σ^2 . We know that

$$\begin{aligned} |\mathcal{R}_\sigma^2| &= 2 \left| \int_{\mathfrak{J}} \int_0^\sigma \chi(t_1)(\omega_1 - \omega_2)u' \zeta dt ds + \int_{\mathfrak{J}} \int_{s-\sigma}^s \chi(t_1)(\omega_1 - \omega_2)u' \zeta dt ds \right| \\ &\leq C \left\{ \int_0^\sigma \left(\int_{\mathfrak{J}} |\omega_1 - \omega_2| dt \right) \iota |u'| ds + \int_{s-\sigma}^s \left(\int_{\mathfrak{J}} |\omega_1 - \omega_2| dt \right) \iota |u'| ds \right\} \\ &:= C(\mathcal{R}_\sigma^{2,1} + \mathcal{R}_\sigma^{2,2}) \end{aligned} \tag{19}$$

from the definition of ι' , where

$$C = \sup_{(t_1, t_2) \in \mathfrak{J}} |\chi(t_1) \zeta(t_1, t_2)|.$$

We have

$$\omega_1 - \omega_2 \in C^0([0, M]; L^1(\mathfrak{J}))$$

(see [29, Proposition 2.1]), $\iota \in C^0([0, s])$, $\iota(0) = 0$ and ι is uniformly bounded independently of σ .

Notice that

$$t \mapsto \left(\int_{\mathfrak{J}} |\omega_1 - \omega_2| dt \right) \iota$$

is right-continuous and vanishes at 0.

So

$$\lim_{\sigma \rightarrow 0} \mathcal{R}_\sigma^{2,1} = 0. \tag{20}$$

Similarly,

$$\lim_{\sigma \rightarrow 0} \mathcal{R}_\sigma^{2,2} = 0. \tag{21}$$

By letting $\sigma \rightarrow 0$ in (19) and applying (20)–(21), we obtain

$$\lim_{\sigma \rightarrow 0} \mathcal{R}_\sigma^2 = 0. \tag{22}$$

Combining (17), (18) and (22), we obtain

$$0 = \int_0^s \int_{\mathfrak{J}} \chi(t_1) (\omega_1 - \omega_2) \zeta_{t_2} dt ds := F(s), \tag{23}$$

where $s \in [0, M]$.

Since

$$\int_{\mathfrak{J}} \chi(t_1) (\omega_1 - \omega_2) \zeta_{t_2} dt$$

is continuous on $[0, M]$ (see [30]), we deduce that

$$F'(s) = 0$$

for all $s \in [0, M]$ from (23).

So

$$\int_{\mathfrak{J}} \chi(t_1) (\omega_1 - \omega_2)(\cdot, s) \zeta_{t_2} dt = 0 \quad \forall t \in [0, M], \forall \zeta \in D(\mathfrak{J}), \tag{24}$$

which yields

$$\int_{\mathbb{R}^2} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon \zeta_{t_2} dt = 0. \tag{25}$$

If we choose

$$\min\left(\frac{((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+}{\sigma}, 1\right) \zeta$$

as the test functions in (25), then we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon \zeta_{t_2} \min\left(\frac{((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+}{\sigma}, 1\right) dt \\ &\quad + \int_{\mathbb{R}^2} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon \min\left(\frac{((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+}{\sigma}, 1\right) \zeta_{t_2} dt \\ &:= S_\sigma^1 + S_\sigma^2. \end{aligned} \tag{26}$$

By applying the Lebesgue theorem to S_σ^1 , we obtain

$$\lim_{\sigma \rightarrow 0} S_\sigma^1 = \int_{\mathbb{J}} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ \zeta_{t_2} dt \tag{27}$$

and

$$\begin{aligned} S_\sigma^2 &= \frac{1}{2\sigma} \int_{\mathbb{R}^2} \chi(t_1) \{ \min((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+, \sigma \}^2 \zeta_{t_2} dt \\ &= -\frac{1}{2\sigma} \int_{\mathbb{R}^2} \chi(t_1) \zeta_{t_2} \min((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+, \sigma \}^2 dt, \end{aligned}$$

which yield

$$\lim_{\sigma \rightarrow 0} S_\sigma^2 = 0. \tag{28}$$

Put

$$\mathbb{J}_\varrho := \mathbb{J} \cup \{x \in \mathbb{R}^N \setminus \mathbb{J} : \text{dist}(t, \mathbb{J}_1) < \varrho\},$$

and let (δ_0, ω_0) and $(\delta_\varrho, \omega_\varrho)$ be the principal eigenvalue associated to $-\Delta$ operator in the domains \mathbb{J} and \mathbb{J}_ϱ , respectively (see [31]).

It is obvious that

$$\|\omega_0\|_{L^\infty(\mathbb{J})} = 1, \quad \|\omega_\varrho\|_{L^\infty(\mathbb{J}_\varrho)} = 1.$$

By the definitions of ω_0 and ω_ϱ , we obtain

$$\omega_0(t) = 0, \quad \partial \omega_0(t) < 0,$$

where $t \in \partial\mathfrak{J} = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, and

$$\omega_\varrho(t) > 0, \tag{29}$$

where $t \in \overline{\Gamma}_1$.

Put

$$\eta_0(t) := -\partial\omega_0(t) > 0 \quad \forall t \in \partial\mathfrak{J}, \tag{30}$$

$$\underline{\eta}_0 := \min_{t \in \overline{\Gamma}_1} \eta_0(t) > 0, \quad \bar{\eta}_0 := \|\eta_0(t)\|_{L^\infty(\overline{\Gamma}_1)}, \tag{31}$$

$$\bar{\omega} := \|\omega(t)\|_{L^\infty(\overline{\Gamma}_1)} > 0, \quad \chi := \|\chi(t)\|_{L^\infty(\overline{\Gamma}_1)} \geq 0. \tag{32}$$

Put

$$0 < \varepsilon < \frac{\underline{\eta}_0}{1 + \chi} \tag{33}$$

and

$$0 < \frac{1}{k_0} < \underline{\eta}_0 - \varepsilon(1 + \chi), \tag{34}$$

respectively.

It is well known that

$$\delta^\varrho < \delta_0, \tag{35}$$

where $\varrho > 0$.

So

$$\lim_{\varrho \downarrow 0} \delta^\varrho = \delta_0. \tag{36}$$

Combining (31), (35) and (36), there exists $\varrho_0 > 0$ satisfying

$$\delta_1 < \delta^\varrho < \delta_0, \tag{37}$$

where $\varrho \in (0, \varrho_0]$.

Thanks to the regularity of ω_ϱ and ω_0 , there exists $\varrho_1 \in (0, \varrho_0]$ satisfying

$$\partial\omega_\varrho(t) \leq \partial\omega_0(t) + \varepsilon, \tag{38}$$

$$\omega_\varrho(t) \leq \omega_0(t) + \varepsilon = \varepsilon, \tag{39}$$

where $\varrho \in (0, \varrho_1]$ and $t \in \overline{\Gamma}_1$.

Let $\bar{x}_\varrho \in \overline{\Gamma}_1$ satisfy

$$\omega_\varrho(\bar{x}_\varrho) = \|\omega_\varrho\|_{L^\infty(\overline{\Gamma}_1)}. \tag{40}$$

It follows that there exists $\bar{z} \in \bar{\Gamma}_1$ satisfying

$$\omega_\varrho(\bar{x}_\varrho) = \|\omega_\varrho\|_{L^\infty(\bar{\Gamma}_1)} = \eta_0(\bar{z})\varrho + o(\varrho) \tag{41}$$

from (39) and (40).

Put

$$g_\varrho := C(\varrho)\omega_\varrho, \tag{42}$$

where

$$C = C(\varrho) := \frac{1}{(k_0\bar{b}\tilde{\gamma})^{\frac{1}{q-1}}(\bar{\eta}_0\varrho)^{\frac{q}{q-1}}} > 0. \tag{43}$$

There exists $\varrho_2 \in (0, \varrho_1]$ such that

$$f_\varrho := g_\varrho|_{\bar{\Gamma}} = C(\varrho)\omega_\varrho|_{\bar{\Gamma}}$$

is a positive strict subsolution of (1) for each $\varrho \in (0, \varrho_2]$ and $\lambda \in (\delta^\varrho, \delta_0)$.

It follows that

$$(-\Delta - \lambda)f_\varrho = C(\varrho)(\delta^\varrho - \lambda)\omega_\varrho < 0 \tag{44}$$

for each $\varrho \in (0, \varrho_1]$ and $\lambda \in (\delta^\varrho, \delta_0)$.

It follows that

$$\begin{aligned} & (\partial + \chi(t))f_\varrho(t) + \tilde{\gamma}\omega(t)f_\varrho^q(t) \\ &= C(\partial\omega_\varrho(t) + \chi(t)\omega_\varrho(t) + \tilde{\gamma}\omega(t)C^{q-1}\omega_\varrho^q(t)) \\ &\leq C(\partial\omega_0(t) + \varepsilon(1 + \chi) + \tilde{\gamma}\bar{b}C^{q-1}\omega_\varrho^q(y_\varrho)) \\ &= C(-\eta_0(t) + \varepsilon(1 + \chi) + \tilde{\gamma}\bar{b}C^{q-1}(\eta_0(z_0)\varrho + o(\varrho))^q) \\ &\leq C(-\underline{\eta}_0 + \varepsilon(1 + \chi) + \tilde{\gamma}\bar{b}C^{q-1}(\bar{\eta}_0\varrho + o(\varrho))^q) \\ &= C(-\underline{\eta}_0 + \varepsilon(1 + \chi) + \tilde{\gamma}\bar{b}C^{q-1}(\bar{\eta}_0^q\varrho^q + o(\varrho^q))) \\ &= C\left(-\underline{\eta}_0 + \varepsilon(1 + \chi) + \frac{1}{k_0} + \frac{1}{k_0\bar{\eta}_0^q} \frac{o(\varrho^q)}{\varrho^q}\right) \end{aligned}$$

from (30), (31), (32), (38), (39), (40), (41) and (43), which, together with (34), shows that there exists $\varrho_2 \in (0, \varrho_1]$ such that

$$(\partial + \chi(t))f_\varrho(t) + \tilde{\gamma}\omega(t)f_\varrho^q(t) \leq C\left(-\underline{\eta}_0 + \varepsilon(1 + \chi) + \frac{1}{k_0} + \frac{1}{k_0\bar{\eta}_0^q} \frac{o(\varrho^q)}{\varrho^q}\right) < 0$$

for each $\varrho \in (0, \varrho_2]$ and $t \in \bar{\Gamma}_1$.

And hence

$$\partial f_\varrho + \chi(t)f_\varrho + \tilde{\gamma}\omega(t)f_\varrho^q < 0, \tag{45}$$

where $t \in \bar{\Gamma}_1$ and $\varrho \in (0, \varrho_2]$.

So

$$f_\varrho|_{\Gamma_0} = C(\varrho)\omega_\varrho|_{\Gamma_0} = 0. \tag{46}$$

Combining (44), (45) and (46), we have

$$\begin{aligned} (-\Delta - \lambda)f_\varrho &< 0 \quad \text{in } \mathfrak{J}, \\ f_\varrho &= 0 \quad \text{on } \Gamma_0, \\ \partial f_\varrho + \chi(t)f_\varrho + \tilde{\gamma}\omega(t)f_\varrho^q &< 0 \quad \text{on } \Gamma_1 \end{aligned} \tag{47}$$

for $\varrho \in (0, \varrho_2]$ and $\lambda \in (\delta^\varepsilon, \delta_0)$.

It follows from (29) that there exists $\tilde{y}_\varrho \in \Gamma_1$ satisfying

$$\min_{t \in \Gamma_1} \omega_\varrho(t) = \omega_\varrho(\tilde{y}_\varrho) > 0 \tag{48}$$

for each $\varrho \in (0, \varrho_2]$ and there exists $\underline{z} \in \Gamma_1$ satisfying

$$\omega_\varrho(\tilde{y}_\varrho) = \eta_0(\underline{z})\varrho + o(\varrho). \tag{49}$$

Combining (48) and (49), we obtain

$$f_\varrho(t) = C\omega_\varrho(t) \geq C\omega_\varrho(\tilde{y}_\varrho) = \frac{\eta_0(\underline{z})\varrho + o(\varrho)}{(k_0\bar{b}\tilde{\gamma})^{\frac{1}{q-1}}(\bar{\eta}_0\varrho)^{\frac{q}{q-1}}} \geq \frac{\underline{\eta}_0\varrho + o(\varrho)}{(k_0\bar{b}\tilde{\gamma})^{\frac{1}{q-1}}(\bar{\eta}_0\varrho)^{\frac{q}{q-1}}}$$

for each $t \in \Gamma_1$ and $\varrho \in (0, \varrho_2]$, which yields

$$\begin{aligned} \liminf_{\varrho \downarrow 0} f_\varrho(t) &\geq \lim_{\varrho \downarrow 0} \frac{\underline{\eta}_0\varrho + o(\varrho)}{k_0^{\frac{1}{q-1}}\tilde{\gamma}^{\frac{1}{q-1}}\bar{b}^{\frac{1}{q-1}}\bar{\eta}_0^{\frac{q}{q-1}}\varrho^{\frac{q}{q-1}}} \\ &= \lim_{\varrho \downarrow 0} \frac{\underline{\eta}_0}{k_0^{\frac{1}{q-1}}\tilde{\gamma}^{\frac{1}{q-1}}\bar{b}^{\frac{1}{q-1}}\bar{\eta}_0^{\frac{q}{q-1}}\varrho^{\frac{1}{q-1}}} = \infty \end{aligned}$$

uniformly on Γ_1 .

Passing to the limit as $\delta \rightarrow 0$ in (26) and using (27)–(28), we have

$$\int_{\mathfrak{J}} \chi(t_1)((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ \zeta_{t_2} dt = 0. \tag{50}$$

If we choose ζ_{t_2} as a test function in (50), then we have

$$\begin{aligned} \int_{\mathfrak{J}} \chi(t_1)((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ \zeta dt + \int_{\mathfrak{J}} \chi(t_1)((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ \zeta_{t_2} dt &= 0, \\ \forall t \in [0, M], \forall \zeta \in D(\mathfrak{J}), \zeta \geq 0, \quad d(\text{supp}(\zeta), \partial\mathfrak{J}) &= \varepsilon_0. \end{aligned} \tag{51}$$

Define the following functions h and g in (t_1^1, t_1^2) (resp. (t_2^1, t_2^2)):

$$h(t_1) = \begin{cases} 2\left(\frac{t_1-t_1^1}{\delta}\right)^2 & \text{if } t_1 \in [t_1^1, t_1^1 + \frac{\delta}{2}], \\ 1 - 2\left(1 - \frac{t_1-t_1^1}{\delta}\right)^2 & \text{if } t_1 \in (t_1^1 + \frac{\delta}{2}, t_1^1 + \delta], \\ 1 & \text{if } t_1 \in (t_1^1 + \delta, t_1^2 - \delta), \\ 1 - 2\left(1 - \frac{t_1^2-t_1}{\delta}\right)^2 & \text{if } t_1 \in (t_1^2 - \delta, t_1^2 - \frac{\delta}{2}], \\ 2\left(\frac{t_1^2-t_1}{\delta}\right)^2 & \text{if } t_1 \in (t_1^2 - \frac{\delta}{2}, t_1^2], \end{cases}$$

and

$$g(t_2) = \begin{cases} 2\left(\frac{t_2-t_2^1}{\delta}\right)^2 & \text{if } t_2 \in [t_2^1, t_2^1 + \frac{\delta}{2}], \\ 1 - 2\left(1 - \frac{t_2-t_2^1}{\delta}\right)^2 & \text{if } t_2 \in (t_2^1 + \frac{\delta}{2}, t_2^1 + \delta], \\ 1 & \text{if } t_2 \in (t_2^1 + \delta, t_2^2 - \delta), \\ 1 - 2\left(1 - \frac{t_2^2-t_2}{\delta}\right)^2 & \text{if } t_2 \in (t_2^2 - \delta, t_2^2 - \frac{\delta}{2}], \\ 2\left(\frac{t_2^2-t_2}{\delta}\right)^2 & \text{if } t_2 \in (t_2^2 - \frac{\delta}{2}, t_2^2], \end{cases}$$

respectively (see [25]), where $t_1^1, t_1^2 \in (0, W)$ and $t_2^1, t_2^2 \in (0, w)$ satisfying that $t_1^1 < t_1^2, t_2^1 < t_2^2$,

$$d(0, t_1^1) = d(0, t_2^1) = d(L, t_1^2) = d(l, t_2^2) = \varepsilon_0$$

and δ is a positive real number.

We have

$$h(t_1^1) = h(t_1^2) = g(t_2^1) = g(t_2^2) = 0, \\ h \in C^2([t_1^1, t_1^2]), \quad g \in C^2([t_2^1, t_2^2]),$$

and $h, g \geq 0$.

If we set

$$\mathfrak{J}_{\varepsilon_0} = (t_1^1, t_1^2) \times (t_2^1, t_2^2),$$

then we see that $hg^2 \in C^2(\overline{\mathfrak{J}_{1,2}})$ and $hg^2 \geq 0$.

Let us extend hg^2 outside $\mathfrak{J}_{\varepsilon_0}$ by 0 and still denote by hg^2 this function. We obtain

$$0 = \int_{\mathfrak{J}_{\varepsilon_0}} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_{\varepsilon}^+ hg^2 dt \\ + 2 \int_{\mathfrak{J}_{\varepsilon_0}} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_{\varepsilon}^+ t_2 hgg' dt \\ := \mathcal{N}_{\delta}^1 + \mathcal{N}_{\delta}^2 \tag{52}$$

by choosing $\zeta = hg^2$ as a test function in (51).

We have

$$\lim_{\delta \rightarrow 0} \mathcal{N}_{\delta}^1 = \int_{\mathfrak{J}_{\varepsilon_0}} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_{\varepsilon}^+ dt \tag{53}$$

and

$$\begin{aligned}
 |\mathcal{N}_\delta^2| &= 2 \left| \int_{t_1^1}^{t_1^2} \int_{t_2^1}^{t_2^1+\delta} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ t_2 h g g' dt \right. \\
 &\quad \left. + \int_{t_1^1}^{t_1^2} \int_{t_2^2-\delta}^{t_2^2} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ t_2 h g g' dt \right| \\
 &\leq C \left\{ \int_{t_1^1}^{t_1^2} \int_{t_2^1}^{t_2^1+\delta} ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ |g'| dt \right. \\
 &\quad \left. + \int_{t_1^1}^{t_1^2} \int_{t_2^2-\delta}^{t_2^2} ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ |g'| dt \right\} \\
 &:= C(\mathcal{N}_\delta^{2,1} + \mathcal{N}_\delta^{2,2}), \tag{54}
 \end{aligned}$$

where

$$C = \sup_{(t_1, t_2) \in \mathcal{J}} (\chi(t_1) t_2 h(t_1)).$$

Since the function

$$t_2 \mapsto \left(\int_{t_1^1}^{t_1^2} ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ dt_1 \right) g$$

is right-continuous and vanishes at t_2^1 , uniformly bounded independently of δ and $|g'| \sim \frac{1}{\delta}$, we know that

$$\lim_{\delta \rightarrow 0} \mathcal{N}_\delta^{2,1} = 0. \tag{55}$$

Similarly

$$\lim_{\delta \rightarrow 0} \mathcal{N}_\delta^{2,2} = 0. \tag{56}$$

It follows from (55)–(56) that

$$\lim_{\delta \rightarrow 0} \mathcal{N}_\delta^2 = 0 \tag{57}$$

by letting $\delta \rightarrow 0$ in (54).

Similarly, it follows from (53) and (57) that

$$\int_{\mathcal{J}_{\varepsilon_0}} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))_\varepsilon^+ dt = 0 \tag{58}$$

by passing to the limit as $\delta \rightarrow 0$ in (52), where $t \in [0, M]$.

Finally, it follows that

$$\int_{\mathcal{J}_{\varepsilon_0}} \chi(t_1) ((\omega_1 - \omega_2)(\cdot, s))^+ dt = 0$$

by letting $\varepsilon \rightarrow 0$ in (58), where $t \in [0, M]$.

So

$$\int_{\mathcal{J}} \chi(t_1)((\omega_1 - \omega_2)(\cdot, s))^+ dt = 0,$$

where $t \in [0, M]$, which yields

$$\chi(t_1)((\omega_1 - \omega_2)(\cdot, s))^+ = 0 \quad \text{a.e. in } \mathcal{J}$$

for all $t \in [0, M]$.

Thanks to (2), we deduce that

$$((\omega_1 - \omega_2)(\cdot, s))^+ = 0 \quad \text{a.e. in } \mathcal{J}$$

for all $t \in [0, M]$ (see [32]).

So

$$\omega_1 \leq \omega_2 \quad \text{a.e. in } P.$$

By exchanging the roles of ω_1 and ω_2 , we obtain

$$\omega_2 \leq \omega_1 \quad \text{a.e. in } P.$$

We conclude that

$$\omega_2 = \omega_1 \quad \text{a.e. in } P.$$

Hence, (14) holds. If we combine (13) and (14), we see that the solution of problem (1) associated with the initial data ω_0 is unique. □

4 Examples

In this section, two boundary value problems involving nonlocal integral boundary conditions will be tested by using the present method.

Example 4.1 Consider the boundary value problem

$$({}^c D^{5/2} + 3{}^c D^{3/2} + 2{}^c D^{1/2})x(t) = \frac{A}{\sqrt{t^2 + 49}}(\cos x + \tan^{-1} t), \quad 0 < t < 1, \tag{59}$$

$$x(0) = 0, \quad x(1/3) = 0, \quad x(1) = \int_0^{1/5} x(s) ds. \tag{60}$$

Here, $\delta = 1/2$, $\sigma = 3/5$, $\xi = 1/3$, $p_2 = 1$, $p_1 = 3$, $p_0 = 2$, $\lambda = 1$, A is a positive constant and

$$f(t, x) = \frac{A}{\sqrt{t^2 + 49}}(\cos x + \tan^{-1} t).$$

Clearly the constants p_2 , p_1 , and p_0 satisfy the condition of Lemma 2.1, and

$$|f(t, x) - f(t, y)| \leq A|x - y|/7,$$

where $\ell = A/7$. Using the given values, we find $\alpha \approx 0.44269$ and $\alpha_1 \approx 0.21725$, It is easy to check that

$$|f(t, x)| \leq \frac{A(2 + \pi)}{2\sqrt{t^2 + 49}} = \theta(t)$$

and $\ell\alpha_1 < 1$ when $A < 32.22094$. As all the conditions of Theorem 3.1 are satisfied the problem (59)–(60) has at least one solution on $[0, 1]$. On the other hand, $\ell\alpha < 1$ whenever $A < 15.81242$ and thus there exists a unique solution for the problem (59)–(60) on $[0, 1]$ by Theorem 3.1.

Example 4.2 Consider the boundary value problem

$$({}^c D^{5/2} + 3{}^c D^{3/2} + 2{}^c D^{1/2})x(t) = \frac{1}{4\pi} \sin(2\pi x) + \frac{|x|^2}{1 + |x|^2}, \quad 0 < t < 1, \tag{61}$$

$$x(0) = 0, \quad x(1/3) = 0, \quad x(1) = \int_0^{1/5} x(s) ds. \tag{62}$$

Here, $\delta = 1/2, \sigma = 3/5, \xi = 1/3, p_1^2 - 4p_2p_0 = 1 > 0, \lambda = 1$, and

$$f(t, x) = \frac{1}{4\pi} \sin(2\pi x) + \frac{|x|^2}{1 + |x|^2}.$$

Clearly

$$|f(t, x)| \leq \left| \frac{1}{4\pi} \sin(2\pi x) + \frac{|x|^2}{1 + |x|^2} \right| \leq \frac{1}{2} \|x\| + 1,$$

where $g(t) = 1, \psi(\|x\|) = \frac{1}{2} \|x\| + 1$.

Then, by using the condition (A4), we find that $K > 0.56853$ (we have used $\alpha = 0.44269$). Thus, the conclusion of Theorem 3.1 applies to problem (61)–(62).

5 Conclusion

This paper was devoted to modifying the Schrödinger-type identity related to the singular boundary value problem. We also presented some mathematical consequences of the method including a stability result. The main technical tools used to develop the mathematical analysis are local and global bifurcation, monotonicity techniques, fixed point theory in b -metric spaces and the maximum principle approach with respect to the Schrödinger operator. As an application, the uniqueness of solutions for singular boundary value problem for the Schrödinger equation was proved.

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