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Determination of differential pencils with impulse from interior spectral data

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Abstract

In this paper, we are concerned with the inverse spectral problems for differential pencils defined on $[0,\pi]$ with an interior discontinuity. We prove that two potential functions are determined uniquely by one spectrum and a set of values of eigenfunctions at some interior point $b \in (0,\pi)$ in the situation of $b = \pi/2$ and $b \neq \pi/2$. For the latter, we need the knowledge of a part of the second spectrum.

MSC: Primary 34A55; 34L05; secondary 34L40

Keywords: Differential pencil; Discontinuous condition; Inverse problem; Interior spectral data; Uniqueness theorem

1 Introduction

We consider the quadratic pencils of Sturm–Liouville operator $L(p,q;h,H;a)$ of the form

$$Ly := -y'' + \left[2\lambda p(x) + q(x)\right]y = \lambda^2 y, \quad x \in [0,\pi/2) \cup (\pi/2,\pi], \tag{1.1}$$

with the boundary conditions

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.2}$$

and with the discontinuous conditions

$$\begin{cases} y(\pi/2 + 0) = ay(\pi/2 - 0), \\ y'(\pi/2 + 0) = a^{-1}y'(\pi/2 - 0), \end{cases} \tag{1.3}$$

where λ is the spectral parameter, $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2(0,\pi)$ are real-valued functions, $h, H \in \mathbb{R}$, and $a \in \mathbb{R}^+/\{1\}$.

Differential equations with potentials depending nonlinearly on the spectral parameter appear frequently in various models of classical mechanics and quantum (see [5–7, 11, 18, 22] and the references therein). For instance, the evolution equations which are used to model interactions between colliding relativistic spineless particles can be reduced to the form (1.1), here the parameter λ^2 can be regarded as the energy of this system (see [5, 11, 18]).

Boundary value problems with discontinuity inside the interval often appear in mathematics, physics, geophysics, mechanics, and other branches of natural properties (see [1, 3, 4, 8, 10, 13–17, 20] and the references therein). The well-known work [4] is the first result about discontinuous inverse eigenvalue problems for the Sturm–Liouville problems, i.e., $p(x) \equiv 0$ in (1.1). Direct and inverse problems for differential pencils with impulse on a finite interval have been investigated in [2].

Inverse spectral problems consist in recovering operators from their spectral characteristics. The interior spectral data used for reconstructing the differential operators contains the known eigenvalues and some information on eigenfunctions at some interior point in the defined interval. The similar problems for the Sturm–Liouville operators and differential pencils were considered in [12, 19, 21].

The aim of this paper is to recover the pencils $L(p,q;h,H;a)$ uniquely from some eigenvalues and information on eigenfunctions at the interior point $b \in (0,\pi)$. As far as we know, the inverse problem for interior spectra data of quadratic pencils with impulse has not been considered before. Note that the obtained results here are new and they are a generalization of the well-known one for the classical Sturm–Liouville operator, which was studied in [12], for a special case that $p(x) \equiv 0$ and $a = 1$. The results in this paper are also a generalization of theorems in [21], where the authors considered the special case that $a = 1$ and assumed either $p(x)$ or $q(x)$ is known a priori, which is unnecessary. The technique we used is similar to those used in [12, 19, 21].

2 Main results

It is known [2] that the spectrum of the pencils $L(p,q;h,H;a)$ consists of simple, real eigenvalues λ_n, $n \in \mathbb{Z}$ under the additional assumption that

$$\int_0^\pi \left\{ \left|y'(x)\right|^2 + q(x)\left|y(x)\right|^2 \right\} \mathrm{d}x > 0$$

for all $y(x) \in W_2^2[0,\pi/2) \cup (\pi/2,\pi]$ such that $y(x) \not\equiv 0$ and $y'(0)\overline{y(0)} - y'(\pi)\overline{y(\pi)} = 0$. The sequence $\{\lambda_n\}_{-\infty}^{\infty}$ satisfies the classical asymptotic form [2]

$$\lambda_n = n + \frac{\omega}{\pi} + O\left(\frac{1}{n}\right), \quad |n| \to \infty, \tag{2.1}$$

where

$$\omega = \int_0^\pi p(t)\,\mathrm{d}t + (-1)^n \arcsin A,$$
$$A = \frac{a - a^{-1}}{a + a^{-1}} \sin\left(\int_{\pi/2}^\pi p(t)\,\mathrm{d}t\right).$$

Denote by $y_n(x)$ the eigenfunction corresponding to the eigenvalue λ_n. Together with $L(p,q;h,H;a)$, let us consider another differential pencil $\tilde{L}(\tilde{p},\tilde{q};\tilde{h},\tilde{H};a)$ of the same form but with different coefficients $(\tilde{p}(x),\tilde{q}(x);\tilde{h},\tilde{H})$. It is assumed in what follows that if a certain symbol δ denotes an object related to L, then $\tilde{\delta}$ will denote an analogous object related to $\tilde{L}$.

The main results of this paper are as follows.

Theorem 2.1 *If, for any $n \in \mathbb{Z}$,*

$$\lambda_n = \tilde{\lambda}_n, \qquad \frac{y'(\pi/2-0,\lambda_n)}{y(\pi/2-0,\lambda_n)} = \frac{\tilde{y}'(\pi/2-0,\lambda_n)}{\tilde{y}(\pi/2-0,\lambda_n)}, \tag{2.2}$$

then $p(x) = \tilde{p}(x)$ on $[0,\pi]$, $q(x) = \tilde{q}(x)$ a.e. on $[0,\pi]$, and $h = \tilde{h}$, $H = \tilde{H}$.

Remark 2.2 Note that if $y(x)$ and $z(x)$ are two continuously differentiable functions on $[0,\pi/2) \cup (\pi/2,\pi]$ and satisfy the same discontinuous condition (1.3), then a direct calculation yields that

$$\bigl(y'z - yz'\bigr)(\pi/2-0) = \bigl(y'z - yz'\bigr)(\pi/2+0).$$

Thus, we can replace the condition $y'(\pi/2-0,\lambda_n)/y(\pi/2-0,\lambda_n) = \tilde{y}'(\pi/2-0,\lambda_n)/\tilde{y}(\pi/2-0,\lambda_n)$ by $y'(\pi/2+0,\lambda_n)/y(\pi/2+0,\lambda_n) = \tilde{y}'(\pi/2+0,\lambda_n)/\tilde{y}(\pi/2+0,\lambda_n)$ in Theorem 2.1.

For the case $b \neq \pi/2$, the uniqueness of $(p,q;h,H)$ can be obtained from a part of the second spectrum. We denote by μ_n the eigenvalues of the pencils $L(p,q;h,H_1;a)$, $H_1 \neq H$, $H_1 \in \mathbb{R}$. Let $l(n)$, $r(n)$ be sequences of integers with the properties

$$l(n) = \frac{n}{\sigma_1}(1+\epsilon_{1,n}), \quad 0 < \sigma_1 \leq 1, \epsilon_{1,n} \to 0, \tag{2.3}$$

$$r(n) = \frac{n}{\sigma_2}(1+\epsilon_{2,n}), \quad 0 < \sigma_2 \leq 1, \epsilon_{2,n} \to 0. \tag{2.4}$$

Theorem 2.3 *Let $l(n)$, $r(n)$, and $b \in (\pi/2,\pi)$ be such that*

$$\sigma_1 > \frac{2b}{\pi} - 1, \qquad \sigma_2 > 2 - \frac{2b}{\pi}. \tag{2.5}$$

If, for any $n \in \mathbb{Z}$,

$$\lambda_n = \tilde{\lambda}_n, \qquad \mu_{l(n)} = \tilde{\mu}_{l(n)}, \qquad \frac{y'(b,\lambda_{r(n)})}{y(b,\lambda_{r(n)})} = \frac{\tilde{y}'(b,\lambda_{r(n)})}{\tilde{y}(b,\lambda_{r(n)})}, \tag{2.6}$$

then $p(x) = \tilde{p}(x)$ on $[0,\pi]$, $q(x) = \tilde{q}(x)$ a.e. on $[0,\pi]$, and $h = \tilde{h}$, $H = \tilde{H}$.

3 Preliminaries

We shall first mention some results which will be needed later.

Let the function $\varphi(x,\lambda)$ be the solution of equation (1.1) with the initial-valued conditions

$$\varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = h, \tag{3.1}$$

and the discontinuity conditions (1.3). It is shown in [2] that there exist functions $A(x,t)$ and $B(x,t)$ whose first order partial derivatives are summable on $[0,\pi]$ for each $x \in [0,\pi]$ such that

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x A(x,t)\cos\lambda t\,\mathrm{d}t + \int_0^x B(x,t)\sin\lambda t\,\mathrm{d}t, \tag{3.2}$$

where

$$\varphi_0(x,\lambda) = \begin{cases} \cos(\lambda x - \beta^+(x)), & x \in (0, \frac{\pi}{2}), \\ \alpha^+ \cos(\lambda x - \beta^+(x)) + \alpha^- \cos(\lambda(\pi - x) + \beta^-(x)), & x \in (\frac{\pi}{2}, \pi), \end{cases} \tag{3.3}$$

and

$$\alpha^\pm = \frac{1}{2}\bigl(a \pm a^{-1}\bigr), \qquad \beta^+(x) = \int_0^x p(t)\,\mathrm{d}t, \qquad \beta^-(x) = \int_{\pi/2}^x p(t)\,\mathrm{d}t. \tag{3.4}$$

It follows from (3.2) and (3.3) that the characteristic function of the pencil $L(p,q;h,H;a)$ can be reduced to $\Delta(\lambda)$, where

$$\begin{aligned} \Delta(\lambda) &= \varphi'(\pi,\lambda) + H\varphi(\pi,\lambda) \\ &= \lambda\bigl[\alpha^- \sin\bigl(\beta^-(\pi)\bigr) - \alpha^+ \sin\bigl(\lambda\pi - \beta^+(\pi)\bigr)\bigr] + O\bigl(e^{|\operatorname{Im}\lambda|\pi}\bigr). \end{aligned} \tag{3.5}$$

Denote by $G_\delta = \{\lambda : |\lambda - n - \omega/\pi| \geq \delta, n \in \mathbb{Z}\}$ with fixed $\delta > 0$. Then there exists a constant $C_\delta > 0$ such that

$$\bigl|\Delta(\lambda)\bigr| \geq C_\delta|\lambda| \exp\bigl(|\operatorname{Im}\lambda|\pi\bigr) \quad \text{for } \lambda \in G_\delta. \tag{3.6}$$

Moreover, for the solutions $\varphi(x,\lambda)$ and $\tilde{\varphi}(x,\lambda)$ of the operators L and $\tilde{L}$, respectively, using (3.2)–(3.4), and by extending the range of $A(x,t)$, $\tilde{A}(x,t)$ evenly with respect to the argument t and $B(x,t)$, $\tilde{B}(x,t)$ oddly with respect to the argument t, and by some straightforward calculations, we infer that there exist functions $R_1(x,t)$ and $R_2(x,t)$ whose first order partial derivatives are summable on $[0,\pi]$ for each $x \in [0,\pi]$ such that, for $0 < x < \pi/2$,

$$\begin{aligned} \varphi(x,\lambda)\tilde{\varphi}(x,\lambda) &= \frac{1}{2}\bigl[\cos\bigl(2\lambda x - \theta_1^+(x)\bigr) + \cos\bigl(\theta_1^-(x)\bigr)\bigr] \\ &\quad + \int_0^x \bigl[R_1(x,t)e^{2i\lambda t} + R_2(x,t)e^{-2i\lambda t}\bigr]\,\mathrm{d}t, \end{aligned} \tag{3.7}$$

and for $\pi/2 < x < \pi$,

$$\begin{aligned} \varphi(x,\lambda)\tilde{\varphi}(x,\lambda) &= \frac{(\alpha^+)^2}{2}\bigl[\cos\bigl(2\lambda x - \theta_1^+(x)\bigr) + \cos\bigl(\theta_1^-(x)\bigr)\bigr] \\ &\quad + \frac{(\alpha^-)^2}{2}\bigl[\cos\bigl(2\lambda(\pi - x) + \theta_2^+(x)\bigr) + \cos\bigl(\theta_2^-(x)\bigr)\bigr] \\ &\quad + \frac{\alpha^+\alpha^-}{2}\bigl[\cos\bigl(\lambda(2x-\pi) - \beta^+(x) - \tilde{\beta}^-(x)\bigr) + \cos\bigl(\lambda\pi - \beta^+(x) + \tilde{\beta}^-(x)\bigr)\bigr] \\ &\quad + \frac{\alpha^+\alpha^-}{2}\bigl[\cos\bigl(\lambda(2x-\pi) - \beta^-(x) - \tilde{\beta}^+(x)\bigr) + \cos\bigl(\lambda\pi + \beta^-(x) - \tilde{\beta}^+(x)\bigr)\bigr] \\ &\quad + \int_0^x \bigl[R_1(x,t)e^{2i\lambda t} + R_2(x,t)e^{-2i\lambda t}\bigr]\,\mathrm{d}t, \end{aligned} \tag{3.8}$$

where

$$\theta_1^\pm(x) = \beta^+(x) \pm \tilde{\beta}^+(x), \qquad \theta_2^\pm(x) = \beta^-(x) \pm \tilde{\beta}^-(x). \tag{3.9}$$

4 Proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1.

Proof of Theorem 2.1 Let $\varphi(x,\lambda)$ be the solution of equation (1.1) satisfying the initial-valued conditions (3.1) and the discontinuity conditions (1.3). Let $\tilde{\varphi}(x,\lambda)$ be the solution of the equation

$$-\tilde{\varphi}''(x,\lambda)+\left[2\lambda\tilde{p}(x)+\tilde{q}(x)\right]\tilde{\varphi}(x,\lambda)=\lambda^2\tilde{\varphi}(x,\lambda) \tag{4.1}$$

with the initial-valued conditions

$$\tilde{\varphi}(0,\lambda)=1, \qquad \tilde{\varphi}'(0,\lambda)=\tilde{h} \tag{4.2}$$

and the discontinuity conditions (1.3). Multiplying (1.1) by $\tilde{\varphi}(x,\lambda)$ and (4.1) by $\varphi(x,\lambda)$, respectively, and subtracting, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tilde{\varphi}(x,\lambda)\varphi'(x,\lambda)-\tilde{\varphi}'(x,\lambda)\varphi(x,\lambda)\right]=\left[2\lambda(p-\tilde{p})(x)+(q-\tilde{q})(x)\right]\varphi(x,\lambda)\tilde{\varphi}(x,\lambda). \tag{4.3}$$

Integrating the above equality from 0 to $\pi/2$ with respect to x, using the initial conditions at $x=0$, we have

$$\begin{aligned}&\int_0^{\pi/2}\left[2\lambda(p-\tilde{p})+(q-\tilde{q})\right](\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x+(h-\tilde{h})\\&\quad=\tilde{\varphi}\left(\frac{\pi}{2}-0,\lambda\right)\varphi'\left(\frac{\pi}{2}-0,\lambda\right)-\tilde{\varphi}'\left(\frac{\pi}{2}-0,\lambda\right)\varphi\left(\frac{\pi}{2}-0,\lambda\right).\end{aligned}$$

Denote

$$P(x)=p(x)-\tilde{p}(x), \qquad Q(x)=q(x)-\tilde{q}(x)$$

and

$$H(\lambda)=h-\tilde{h}+2\lambda\int_0^{\pi/2}P(x)(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x+\int_0^{\pi/2}Q(x)(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x. \tag{4.4}$$

It follows from (3.2)–(3.3) and (3.7) that $H(\lambda)$ is an entire function of exponential type, and there are some positive constants C_1 and C_2 such that

$$\left|H(\lambda)\right|\le\left(C_1+C_2|\lambda|\right)\exp\left(|\operatorname{Im}\lambda|\pi\right) \quad \text{for all } \lambda\in\mathbb{C}. \tag{4.5}$$

From assumption (2.2) we have

$$\tilde{\varphi}\left(\frac{\pi}{2}-0,\lambda_n\right)\varphi'\left(\frac{\pi}{2}-0,\lambda_n\right)-\tilde{\varphi}'\left(\frac{\pi}{2}-0,\lambda_n\right)\varphi\left(\frac{\pi}{2}-0,\lambda_n\right)=0,$$

which means

$$H(\lambda_n)=0, \quad n\in\mathbb{Z}. \tag{4.6}$$

Define

$$W(\lambda) = \frac{H(\lambda)}{\Delta(\lambda)}, \tag{4.7}$$

which is an entire function from the above arguments, and it follows from (3.6) and (4.5) that

$$W(\lambda) = O(1)$$

for sufficiently large $|\lambda|$, $\lambda \in G_\delta$. Thus, by Liouville's theorem [9], we obtain for all $\lambda \in \mathbb{C}$ that

$$W(\lambda) = C,$$

where C is a constant, this together with (4.7) further gives that

$$H(\lambda) = C\Delta(\lambda) \quad \text{for all } \lambda \in \mathbb{C}. \tag{4.8}$$

Let us show that the constant $C = 0$. Based on (4.7) and (3.5), we can rewrite the equation $H(\lambda) = C\Delta(\lambda)$ in the form

$$\begin{aligned} &h - \tilde{h} + 2\lambda \int_0^{\pi/2} P(x)(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x + \int_0^{\pi/2} Q(x)(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x \\ &\quad = C\lambda\bigl[\alpha^- \sin\bigl(\beta^-(\pi)\bigr) - \alpha^+ \sin\bigl(\lambda\pi - \beta^+(\pi)\bigr)\bigr] + O\bigl(e^{|\operatorname{Im}\lambda|\pi}\bigr), \end{aligned}$$

that is,

$$\begin{aligned} &\frac{h - \tilde{h}}{\lambda} + 2\int_0^{\pi/2} P(x)(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x + \int_0^{\pi/2} \frac{Q(x)}{\lambda}(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x \\ &\quad = C\bigl[\alpha^- \sin\bigl(\beta^-(\pi)\bigr) - \alpha^+ \sin\bigl(\lambda\pi - \beta^+(\pi)\bigr)\bigr] + O\biggl(\frac{e^{|\operatorname{Im}\lambda|\pi}}{\lambda}\biggr). \end{aligned}$$

By use of Riemann–Lebesgue lemma [9], we see that the limit of the left-hand side of the above equality exists as $\lambda \to \infty$, $\lambda \in \mathbb{R}$. Thus we obtain that $C = 0$. So we have from (4.8) that

$$H(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{C}. \tag{4.9}$$

Substituting (3.7) into (4.4), we get

$$\begin{aligned} H(\lambda) &= h - \tilde{h} + 2\lambda \int_0^{\pi/2} P(x)(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x + \int_0^{\pi/2} Q(x)(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x \\ &= h - \tilde{h} + \frac{1}{2}\int_0^{\pi/2} Q(x)\bigl[\cos\bigl(2\lambda x - \theta_1^+(x)\bigr) + \cos\bigl(\theta_1^-(x)\bigr)\bigr]\,\mathrm{d}x \\ &\quad + \int_0^{\pi/2} Q(x)\int_0^x \bigl[R_1(x,t)e^{2i\lambda t} + R_2(x,t)e^{-2i\lambda t}\bigr]\,\mathrm{d}t\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+ \lambda \int_0^{\pi/2} P(x)\big[\cos\big(2\lambda x - \theta_1^+(x)\big) + \cos\big(\theta_1^-(x)\big)\big]\,\mathrm{d}x \\
&+ 2\lambda \int_0^{\pi/2} P(x) \int_0^x \big[R_1(x,t)e^{2i\lambda t} + R_2(x,t)e^{-2i\lambda t}\big]\,\mathrm{d}t\,\mathrm{d}x \\
= {}& h - \tilde{h} + \frac{1}{2}\int_0^{\pi/2} Q(x)\cos\big(\theta_1^-(x)\big)\,\mathrm{d}x + \lambda \int_0^{\pi/2} P(x)\cos\big(\theta_1^-(x)\big)\,\mathrm{d}x \\
&+ \frac{1}{2}\int_0^{\pi/2} Q_1(t)e^{2i\lambda t}\,\mathrm{d}t + \frac{1}{2}\int_0^{\pi/2} Q_2(t)e^{-2i\lambda t}\,\mathrm{d}t \\
&+ \lambda \int_0^{\pi/2} P_1(t)e^{2i\lambda t}\,\mathrm{d}t + \lambda \int_0^{\pi/2} P_2(t)e^{-2i\lambda t}\,\mathrm{d}t,
\end{aligned}
$$

where

$$
\begin{cases}
Q_1(t) = \frac{1}{2}Q(t)e^{-i\theta_1^+(t)} + 2\int_t^{\pi/2} Q(x)R_1(x,t)\,\mathrm{d}x, \\
Q_2(t) = \frac{1}{2}Q(t)e^{i\theta_1^+(t)} + 2\int_t^{\pi/2} Q(x)R_2(x,t)\,\mathrm{d}x, \\
P_1(t) = \frac{1}{2}P(t)e^{-i\theta_1^+(t)} + 2\int_t^{\pi/2} P(x)R_1(x,t)\,\mathrm{d}x, \\
P_2(t) = \frac{1}{2}P(t)e^{i\theta_1^+(t)} + 2\int_t^{\pi/2} P(x)R_2(x,t)\,\mathrm{d}x.
\end{cases}
\tag{4.10}
$$

Moreover, by use of the Riemann–Lebesgue lemma as $\lambda \to \infty$, $\lambda \in \mathbb{R}$, we obtain from the fact $H(\lambda) \equiv 0$ that

$$
\int_0^{\pi/2} P(x)\cos\big(\theta_1^-(x)\big)\,\mathrm{d}x = 0.
$$

Thus, the function $H(\lambda)$ can be reduced as

$$
\begin{aligned}
H(\lambda) = {}& h - \tilde{h} + \frac{1}{2}\int_0^{\pi/2} Q(x)\cos\big(\theta_1^-(x)\big)\,\mathrm{d}x \\
&+ \frac{1}{2}\int_0^{\pi/2} Q_1(t)e^{2i\lambda t}\,\mathrm{d}t + \frac{1}{2}\int_0^{\pi/2} Q_2(t)e^{-2i\lambda t}\,\mathrm{d}t \\
&+ \lambda \int_0^{\pi/2} P_1(t)e^{2i\lambda t}\,\mathrm{d}t + \lambda \int_0^{\pi/2} P_2(t)e^{-2i\lambda t}\,\mathrm{d}t.
\end{aligned}
\tag{4.11}
$$

Integrating by parts in (4.11), we have from (4.10) that

$$
\begin{aligned}
H(\lambda) = {}& h - \tilde{h} + \frac{1}{2}\int_0^{\pi/2} Q(x)\cos\big(\theta_1^-(x)\big)\,\mathrm{d}x \\
&+ \frac{1}{2}\int_0^{\pi/2} Q_1(t)e^{2i\lambda t}\,\mathrm{d}t + \frac{1}{2}\int_0^{\pi/2} Q_2(t)e^{-2i\lambda t}\,\mathrm{d}t \\
&+ \frac{1}{2}P\left(\frac{\pi}{2}\right)\sin\left(\lambda\pi - \theta_1^+\left(\frac{\pi}{2}\right)\right) + \frac{i}{2}\big(P_1(0) - P_2(0)\big) \\
&+ \frac{i}{2}\int_0^{\pi/2} P_1'(t)e^{2i\lambda t}\,\mathrm{d}t - \frac{i}{2}\int_0^{\pi/2} P_2'(t)e^{-2i\lambda t}\,\mathrm{d}t.
\end{aligned}
$$

Again, by use of the Riemann–Lebesgue lemma as $\lambda \to \infty$, $\lambda \in \mathbb{R}$, we obtain from the fact $H(\lambda) \equiv 0$ that

$$P\left(\frac{\pi}{2}\right) = 0, \tag{4.12}$$

$$h - \tilde{h} + \frac{1}{2}\int_0^{\pi/2} Q(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x + \frac{i}{2}\bigl(P_1(0) - P_2(0)\bigr) = 0, \tag{4.13}$$

and

$$\int_0^{\pi/2}\bigl[Q_1(t) + iP_1'(t)\bigr]e^{2i\lambda t}\,\mathrm{d}t + \int_0^{\pi/2}\bigl[Q_2(t) - iP_2'(t)\bigr]e^{-2i\lambda t}\,\mathrm{d}t = 0. \tag{4.14}$$

Because the exponential system $\{(e^{2i\lambda t}, e^{-2i\lambda t})^T : \lambda \in \mathbb{R}\}$ is complete in $(L^2(0,\pi/2))^2$, consequently,

$$Q_1(t) + iP_1'(t) = 0 = Q_2(t) - iP_2'(t) \quad \text{for } t \in (0,\pi/2). \tag{4.15}$$

Substituting (4.10) into (4.15), together with (4.12), we have

$$\begin{cases} Q(t)e^{-i\theta_1^+(t)} + [(\theta_1^+(t))'e^{-i\theta_1^+(t)} - 4iR_1(t,t)]P(t) + ie^{-i\theta_1^+(t)}P'(t) \\ \quad + 4\int_t^{\pi/2} Q(x)R_1(x,t)\,\mathrm{d}x + 4i\int_t^{\pi/2} P(x)\frac{\partial}{\partial t}R_1(x,t)\,\mathrm{d}x = 0, \\ Q(t)e^{i\theta_1^+(t)} + [(\theta_1^+(t))'e^{i\theta_1^+(t)} + 4iR_2(t,t)]P(t) - ie^{i\theta_1^+(t)}P'(t) \\ \quad + 4\int_t^{\pi/2} Q(x)R_2(x,t)\,\mathrm{d}x - 4i\int_t^{\pi/2} P(x)\frac{\partial}{\partial t}R_2(x,t)\,\mathrm{d}x = 0, \\ P(t) + \int_t^{\pi/2} P'(x)\,\mathrm{d}x = 0. \end{cases} \tag{4.16}$$

Define

$$F(t) = \bigl(Q(t), P(t), P'(t)\bigr)^T,$$

$$K_1(t) = \begin{pmatrix} e^{-i\theta_1^+(t)} & (\theta_1^+(t))'e^{-i\theta_1^+(t)} - 4iR_1(t,t) & ie^{-i\theta_1^+(t)} \\ e^{i\theta_1^+(t)} & (\theta_1^+(t))'e^{i\theta_1^+(t)} + 4iR_2(t,t) & -ie^{i\theta_1^+(t)} \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$K_2(x,t) = \begin{pmatrix} 4R_1(x,t) & 4i\frac{\partial}{\partial t}R_1(x,t) & 0 \\ 4R_2(x,t) & -4i\frac{\partial}{\partial t}R_2(x,t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Equation (4.16) can readily be reduced to a vector form

$$K_1(t)F(t) + \int_t^{\pi/2} K_2(x,t)F(x)\,\mathrm{d}x = 0 \quad \text{for } t \in (0,\pi/2). \tag{4.17}$$

Because $\det K_1(t) = 2i \neq 0$, (4.17) can be rewritten as

$$F(t) + \int_t^{\pi/2} K_1^{-1}(t)K_2(x,t)F(x)\,\mathrm{d}x = 0 \quad \text{for } t \in (0,\pi/2).$$

This is a homogeneous Volterra integral equation and the kernel function is $K_1^{-1}(t)K_2(x,t)$, its solution is identically zero. Thus, we have

$$F(t) = \mathbf{0} \quad \text{a.e. on } [0,\pi/2],$$

which yields that

$$Q(t) = 0 = P(t) \quad \text{a.e. on } [0,\pi/2].$$

Therefore, we obtain

$$p(x) = \tilde{p}(x) \quad \text{on } [0,\pi/2], \qquad q(x) = \tilde{q}(x) \quad \text{a.e. on } [0,\pi/2].$$

Moreover, from (4.10) and (4.13), it is obvious that

$$h = \tilde{h}.$$

To prove that

$$p(x) = \tilde{p}(x) \quad \text{on } [\pi/2,\pi], \qquad q(x) = \tilde{q}(x) \quad \text{a.e. on } [\pi/2,\pi], \qquad H = \tilde{H}, \tag{4.18}$$

we should repeat the earlier argument for the supplementary problem

$$\begin{cases} -y'' + [2\lambda p_1(x) + q_1(x)]y = \lambda^2 y, \quad x \in [0,\pi/2) \cup (\pi/2,\pi], \\ y'(0) - Hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0, \\ y(\pi/2+0) = a^{-1}y(\pi/2-0), \\ y'(\pi/2+0) = ay'(\pi/2-0), \end{cases}$$

where $q_1(x) = q(\pi - x)$ and $p_1(x) = p(\pi - x)$. Thus, we obtain $Q(\pi - t) = 0 = P(\pi - t)$ a.e. on $[0,\pi/2]$, and $H = \tilde{H}$, that is, (4.18) holds. The proof is complete. □

5 Proof of Theorem 2.3

To prove Theorem 2.3, we need the following lemma.

Lemma 5.1 *Let $m(n)$ be a sequence of integers such that*

$$m(n) = \frac{n}{\sigma}(1 + \epsilon_n), \quad 0 < \sigma \le 1, \epsilon_n \to 0. \tag{5.1}$$

(i) *Let $b \in (0,\pi/2)$ satisfy $\sigma > 2b/\pi$. If, for any $n \in \mathbb{Z}$,*

$$\lambda_{m(n)} = \tilde{\lambda}_{m(n)}, \qquad \frac{y'(b,\lambda_{m(n)})}{y(b,\lambda_{m(n)})} = \frac{\tilde{y}'(b,\lambda_{m(n)})}{\tilde{y}(b,\lambda_{m(n)})}, \tag{5.2}$$

then $p(x) = \tilde{p}(x)$ on $[0,b]$, $q(x) = \tilde{q}(x)$ a.e. on $[0,b]$, and $h = \tilde{h}$.

(ii) *Let $b \in (\pi/2, \pi)$ satisfy $\sigma > 2 - 2b/\pi$. If, for any $n \in \mathbb{Z}$,*

$$\lambda_{m(n)} = \tilde{\lambda}_{m(n)}, \qquad \frac{y'(b, \lambda_{m(n)})}{y(b, \lambda_{m(n)})} = \frac{\tilde{y}'(b, \lambda_{m(n)})}{\tilde{y}(b, \lambda_{m(n)})}, \tag{5.3}$$

then $p(x) = \tilde{p}(x)$ on $[b, \pi]$, $q(x) = \tilde{q}(x)$ a.e. on $[b, \pi]$, and $H = \tilde{H}$.

Proof (i) Integrating equation (4.3) from 0 to b with respect to x, using the initial conditions at $x = 0$, we obtain

$$\int_0^b \bigl[2\lambda(p - \tilde{p}) + (q - \tilde{q})\bigr](\varphi\tilde{\varphi})(x, \lambda)\,dx + (h - \tilde{h}) = \tilde{\varphi}(b, \lambda)\varphi'(b, \lambda) - \tilde{\varphi}'(b, \lambda)\varphi(b, \lambda).$$

Define

$$H_1(\lambda) = h - \tilde{h} + 2\lambda \int_0^b (p - \tilde{p})(x)(\varphi\tilde{\varphi})(x, \lambda)\,dx + \int_0^b (q - \tilde{q})(x)(\varphi\tilde{\varphi})(x, \lambda)\,dx. \tag{5.4}$$

From assumption (5.2) we have

$$\tilde{\varphi}(b, \lambda_{m(n)})\varphi'(b, \lambda_{m(n)}) - \tilde{\varphi}'(b, \lambda_{m(n)})\varphi(b, \lambda_{m(n)}) = 0,$$

which means

$$H_1(\lambda_{m(n)}) = 0, \quad n \in \mathbb{Z}. \tag{5.5}$$

Next, we shall show that $H_1(\lambda) \equiv 0$ on the whole λ-plane. From (5.4) and (3.7) one has

$$\bigl|H_1(\lambda)\bigr| \le (C_1 + C_2 r)e^{2br|\sin\theta|} \tag{5.6}$$

for some positive constants C_1 and C_2, where $\lambda = re^{i\theta}$. Moreover, we see that the entire function $H_1(\lambda)$ is a function of exponential type $\le 2b$. Define the indicator of function $H_1(\lambda)$ by

$$h(\theta) = \limsup_{r \to \infty} \frac{\ln|H_1(re^{i\theta})|}{r}. \tag{5.7}$$

One obtains the following estimate from (5.6) and (5.7):

$$h(\theta) \le 2b|\sin\theta|.$$

Let us denote by $n(r)$ the number of zeros of $H_1(\lambda)$ in the disk $|\lambda| \le r$. From equation (5.5), the assumption of this lemma, and the known asymptotic expression (2.1) of the eigenvalues λ_n, we have the following estimate for the number of zeros of $H_1(\lambda)$ in the disk $|\lambda| \le r$:

$$n(r) = 1 + 2\bigl[\sigma r\bigl(1 + \epsilon(r)\bigr)\bigr] = 2\sigma r\bigl(1 + \epsilon(r)\bigr).$$

Here $\epsilon(r) \to 0$ for $r \to \infty$ and $[x]$ is the integer part of x. It follows that in the case under consideration

$$\lim_{r\to\infty} \frac{n(r)}{r} = 2\sigma > \frac{4b}{\pi} = \frac{b}{\pi} \int_0^{2\pi} |\sin\theta| \,\mathrm{d}\theta \geq \frac{1}{2\pi} \int_0^{2\pi} h(\theta)\,\mathrm{d}\theta. \tag{5.8}$$

To complete the proof, we have to recall the following theorem [9, p. 173]: The set of zeros of every entire function of the exponential type, not identically zero, satisfy the inequality

$$\liminf_{r\to\infty} \frac{n(r)}{r} \leq \frac{1}{2\pi} \int_0^{2\pi} h(\theta)\,\mathrm{d}\theta. \tag{5.9}$$

Inequalities (5.8) and (5.9) imply that $H_1(\lambda) \equiv 0$ on the whole λ-plane. As already mentioned, if $H_1(\lambda) \equiv 0$, then repeating the last part of the proof of Theorem 2.1 (from (4.9) to (4.18)), we have that the conclusion of this lemma is true.

(ii) To prove that

$$p(x) = \tilde{p}(x) \quad \text{on } [b,\pi], \qquad q(x) = \tilde{q}(x) \quad \text{a.e. on } [b,\pi], \qquad H = \tilde{H},$$

we will consider the supplementary problem $\hat{L}$

$$\begin{cases} -y'' + [2\lambda p_1(x) + q_1(x)]y = \lambda^2 y, \quad x \in [0,\pi/2) \cup (\pi/2,\pi], \\ y'(0) - Hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0, \\ y(\pi/2+0) = a^{-1}y(\pi/2-0), \\ y'(\pi/2+0) = ay'(\pi/2-0), \end{cases}$$

where $q_1(x) = q(\pi - x)$ and $p_1(x) = p(\pi - x)$. A direct calculation implies that $\hat{y}_n(x) := y_n(\pi - x)$ is the solution to the supplementary problem $\hat{L}$ and $\hat{y}_n(\pi - b) = y_n(b)$. Note that $\pi - b \in (0,\pi/2)$. Thus, the assumption conditions for $\hat{L}$ in case (i) are still satisfied. Repeating the above arguments, we can obtain the proof of this lemma. □

Now we can give the proof of Theorem 2.3.

Proof of Theorem 2.3 Firstly, let us note that based on the condition

$$\lambda_{r(n)} = \tilde{\lambda}_{r(n)}, \qquad \frac{\varphi'(b,\lambda_{r(n)})}{\varphi(b,\lambda_{r(n)})} = \frac{\tilde{\varphi}'(b,\lambda_{r(n)})}{\tilde{\varphi}(b,\lambda_{r(n)})}, \tag{5.10}$$

it follows from Lemma 5.1 that $p(x) = \tilde{p}(x)$ on $[b,\pi]$, $q(x) = \tilde{q}(x)$ a.e. on $[b,\pi]$, and $H = \tilde{H}$. Thus it needs to be proved that $p(x) = \tilde{p}(x)$ on $[0,b]$, $q(x) = \tilde{q}(x)$ a.e. on $[0,b]$, and $h = \tilde{h}$.

For the case of $b \in (\pi/2,\pi)$, integrating equation (4.3) from 0 to b with respect to x, using the initial conditions at $x = 0$, we obtain

$$\begin{aligned} H_2(\lambda) &:= (h - \tilde{h}) + \int_0^b \big[2\lambda(p - \tilde{p}) + (q - \tilde{q})\big](\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x \\ &= \tilde{\varphi}(b,\lambda)\varphi'(b,\lambda) - \tilde{\varphi}'(b,\lambda)\varphi(b,\lambda) \\ &\quad + \big[\tilde{\varphi}(x,\lambda)\varphi'(x,\lambda) - \tilde{\varphi}'(x,\lambda)\varphi(x,\lambda)\big]\Big|_{\pi/2+0}^{\pi/2-0}. \end{aligned} \tag{5.11}$$

We will finish the remainder of the proof by the following three steps.

Step 1: To prove that $H_2(\lambda) \equiv 0$ for $\lambda \in \mathbb{C}$. Since $H = \tilde{H}$, we get that the eigenfunctions $\varphi(x,\lambda_n)$ and $\tilde{\varphi}(x,\lambda_n)$ satisfy the same boundary condition at $x = \pi$, we have from the conclusion of $p(x) = \tilde{p}(x)$ on $[b,\pi]$, $q(x) = \tilde{q}(x)$ a.e. on $[b,\pi]$ that

$$\varphi(x,\lambda_n) = \gamma_n \tilde{\varphi}(x,\lambda_n) \quad \text{on } [b,\pi], n \in \mathbb{Z}, \tag{5.12}$$

where γ_n are constants. Since the functions $\varphi(x,\lambda_n)$ and $\tilde{\varphi}(x,\lambda_n)$ satisfy the same discontinuous conditions at $x = \pi/2$ (see (1.3)), we infer by a direct calculation that

$$\bigl[\tilde{\varphi}\varphi' - \tilde{\varphi}'\varphi\bigr](\pi/2 + 0,\lambda_n) = \bigl[\tilde{\varphi}\varphi' - \tilde{\varphi}'\varphi\bigr](\pi/2 - 0,\lambda_n).$$

Hence we obtain from (5.11) and (5.12) that

$$H_2(\lambda_n) = 0, \quad n \in \mathbb{Z}. \tag{5.13}$$

Moreover, for the same reason, we also obtain that $H_2(\mu_{l(n)}) = 0$, $n \in \mathbb{Z}$. According to the asymptotic expression for eigenvalues λ_n and μ_n, see (2.1), counting the number of λ_n and μ_n located inside the disc of radius r, we obtain $1 + 2[r + O(1)]$ of λ_n's and $1 + 2[\sigma_1 r + O(1)]$ of μ_n's. Hence the total number of λ_n's and μ_n's in the disc is

$$n(r) = 2 + 2\bigl[r(\sigma_1 + 1) + O(1)\bigr]$$

and

$$\lim_{r\to\infty} \frac{n(r)}{r} = 2(\sigma_1 + 1). \tag{5.14}$$

Repeating the last part of the proof of (i) of Lemma 5.1, with the help of the condition $\sigma_1 > 2b/\pi - 1$, we have that inequality (5.9) does not hold, which means $H_2(\lambda) \equiv 0$ on the whole λ-plane.

Step 2: To obtain the integral equation (5.30). We have from (3.7), (3.8), and (5.11)

$$\begin{aligned}
H_2(\lambda) &= (h - \tilde{h}) + 2\lambda \int_0^b P(x)(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x + \int_0^b Q(x)(\varphi\tilde{\varphi})(x,\lambda)\,\mathrm{d}x \\
&= h - \tilde{h} + \frac{1}{2}\int_0^{\pi/2} Q(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x + \frac{(\alpha^+)^2}{2}\int_{\pi/2}^b Q(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x \\
&\quad + \frac{(\alpha^-)^2}{2}\int_{\pi/2}^b Q(x)\cos\bigl(\theta_2^-(x)\bigr)\,\mathrm{d}x + \lambda\bigl(\alpha^-\bigr)^2\int_{\pi/2}^b P(x)\cos\bigl(\theta_2^-(x)\bigr)\,\mathrm{d}x \\
&\quad + \frac{\alpha^+\alpha^-}{2}\int_{\pi/2}^b Q(x)\bigl[\cos\bigl(\lambda\pi - \beta^+(x) + \tilde{\beta}^-(x)\bigr) + \cos\bigl(\lambda\pi - \tilde{\beta}^+(x) + \beta^-(x)\bigr)\bigr]\,\mathrm{d}x \\
&\quad + \lambda\int_0^{\pi/2} P(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x + \lambda\bigl(\alpha^+\bigr)^2\int_{\pi/2}^b P(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x \\
&\quad + \lambda\alpha^+\alpha^-\int_{\pi/2}^b P(x)\bigl[\cos\bigl(\lambda\pi - \beta^+(x) + \tilde{\beta}^-(x)\bigr) + \cos\bigl(\lambda\pi - \tilde{\beta}^+(x) + \beta^-(x)\bigr)\bigr]\,\mathrm{d}x \\
&\quad + \int_0^{\pi/2}\bigl[f_1(x)e^{2i\lambda x} + f_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+ (\alpha^+)^2 \int_{\pi/2}^{b} \bigl[f_1(x)e^{2i\lambda x} + f_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\
&+ \int_{\pi/2}^{b} \bigl[f_3(x)e^{2i\lambda(\pi-x)} + f_4(x)e^{-2i\lambda(\pi-x)}\bigr]\,\mathrm{d}x \\
&+ \int_{\pi/2}^{b} \bigl[\bigl(f_5(x)+f_7(x)\bigr)e^{2i\lambda(x-\pi/2)} + \bigl(f_6(x)+f_8(x)\bigr)e^{-2i\lambda(x-\pi/2)}\bigr]\,\mathrm{d}x \\
&+ \int_0^b e^{2i\lambda t}\int_t^b Q(x)R_1(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^b e^{-2i\lambda t}\int_t^b Q(x)R_2(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&+ 2\lambda \int_0^{\pi/2} \bigl[g_1(x)e^{2i\lambda x} + g_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\
&+ 2\lambda (\alpha^+)^2 \int_{\pi/2}^{b} \bigl[g_1(x)e^{2i\lambda x} + g_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\
&+ 2\lambda \int_{\pi/2}^{b} \bigl[g_3(x)e^{2i\lambda(\pi-x)} + g_4(x)e^{-2i\lambda(\pi-x)}\bigr]\,\mathrm{d}x \\
&+ 2\lambda \int_{\pi/2}^{b} \bigl[\bigl(g_5(x)+g_7(x)\bigr)e^{2i\lambda(x-\pi/2)} + \bigl(g_6(x)+g_8(x)\bigr)e^{-2i\lambda(x-\pi/2)}\bigr]\,\mathrm{d}x \\
&+ 2\lambda \int_0^b e^{2i\lambda t}\int_t^b P(x)R_1(x,t)\,\mathrm{d}x\,\mathrm{d}t + 2\lambda \int_0^b e^{-2i\lambda t}\int_t^b P(x)R_2(x,t)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where

$$
\begin{cases}
f_1(x) = \frac{Q(x)}{4}e^{-i\theta_1^+(x)}, \\
f_2(x) = \frac{Q(x)}{4}e^{i\theta_1^+(x)}, \\
f_3(x) = \frac{(\alpha^-)^2}{4}Q(x)e^{i\theta_2^+(x)}, \\
f_4(x) = \frac{(\alpha^-)^2}{4}Q(x)e^{-i\theta_2^+(x)}, \\
f_5(x) = \frac{\alpha^+\alpha^-}{4}Q(x)e^{-i(\beta^+(x)+\tilde{\beta}^-(x))}, \\
f_6(x) = \frac{\alpha^+\alpha^-}{4}Q(x)e^{i(\beta^+(x)+\tilde{\beta}^-(x))}, \\
f_7(x) = \frac{\alpha^+\alpha^-}{4}Q(x)e^{-i(\tilde{\beta}^+(x)+\beta^-(x))}, \\
f_8(x) = \frac{\alpha^+\alpha^-}{4}Q(x)e^{i(\tilde{\beta}^+(x)+\beta^-(x))};
\end{cases}
\tag{5.15}
$$

and

$$
\begin{cases}
g_1(x) = \frac{P(x)}{4}e^{-i\theta_1^+(x)}, \\
g_2(x) = \frac{P(x)}{4}e^{i\theta_1^+(x)}, \\
g_3(x) = \frac{(\alpha^-)^2}{4}P(x)e^{i\theta_2^+(x)}, \\
g_4(x) = \frac{(\alpha^-)^2}{4}P(x)e^{-i\theta_2^+(x)}, \\
g_5(x) = \frac{\alpha^+\alpha^-}{4}P(x)e^{-i(\beta^+(x)+\tilde{\beta}^-(x))}, \\
g_6(x) = \frac{\alpha^+\alpha^-}{4}P(x)e^{i(\beta^+(x)+\tilde{\beta}^-(x))}, \\
g_7(x) = \frac{\alpha^+\alpha^-}{4}P(x)e^{-i(\tilde{\beta}^+(x)+\beta^-(x))}, \\
g_8(x) = \frac{\alpha^+\alpha^-}{4}P(x)e^{i(\tilde{\beta}^+(x)+\beta^-(x))}.
\end{cases}
\tag{5.16}
$$

Moreover, from $H_2(\lambda) \equiv 0$ on the whole λ-plane and by use of the Riemann–Lebesgue lemma as $\lambda \to \infty$ for $\lambda \in \mathbb{R}$, we obtain that

$$\begin{aligned}
&\int_0^{\pi/2} P(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x + \bigl(\alpha^+\bigr)^2 \int_{\pi/2}^b P(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x \\
&\quad + \bigl(\alpha^-\bigr)^2 \int_{\pi/2}^b P(x)\cos\bigl(\theta_2^-(x)\bigr)\,\mathrm{d}x = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\pi/2}^b P(x)\bigl[\cos\bigl(\beta^+(x) - \tilde{\beta}^-(x)\bigr) + \cos\bigl(\tilde{\beta}^+(x) - \beta^-(x)\bigr)\bigr]\,\mathrm{d}x = 0, \\
&\int_{\pi/2}^b P(x)\bigl[\sin\bigl(\beta^+(x) - \tilde{\beta}^-(x)\bigr) + \sin\bigl(\tilde{\beta}^+(x) - \beta^-(x)\bigr)\bigr]\,\mathrm{d}x = 0.
\end{aligned}$$

Hence, the function $H_2(\lambda)$ can be rewritten as

$$\begin{aligned}
H_2(\lambda) = {}& h - \tilde{h} + \frac{1}{2}\int_0^{\pi/2} Q(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x + \frac{(\alpha^+)^2}{2}\int_{\pi/2}^b Q(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x \\
&+ \frac{\alpha^+\alpha^-}{2}\int_{\pi/2}^b Q(x)\bigl[\cos\bigl(\lambda\pi - \beta^+(x) + \tilde{\beta}^-(x)\bigr) + \cos\bigl(\lambda\pi - \tilde{\beta}^+(x) + \beta^-(x)\bigr)\bigr]\,\mathrm{d}x \\
&+ \frac{(\alpha^-)^2}{2}\int_{\pi/2}^b Q(x)\cos\bigl(\theta_2^-(x)\bigr)\,\mathrm{d}x + \int_0^{\pi/2}\bigl[f_1(x)e^{2i\lambda x} + f_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\
&+ \bigl(\alpha^+\bigr)^2\int_{\pi/2}^b \bigl[f_1(x)e^{2i\lambda x} + f_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\
&+ \int_{\pi/2}^b \bigl[f_3(x)e^{2i\lambda(\pi-x)} + f_4(x)e^{-2i\lambda(\pi-x)}\bigr]\,\mathrm{d}x \\
&+ \int_{\pi/2}^b \bigl[\bigl(f_5(x) + f_7(x)\bigr)e^{2i\lambda(x-\pi/2)} + \bigl(f_6(x) + f_8(x)\bigr)e^{-2i\lambda(x-\pi/2)}\bigr]\,\mathrm{d}x \\
&+ \int_0^b e^{2i\lambda t}\int_t^b Q(x)R_1(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^b e^{-2i\lambda t}\int_t^b Q(x)R_2(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&+ 2\lambda\int_0^{\pi/2}\bigl[g_1(x)e^{2i\lambda x} + g_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\
&+ 2\lambda\bigl(\alpha^+\bigr)^2\int_{\pi/2}^b \bigl[g_1(x)e^{2i\lambda x} + g_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\
&+ 2\lambda\int_{\pi/2}^b \bigl[g_3(x)e^{2i\lambda(\pi-x)} + g_4(x)e^{-2i\lambda(\pi-x)}\bigr]\,\mathrm{d}x \\
&+ 2\lambda\int_{\pi/2}^b \bigl[\bigl(g_5(x) + g_7(x)\bigr)e^{2i\lambda(x-\pi/2)} + \bigl(g_6(x) + g_8(x)\bigr)e^{-2i\lambda(x-\pi/2)}\bigr]\,\mathrm{d}x \\
&+ 2\lambda\int_0^b e^{2i\lambda t}\int_t^b P(x)R_1(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&+ 2\lambda\int_0^b e^{-2i\lambda t}\int_t^b P(x)R_2(x,t)\,\mathrm{d}x\,\mathrm{d}t. \qquad (5.17)
\end{aligned}$$

Specifically, with variable substitution, we can rewrite the integration of the functions $f_j(x)$ for $j = \overline{3,8}$ and $g_k(x)$ for $k = \overline{3,8}$, such as

$$\begin{cases} \int_{\pi/2}^{b} f_3(x)e^{2i\lambda(\pi-x)}\,\mathrm{d}x = \int_{\pi-b}^{\pi/2} f_3(\pi-x)e^{2i\lambda x}\,\mathrm{d}x; \\ \int_{\pi/2}^{b} f_5(x)e^{2i\lambda(x-\pi/2)}\,\mathrm{d}x = \int_{0}^{b-\pi/2} f_5(x+\frac{\pi}{2})e^{2i\lambda x}\,\mathrm{d}x. \end{cases} \tag{5.18}$$

Thus equations (5.17)–(5.18) imply that

$$\begin{aligned} H_2(\lambda) = {} & h - \tilde{h} + \frac{1}{2}\int_0^{\pi/2} Q(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x + \frac{(\alpha^+)^2}{2}\int_{\pi/2}^{b} Q(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x \\ & + \frac{\alpha^+\alpha^-}{2}\int_{\pi/2}^{b} Q(x)\bigl[\cos\bigl(\lambda\pi - \beta^+(x) + \tilde{\beta}^-(x)\bigr) + \cos\bigl(\lambda\pi - \tilde{\beta}^+(x) + \beta^-(x)\bigr)\bigr]\,\mathrm{d}x \\ & + \frac{(\alpha^-)^2}{2}\int_{\pi/2}^{b} Q(x)\cos\bigl(\theta_2^-(x)\bigr)\,\mathrm{d}x + \int_0^b \bigl[F_1(x)e^{2i\lambda x} + F_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\ & + \int_0^b e^{2i\lambda x}\int_x^b Q(t)R_1(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_0^b e^{-2i\lambda x}\int_x^b Q(t)R_2(t,x)\,\mathrm{d}t\,\mathrm{d}x \\ & + 2\lambda\int_0^b \bigl[G_1(x)e^{2i\lambda x} + G_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\ & + 2\lambda\int_0^b e^{2i\lambda x}\int_x^b P(t)R_1(t,x)\,\mathrm{d}t\,\mathrm{d}x + 2\lambda\int_0^b e^{-2i\lambda x}\int_x^b P(t)R_2(t,x)\,\mathrm{d}t\,\mathrm{d}x, \end{aligned} \tag{5.19}$$

where $F_j(x)$ and $G_j(x)$ for $j = 1,2$ have the following form: If $\pi/2 < b < 3\pi/4$

$$F_1(x) = \begin{cases} f_1(x) + f_5(\pi/2+x) + f_7(\pi/2+x), & x \in [0, b-\pi/2], \\ f_1(x), & x \in [b-\pi/2, \pi-b], \\ f_1(x) + f_3(\pi-x), & x \in [\pi-b, \pi/2], \\ (\alpha^+)^2 f_1(x), & x \in [\pi/2, b]; \end{cases} \tag{5.20}$$

$$F_2(x) = \begin{cases} f_2(x) + f_6(\pi/2+x) + f_8(\pi/2+x), & x \in [0, b-\pi/2], \\ f_2(x), & x \in [b-\pi/2, \pi-b], \\ f_2(x) + f_4(\pi-x), & x \in [\pi-b, \pi/2], \\ (\alpha^+)^2 f_2(x), & x \in [\pi/2, b]; \end{cases} \tag{5.21}$$

$$G_1(x) = \begin{cases} g_1(x) + g_5(\pi/2+x) + g_7(\pi/2+x), & x \in [0, b-\pi/2], \\ g_1(x), & x \in [b-\pi/2, \pi-b], \\ g_1(x) + g_3(\pi-x), & x \in [\pi-b, \pi/2], \\ (\alpha^+)^2 g_1(x), & x \in [\pi/2, b]; \end{cases} \tag{5.22}$$

$$G_2(x) = \begin{cases} g_2(x) + g_6(\pi/2+x) + g_8(\pi/2+x), & x \in [0, b-\pi/2], \\ g_2(x), & x \in [b-\pi/2, \pi-b], \\ g_2(x) + g_4(\pi-x), & x \in [\pi-b, \pi/2], \\ (\alpha^+)^2 g_2(x), & x \in [\pi/2, b]. \end{cases} \tag{5.23}$$

If $3\pi/4 \le b < \pi$

$$F_1(x) = \begin{cases} f_1(x) + f_5(\pi/2 + x) + f_7(\pi/2 + x), & x \in [0, \pi - b], \\ f_1(x) + f_3(\pi - x) + f_5(\pi/2 + x) + f_7(\pi/2 + x), & x \in [\pi - b, b - \pi/2], \\ f_1(x) + f_3(\pi - x), & x \in [b - \pi/2, \pi/2], \\ (\alpha^+)^2 f_1(x), & x \in [\pi/2, b]; \end{cases} \tag{5.24}$$

$$F_2(x) = \begin{cases} f_2(x) + f_6(\pi/2 + x) + f_8(\pi/2 + x), & x \in [0, \pi - b], \\ f_2(x) + f_4(\pi - x) + f_6(\pi/2 + x) + f_8(\pi/2 + x), & x \in [\pi - b, b - \pi/2], \\ f_2(x) + f_4(\pi - x), & x \in [b - \pi/2, \pi/2], \\ (\alpha^+)^2 f_2(x), & x \in [\pi/2, b]; \end{cases} \tag{5.25}$$

$$G_1(x) = \begin{cases} g_1(x) + g_5(\pi/2 + x) + g_7(\pi/2 + x), & x \in [0, \pi - b], \\ g_1(x) + g_3(\pi - x) + g_5(\pi/2 + x) + g_7(\pi/2 + x), & x \in [\pi - b, b - \pi/2], \\ g_1(x) + g_3(\pi - x), & x \in [b - \pi/2, \pi/2], \\ (\alpha^+)^2 g_1(x), & x \in [\pi/2, b]; \end{cases} \tag{5.26}$$

$$G_2(x) = \begin{cases} g_2(x) + g_6(\pi/2 + x) + g_8(\pi/2 + x), & x \in [0, \pi - b], \\ g_2(x) + g_4(\pi - x) + g_6(\pi/2 + x) + g_8(\pi/2 + x)], & x \in [\pi - b, b - \pi/2], \\ g_2(x) + g_4(\pi - x), & x \in [b - \pi/2, \pi/2], \\ (\alpha^+)^2 g_2(x), & x \in [\pi/2, b]. \end{cases} \tag{5.27}$$

Integrating by parts in (5.19), we have

$$\begin{aligned} H_2(\lambda) = {} & h - \tilde{h} + \frac{1}{2}\int_0^{\pi/2} Q(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x + \frac{(\alpha^+)^2}{2}\int_{\pi/2}^{b} Q(x)\cos\bigl(\theta_1^-(x)\bigr)\,\mathrm{d}x \\ & + \frac{\alpha^+\alpha^-}{2}\int_{\pi/2}^{b} Q(x)\bigl[\cos\bigl(\lambda\pi - \beta^+(x) + \tilde{\beta}^-(x)\bigr) + \cos\bigl(\lambda\pi - \tilde{\beta}^+(x) + \beta^-(x)\bigr)\bigr]\,\mathrm{d}x \\ & + \frac{(\alpha^-)^2}{2}\int_{\pi/2}^{b} Q(x)\cos\bigl(\theta_2^-(x)\bigr)\,\mathrm{d}x + \int_0^b \bigl[F_1(x)e^{2i\lambda x} + F_2(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\ & + \int_0^b e^{2i\lambda x}\int_x^b Q(t)R_1(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_0^b e^{-2i\lambda x}\int_x^b Q(t)R_2(t,x)\,\mathrm{d}t\,\mathrm{d}x \\ & + i\bigl[G_1(0) - G_2(0)\bigr] + i\bigl[G_2(b)e^{-2i\lambda b} - G_1(b)e^{2i\lambda b}\bigr] \\ & + i\int_0^b P(x)\bigl[R_1(x,0) - R_2(x,0)\bigr]\,\mathrm{d}x + i\int_0^b \bigl[G_1'(x)e^{2i\lambda x} - G_2'(x)e^{-2i\lambda x}\bigr]\,\mathrm{d}x \\ & + i\int_0^b e^{2i\lambda x}\biggl[\int_x^b P(t)\frac{\partial}{\partial x}R_1(t,x)\,\mathrm{d}t - P(x)R_1(x,x)\biggr]\,\mathrm{d}x \\ & - i\int_0^b e^{-2i\lambda x}\biggl[\int_x^b P(t)\frac{\partial}{\partial x}R_2(t,x)\,\mathrm{d}t - P(x)R_2(x,x)\biggr]\,\mathrm{d}x. \end{aligned}$$

Moreover, from $H_2(\lambda) \equiv 0$ on the whole λ-plane and by use of the Riemann–Lebesgue lemma as $\lambda \to \infty$ for $\lambda \in \mathbb{R}$, we obtain that

$$G_2(b) = G_1(b) = 0, \tag{5.28}$$

and

$$\begin{cases} h - \tilde{h} + i[G_2(0) - G_1(0)] + \frac{1}{2}\int_0^{\pi/2} Q(x)\cos(\theta_1^-(x))\,\mathrm{d}x \\ \quad + i\int_0^b P(x)[R_1(x,0) - R_2(x,0)]\,\mathrm{d}x + \frac{(\alpha^+)^2}{2}\int_{\pi/2}^b Q(x)\cos(\theta_1^-(x))\,\mathrm{d}x \\ \quad + \frac{(\alpha^-)^2}{2}\int_{\pi/2}^b Q(x)\cos(\theta_2^-(x))\,\mathrm{d}x = 0, \\ \int_{\pi/2}^b Q(x)[\cos(\beta^+(x) - \tilde{\beta}^-(x)) + \cos(\tilde{\beta}^+(x) - \beta^-(x))]\,\mathrm{d}x = 0, \\ \int_{\pi/2}^b Q(x)[\sin(\beta^+(x) - \tilde{\beta}^-(x)) + \sin(\tilde{\beta}^+(x) - \beta^-(x))]\,\mathrm{d}x = 0. \end{cases} \tag{5.29}$$

Hence, $H_2(\lambda)$ can be rewritten as

$$\begin{aligned} H_2(\lambda) = {} & \int_0^b \big[F_1(x)e^{2i\lambda x} + F_2(x)e^{-2i\lambda x}\big]\,\mathrm{d}x + \int_0^b e^{2i\lambda x}\int_x^b Q(t)R_1(t,x)\,\mathrm{d}t\,\mathrm{d}x \\ & + \int_0^b e^{-2i\lambda x}\int_x^b Q(t)R_2(t,x)\,\mathrm{d}t\,\mathrm{d}x + i\int_0^b \big[G_1'(x)e^{2i\lambda x} - G_2'(x)e^{-2i\lambda x}\big]\,\mathrm{d}x \\ & + i\int_0^b e^{2i\lambda x}\bigg[\int_x^b P(t)\frac{\partial}{\partial x}R_1(t,x)\,\mathrm{d}t - P(x)R_1(x,x)\bigg]\,\mathrm{d}x \\ & - i\int_0^b e^{-2i\lambda x}\bigg[\int_x^b P(t)\frac{\partial}{\partial x}R_2(t,x)\,\mathrm{d}t - P(x)R_2(x,x)\bigg]\,\mathrm{d}x. \end{aligned} \tag{5.30}$$

Step 3: To prove that

$$F(x) = \big(Q(x), P(x), P'(x)\big)^T = \mathbf{0}, \quad x \in [0,b].$$

Since $H_2(\lambda) = 0$, it follows from (5.30) and the completeness of the vector functions $\{(e^{2i\lambda x}, e^{-2i\lambda x})^T : \lambda \in \mathbb{R}\}$ in $(L^2(0,b))^2$ that, for $x \in [0,b]$,

$$F_1(x) + \int_x^b Q(t)R_1(t,x)\,\mathrm{d}t + iG_1'(x) - iP(x)R_1(x,x) + i\int_x^b P(t)\frac{\partial}{\partial x}R_1(t,x)\,\mathrm{d}t = 0 \tag{5.31}$$

and

$$F_2(x) + \int_x^b Q(t)R_2(t,x)\,\mathrm{d}t - iG_2'(x) + iP(x)R_2(x,x) - i\int_x^b P(t)\frac{\partial}{\partial x}R_2(t,x)\,\mathrm{d}t = 0. \tag{5.32}$$

Specially, according to the definitions of $G_1(x)$ and $G_2(x)$ (see (5.22) and (5.23), or (5.26) and (5.27)), we infer from (5.28) that

$$P(b) = 0. \tag{5.33}$$

The forms of $F_j(x)$ and $G_j(x)$ for $j = 1,2$ will help us to obtain that $F(x) = \mathbf{0}$ on $[0,b]$. We only consider the case $3\pi/4 \le b < \pi$, the other case $\pi/2 < b < 3\pi/4$ can be treated similarly. According to (5.24)–(5.27), we see from (5.31)–(5.32) that, for $x \in [\pi/2, b]$,

$$\begin{aligned} & \big(\alpha^+\big)^2 f_1(x) + \int_x^b Q(t)R_1(t,x)\,\mathrm{d}t + i\big(\alpha^+\big)^2 g_1'(x) - iP(x)R_1(x,x) \\ & \quad + i\int_x^b P(t)\frac{\partial}{\partial x}R_1(t,x)\,\mathrm{d}t = 0 \end{aligned}$$

and

$$\left(\alpha^+\right)^2 f_2(x) + \int_x^b Q(t)R_2(t,x)\,\mathrm{d}t - i\left(\alpha^+\right)^2 g_2'(x) + iP(x)R_2(x,x)$$
$$- i\int_x^b P(t)\frac{\partial}{\partial x}R_2(t,x)\,\mathrm{d}t = 0,$$

which together with (5.15)–(5.16) further give that

$$\begin{cases} (\alpha^+)^2 Q(x)e^{-i\theta_1^+(x)} + [(\alpha^+)^2(\theta_1^+(x))'e^{-i\theta_1^+(x)} - 4iR_1(x,x)]P(x) \\ \quad + i(\alpha^+)^2 P'(x)e^{-i\theta_1^+(x)} + 4\int_x^b Q(t)R_1(t,x)\,\mathrm{d}t + 4i\int_x^b P(t)\frac{\partial}{\partial x}R_1(t,x)\,\mathrm{d}t = 0, \\ (\alpha^+)^2 Q(x)e^{i\theta_1^+(x)} + [(\alpha^+)^2(\theta_1^+(x))'e^{i\theta_1^+(x)} + 4iR_2(x,x)]P(x) \\ \quad - i(\alpha^+)^2 P'(x)e^{i\theta_1^+(x)} + 4\int_x^b Q(t)R_2(t,x)\,\mathrm{d}t - 4i\int_x^b P(t)\frac{\partial}{\partial x}R_2(t,x)\,\mathrm{d}t = 0. \end{cases} \tag{5.34}$$

Define

$$K_1(x) = \begin{pmatrix} (\alpha^+)^2 e^{-i\theta_1^+(x)} & (\alpha^+)^2(\theta_1^+(x))'e^{-i\theta_1^+(x)} - 4iR_1(x,x) & i(\alpha^+)^2 e^{-i\theta_1^+(x)} \\ (\alpha^+)^2 e^{i\theta_1^+(x)} & (\alpha^+)^2(\theta_1^+(x))'e^{i\theta_1^+(x)} + 4iR_2(x,x) & -i(\alpha^+)^2 e^{i\theta_1^+(x)} \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$K_2(t,x) = \begin{pmatrix} 4R_1(t,x) & 4i\frac{\partial}{\partial x}R_1(t,x) & 0 \\ 4R_2(t,x) & -4i\frac{\partial}{\partial x}R_2(t,x) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.35}$$

Equations (5.34) and (5.33) can readily be reduced to a vector form

$$K_1(x)F(x) + \int_x^b K_2(t,x)F(t)\,\mathrm{d}t = 0 \quad \text{for } x \in [\pi/2, b]. \tag{5.36}$$

Because $\det K_1(x) = 2i(\alpha^+)^4 \neq 0$, (5.36) can be rewritten as

$$F(x) + \int_x^b K_1^{-1}(x)K_2(t,x)F(t)\,\mathrm{d}t = 0 \quad \text{for } x \in [\pi/2, b].$$

This is a homogeneous Volterra integral equation and the kernel function is $K_1^{-1}(x)K_2(t,x)$, its solution is identically zero. Thus, we have

$$F(x) = \mathbf{0} \quad \text{a.e. on } x \in [\pi/2, b].$$

When $x \in [b-\pi/2, \pi/2]$, it follows that $\pi - x \in [\pi/2, 3\pi/2 - b]$. Thus $f_3(\pi - x) = f_4(\pi - x) = g_3(\pi - x) = g_4(\pi - x) = 0$ for almost all $x \in [b-\pi/2, \pi/2]$. Based on (5.24)–(5.27) and (5.15)–(5.16), and further $F(x) = \mathbf{0}$ for all $x \in [\pi/2, b]$, we have from (5.31)–(5.32) that

$$K_1(x)F(x) + \int_x^{\pi/2} K_2(t,x)F(t)\,\mathrm{d}t = 0 \quad \text{for } x \in [b - \pi/2, \pi/2], \tag{5.37}$$

where $K_2(t,x)$ is defined as (5.35) and

$$K_1(x) = \begin{pmatrix} e^{-i\theta_1^+(x)} & (\theta_1^+(x))'e^{-i\theta_1^+(x)} - 4iR_1(x,x) & ie^{-i\theta_1^+(x)} \\ e^{i\theta_1^+(x)} & (\theta_1^+(x))'e^{i\theta_1^+(x)} + 4iR_2(x,x) & -ie^{i\theta_1^+(x)} \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.38}$$

Because of $\det K_1(x) = 2i \neq 0$, (5.37) and the kernel function $K_1^{-1}(x)K_2(t,x)$ imply

$$F(x) = \mathbf{0} \quad \text{a.e. on } x \in [b-\pi/2, \pi/2].$$

When $x \in [\pi - b, b - \pi/2]$, it follows that $x + \pi/2, \pi - x \in [3\pi/2 - b, b] \subset [b - \pi/2, b]$. Thus $f_3(\pi - x) = f_4(\pi - x) = 0 = f_j(x + \pi/2)$ and $g_3(\pi - x) = g_4(\pi - x) = 0 = g_j(x + \pi/2)$ for all $x \in [\pi - b, b - \pi/2]$ and $j = \overline{5,8}$. By (5.24)–(5.27) and (5.15)–(5.16), and further $F(x) = \mathbf{0}$ for $x \in [b - \pi/2, b]$, we have also from (5.31)–(5.32) that

$$K_1(x)F(x) + \int_x^{b-\pi/2} K_2(t,x)F(t)\,\mathrm{d}t = 0 \quad \text{for } x \in [\pi - b, b - \pi/2],$$

where $K_1(x)$ is defined as (5.38) and $K_2(t,x)$ is defined as (5.35). The above equation and the kernel function $K_1^{-1}(x)K_2(t,x)$ imply that

$$F(x) = \mathbf{0} \quad \text{a.e. on } x \in [\pi - b, b - \pi/2].$$

When $x \in [0, \pi - b]$, it follows that $x + \pi/2 \in [\pi/2, 3\pi/2 - b]$. Thus $f_j(x + \pi/2) = g_j(x + \pi/2) = 0$ for $x \in [0, \pi - b]$ and $j = \overline{5,8}$. By virtue of (5.24)–(5.27) and (5.15)–(5.16), and further $F(x) = \mathbf{0}$ for $x \in [\pi - b, b]$, we have also from (5.31)–(5.32) that

$$K_1(x)F(x) + \int_x^{\pi-b} K_2(t,x)F(t)\,\mathrm{d}t = 0 \quad \text{for } x \in [0, \pi - b],$$

where $K_1(x)$ is defined as (5.38) and $K_2(t,x)$ is defined as (5.35). The above equation and the kernel function $K_1^{-1}(x)K_2(t,x)$ imply that

$$F(x) = \mathbf{0} \quad \text{a.e. on } x \in [0, \pi - b].$$

Therefore, in the case $3\pi/4 \leq b < \pi$, we have $F(x) = \mathbf{0}$ on $[0,b]$, that is, $p(x) = \tilde{p}(x)$ on $[0,b]$, $q(x) = \tilde{q}(x)$ a.e. on $[0,b]$. This together with (5.29) further implies

$$h = \tilde{h}.$$

Consequently, $p(x) = \tilde{p}(x)$ on $[0,\pi]$, $q(x) = \tilde{q}(x)$ a.e. on $[0,\pi]$, $h = \tilde{h}$, and $H = \tilde{H}$. This completes the proof of the theorem. □

6 Conclusion

Inverse spectral problems consist in recovering operators from their spectral characteristics. The interior spectral data used for reconstructing the differential operators contains the known eigenvalues and some information on eigenfunctions at some interior point in the defined interval. Our research here mainly focuses on the inverse problem for interior spectra data of quadratic pencils with impulse inside the defined interval, which has not been considered before as far as we known. With the help of the known interior data, we prove two uniqueness theorems for the pencils $L(p, q; h, H; a)$, which are the generalization of the known results in [12] and [21].

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