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Ground state solutions of Nehari–Pohožaev type for a kind of nonlinear problem with general nonlinearity and nonlocal convolution term

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Abstract

In this paper, we consider the following nonlinear problem with general nonlinearity and nonlocal convolution term:

$$\begin{cases} -\Delta u + V(x)u + (I_\alpha * |u|^q)|u|^{q-2}u = f(u), & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $\alpha \in (0, 3)$, $q \in [1 + \frac{\alpha}{3}, 3 + \alpha)$, $I_\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$ is the Riesz potential, $V \in \mathcal{C}(\mathbb{R}^3, [0, \infty))$, $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $F(t) = \int_0^t f(s) ds$ satisfies $\lim_{|t| \rightarrow \infty} F(t)/|t|^\sigma = \infty$ with $\sigma = \min\{2, \frac{2\beta+2}{\beta}\}$ where $\beta = \frac{\alpha+2}{2(q-1)}$. By using new analytic techniques and new inequalities, we prove the above system admits a ground state solution under mild assumptions on V and f .

Keywords: Schrödinger–Poisson equation; Nehari–Pohožaev type; Ground state solutions; Convolution nonlinearity

1 Introduction

In this paper we are concerned with the existence of ground state solutions for the nonlinear problem:

$$\begin{cases} -\Delta u + V(x)u + (I_\alpha * |u|^q)|u|^{q-2}u = f(u), & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \quad (1.1)$$

where $\alpha \in (0, 3)$, $q \in [1 + \frac{\alpha}{3}, 3 + \alpha)$, $V \in \mathcal{C}(\mathbb{R}^3, [0, \infty))$, $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $I_\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, 3)$, given by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{with } A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}, x \in \mathbb{R}^3 \setminus \{0\}.$$

When $q = 2$ and $\alpha = 2$, the convolution term of (1.1) can be considered as analogous to the internal potential ϕ in Schrödinger–Poisson system like the following equation:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \quad (1.2)$$

System (1.2) was first introduced in [1] as a model describing solitary waves for the non-linear stationary Schrödinger equation interacting with the electrostatic field. We note that system (1.2) is also called a Schrödinger–Maxwell equation system, for more details on the physical aspects of this problem, we refer to [1] and the references therein.

In recent years, the existence, multiplicity and concentration of nontrivial solutions of (1.2) have been the subject of extensive mathematical studies, for example, [2–8]. Problem (1.2) with $V(x) \equiv 1$ or being radially symmetric, has been widely studied under various conditions on f ; see for example [9–17]. When $V = 1$ and $f(u) = |u|^{p-2}u$, by introducing a new manifold that is defined by a condition which is a combination of the Nehari equation and the Pohožaev equality, Ruiz [16] showed that (1.2) admits a positive radial solution if $3 < p < 6$, but does not have a nontrivial solution for $2 < p \leq 3$. Under the same assumptions, based on Ruiz’s approach in [16], Azzollini and Pomponio [3] obtained the existence of ground state solutions for (1.2) by using a concentration-compactness argument. When $f(u) = |u|^{p-2}u$ and V satisfies the following assumptions:

(V1) $V \in C(\mathbb{R}^3, [0, \infty))$ and $V_\infty := \lim_{|y| \rightarrow \infty} V(y) \geq V(x)$ for $x \in \mathbb{R}^3$;

(V2') $V(x)$ is weakly differentiable, and satisfies $\nabla V(x) \cdot x \in L^\infty(\mathbb{R}^3) \cup L^{3/2}(\mathbb{R}^3)$, and

$$2V(x) + \nabla V(x) \cdot x \geq 0 \quad \text{a.e. } x \in \mathbb{R}^3,$$

Zhao and Zhao [8] established the existence of ground state solutions for (1.2) by using the Jeanjean’s monotonicity trick [18]. In a recent paper [19], Tang and Chen introduced some new tricks to generalize and improve the results in [3, 8, 16] to the more general case where V satisfies (V1) and (V2') and f satisfies the following assumptions:

(F1) $f \in C(\mathbb{R}, \mathbb{R})$, and there exist constants $C > 0$ and $q \in (2, 6)$ such that

$$|f(t)| \leq C(1 + |t|^{q-1}), \quad \forall t \in \mathbb{R};$$

(F2) $f(t) = o(t)$ as $t \rightarrow 0$;

(F3) $\lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|^3} = \infty$, where $F(t) = \int_0^t f(s) ds$;

(F4) $[2f(t)t - 3F(t)]/t^3$ is nondecreasing on $(-\infty, 0) \cup (0, +\infty)$.

There is also other work about ground state solutions for (1.2); we refer to [20, 21]. Motivated by the above work and [22–30], in the present paper, we shall extend the results concerning the existence of ground state solutions for (1.2) in [23] to (1.1). Compared with (1.2), it is more difficult to deal with (1.1) for the reason that $q \in [1 + \frac{\alpha}{3}, 3 + \alpha)$. Because of the changing of q , the competing effect of $(I_\alpha * |u|^q)|u|^{q-2}u$ and $f(u)$ is also changing.

For any $\varepsilon > 0$, it follows from (F1) and (F2) that there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon |t| + C_\varepsilon |t|^{p-1}, \quad \forall t \in \mathbb{R}. \quad (1.3)$$

Under assumptions (V1), (F1), (F2) and (1.3), the functional

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2] dx + \frac{1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx - \int_{\mathbb{R}^3} F(u) dx \quad (1.4)$$

is well defined in $H^1(\mathbb{R}^3)$ and $\Phi \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$, where $F(t) = \int_0^t f(s) ds$. Moreover, for any $u, v \in H^1(\mathbb{R}^3)$,

$$\begin{aligned} \langle \Phi'(u), v \rangle &= \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + V(x)uv] dx \\ &\quad + \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^{q-2} uv dx - \int_{\mathbb{R}^3} f(u)v dx. \end{aligned} \quad (1.5)$$

Hence, the solution of (1.1) are critical points of $\Phi(u)$. A solution is called a ground state solution if its energy is minimal among all nontrivial solutions.

In this paper, let $\beta = \frac{\alpha+2}{2(q-1)}$, and in addition to (V1), (F1) and (F2), we also need to introduce the following assumptions:

(V2) $V \in C^1(\mathbb{R}^3)$, the set $\{x \in \mathbb{R}^3 : |\nabla V(x) \cdot x| \geq \epsilon\}$ has finite Lebesgue measure for every $\epsilon > 0$, and the function $t \mapsto t^2[(2\beta-3)V(tx) - \nabla V(tx) \cdot (tx)]$ is increasing on $(0, +\infty)$ for every $x \in \mathbb{R}^3$;

(V3) $V \in C^1(\mathbb{R}^3)$, $\nabla V(x) \cdot x \in L^\infty(\mathbb{R}^3)$ and there exists $\varrho > 0$ such that

$$2V(x) + \nabla V(x) \cdot x \geq \varrho, \quad \forall x \in \mathbb{R}^3;$$

(V3') $V \in C^1(\mathbb{R}^3)$, $\nabla V(x) \cdot x \in L^\infty(\mathbb{R}^3)$, $2V(x) + \nabla V(x) \cdot x \geq \varrho$, $\forall x \in \mathbb{R}^3$ and there exists $\mu > \frac{2\beta+2}{\beta}$ such that

$$f(t)t - \mu F(t) \geq 0, \quad \forall t \in \mathbb{R};$$

(F5) $\lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|^{\frac{2\beta+2}{\beta}}} = \infty$;

(F6) $[\beta f(t)t - 3F(t)]/|t|^{\frac{\beta+2}{\beta}}$ is nondecreasing on both $(-\infty, 0)$ and $(0, +\infty)$.

To state our results, we define the Nehari–Pohožaev manifold as follows:

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J(u) := \beta \langle \Phi'(u), u \rangle - \mathcal{P}(u) = 0\}, \quad (1.6)$$

where

$$\begin{aligned} \mathcal{P}(u) &= \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u^2 dx + \frac{3+\alpha}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx \\ &\quad - 3 \int_{\mathbb{R}^3} F(u) dx \end{aligned} \quad (1.7)$$

which is associated with the Pohožaev identity $\mathcal{P}(u) = 0$ of (1.1) that can be obtained by the same argument as in [13, 31]. And

$$\begin{aligned} J(u) &= \frac{2\beta-1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [(2\beta-3)V(x) - \nabla V(x) \cdot x] u^2 dx \\ &\quad + \frac{2\beta-1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx - \int_{\mathbb{R}^3} [\beta f(u)u - 3F(u)] dx. \end{aligned} \quad (1.8)$$

Throughout the paper we use the following notations:

- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx, \quad \|u\| = (u, u)^{\frac{1}{2}}, \quad \forall u, v \in H^1(\mathbb{R}^3).$$

- $L^s(\mathbb{R}^3)$ ($1 \leq s < \infty$) denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^3} |u|^s dx)^{1/s}$;
- for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, $u_t(x) := u(tx)$ for $t > 0$;
- for any $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$;
- $C_1, C_2, \dots$ denotes positive constants possibly different in different places.

Next, we state the main results of this paper.

Theorem 1.1 *Assume that V and f satisfy (V1), (V2), (F1), (F2), (F5) and (F6) hold. Then problem (1.1) has a ground state solution $u_0 \in H^1(\mathbb{R}^3)$ such that $\Phi(u_0) = \inf_{\mathcal{M}} \Phi = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t>0} \Phi(t^\beta u_t) > 0$.*

Theorem 1.2 *Assume that V and f satisfy (V1), (V3), (F1), (F2), (F5) and (F6) hold. Then problem (1.1) has a positive ground state solution.*

Theorem 1.3 *Assume that V and f satisfy (V1), (V3'), (F1), (F2), (F5) and (F6) hold. Then problem (1.1) has a positive ground state solution.*

Inspired by [19, 32], we shall prove Theorem 1.1 following this scheme:

Step (i) we verify $\mathcal{M} \neq \emptyset$ and establish the minimax characterization of

$$m = \inf_{\mathcal{M}} \Phi > 0;$$

Step (ii) we prove that m can be obtained;

Step (iii) we show that the minimizer of Φ on $\mathcal{M}$ is a critical point.

Although we mainly follow the procedure of [19, 32], we have to face many new difficulties due to the mutual competing effect between $f(u)$ and $(I_\alpha * |u|^q)|u|^{q-2}u$. More precisely, in Step (i), we first establish a key inequality in Lemma 2.2 by using some properties about nonlinearity term, and also we use Nehari–Pohožaev manifold to remove the influence of the term $(I_\alpha * |u|^q)|u|^{q-2}u$ which needs a kind of computational technique; it is worth mentioning that [23, Proposition 2.7] gives us an excellent numerical property as regards $V(x)$ which is convenient for us to eliminate the potential term $V(x)$ in some inequalities. Then we construct a saddle point structure with respect to the fibre $\{t^\beta u_t : t > 0\} \subset H^1(\mathbb{R}^3)$ for $u \in H^1(\mathbb{R}^3) \setminus \{0\}$; see Lemma 2.6, finally based on these constructions we obtain the minimax characterization of m ; see Lemma 2.8. In Step (ii), we first choose a minimizing sequence $\{u_n\}$ of Φ on $\mathcal{M}$, and show that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, then with the help of the key inequality established in Lemma 2.2 and a concentration-compactness argument, we prove that there exist $\hat{u} \in H^1(\mathbb{R}^3)$ and $\hat{t} > 0$ such that $u_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^3)$ up to translations and extraction of a subsequence, and $\hat{t}^\beta \hat{u}_{\hat{t}} \in \mathcal{M}$ is a minimizer of $\inf_{\mathcal{M}} \Phi$; see Lemma 2.13 and Lemma 2.14. Step (iii) is similar to [19, Lemma 2.10].

Motivated by [8, 19], we use the Jeanjean's monotonicity trick [18] to prove Theorems 1.2 and 1.3, which can help us to construct a bounded (PS) sequence. The difficulty in the proof is to overcome the lack of compactness, and a more careful analysis is needed to consider the relationship between the mountain pass level for Φ and the least energy of the functional associated "limit problem" of (1.1) which is used to recover the compactness; see Lemma 3.4. By using Theorem 1.1 and applying the global compactness lemma

and (V1) and (V3) (or (V3')), we can prove Theorems 1.2 and 1.3; see Lemma 3.5 and Lemma 3.6.

2 Proof of Theorem 1.1

First, by a simple calculation, we establish some key inequalities.

Lemma 2.1 *Assume that (F1) and (F6) hold, Then*

$$\frac{1}{t^3}F(t^\beta\tau) - F(\tau) + \frac{1-t^{2\beta-1}}{2\beta-1}[\beta f(\tau)\tau - 3F(\tau)] \geq 0, \quad \forall t \geq 0, \tau \in \mathbb{R}. \quad (2.1)$$

Proof It is evident that (2.1) holds for $\tau = 0$. For $\tau \neq 0$, let

$$g(t) = \frac{1}{t^3}F(t^\beta\tau) - F(\tau) + \frac{1-t^{2\beta-1}}{2\beta-1}[\beta f(\tau)\tau - 3F(\tau)] \geq 0. \quad (2.2)$$

Then from (F4), one has

$$\begin{aligned} g'(t) &= -\frac{3}{t^4}F(t^\beta\tau) + \frac{\beta}{t^4}f(t^\beta\tau)t^\beta\tau - t^{2\beta-2}[\beta f(\tau)\tau - 3F(\tau)] \\ &= t^{2\beta-2} \left[\frac{\beta f(t^\beta\tau)t^\beta\tau - 3F(t^\beta\tau)}{t^{2\beta+2}} - \beta f(\tau)\tau + 3F(\tau) \right] \\ &\begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1. \end{cases} \end{aligned} \quad (2.3)$$

It follows that $g(t) \geq g(1) = 0$ for $t \geq 0$. This, together with (2.2) implies (2.1) holds. $\square$

Define

$$\begin{aligned} \kappa(x, t) &= 2V(x) - (2\beta - 1)t^{2\beta-3}V(t^{-1}x) + (2\beta - 3)t^{2\beta-1}V(x) \\ &\quad + [1 - t^{2\beta-1}\nabla V(x) \cdot x], \quad \forall x \in \mathbb{R}^3, \forall t > 0. \end{aligned} \quad (2.4)$$

It is easy to check that (V2) implies

$$\kappa(x, t) > 0, \quad \forall x \in \mathbb{R}^3, \forall t \in (0, 1) \cup (1, \infty). \quad (2.5)$$

Lemma 2.2 *Assume that (V1), (V4), (F1), (F2) and (F6) hold, Then*

$$\begin{aligned} \Phi(u) &\geq \Phi(t^\beta u_t) + \frac{1-t^{2\beta-1}}{2\beta-1}J(u) + \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} \kappa(x, t)u^2, \\ &\quad \forall u \in H^1(\mathbb{R}^3), \forall t > 0. \end{aligned} \quad (2.6)$$

Proof Note that

$$\begin{aligned} \Phi(t^\beta u_t) &= \frac{t^{2\beta-1}}{2} \|\nabla u\|_2^2 + \frac{t^{2\beta-3}}{2} \int_{\mathbb{R}^3} V(t^{-1}x)u^2 dx \\ &\quad + \frac{t^{2\beta-1}}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q)|u|^q dx - \frac{1}{t^3} \int_{\mathbb{R}^3} F(t^\beta u) dx. \end{aligned} \quad (2.7)$$

Thus, by (1.4), (1.8), (2.1), (2.5) and (2.7), one has

$$\begin{aligned}
 & \Phi(u) - \Phi(t^\beta u_t) \\
 &= \frac{1-t^{2\beta-1}}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [V(x) - t^{2\beta-3} V(t^{-1}x)] u^2(x) dx \\
 & \quad + \frac{1-t^{2\beta-1}}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx + \int_{\mathbb{R}^3} [t^{-3} F(t^\beta u) - F(u)] dx \\
 &= \frac{1-t^{2\beta-1}}{2\beta-1} \left\{ \frac{2\beta-1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [(2\beta-3)V(x) - \nabla V(x) \cdot x] u^2 dx \right\} \\
 & \quad + \frac{1-t^{2\beta-1}}{2\beta-1} \left\{ \frac{2\beta-1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx - \int_{\mathbb{R}^3} [\beta f(u)u - 3F(u)] dx \right\} \\
 & \quad + \int_{\mathbb{R}^3} \left\{ \frac{1}{t^3} F(t^\beta u) - F(u) + \frac{1-t^{2\beta-1}}{2\beta-1} [\beta f(u)u - 3F(u)] \right\} dx \\
 & \quad + \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} \kappa(x, t) u^2 dx \\
 & \geq \frac{1-t^{2\beta-1}}{2\beta-1} J(u) + \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} \kappa(x, t) u^2 dx. \tag{2.8}
 \end{aligned}$$

This shows that (2.6) holds. $\square$

Remark that (2.6) with $t \rightarrow 0$ implies

$$\Phi(u) \geq \frac{1}{2\beta-1} J(u) + \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} [2V(x) + \nabla V(x) \cdot x] u^2 dx, \quad \forall u \in H^1(\mathbb{R}^3). \tag{2.9}$$

To overcome the lack of compactness of Sobolev space embedding in $\mathbb{R}^3$, we define the following energy functional:

$$\begin{aligned}
 \Phi^\infty(u) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2) dx + \frac{1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx \\
 & \quad - \int_{\mathbb{R}^3} F(u) dx. \tag{2.10}
 \end{aligned}$$

Corresponding to (1.6) and (1.8), we define

$$\mathcal{M}^\infty := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J^\infty(u) = 0\} \tag{2.11}$$

and

$$\begin{aligned}
 J^\infty(u) &:= \frac{2\beta-1}{2} \|\nabla u\|_2^2 + \frac{2\beta-3}{2} V_\infty \|u\|_2^2 + \frac{2\beta-1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx \\
 & \quad - \int_{\mathbb{R}^3} [\beta f(u)u - 3F(u)] dx. \tag{2.12}
 \end{aligned}$$

From Lemma 2.2, we have the following two corollaries.

Corollary 2.3 Assume that (F1), (F2) and (F6) hold. Then

$$\begin{aligned}\Phi^\infty(u) &\geq \Phi^\infty(t^\beta u_t) + \frac{1-t^{2\beta-1}}{2\beta-1} J^\infty(u) \\ &\quad + \frac{(2\beta-1)(1-t^{2\beta-3}) - (2\beta-3)(1-t^{2\beta-1})}{2(2\beta-1)} V_\infty \|u\|_2^2, \\ \forall u &\in H^1(\mathbb{R}^3), \forall t \geq 0.\end{aligned}\quad (2.13)$$

Corollary 2.4 Assume that (V1), (V2), (F1) and (F6) hold. Then for $u \in \mathcal{M}$

$$\Phi(u) = \max_{t>0} \Phi(t^\beta u_t). \quad (2.14)$$

From [23, Proposition 2.7], we can obtain the following lemma.

Lemma 2.5 Assume that (V1) and (V2) hold. Then there exist two constants $\rho_1, \rho_2 > 0$ such that

$$2V(x) + \nabla V(x) \cdot x \geq \rho_1 \quad (2.15)$$

and

$$(2\beta-3)V(x) - \nabla V(x) \cdot x \geq \rho_2. \quad (2.16)$$

Lemma 2.6 Assume that (V1), (V2), (F1), (F2) and (F6) hold. Then, for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u^\beta u_{t_u} \in \mathcal{M}$.

Proof Let $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ be fixed and define a function $\zeta(t) := \Phi(t^\beta u_t)$ on $(0, \infty)$. Clearly, by (1.8) and (2.5), we have

$$\begin{aligned}\zeta'(t) = 0 &\Leftrightarrow \frac{2\beta-1}{2} t^{2\beta-2} \|\nabla u\|_2^2 \\ &\quad + \frac{t^{2\beta-4}}{2} \int_{\mathbb{R}^3} [(2\beta-3)V(t^{-1}x) - \nabla V(t^{-1}x) \cdot (t^{-1}x)] u^2 dx \\ &\quad + \frac{2\beta-1}{2q} t^{2\beta-2} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx \\ &\quad - \int_{\mathbb{R}^3} [\beta f(t^\beta u) u t^{\beta-4} - 3F(t^\beta u) t^{-4}] dx \\ &= 0 \Leftrightarrow J(t^\beta u_t) = 0 \Leftrightarrow t^\beta u_t \in \mathcal{M}.\end{aligned}\quad (2.17)$$

By (V1), (F1) and (F5), we have $\lim_{t \rightarrow 0^+} \zeta'(t) = 0$, $\zeta'(t) > 0$ for $t > 0$ small and $\zeta'(t) < 0$ for t large. Therefore $\max_{t \in [0, +\infty)} \zeta(t)$ is obtained at $t_u > 0$ so that $\zeta'(t_u) = 0$ and $t_u^\beta u_{t_u} \in \mathcal{M}$.

Next we claim that t_u is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. In fact, for any given $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, let $t_1, t_2 > 0$ such that $t_1^\beta u_{t_1}, t_2^\beta u_{t_2} \in \mathcal{M}$. Then $J(t_1^\beta u_{t_1}) = J(t_2^\beta u_{t_2}) = 0$. Jointly with (2.9),

we have

$$\begin{aligned}\Phi(t_1^\beta u_{t_1}) &\geq \Phi(t_2^\beta u_{t_2}) \\ &\quad + \frac{1}{2\beta-1} \frac{t_2^{2\beta-1} - t_1^{2\beta-1}}{t_2^{2\beta-1}} J(t_1^\beta u_{t_1}) + \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} \kappa\left(x, \frac{t_2}{t_1}\right) u^2 dx \\ &\geq \Phi(t_2^\beta u_{t_2}) + \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} \kappa\left(x, \frac{t_2}{t_1}\right) u^2 dx,\end{aligned}\quad (2.18)$$

$$\begin{aligned}\Phi(t_2^\beta u_{t_2}) &\geq \Phi(t_1^\beta u_{t_1}) + \frac{1}{2\beta-1} \frac{t_1^{2\beta-1} - t_2^{2\beta-1}}{t_1^{2\beta-1}} J(t_2^\beta u_{t_2}) \\ &\quad + \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} \kappa\left(x, \frac{t_1}{t_2}\right) u^2 dx \\ &\geq \Phi(t_1^\beta u_{t_1}) + \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} \kappa\left(x, \frac{t_1}{t_2}\right) u^2 dx.\end{aligned}\quad (2.19)$$

Combining (2.18) and (2.19), we have $t_1 = t_2$. Therefore, $t_u > 0$ is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. $\square$

Corollary 2.7 Assume that (F1), (F5) and (F6) hold. Then, for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u^\beta u_{t_u} \in \mathcal{M}^\infty$.

Combining Corollary 2.4 with Lemma 2.6, we have the following lemma.

Lemma 2.8 Assume that (V1), (V2), (F1), (F2), (F5) and (F6) hold. Then

$$\inf_{u \in \mathcal{M}} \Phi(u) = m = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} \Phi(t^\beta u_t).$$

Lemma 2.9 Assume that (F1) and (F2) hold. If $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, then along a subsequence of $\{u_n\}$

$$\lim_{n \rightarrow \infty} \sup_{\varphi \in H^1(\mathbb{R}^3), \|\varphi\| \leq 1} \left| \int_{\mathbb{R}^3} [f(u_n) - f(u_n - \bar{u}) - f(\bar{u})] \varphi dx \right| = 0. \quad (2.20)$$

From [33, Proposition 4.3], we can obtain the following Lemma.

Lemma 2.10 Let $\alpha \in (0, 3)$ and $q \in [1, 3 + \alpha)$ hold. If $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, then along a subsequence of $\{u_n\}$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |(I_\alpha * |u_n|^q) |u_n|^q - (I_\alpha * |u_n - \bar{u}|^q) |u_n - \bar{u}|^q - (I_\alpha * |\bar{u}|^q) |\bar{u}|^q| = 0. \quad (2.21)$$

Lemma 2.11 Assume that (V1), (V2), (F1), (F5) and (F6) hold. Then

- (i) there exists $\rho > 0$ such that $\|u\| \geq \rho$, $\forall u \in \mathcal{M}$;
- (ii) $m = \inf_{\mathcal{M}} \Phi > 0$.

Proof (i) Since $J(u) = 0$, $\forall u \in \mathcal{M}$, by (F1), (F2), (1.8), (2.16) and due to the Sobolev embedding theorem, one has

$$\begin{aligned} \frac{\min\{2\beta-1, \rho_2\}}{2} \|u\|^2 &\leq \frac{2\beta-1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [(2\beta-3)V(x) - \nabla V(x) \cdot x] u^2 dx \\ &\quad + \frac{2\beta-1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx \\ &= \int_{\mathbb{R}^3} [\beta f(u)u - 3F(u)] dx \\ &\leq \frac{\min\{2\beta-1, \rho_2\}}{4} \|u\|^2 + C_1 \|u\|^p, \end{aligned} \quad (2.22)$$

which implies

$$\|u\| \geq \rho := \left(\frac{\min\{2\beta-1, \rho_2\}}{4C_1} \right)^{\frac{1}{p-2}}, \quad \forall u \in \mathcal{M}. \quad (2.23)$$

(ii) Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \rightarrow m$. There are two possible cases:

(1) $\inf_{n \in \mathbb{N}} \|u_n\|_2 > 0$ and

(2) $\inf_{n \in \mathbb{N}} \|u_n\|_2 = 0$.

Case (1) $\inf_{n \in \mathbb{N}} \|u_n\|_2 := \varrho_1 > 0$. In this case, by (2.9) and (2.15), one has

$$m + o(1) = \Phi(u_n) = \Phi(u_n) - \frac{1}{2\beta-1} J(u_n) \geq \frac{\rho_1}{2(2\beta-1)} \varrho_1^2. \quad (2.24)$$

Case (2) $\inf_{n \in \mathbb{N}} \|u_n\|_2 := 0$, by (2.23), passing to a subsequence, we have

$$\|u_n\|_2 \rightarrow 0, \quad \|\nabla u_n\|_2 \geq \frac{1}{2} \rho. \quad (2.25)$$

Note that (F1) implies that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(t)| \leq C_\varepsilon |t|^2 + \varepsilon |t|^6, \quad \forall t \in \mathbb{R}. \quad (2.26)$$

By (2.26) and the Sobolev embedding inequality, we have

$$\int_{\mathbb{R}^3} F(u) dx \leq C_2 \|u\|_2^2 + \frac{1}{2} S^3 \|u\|_6^6 \leq C_2 \|u\|_2^2 + \frac{1}{4} \|\nabla u\|_2^6. \quad (2.27)$$

Let $t_n = \|\nabla u_n\|_2^{-\frac{2}{2\beta-1}}$, then (2.25) implies that $\{t_n\}$ is bounded. Since $J(u_n) = 0$, it follows from (2.6), (2.7), (2.25) and (2.27) that

$$\begin{aligned} m + o(1) &= \Phi(u_n) \geq \Phi(t_n^\beta u_n) \\ &= \frac{t_n^{2\beta-1}}{2} \|\nabla u_n\|_2^2 + \frac{t_n^{2\beta-3}}{2} \int_{\mathbb{R}^3} V(t^{-1}x) u_n^2 dx \\ &\quad + \frac{t_n^{2\beta-1}}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u_n|^q) |u_n|^q dx \\ &\quad - t_n^3 \int_{\mathbb{R}^3} F(t_n^\beta u_n) dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{t_n^{2\beta-1}}{2} \|\nabla u\|_2^2 - \frac{t_n^{6\beta-3}}{4} C_2 \|\nabla u\|_2^6 \\
&= \frac{t_n^{2\beta-1}}{4} \|\nabla u\|_2^2 \left[2 - \left(t_n^{2\beta-1} \|\nabla u\|_2^2 \right)^2 \right] = \frac{1}{4} + o(1).
\end{aligned} \tag{2.28}$$

Case (1) and Case (2) show that $m = \inf_{\mathcal{M}} \Phi > 0$. $\square$

Lemma 2.12 Assume that (V1), (F1) and (F2) hold. If $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, then along a subsequence

$$\begin{aligned}
\Phi(u_n) &= \Phi(\bar{u}) + \Phi(u_n - \bar{u}) + o(1), \quad J(u_n) = J(\bar{u}) + J(u_n - \bar{u}) + o(1), \\
\Phi'(u_n) &= \Phi'(\bar{u}) + \Phi'(u_n - \bar{u}) + o(1), \\
\langle \Phi'(u_n), u_n \rangle &= \langle \Phi'(\bar{u}), \bar{u} \rangle + \langle \Phi'(u_n - \bar{u}), (u_n - \bar{u}) \rangle + o(1).
\end{aligned} \tag{2.29}$$

Lemma 2.13 Assume that (V1), (V2), (F1), (F2), (F5) and (F6) hold. Then $m^\infty := \inf_{\mathcal{M}^\infty} \Phi^\infty \geq m$.

Proof In view of Lemma 2.2 and Corollary 2.4, we have $\mathcal{M}^\infty \neq \emptyset$. Arguing indirectly, we assume that $m > m^\infty$. Let $\varepsilon := m - m^\infty$, then there exists u_ε^∞ such that

$$u_\varepsilon^\infty \in \mathcal{M}^\infty \quad \text{and} \quad m^\infty + \frac{\varepsilon}{2} > \Phi^\infty(u_\varepsilon^\infty). \tag{2.30}$$

In view of Corollary 2.7, there exists $t_\varepsilon > 0$ such that $t_\varepsilon^\beta(u_\varepsilon^\infty)_{t_\varepsilon} \in \mathcal{M}$. Thus, it follows from (V1), (1.4), (2.10), (2.13) and (2.30) that

$$m^\infty + \frac{\varepsilon}{2} > \Phi^\infty(u_\varepsilon^\infty) \geq \Phi^\infty(t_\varepsilon^\beta(u_\varepsilon^\infty)_{t_\varepsilon}) \geq \Phi(t_\varepsilon^\beta(u_\varepsilon^\infty)_{t_\varepsilon}) \geq m. \tag{2.31}$$

This contradiction shows that $m^\infty > m$. $\square$

Lemma 2.14 Assume that (V1), (V2), (F1), (F2), (F5) and (F6) hold. Then m is obtained.

Proof In view of Lemma 2.6 and Lemma 2.11, we have $\mathcal{M} \neq \emptyset$ and $m > 0$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \rightarrow m$. Since $J(u_n) = 0$, it follows from (2.9) and (2.15) that

$$\begin{aligned}
m + o(1) &= \Phi(u_n) = \Phi(u_n) - \frac{1}{2(2\beta-1)} J(u_n) \\
&\geq \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} [2V(x) + \nabla V(x) \cdot x] u_n^2 dx \\
&\geq \frac{\rho_1}{2(2\beta-1)} \|u_n\|_2^2.
\end{aligned} \tag{2.32}$$

This shows that $\|u_n\|_2$ is bounded. Next, we prove that $\|\nabla u_n\|_2$ is also bounded. Arguing by contradiction, suppose that $\|\nabla u_n\|_2 \rightarrow \infty$. By (1.3) and the Sobolev embedding inequality, for $u \in H^1(\mathbb{R}^3)$ one has

$$\int_{\mathbb{R}^3} F(u) dx \leq C_3 \|u\|_2^2 + \frac{1}{2(8m)^2} S^3 \|u\|_6^6 \leq C_3 \|u\|_2^2 + \frac{1}{4(8m)^2} \|\nabla u\|_2^6. \tag{2.33}$$

Let $t_n = (8m/\|\nabla u_n\|_2^2)^{\frac{1}{2\beta-1}}$. Since $J(u_n) = 0$, it follows from (1.4), (2.6) and (2.32) that

$$\begin{aligned} m + o(1) &= \Phi(u_n) \geq \Phi(t_n^\beta(u_n)_{t_n}) \\ &= \frac{t_n^{2\beta-1}}{2} \|\nabla u_n\|_2^2 + \frac{t_n^{2\beta-3}}{2} \int_{\mathbb{R}^3} V(t_n^{-1}x) u_n^2 dx \\ &\quad + \frac{t_n^{2\beta-1}}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u_n|^q) |u_n|^q dx - t_n^{-3} \int_{\mathbb{R}^3} F(t_n^\beta u_n) \\ &\geq \frac{t_n^{2\beta-1}}{2} \|\nabla u_n\|_2^2 - \frac{C_3 t_n^{2\beta}}{t_n^3} \|u_n\|_2^2 - \frac{1}{4(8m)^2} t_n^{6\beta-3} \|\nabla u_n\|_2^6 \\ &= \frac{t_n^{2\beta-1}}{2} \|\nabla u_n\|_2^2 \left[1 - \frac{1}{2} \left(\frac{t_n^{2\beta-1} \|\nabla u_n\|_2^2}{8m} \right)^2 \right] + o(1) \\ &= 2m + o(1). \end{aligned} \quad (2.34)$$

This contradiction shows that $\{\|\nabla u_n\|_2^2\}$ is also bounded, and so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$. Then $u_n \rightarrow \bar{u}$ in $L_{\text{loc}}^s(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n \rightarrow \bar{u}$ a.e. in $\mathbb{R}^3$. There are two possible cases: (i) $\bar{u} = 0$ and (ii) $\bar{u} \neq 0$.

Case (i) $\bar{u} = 0$. i.e. $u_n \rightarrow 0$ in $H^1(\mathbb{R}^3)$. Then $u_n \rightarrow 0$ in $L_{\text{loc}}^s(\mathbb{R}^3)$ for $2 \leq s < 2^*$ and $u_n \rightarrow 0$ a.e. in $\mathbb{R}^3$. Using (V1) and (V2), it is easy to show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} [V_\infty - V(x)] u_n^2 dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \nabla V(x) \cdot x u_n^2 dx = 0 \quad (2.35)$$

From (1.4), (1.8), (2.10), (2.12) and (2.35), one can get

$$\Phi^\infty(u_n) \rightarrow m, \quad J^\infty(u_n) \rightarrow 0. \quad (2.36)$$

Note that (F1) and (F2) imply that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(t)| \leq \varepsilon |t|^2 + C_\varepsilon |t|^p, \quad \forall t \in \mathbb{R}. \quad (2.37)$$

By (1.3), (1.8), (2.10), (2.37) and Lemma 2.11(i), one has

$$\begin{aligned} \frac{\min\{\rho_2, 3\}}{2} \rho^2 &\leq \frac{2\beta-1}{2} \|\nabla u_n\|_2^2 + \int_{\mathbb{R}^3} \left[\frac{2\beta-3}{2} V(x) - \frac{1}{2} \nabla V(x) \cdot x \right] u_n^2 dx \\ &\quad + \frac{2\beta-1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u_n|^q) |u_n|^q dx \\ &= \int_{\mathbb{R}^3} [\beta f(u_n) u_n - 3F(u)] dx \\ &\leq C_4 (\varepsilon |u_n|_2^2 + C_\varepsilon |u_n|_p^p). \end{aligned} \quad (2.38)$$

Using (2.38) and Lion's concentration compactness principle [34, Lemma 1.21], we can prove that there exists $\delta > 0$ and $y_n \in \mathbb{R}^3$, such that $\int_{B_1(y_n)} |u_n|^2 dx > \delta$. Let $\hat{u}_n(x) = u_n(x + y_n)$. Then we have $\|\hat{u}_n\| = \|u_n\|$ and

$$J^\infty(\hat{u}_n) = o(1), \quad \Phi^\infty(\hat{u}_n) \rightarrow m, \quad \int_{B_1(0)} |\hat{u}_n|^2 dx > \delta. \quad (2.39)$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u}, & \text{in } H^1(\mathbb{R}^3); \\ \hat{u}_n \rightarrow \hat{u}, & \text{in } L^s_{loc}(\mathbb{R}^3), \forall s \in [1, 6); \\ \hat{u}_n \rightarrow \hat{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \quad (2.40)$$

Let $w_n = \hat{u}_n - \hat{u}$. then (2.40) and Lemma 2.12 yield

$$\Phi^\infty(\hat{u}_n) = \Phi^\infty(\hat{u}) + \Phi^\infty(w_n) + o(1), \quad J^\infty(\hat{u}_n) = J^\infty(\hat{u}) + J^\infty(w_n) + o(1). \quad (2.41)$$

We define the functional $\Psi_\infty : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ for $u \in H^1(\mathbb{R}^3)$ by

$$\begin{aligned} \Psi^\infty(u) &= \Phi^\infty(u) - \frac{1}{2\beta - 1} J^\infty(u) \\ &= \frac{V_\infty}{2\beta - 1} \|u\|_2^2 + \frac{1}{2(2\beta - 1)} \int_{\mathbb{R}^3} [2\beta f(u)u - (4\beta + 4)F(u)] dx. \end{aligned} \quad (2.42)$$

From (2.5), (2.7), (2.39), (2.41) and (2.42), one has

$$\Psi^\infty(w_n) = m - \Psi^\infty(\hat{u}) + o(1), \quad J^\infty(w_n) = -J^\infty(\hat{u}) + o(1). \quad (2.43)$$

If there exists a subsequence $\{w_{n_i}\}$ of w_n such that $w_{n_i} = 0$, then we have

$$\Phi^\infty(\hat{u}) = m, \quad J^\infty(\hat{u}) = 0. \quad (2.44)$$

Next, we assume that $w_n \neq 0$, we claim that $J^\infty(w_n) \leq 0$. Otherwise, if $J^\infty(\hat{u}) > 0$, then (2.43) implies $J^\infty(w_n) < 0$ for large n . In view of Corollary 2.7, there exists $t_n > 0$ such that $t_n^\beta(w_n)_{t_n} \in \mathcal{M}^\infty$ for large n . From (2.5), (2.12), (2.13), (2.43) and Lemma 2.13, we obtain

$$\begin{aligned} m - \Psi^\infty(\hat{u}) + o(1) &= \Psi^\infty(w_n) \\ &= \Phi^\infty(w_n) - \frac{1}{2\beta - 1} J^\infty(w_n) \\ &\geq \Phi^\infty(t_n^\beta(w_n)_{t_n}) - \frac{t_n^{2\beta-1}}{2\beta - 1} J^\infty(w_n) \\ &\quad + \left[\frac{(2\beta - 1)(1 - t_n^{2\beta-3}) - (2\beta - 3)(1 - t_n^{2\beta-3})V_\infty}{2(2\beta - 1)} \right] \|\nabla w_n\|_2^2 \\ &\geq m^\infty - \frac{t_n^{2\beta-1}}{2\beta - 1} J^\infty(w_n) \\ &\quad + \left[\frac{(2\beta - 1)(1 - t_n^{2\beta-3}) - (2\beta - 3)(1 - t_n^{2\beta-3})V_\infty}{2(2\beta - 1)} \right] \|\nabla w_n\|_2^2 \\ &\geq m, \end{aligned} \quad (2.45)$$

which is a contradiction due to $\Psi^\infty(\hat{u}) > 0$. This shows that $J^\infty(\hat{u}) \leq 0$. In view of Lemma 2.6, there exists $t_\infty > 0$ such that $t_\infty^\beta \hat{u}_{t_\infty} \in \mathcal{M}^\infty$. By (2.5), (2.7), (2.36), (2.39), (2.42), Fatou's

lemma and Lemma 2.13, we have

$$\begin{aligned}
 m &= \lim_{n \rightarrow \infty} \left[\Phi^\infty(\hat{u}_n) - \frac{1}{2\beta-1} J^\infty(\hat{u}_n) \right] \\
 &\geq \Psi^\infty(\hat{u}_n) \geq \Psi^\infty(\hat{u}) = \Phi^\infty(\hat{u}) - \frac{1}{2\beta-1} J^\infty(\hat{u}) \\
 &\geq \Phi^\infty(t_\infty^\beta \hat{u}_{t_\infty}) - \frac{t_\infty^{2\beta-1}}{2\beta-1} J^\infty(\hat{u}) \\
 &\quad + \left[\frac{(2\beta-1)(1-t_\infty^{2\beta-3}) - (2\beta-3)(1-t_\infty^{2\beta-3})V_\infty}{2(2\beta-1)} \right] \|\hat{u}\|_2^2 \\
 &\geq m^\infty - \frac{t_\infty^{2\beta-1}}{2\beta-1} J^\infty(\hat{u}) + \left[\frac{(2\beta-1)(1-t_\infty^{2\beta-3}) - (2\beta-3)(1-t_\infty^{2\beta-3})V_\infty}{2(2\beta-1)} \right] \|\hat{u}\|_2^2 \\
 &\geq m,
 \end{aligned} \tag{2.46}$$

which implies (2.44) holds also. In view of Lemma 2.6, there exists $\hat{t} > 0$ such that $\hat{t}^\beta \hat{u}_{\hat{t}} \in \mathcal{M}$, moreover, it follows from (V1), (1.3), (2.7), (2.44) and Corollary 2.4 that

$$m \leq \Phi(\hat{t}^\beta \hat{u}_{\hat{t}}) \leq \Phi^\infty(\hat{t}^\beta \hat{u}_{\hat{t}}) \leq \Phi^\infty(\hat{u}) = m. \tag{2.47}$$

This shows that m is obtained at $\hat{t}^\beta \hat{u}_{\hat{t}} \in \mathcal{M}$.

Case (ii) $\bar{u} \neq 0$. In this case, analogous to the proof of (2.44), by using Φ and J instead of Φ^∞ and J^∞ , we can deduce that $\Phi(\bar{u}) = m$ and $J(\bar{u}) = 0$. $\square$

In the same way as [19] or [35], we can obtain the following lemma.

Lemma 2.15 *Assume that (V1), (V2), (F1), (F2), (F5) and (F6) hold. If $\bar{u} \in \mathcal{M}$ and $\Phi(\bar{u}) = m$, then $\bar{u}$ is a critical point of Φ .*

Proof of Theorem 1.1 In view of Lemma 2.14 and 2.15, there exists $\bar{u} \in \mathcal{M}$ such that

$$\Phi(\bar{u}) = m = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t>0} \Phi(t^\beta u_t), \quad \Phi'(\bar{u}) = 0. \tag{2.48}$$

This shows that $\bar{u}$ is a ground state solution of (1.1) such that $\Phi(\bar{u}) = m = \inf_{\mathcal{M}} \Phi$. $\square$

3 Proofs of Theorem 1.2 and 1.3

Since we are looking for positive solutions to (1.1), without loss of generality, we suppose that $f(t) = 0$ for $t < 0$ in this section.

To use the Jeanjean's monotonicity trick [18, Theorem 1.1], for $\lambda \in [1/2, 1]$ we introduce two families of $\mathcal{C}^1$ -functions on $H^1(\mathbb{R}^3)$ defined by

$$\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx + \frac{1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx - \lambda \int_{\mathbb{R}^3} F(u) dx \tag{3.1}$$

and

$$\Phi_\lambda^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2) dx + \frac{1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx - \lambda \int_{\mathbb{R}^3} F(u) dx. \tag{3.2}$$

In the same way as [13, 31], we can obtain the following lemma.

Lemma 3.1 Assume that (V1), (V3) (or(V3')), (F1), (F2), (F5) and (F6) hold. Let u be a critical point of Φ_λ in $H^1(\mathbb{R}^3)$, then we have the following Pohožaev type identity:

$$P_\lambda(u) := \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u^2 dx + \frac{3+\alpha}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx - 3\lambda \int_{\mathbb{R}^3} F(u) dx. \quad (3.3)$$

We set $J_\lambda(u) := \beta \langle \Phi'_\lambda(u), u \rangle - P_\lambda(u)$, then, for $\lambda \in [1/2, 1]$,

$$J_\lambda(u) = \frac{2\beta-1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [(2\beta-3)V(x) - \nabla V(x) \cdot x] u^2 dx + \frac{2\beta-1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx - \lambda \int_{\mathbb{R}^3} [\beta f(u)u - 3F(u)] dx. \quad (3.4)$$

Correspondingly, for $\lambda \in [1/2, 1]$ we also let

$$J_\lambda^\infty(u) = \frac{2\beta-1}{2} \|\nabla u\|_2^2 + \frac{(2\beta-3)V_\infty}{2} \|u\|_2^2 + \frac{2\beta-1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u|^q) |u|^q dx - \lambda \int_{\mathbb{R}^3} [\beta f(u)u - 3F(u)] dx. \quad (3.5)$$

Set

$$\mathcal{M}_\lambda^\infty := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J_\lambda^\infty(u) = 0\}, \quad m_\lambda^\infty := \inf_{\mathcal{M}_\lambda^\infty} \Phi_\lambda^\infty. \quad (3.6)$$

By Corollary 2.3, we have the following lemma.

Lemma 3.2 Assume that (F1), (F5) and (F6) hold. Then

$$\begin{aligned} \Phi_\lambda^\infty(u) &\geq \Phi_\lambda^\infty(t^\beta u_t) + \frac{1-t^{2\beta-1}}{2\beta-1} J_\lambda^\infty(u) \\ &\quad + \frac{(2\beta-1)(1-t^{2\beta-3}) - (2\beta-3)(1-t^{2\beta-1})}{2(2\beta-1)} V_\infty \|u\|_2^2, \\ &\quad \forall v \in H^1(\mathbb{R}^3), t > 0. \end{aligned} \quad (3.7)$$

Since $f(t) = 0$ for $t < 0$, from Theorem 1.1, the strong maximum principle and standard arguments, we can deduce that Φ_1^∞ has a minimizer $u_1^\infty > 0$ on $\mathcal{M}_1^\infty$, i.e.

$$u_1^\infty \in \mathcal{M}_1^\infty, \quad (\Phi_1^\infty)'(u_1^\infty) = 0 \quad \text{and} \quad m_1^\infty = \Phi_1^\infty(u_1^\infty). \quad (3.8)$$

Lemma 3.3 Under the assumptions of Theorem 1.2 or Theorem 1.3, we have

(i) There exists $T > 0$ independent of λ such that $\Phi_\lambda(T^\beta(u_1^\infty)_T) < 0$ for all $\lambda \in [1/2, 1]$;

(ii) there exists a positive constant κ_0 independent of λ such that, for all $\lambda \in [1/2, 1]$,

$$c_\lambda = \inf_{\gamma \in \tau} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) \geq \kappa_0 > \max\{\Phi_\lambda(0), \Phi_\lambda(T^\beta(u_1^\infty)_T)\}, \quad (3.9)$$

where $\tau = \{\gamma \in C([0, 1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = T^\beta(u_1^\infty)_T\}$;

(iii) c_λ and m_λ^∞ are non-increasing on $\lambda \in [1/2, 1]$.

The proof of Lemma 3.3 is standard, so we omit it.

Lemma 3.4 Under the assumptions of Theorem 1.2 or Theorem 1.3, there exists $\bar{\lambda} \in [1/2, 1)$ such that $c_\lambda < m_\lambda^\infty$ for $\lambda \in (\bar{\lambda}, 1]$.

Proof It is easy to see that $\Phi_\lambda(t^\beta(u_1^\infty)_t)$ is continuous on $t \in (0, \infty)$. Hence for any $\lambda \in [1/2, 1)$, we can choose $t_\lambda \in (0, T)$ such that $\Phi_\lambda(t_\lambda^\beta(u_1^\infty)_{t_\lambda}) = c_\lambda$. Let $\beta_0 = \inf_{\lambda \in [1/2, 1]} t_\lambda$. If $\beta_0 = 0$, then there exists a sequence $\{\lambda_n\} \subset [1/2, 1]$ such that $\lambda_n \rightarrow \lambda_0 \in [1/2, 1]$ and $t_{\lambda_n} \rightarrow 0$, and so by (3.1) and Lemma 3.3(iii), one has

$$0 < c_1 \leq c_{\lambda_n} \leq \Phi_{\lambda_n}(t_{\lambda_n}^\beta(u_1^\infty)_{t_{\lambda_n}}) = o(1). \quad (3.10)$$

This contradiction shows $\beta_0 > 0$. Thus $0 < \beta_0 \leq t_\lambda < T$ for all $\lambda \in [1/2, 1]$. Let

$$\bar{\lambda} := \max\left\{\frac{1}{2}, 1 - \frac{\beta_0^{2\beta-3} T^3 \min_{\beta_0 \leq s \leq T} \int_{\mathbb{R}^3} [V_\infty - V(sx)] |u_1^\infty|^2 dx}{2 \int_{\mathbb{R}^3} F(T^\beta u_1^\infty) dx}\right\}. \quad (3.11)$$

Then $1/2 \leq \bar{\lambda} < 1$. From (3.1), (3.2), (3.7), (3.11) and Lemma 3.3(iii), we derive

$$\begin{aligned} m_\lambda^\infty &\geq m_1^\infty = \Phi_1^\infty(u_1^\infty) \geq \Phi_1^\infty(t_\lambda^\beta(u_1^\infty)_{t_\lambda}) \\ &= \Phi_\lambda(t_\lambda^\beta(u_1^\infty)_{t_\lambda}) - \frac{(1-\lambda)}{t_\lambda^3} \int_{\mathbb{R}^3} F(t_\lambda^\beta u_1^\infty) dx + \frac{t_\lambda^{2\beta-3}}{2} \int_{\mathbb{R}^3} [V_\infty - V(t_\lambda x)] |u_1^\infty|^2 dx \\ &\geq c_\lambda - \frac{(1-\lambda)}{T^3} \int_{\mathbb{R}^3} F(T^\beta u_1^\infty) dx + \frac{\beta_0^{2\beta-3}}{2} \min_{\beta_0 \leq s \leq T} \int_{\mathbb{R}^3} [V_\infty - V(sx)] |u_1^\infty|^2 dx \\ &> c_\lambda, \quad \forall \lambda \in (\bar{\lambda}, 1]. \end{aligned} \quad (3.12)$$

□

Lemma 3.5 Under the assumptions of Theorem 1.2 or Theorem 1.3, for almost every $\lambda \in (\bar{\lambda}, 1]$, there exists $u_\lambda \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$\Phi'_\lambda(u_\lambda) = 0, \quad \Phi_\lambda(u_\lambda) = c_\lambda. \quad (3.13)$$

Proof In view of the Jeanjean's monotonicity trick [36, Theorem 1.1] and Lemma 3.3, for almost every $\lambda \in [1/2, 1]$, there exists a bounded sequence $\{u_n(\lambda)\} \subset H^1(\mathbb{R}^3)$, for simplicity, we denote it by $\{u_n\}$ instead of $\{u_n(\lambda)\}$ such that

$$\Phi_\lambda(u_n) \rightarrow c_\lambda, \quad \|\Phi'_\lambda(u_n)\| \rightarrow 0. \quad (3.14)$$

Using Lemma 2.12, we can deduce that there exists $u_\lambda \in H^1(\mathbb{R}^3)$, an integer $l \in \mathbb{N} \cup \{0\}$, a sequence $\{y_n^k\} \subset \mathbb{R}^3$ and $w^k \in H^1(\mathbb{R}^3)$ for $1 \leq k \leq l$ such that $u_n \rightharpoonup u_\lambda$ in $H^1(\mathbb{R}^3)$, $\Phi'_\lambda(u_\lambda) = 0$, $(\Phi_\lambda^\infty)'(w^k) = 0$ and $\Phi_\lambda^\infty(w^k) \geq m_\lambda^\infty$ for $1 \leq k \leq l$,

$$\left\| u_n - u_\lambda - \sum_{k=1}^l w^k(\cdot + y_n^k) \right\| \rightarrow 0 \quad \text{and} \quad \Phi_\lambda(u_n) \rightarrow \Phi_\lambda(u_\lambda) + \sum_{k=1}^l \Phi_\lambda^\infty(w^k). \quad (3.15)$$

Since $\Phi'_\lambda(u_\lambda) = 0$, $J_\lambda(u_\lambda) = 0$. It follows from (3.1) and (3.4) that

$$\begin{aligned} \Phi_\lambda(u_\lambda) &= \Phi_\lambda(u_\lambda) - \frac{1}{2\beta-1} J_\lambda(u_\lambda) \\ &= \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} [2V(x) + \nabla V(x) \cdot x] u_\lambda^2 dx \\ &\quad + \frac{\lambda}{2\beta-1} \int_{\mathbb{R}^3} [\beta f(u_\lambda) u_\lambda - (2\beta+2)F(u_\lambda)] dx \geq 0. \end{aligned} \quad (3.16)$$

If $l \neq 0$, then

$$c_\lambda = \lim_{n \rightarrow \infty} \Phi_\lambda(u_n) = \Phi_\lambda(u_\lambda) + \sum_{i=1}^l \Phi_\lambda^\infty(w^i) \geq m_\lambda^\infty, \quad \forall \lambda \in (\bar{\lambda}, 1], \quad (3.17)$$

which is a contradiction by Lemma 3.4. Thus $l = 0$, and (3.15) implies that $u_n \rightarrow u_\lambda$ in $H^1(\mathbb{R}^3)$ and $\Phi_\lambda(u_\lambda) = c_\lambda$ for almost every $\lambda \in (\bar{\lambda}, 1]$. $\square$

Lemma 3.6 *Under the assumptions of Theorem 1.2 or Theorem 1.3, there exists $\bar{u} > 0$ such that*

$$\Phi'(\bar{u}) = 0, \quad \Phi(\bar{u}) = c_1 > 0. \quad (3.18)$$

Proof Under the assumptions of Theorem 1.2 or 1.3, in view of Lemma 3.5, there exists two sequences of $\{\lambda_n\} \subset (\bar{\lambda}, 1]$ and $\{u_{\lambda_n}\} \subset H^1(\mathbb{R}^3)$, denoted $\{u_n\}$, such that

$$\lambda_n \rightarrow 1, \quad \Phi'_{\lambda_n}(u_n) = 0, \quad \Phi_{\lambda_n}(u_n) = c_{\lambda_n}. \quad (3.19)$$

By (3.1), (3.4), (3.19) and Lemma 3.3(iii), one has

$$\begin{aligned} c_{1/2} \geq c_{\lambda_n} &= \Phi_{\lambda_n}(u_n) - \frac{1}{2\beta-1} J_{\lambda_n}(u_n) \\ &= \frac{1}{2(2\beta-1)} \int_{\mathbb{R}^3} [2V(x) + \nabla V(x) \cdot x] u_n^2 dx \\ &\quad + \frac{\lambda_n}{2\beta-1} \int_{\mathbb{R}^3} [\beta f(u_n) u_n - (2\beta+2)F(u_n)] dx. \end{aligned} \quad (3.20)$$

To prove the boundedness of $\{\|u_n\|\}$, we distinguish two cases: (1) (V3) holds; (2) (V3') holds.

Case (1) (V3) holds. In this case, by (F6) and (3.20), one has

$$c_{1/2} \geq \frac{\rho_1}{2(2\beta-1)} \|u_n\|_2^2, \quad (3.21)$$

which implies that $\{\|u_n\|_2\}$ is bounded. Next, we prove that $\{\|\nabla u_n\|_2\}$ is also bounded. Arguing by contradiction, suppose that $\|\nabla u_n\|_2 \rightarrow \infty$. By (V1), (V3), (3.21) and Lemma 3.3(iii), one has

$$c_{\lambda_n} + \int_{\mathbb{R}^3} [(2\beta - 2)V_\infty - (2\beta - 2)V(x) - |\nabla V(x) \cdot x|] u_n^2 \leq M_0, \quad (3.22)$$

for some constant M_0 . Let $t_n = \min\{1, 2(M_0/\|\nabla u_n\|_2^2)^{1/2\beta-1}\}$, then $t_n \rightarrow 0$. Thus, it follows from (3.1), (3.2), (3.4), (3.5) and (3.22) that

$$\begin{aligned} \Phi_{\lambda_n}^\infty(t_n^\beta(u_n)_{t_n}) &\leq \Phi_{\lambda_n}^\infty(u_n) - \frac{1 - t_n^{2\beta-1}}{2\beta - 1} J_{\lambda_n}^\infty(u_n) \\ &= \Phi_{\lambda_n}(u_n) + \frac{1}{2} \int_{\mathbb{R}^3} [V_\infty - V(x)] u_n^2 dx \\ &\quad - \frac{1 - t_n^{2\beta-1}}{2\beta - 1} \left\{ J_{\lambda_n}(u_n) + \frac{1}{2} \int_{\mathbb{R}^3} [(2\beta - 3)V_\infty \right. \\ &\quad \left. - (2\beta - 3)V(x) - |\nabla V(x) \cdot x|] u_n^2 dx \right\} \\ &\leq c_{\lambda_n} + \int_{\mathbb{R}^3} [(2\beta - 2)V_\infty - (2\beta - 2)V(x) - |\nabla V(x) \cdot x|] u_n^2 \leq M_0. \end{aligned} \quad (3.23)$$

Analogous to the proof of (2.34), we can deduce a contradiction by using (3.23). Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ under the assumptions of Theorem 1.2.

Case (2) (V3') holds. In this case, (V3') and (3.20) imply

$$\begin{aligned} c_{1/2} &\geq \frac{\lambda_n}{2\beta - 1} \int_{\mathbb{R}^3} [\beta f(u_n)u_n - (2\beta + 2)F(u_n)] dx \\ &\geq C_5 \left(\mu - \frac{2\beta + 2}{\beta} \right) \int_{\mathbb{R}^3} F(u_n) dx. \end{aligned} \quad (3.24)$$

Then it follows from (V1), (3.1), and (3.24) that

$$\begin{aligned} \frac{\gamma_0}{2} \|u_n\|^2 &\leq \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u_n|^2 + V(x)u_n^2] dx + \frac{1}{2q} \int_{\mathbb{R}^3} (I_\alpha * |u_n|^q) |u_n|^q dx \\ &= \lambda \int_{\mathbb{R}^3} F(u_n) dx \leq C_6, \end{aligned} \quad (3.25)$$

where γ_0 is a positive constant. Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ under the assumptions of Theorem 1.3. Similar to the proof of Lemma 3.5, there exists $\bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that (3.18) holds. Moreover, by the strong maximum principle and a standard argument, we can conclude that $\bar{u} > 0$. $\square$

Proofs of Theorem 1.2 and Theorem 1.3. Let

$$K := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \Phi'(u) = 0\}, \quad \hat{m} := \inf_{u \in K} \Phi(u). \quad (3.26)$$

Then Lemma 3.6 shows that $K \neq \emptyset$ and $\hat{m} \leq c_1$. For any $u \in K$, (1.8), (3.4) and Lemma 3.1 imply $J(u) = J_1(u) = \beta \langle \Phi'(u), u \rangle - \mathcal{P}(u) = 0$. As in (3.16), we have $\Phi(u) = \Phi_1(u) \geq 0$ for any

$u \in K$, and so $\hat{m} \geq 0$. Let $\{u_n\} \subset K$ such that $\Phi'(u_n) = 0$ and $\Phi(u_n) \rightarrow \hat{m}$. In view of Lemma 3.4, $\hat{m} \leq c_1 \leq m_1^\infty$. Similar to the proof of Lemma 3.6, we can deduce that there exists $\hat{u} > 0$ such that $\Phi'(\hat{u}) = 0$ and $\Phi(\hat{u}) = \hat{m}$. This shows that $\hat{u} \in H^1(\mathbb{R}^3)$ is a positive ground state solution of (1.1).

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Authors' contributions

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