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Multiple solutions of fourth-order difference equations with different boundary conditions

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Abstract

In the present paper, a class of fourth-order nonlinear difference equations with Dirichlet boundary conditions or periodic boundary conditions are considered. Based on the invariant sets of descending flow in combination with the mountain pass lemma, we establish a series of sufficient conditions on the existence of multiple solutions for these boundary value problems. In addition, some examples are provided to demonstrate the applicability of our results.

MSC: 39A12; 39A23

Keywords: Multiple solutions; Boundary value problems; Invariant sets of descending flow; Mountain pass lemma

1 Introduction

Given an integer $N > 1$, $[1, N]$ denotes the discrete interval $\{1, 2, \dots, N\}$. Consider the fourth-order nonlinear difference equation

$$\Delta^4 x(n-2) = f(n, x(n)), \quad n \in [1, N], \quad (1.1)$$

with Dirichlet boundary conditions

$$x(-1) = x(0) = 0 = x(N+1) = x(N+2) \quad (1.2)$$

or periodic boundary conditions

$$\Delta^i x(-1) = \Delta^i x(N-1), \quad i = 0, 1, 2, 3. \quad (1.3)$$

Here $f \in C([1, N] \times \mathbf{R}, \mathbf{R})$, $F(n, x) = \int_0^x f(n, s) ds$ and $F(n, 0) = 0$. Δ is the forward difference operator and $\Delta x(n) = x(n+1) - x(n)$, $\Delta^0 x(n) = x(n)$. For $i \geq 1$, $\Delta^i x(n) = \Delta(\Delta^{i-1} x(n))$.

Equation (1.1) can be regarded as a discrete analogue of the continuous versions equation

$$x^{(4)}(t) = f(t, x(t)), \quad t \in \mathbf{R},$$

which is used to describe the stationary states of the deflection of an elastic beam [1]. As to difference equation (1.1), [2] establishes the existence of periodic solutions with mini-

mal period by employing variational techniques and the linking theorem. Using Dancer's global bifurcation theorem, [3] shows the existence and multiplicity of positive solutions of (1.1) in the form of

$$\Delta^4 x(n-2) = \lambda h(n)f(x(n)), \quad n \in [2, N].$$

With the rapid development of the technique of computers and the theory of nonlinear difference equations, difference equations have been widely used to study discrete models in many fields such as finance insurance, computing, electrical circuit analysis, dynamical systems, physical field, and biology; see [4, 5] and the references therein. Importantly, much literature and many monographs deal with problems of the existence and multiplicity of solutions by using various methods, such as critical point theory [6–8], topological degree theory [9], fixed-point index theory [10]. Recently, some literature [11, 12] studied solutions of ϕ_c -Laplacian difference equations. For more research on solutions of difference equations, we can refer to [13–21, 28–31].

To the best of our knowledge, there are few studies on sign-changing solutions of fourth-order difference equations. In 2015, He, Zhou et al. [22] obtained the existence of sign-changing solutions for the following periodic boundary value problem:

$$\begin{cases} -\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = f(n, x(n)), & n \in [1, N], \\ x(0) = x(N), & \Delta x(0) = \Delta x(N), \end{cases}$$

by applying invariant sets of descending flow. Furthermore, [23–25] deal with other second-order nonlinear boundary value problems and achieve sign-changing solutions in a similar way to [22].

Motivated by the above reasons, the aims of this paper are as follows. Based on the invariant sets of descending flow, the mountain pass lemma and variational methods, we establish a series of sufficient conditions on the existence of multiple solutions including positive solutions, negative solutions and sign-changing solutions for Dirichlet boundary value problems (1.1) with (1.2) and periodic boundary value problems (1.1) with (1.3). To demonstrate the applicability of our results, some examples are provided. Here and hereafter, a positive (negative) solution $x(n)$ to (1.1) with (1.2) or (1.1) with (1.3) is a sequence $\{x(n)\}$ that satisfies Eq. (1.1) and the boundary conditions (1.2) or (1.3) with $x(n) > 0$ ($x(n) < 0$) for all $n \in [1, N]$. While $\{x(n)\}$ includes both positive and negative components, we call $x(n)$ is a sign-changing solution.

Next, we give some known results which are critical for the proofs of our main results.

Definition 1.1 ([26]) Let H be a Banach space. The functional $I \in C^1(H, \mathbf{R})$ is said to satisfy the Palais–Smale condition ((PS) condition for short) if any sequence $\{x_n\} \subset H$ satisfying $|I(x_n)| \leq c$ for some $c > 0$ and $I'(x_n) \rightarrow 0$ as $n \rightarrow \infty$ possesses a convergent subsequence.

Definition 1.2 ([26]) Let H be a Banach space, the functional $I \in C^1(H, \mathbf{R})$. If any sequence $\{x_n\} \subset H$ with $I(x_n) \rightarrow c$ for some $c \in \mathbf{R}$ and $(1 + \|x_n\|)\|I'(x_n)\| \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence in H , then we say that I satisfies the Cerami condition ((C) condition for short).

Let B_r denote the open ball in H with radius r and center 0, and ∂B_r be its boundary.

Lemma 1.1 (Mountain pass lemma [1]) *Let H be a real Banach space, and $I \in C^1(H, \mathbf{R})$ satisfies the (PS) condition. If $I(0) = 0$ and the following conditions hold:*

- (i) *there exist constants $r > 0$ and $\rho > 0$ such that $I(x) \geq \rho$ for all $x \in \partial B_r$;*
- (ii) *there exists $x_0 \in H \setminus B_r$ such that $I(x_0) \leq 0$,*

then I has a critical value $c \geq \rho$, and c can be characterized as

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} I(h(s)),$$

here

$$\Gamma = \{h \in C([0, 1], H) | h(0) = 0, h(1) = x_0\}.$$

Lemma 1.2 ([27]) *Let $I \in C^1(H, \mathbf{R})$ be a functional defined on a Hilbert space H which satisfies the (PS) condition and $I'(x) = x - S(x)$ for all $x \in H$. If there are open convex subsets D_1 and D_2 of H satisfying $S(\partial D_1) \subset D_1$, $S(\partial D_2) \subset D_2$ and $D_1 \cap D_2 \neq \emptyset$, and, moreover, there is a path $h: [0, 1] \rightarrow H$ such that*

$$h(0) \in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1,$$

and

$$\inf_{x \in \overline{D_1} \cap \overline{D_2}} I(x) > \sup_{\tau \in [0,1]} I(h(\tau)),$$

then I has at least four critical points, one in $H \setminus (\overline{D_1} \cup \overline{D_2})$, one in $D_1 \setminus \overline{D_2}$, one in $D_1 \cap D_2$, and one in $D_2 \setminus \overline{D_1}$.

Remark 1.1 Theorem 5.1 in [26] tells us that the (PS) condition can be substituted by the weaker (C) condition in Lemma 1.2.

The rest of the paper is organized as follows. In Sect. 2, BVP (1.1) with (1.2) is considered, and a series of sufficient conditions are established to ensure the existence of multiple solutions including positive solutions, negative solutions and sign-changing solutions by variational methods together with invariant sets of descending flow. In a similar way, Sect. 3 achieves some results on BVP (1.1) with (1.3). Finally, three examples to illustrate the applicability of our theoretical results are provided in Sect. 4.

2 Multiple solutions for BVP (1.1) with (1.2)

Given a constant $m \geq 0$, define the inner product of M_1 as

$$\langle x, y \rangle_m = \sum_{n=0}^{N+1} \Delta^2 x(n-1) \Delta^2 y(n-1) + m \sum_{n=1}^N x(n) y(n), \quad \forall x, y \in M_1,$$

where $M_1 = \{x : [-1, N+2] \rightarrow \mathbf{R} \mid x(-1) = x(0) = 0 = x(N+1) = x(N+2)\}$. Then M_1 is an N -dimensional Hilbert space and the induced norm is

$$\|x\|_m = \left(\sum_{n=0}^{N+1} |\Delta^2 x(n-1)|^2 + m \sum_{n=1}^N |x(n)|^2 \right)^{\frac{1}{2}}, \quad \forall x \in M_1.$$

Let H be an N -dimensional Hilbert space with the common inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. It follows that M_1 and H are isomorphic, and the norm $\|\cdot\|_m$ is equivalent to $\|\cdot\|$.

For BVP (1.1) with (1.2), we consider the following functional $I_1 : H \rightarrow \mathbf{R}$:

$$I_1(x) = \frac{1}{2} \sum_{n=0}^{N+1} |\Delta^2 x(n-1)|^2 - \sum_{n=1}^N F(n, x(n)). \quad (2.1)$$

Write

$$A_1 = \begin{pmatrix} 6 & -4 & 1 & \cdots & 0 & 0 & 0 \\ -4 & 6 & -4 & \cdots & 0 & 0 & 0 \\ 1 & -4 & 6 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 6 & -4 & 1 \\ 0 & 0 & 0 & \cdots & -4 & 6 & -4 \\ 0 & 0 & 0 & \cdots & 1 & -4 & 6 \end{pmatrix}_{N \times N},$$

then (2.1) can be rewritten as

$$I_1(x) = \frac{1}{2} (A_1 x, x) - \sum_{n=1}^N F(n, x(n)),$$

where $x = (x(1), x(2), \dots, x(N))^T \in H$.

It is not difficult to verify that A_1 is a positive definite matrix. Let eigenvalues of A_1 be $\lambda_1, \lambda_2, \dots, \lambda_N$ and let us have $\lambda_j > 0$ ($j = 1, 2, \dots, N$). Without loss of generality, we can assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

Remark 2.1 Obviously, the λ of the linear eigenvalue problem

$$\begin{cases} \Delta^4 x(n-2) = \lambda x(n), & n \in [1, N], \\ x(-1) = x(0) = 0 = x(N+1) = x(N+2), \end{cases} \quad (2.2)$$

corresponding to BVP (1.1) with (1.2), is exactly the eigenvalue of the matrix A_1 .

Now we state the main results of this section.

Theorem 2.1 Assume that:

$$(F_1) \quad \max_{n \in [1, N]} \limsup_{x \rightarrow 0} \left| \frac{f(n, x)}{x} \right| < \lambda_1;$$

(F₂) $\lim_{|x| \rightarrow \infty} \frac{f(n,x)}{x} = l$ where either constant $l > \lambda_2$ or $l = +\infty$ with $|f(n,x)| \leq C(1 + |x|^{s-1})$ for all $n \in [1, N]$ and some $s > 2$, $C > 0$;

(F₃) $\lim_{|x| \rightarrow \infty} [xf(n,x) - 2F(n,x)] = \infty$, $\forall n \in [1, N]$.

Then we have

- (i) if (F₁)–(F₂) are satisfied and $l \in (\lambda_2, +\infty]$ is not an eigenvalue of (2.2), then BVP (1.1) with (1.2) has at least three nontrivial solutions: one is sign-changing, one is positive and one is negative;
- (ii) if (F₁)–(F₃) are satisfied, the conclusion of (i) is true even if l is an eigenvalue of (2.2).

Theorem 2.2 If $\liminf_{|x| \rightarrow \infty} \frac{f(n,x)}{x} > \lambda_1$ and $\limsup_{x \rightarrow 0} \frac{f(n,x)}{x} < \lambda_1$, then BVP (1.1) with (1.2) possesses at least a positive solution and a negative solution. That is to say,

- (i) when $\liminf_{x \rightarrow -\infty} \frac{f(n,x)}{x} > \lambda_1$ and $\limsup_{x \rightarrow 0^-} \frac{f(n,x)}{x} < \lambda_1$, BVP (1.1) with (1.2) admits at least a negative solution;
- (ii) when $\liminf_{x \rightarrow +\infty} \frac{f(n,x)}{x} > \lambda_1$ and $\limsup_{x \rightarrow 0^+} \frac{f(n,x)}{x} < \lambda_1$, BVP (1.1) with (1.2) admits at least a positive solution.

In the following, we devote ourselves to making preparations to verify our main results. Consider the following problem to obtain Green's function of BVP (1.1) with (1.2):

$$\begin{cases} \Delta^4 x(n-2) - m\Delta^2 x(n-1) = h(n), & n \in [1, N], \\ x(-1) = x(0) = 0 = x(N+1) = x(N+2), \end{cases} \quad (2.3)$$

here $h : [1, N] \rightarrow \mathbf{R}$. Define

$$B_1 = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{N \times N},$$

then (2.3) and the system $(A_1 + mB_1)x = h$ are equivalent. Therefore, the unique solution of (2.3) can be expressed by

$$x = (A_1 + mB_1)^{-1}h. \quad (2.4)$$

Noticing that there are two solutions $r_1 = m$ and $r_2 = 0$ for $X(r) = r^2 - mr = 0$,

$$\Delta^4 x(n-2) - m\Delta^2 x(n-1) = (-\Delta^2 L + r_2)(-\Delta^2 L + r_1)x(n) = (-\Delta^2 L + r_1)(-\Delta^2 L + r_2)x(n),$$

where $x = (x(-1), x(0), \dots, x(N+1), x(N+2))$, $Lx(n) = x(n-1)$, $n \in [1, N]$.

Now we have the following lemma.

Lemma 2.1 For $i \in \{1, 2\}$, the unique solution of BVP

$$\begin{cases} -\Delta^2 x(n-1) + r_i x(n) = h(n), & n \in [1, N], \\ x(0) = 0, & x(N+1) = 0, \end{cases} \quad (2.5)$$

has the form

$$x(n) = \sum_{s=1}^N G_i(n, s)h(s), \quad n \in [0, N+1],$$

with

$$G_1(n, s) = \begin{cases} \frac{(p^s - N - p^{N-s+2})(p^n - p^{-n})}{(1-p^2)(p^{N+1} - p^{-N-1})}, & 0 \leq n \leq s \leq N+1, \\ \frac{(p^n - N - p^{N-n+2})(p^s - p^{-s})}{(1-p^2)(p^{N+1} - p^{-N-1})}, & 0 \leq s \leq n \leq N+1, \end{cases}$$

$$G_2(n, s) = \begin{cases} \frac{n(N+1-s)}{N+1}, & 0 \leq n \leq s \leq N+1, \\ \frac{s(N+1-n)}{N+1}, & 0 \leq s \leq n \leq N+1, \end{cases}$$

and $P = \frac{(2+m) + \sqrt{(2+m)^2 - 4}}{2}$.

Proof (i) When $i = 1$, BVP (2.5) corresponds to

$$\begin{cases} -\Delta^2 x(n-1) + mx(n) = h(n), & n \in [1, N], \\ x(0) = 0, & x(N+1) = 0. \end{cases} \quad (2.6)$$

Consider the homogeneous equation of (2.6),

$$\begin{cases} -\Delta^2 x(n-1) + mx(n) = 0, & n \in [1, N], \\ x(0) = 0, & x(N+1) = 0. \end{cases} \quad (2.7)$$

Then the corresponding characteristic equation to (2.7) is

$$P^2 - (2+m)P + 1 = 0,$$

which means that $P_{1,2} = \frac{(2+m) \pm \sqrt{(2+m)^2 - 4}}{2}$ are two different roots of it. Note $P_1 = \frac{1}{P_2}$, so we can denote

$$P = \frac{(2+m) + \sqrt{(2+m)^2 - 4}}{2} \quad \text{and} \quad P^{-1} = \frac{(2+m) - \sqrt{(2+m)^2 - 4}}{2}.$$

Therefore, the general solution of (2.6) can be expressed by

$$x(n) = c_1(n)P^n + c_2(n)P^{-n}. \quad (2.8)$$

Next, to determine coefficients $c_1(n)$ and $c_2(n)$ in (2.8), replacing $x(n)$ in (2.6) with (2.8), we get

$$\begin{cases} \Delta c_1(n-1)P^n + \Delta c_2(n-1)P^{-n} = 0, \\ \Delta c_1(n-1)P^{n+1} + \Delta c_2(n-1)P^{-n-1} = -h(n). \end{cases}$$

Direct computation yields

$$c_1(n) = c_1(0) + \sum_{s=1}^n \frac{h(s)P^{-s+1}}{1-P^2}, \quad c_2(n) = c_2(0) + \sum_{s=1}^n \frac{-h(s)P^{s+1}}{1-P^2}.$$

Thus (2.8) can be rewritten in the form of

$$x(n) = c_1(0)P^n + c_2(0)P^{-n} + \sum_{s=1}^n \frac{P^{-s+1+n} - P^{s+1-n}}{1-P^2} h(s).$$

Applying the boundary conditions $x(0) = 0$ and $x(N+1) = 0$, we have

$$c_2(0) = -c_1(0) \quad \text{and} \quad c_1(0) = \sum_{s=1}^N \frac{P^{s-N} - P^{N-s+2}}{(1-P^2)(P^{N+1} - P^{-N-1})} h(s),$$

then

$$x(n) = \sum_{s=1}^N \frac{(P^{s-N} - P^{N-s+2})(P^n - P^{-n})}{(1-P^2)(P^{N+1} - P^{-N-1})} h(s) + \sum_{s=1}^n \frac{P^{n-s+1} - P^{s-n+1}}{1-P^2} h(s).$$

Denote

$$G_1(n, s) = \begin{cases} \frac{(P^{s-N} - P^{N-s+2})(P^n - P^{-n})}{(1-P^2)(P^{N+1} - P^{-N-1})}, & 0 \leq n \leq s \leq N+1, \\ \frac{(P^{n-N} - P^{N-n+2})(P^s - P^{-s})}{(1-P^2)(P^{N+1} - P^{-N-1})}, & 0 \leq s \leq n \leq N+1, \end{cases}$$

hence the unique solution of (2.6) is in the form of

$$x(n) = \sum_{s=1}^N G_1(n, s)h(s), \quad n \in [0, N+1].$$

(ii) When $i = 2$, the general solution of the BVP

$$\begin{cases} -\Delta^2 x(n-1) = h(n), & n \in [1, N], \\ x(0) = 0, & x(N+1) = 0, \end{cases} \quad (2.9)$$

is given by

$$x(n) = c_1(0) + nc_2(0) + \sum_{s=1}^n (s-n)h(s).$$

Employing the boundary conditions, we get

$$x(n) = \sum_{s=1}^N \frac{n(N+1-s)}{N+1} h(s) + \sum_{s=1}^n (s-n)h(s).$$

Then the unique solution of (2.9) can be written as

$$x(n) = \sum_{s=1}^N G_2(n, s)h(s), \quad n \in [0, N+1],$$

where

$$G_2(n, s) = \begin{cases} \frac{n(N+1-s)}{N+1}, & 0 \leq n \leq s \leq N+1, \\ \frac{s(N+1-n)}{N+1}, & 0 \leq s \leq n \leq N+1. \end{cases} \quad \square$$

Remark 2.2 For $i = 1, 2$ and any $n, s \in [1, N]$, it is easy to verify $G_i(n, s) = G_i(s, n) > 0$.

Combining $\Delta^4 x(n-2) - m\Delta^2 x(n-1) = (-\Delta^2 L + r_2)(-\Delta^2 L + r_1)x(n)$ with Lemma 2.1, induces the following.

Lemma 2.2

$$x(n) = \sum_{s=1}^N \left(\sum_{j=1}^N G_1(n, j) G_2(j, s) \right) h(s) = \sum_{s=1}^N \left(\sum_{j=1}^N G_2(n, j) G_1(j, s) \right) h(s), \quad n \in [1, N],$$

is the unique solution of BVP (2.3).

Proof Making use of Lemma 2.1, we see that both

$$\begin{cases} -\Delta^2 x(n-1) + r_1 x(n) = y(n), & n \in [1, N], \\ x(0) = 0, & x(N+1) = 0, \end{cases}$$

and

$$\begin{cases} -\Delta^2 y(n-1) + r_2 y(n) = h(n), & n \in [1, N], \\ y(0) = 0, & y(N+1) = 0, \end{cases}$$

have exactly one solution, namely,

$$x(n) = \sum_{j=1}^N G_1(n, j) y(j), \quad n \in [0, N+1],$$

and

$$y(n) = \sum_{s=1}^N G_2(n, s) h(s), \quad n \in [0, N+1],$$

respectively.

Furthermore,

$$\Delta^4 x(n-2) - m\Delta^2 x(n-1) = (-\Delta^2 L + r_2)(-\Delta^2 L + r_1)x(n).$$

Thus we find that (2.3) has a unique solution,

$$x(n) = \sum_{j=1}^N G_1(n, j) \left(\sum_{s=1}^N G_2(j, s) h(s) \right) = \sum_{s=1}^N \left(\sum_{j=1}^N G_1(n, j) G_2(j, s) \right) h(s), \quad n \in [1, N].$$

Similarly,

$$x(n) = \sum_{s=1}^N \left(\sum_{j=1}^N G_2(n, j) G_1(j, s) \right) h(s), \quad n \in [1, N].$$

This completes the proof of Lemma 2.2. $\square$

Define $f_m, K_m : H \rightarrow H$ as

$$\begin{aligned} (f_m x)(n) &= f(n, x(n)) - m \Delta^2 x(n-1), \\ (K_m x)(n) &= \sum_{s=1}^N \sum_{j=1}^N G_2(n, j) G_1(j, s) x(s). \end{aligned}$$

In view of the definition of K_m , together with (2.4) and Lemma 2.2, we have

$$K_m = (A_1 + mB_1)^{-1}. \quad (2.10)$$

Remark 2.3 Let the completely continuous operator $S_m : H \rightarrow H$ be $S_m = K_m f_m$. Lemma 2.2 implies that $x = \{x(n)\}_{n=-1}^{N+2}$ satisfies BVP (1.1) with (1.2) if and only if $x = \{x(n)\}_{n=1}^N \in H$ satisfies $S_m(x) = x$.

Lemma 2.3 The functional I_1 defined as (2.1) is Fréchet differentiable on H , and the critical points of I_1 are exactly the fixed points of S_m .

Proof For any $x, y \in H$, using Lagrange mean value theorem, there exists $\theta(n) \in (0, 1)$, $n \in [1, N]$ such that

$$\begin{aligned} I_1(x+y) - I_1(x) &= \frac{1}{2} \sum_{n=0}^{N+1} |\Delta^2(x+y)(n-1)|^2 - \sum_{n=1}^N F(n, (x+y)(n)) \\ &\quad - \frac{1}{2} \sum_{n=0}^{N+1} |\Delta^2 x(n-1)|^2 + \sum_{n=1}^N F(n, x(n)) \\ &= \frac{1}{2} \sum_{n=0}^{N+1} |\Delta^2 y(n-1)|^2 + \sum_{n=0}^{N+1} \Delta^2 x(n-1) \Delta^2 y(n-1) - \sum_{n=1}^N f(n, x(n) + \theta(n)y(n))y(n) \\ &= \frac{1}{2} \|y\|_m^2 - \frac{m}{2} \|y\|^2 + \langle x, y \rangle_m - m \sum_{n=1}^N x(n)y(n) - \sum_{n=1}^N f(n, x(n) + \theta(n)y(n))y(n). \end{aligned}$$

Then

$$\begin{aligned} I_1(x+y) - I_1(x) - \langle x, y \rangle_m + \sum_{n=1}^N [f(n, x(n)) + mx(n)]y(n) \\ = \frac{1}{2} \|y\|_m^2 - \frac{m}{2} \|y\|^2 + \sum_{n=1}^N [f(n, x(n)) - f(n, x(n) + \theta(n)y(n))]y(n). \end{aligned}$$

It follows from the continuity of f that

$$\lim_{\|y\|_m \rightarrow 0} [f(n, x(n)) - f(n, x(n) + \theta(n)y(n))] = 0.$$

Therefore,

$$\lim_{\|y\|_m \rightarrow 0} \frac{I_1(x+y) - I_1(x) - \langle x, y \rangle_m + \sum_{n=1}^N [f(n, x(n)) + mx(n)]y(n)}{\|y\|_m} = 0,$$

which signifies that I_1 is Fréchet differentiable on H and

$$\langle I_1'(x), y \rangle_m = \langle x, y \rangle_m - \sum_{n=1}^N [f(n, x(n)) + mx(n)]y(n). \quad (2.11)$$

Next, we only need to prove $\langle x - S_m(x), y \rangle_m = \langle x, y \rangle_m - \sum_{n=1}^N [f(n, x(n)) + mx(n)]y(n)$ to accomplish the proof of Lemma 2.3.

For any $x, y \in H$, the boundary conditions imply that

$$\begin{aligned} \sum_{n=1}^N \Delta^4 x(n-2)y(n) &= \sum_{n=1}^N [\Delta^2 x(n) - 2\Delta^2 x(n-1) + \Delta^2 x(n-2)]y(n) \\ &= \sum_{n=1}^N \Delta^2 x(n)y(n) - 2 \sum_{n=1}^N \Delta^2 x(n-1)y(n) + \sum_{n=1}^N \Delta^2 x(n-2)y(n) \\ &= \sum_{n=1}^N \Delta^2 x(n-1)\Delta^2 y(n-1) + \Delta^2 x(N)\Delta^2 y(N) + \Delta^2 x(-1)\Delta^2 y(-1) \\ &= \sum_{n=0}^{N+1} \Delta^2 x(n-1)\Delta^2 y(n-1), \end{aligned}$$

which implies that

$$\begin{aligned} \langle S_m(x), y \rangle_m &= \sum_{n=0}^{N+1} \Delta^2 (S_m x)(n-1)\Delta^2 y(n-1) + m \sum_{n=1}^N (S_m x)(n)y(n) \\ &= \sum_{n=1}^N \Delta^4 (S_m x)(n-2)y(n) + m \sum_{n=1}^N (S_m x)(n)y(n) \\ &= \sum_{n=1}^N [\Delta^4 (S_m x)(n-2) + m(S_m x)(n)]y(n) \\ &= \sum_{n=1}^N [f(n, x(n)) + mx(n)]y(n). \end{aligned} \quad (2.12)$$

Then

$$\langle x - S_m(x), y \rangle_m = \langle x, y \rangle_m - \sum_{n=1}^N [f(n, x(n)) + mx(n)]y(n).$$

Hence

$$\langle I'_1(x), y \rangle_m = \langle x - S_m(x), y \rangle_m,$$

which implies that $I'_1(x) = x - S_m(x)$ for all $x \in H$. $\square$

Aiming to prove our main results, we introduce the following basic notations and necessary results. Let

$$\Lambda = \{x \in H : x \geq 0\}, \quad -\Lambda = \{x \in H : x \leq 0\}$$

be two convex cones and

$$D_\varepsilon^+ = \{x \in H : \text{dist}_m(x, \Lambda) < \varepsilon\}, \quad D_\varepsilon^- = \{x \in H : \text{dist}_m(x, -\Lambda) < \varepsilon\}$$

denote two sets with $\text{dist}_m(x, \pm\Lambda) = \inf_{y \in \pm\Lambda} \|x - y\|_m$ and $\varepsilon > 0$ is an arbitrary constant. It is easy to find that $D_\varepsilon^\pm$ are open convex subsets on H with $D_\varepsilon^+ \cap D_\varepsilon^- \neq \emptyset$. Furthermore, $H \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$ contains only sign-changing functions.

Lemma 2.4 *Under hypotheses (F_1) and (F_2) , there exists $\varepsilon_0 > 0$ such that*

$$S_m(\partial D_\varepsilon^-) \subset D_\varepsilon^-, \quad S_m(\partial D_\varepsilon^+) \subset D_\varepsilon^+,$$

hold for $\varepsilon \in (0, \varepsilon_0)$. Let $x \in D_\varepsilon^-$ (D_ε^+) be a nontrivial critical point of I_1 , then x is a negative (positive) solution of BVP (1.1) with (1.2).

Proof The proof of the conclusions of D_ε^- is analogous to that of D_ε^+ , here we prove the case of D_ε^- in detail.

Because of (F_1) and (F_2) , we have, for all $x \in \mathbf{R} \setminus \{0\}$ and $n \in [1, N]$,

$$x(f(n, x) + mx) > 0. \quad (2.13)$$

From the definition of $\|\cdot\|_m$, direct calculation leads to

$$\|x\|_m^2 = \sum_{n=0}^{N+1} |\Delta^2 x(n-1)|^2 + m \sum_{n=1}^N |x(n)|^2 = (A_1 x, x) + m \|x\|^2,$$

which means

$$\sqrt{\lambda_1 + m} \|x\| \leq \|x\|_m \leq \sqrt{\lambda_N + m} \|x\|. \quad (2.14)$$

For any $x \in H$, let $x^+ = \max\{x, 0\}$, $x^- = \min\{x, 0\}$ and $y = S_m(x)$, then

$$\|x^+\| = \inf_{\psi \in -\Lambda} \|x - \psi\| \leq \frac{1}{\sqrt{m + \lambda_1}} \inf_{\psi \in -\Lambda} \|x - \psi\|_m = \frac{1}{\sqrt{m + \lambda_1}} \text{dist}_m(x, -\Lambda). \quad (2.15)$$

It follows from (F_1) and (F_2) that there exists constant $\tau > 0$ such that

$$|f(n, x) + mx| \leq (m + \lambda_1 - \tau)|x| + C|x|^{s-1}, \quad \forall (n, x) \in [1, N] \times \mathbf{R}. \quad (2.16)$$

Note that $y = y^+ + y^-$ and $y^- \in -\Lambda$, so $\text{dist}_m(y, -\Lambda) \leq \|y - y^-\|_m = \|y^+\|_m$. In view of the property of operator S_m together with above inequalities, it yields

$$\begin{aligned} & \text{dist}_m(y, -\Lambda) \|y^+\|_m \\ & \leq \langle y^+, y^+ \rangle_m \\ & = \langle S_m(x^+), y^+ \rangle_m \\ & = \sum_{n=1}^N [f(n, x^+(n)) + mx^+(n)] y^+(n) \\ & \leq (m + \lambda_1 - \tau) \|x^+\| \|y^+\| + C \|x^+\|^{s-1} \|y^+\| \\ & \leq \left(\frac{m + \lambda_1 - \tau}{m + \lambda_1} \text{dist}_m(x, -\Lambda) + \frac{C}{\sqrt{(m + \lambda_1)^s}} (\text{dist}_m(x, -\Lambda))^{s-1} \right) \|y^+\|_m, \end{aligned}$$

that is,

$$\text{dist}_m(y, -\Lambda) \leq \frac{m + \lambda_1 - \tau}{m + \lambda_1} \text{dist}_m(x, -\Lambda) + \frac{C}{\sqrt{(m + \lambda_1)^s}} (\text{dist}_m(x, -\Lambda))^{s-1}.$$

Therefore, there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ and $x \in D_\varepsilon^-$,

$$\text{dist}_m(S_m(x), -\Lambda) \leq \frac{2(m + \lambda_1) - \tau}{2(m + \lambda_1)} \text{dist}_m(x, -\Lambda). \quad (2.17)$$

Due to $\frac{2(m+\lambda_1)-\tau}{2(m+\lambda_1)} < 1$, then

$$S_m(\partial D_\varepsilon^-) \subset D_\varepsilon^-.$$

Furthermore, if $x \in D_\varepsilon^-$ is a nontrivial critical point of I_1 , so $I_1'(x) = x - S_m(x) = 0$, $S_m(x) = x$. Equation (2.17) implies that $\text{dist}_m(x, -\Lambda) = 0$, i.e., $x \in -\Lambda \setminus \{0\}$. According to (2.13), we can show $x(n) < 0$ for all $n \in [1, N]$, which indicates that x is a negative solution of BVP (1.1) with (1.2). $\square$

Lemma 2.5 *Under the assumption (F_2) , if either*

- (i) $l = +\infty$ or
- (ii) $l < +\infty$ is not an eigenvalue of the matrix A_1 ,

then the functional I_1 satisfies the (PS) condition.

Proof Let $\{x_k\} \subset H$ be a sequence such that $|I_1(x_k)| \leq K$ for some $K > 0$ and $I_1'(x_k) \rightarrow 0$ as $k \rightarrow \infty$. Since H is an N -dimensional Hilbert space, it is sufficient to show that $\{x_k\}$ is bounded to see that $\{x_k\}$ has a convergent subsequence.

(i) When $l = +\infty$, we can choose $\eta_1 > 0$ such that, for any $(n, x) \in [1, N] \times \mathbf{R}$, $F(n, x) \geq \lambda_N x^2 - \eta_1$. Then

$$I_1(x_k) \leq -\frac{1}{2} \lambda_N \|x_k\|^2 + \eta_1 N, \quad (2.18)$$

so

$$\frac{1}{2} \lambda_N \|x_k\|^2 \leq K + \eta_1 N,$$

that is,

$$\|x_k\|^2 \leq \frac{2(K + \eta_1 N)}{\lambda_N},$$

which indicates that $\{x_k\}$ is bounded.

(ii) When $l < +\infty$ is not an eigenvalue of matrix A_1 , we complete the proof by contradiction. Suppose there exists a subsequence of $\{x_k\}$ (still denoted by $\{x_k\}$) such that $\rho_k = \|x_k\| \rightarrow +\infty$ as $k \rightarrow \infty$. Set $y_k = \frac{x_k}{\rho_k} \in H$, then $\|y_k\| = 1$. The completeness of H shows that there is $y \in H$ satisfying $y_k \rightarrow y$ as $k \rightarrow \infty$. Put

$$\omega_k = \left(\frac{f(1, x_k(1))}{x_k(1)} y_k(1), \dots, \frac{f(N, x_k(N))}{x_k(N)} y_k(N) \right)^\tau.$$

By $\lim_{|x| \rightarrow \infty} \frac{f(n, x)}{x} = l$ and $I'_1(x_k) = x_k - K_m f_m x_k$, we have

$$\frac{I'_1(x_k)}{\rho_k} = y_k - \frac{1}{\rho_k} K_0 f_0 x_k = y_k - K_0 \omega_k \rightarrow y - K_0 l y, \quad k \rightarrow \infty.$$

Noting that $\frac{I'_1(x_k)}{\rho_k} \rightarrow 0$ as $k \rightarrow \infty$, so $y - K_0 l y = 0$. In view of (2.10), we find that l is an eigenvalue of matrix A_1 , which is a contradiction. Therefore, $\{x_k\}$ is bounded. $\square$

In the following we verify that I_1 satisfies the (C) condition under suitable assumptions.

Lemma 2.6 *Under the assumption (F_3) , the functional I_1 satisfies the (C) condition.*

Proof Let $\{x_k\} \subset H$ be a sequence satisfying $I_1(x_k) \rightarrow c$ for some $c \in \mathbf{R}$ and $(1 + \|x_k\|_m) \|I'_1(x_k)\|_m \rightarrow 0$ as $k \rightarrow \infty$. Due to the finite-dimensionality of H , we only need to test that $\{x_k\}$ is bounded.

(i) Assume $\lim_{|x| \rightarrow \infty} [xf(n, x) - 2F(n, x)] = -\infty$. Because of the boundedness, there is a constant $R_1 > 0$ such that

$$-R_1 \leq I_1(x_k) \leq R_1 \quad \text{and} \quad (1 + \|x_k\|_m) \|I'_1(x_k)\|_m \leq R_1.$$

Applying the Cauchy–Schwarz inequality yields

$$\begin{aligned} -3R_1 &\leq 2I_1(x_k) - (1 + \|x_k\|_m) \|I'_1(x_k)\|_m \\ &\leq 2I_1(x_k) - \langle I'_1(x_k), x_k \rangle_m \\ &= \sum_{n=1}^N [x_k(n)f(n, x_k(n)) - 2F(n, x_k(n))]. \end{aligned} \quad (2.19)$$

We affirm that $\{x_k\}$ is bounded. Otherwise, it possesses a subsequence, still denoted by $\{x_k\}$, such that, for some $n_0 \in [1, N]$, $|x_k(n_0)| \rightarrow +\infty$ as $k \rightarrow \infty$. Then

$$x_k(n_0)f(n_0, x_k(n_0)) - 2F(n_0, x_k(n_0)) \rightarrow -\infty, \quad k \rightarrow \infty.$$

Bearing in mind the continuity of f and assuming (i), there exists a constant $R_2 > 0$ such that

$$xf(n, x) - 2F(n, x) \leq R_2, \quad \forall n \in [1, N], x \in \mathbf{R}.$$

Therefore, when $k \rightarrow \infty$,

$$\begin{aligned} & \sum_{n=1}^N [x_k(n)f(n, x_k(n)) - 2F(n, x_k(n))] \\ & \leq x_k(n_0)f(n_0, x_k(n_0)) - 2F(n_0, x_k(n_0)) + (N-1)R_2 \rightarrow -\infty, \end{aligned}$$

which is contrary to (2.19). Our result is proved.

(ii) Assume $\lim_{|x| \rightarrow \infty} [xf(n, x) - 2F(n, x)] = +\infty$. In this case, the proof that I_1 satisfies (C) condition is similar to the above case and we omit it.

In summary, I_1 satisfies (C) condition under (F_3) and the proof of Lemma 2.6 is completed. $\square$

Lemma 2.7 Let z_1, z_2 denote eigenvectors corresponding to eigenvalues λ_1, λ_2 of matrix A_1 , respectively. If $l > \lambda_2$, then

$$\lim_{\|x\|_m \rightarrow +\infty} I_1(x) = -\infty, \quad \forall x \in H_1 = \text{span}\{z_1, z_2\}.$$

Proof (i) When $l = +\infty$. From (2.18), it is easy to see that, for any $x \in H_1$, $I_1(x) \rightarrow -\infty$ as $\|x\|_m \rightarrow +\infty$.

(ii) Suppose $l \in (\lambda_2, +\infty)$. For any $x \in H_1$, x can be written as $x = \varepsilon_1 z_1 + \varepsilon_2 z_2$. Without loss of generality, we assume that z_1 is orthogonal to z_2 , then $\|x\|^2 = \varepsilon_1^2 \|z_1\|^2 + \varepsilon_2^2 \|z_2\|^2$. Choose ϵ such that $0 < \epsilon < \min\{l - \lambda_i\}$, $i = 1, 2$. In view of $\lim_{|x| \rightarrow +\infty} \frac{f(n, x)}{x} = l$, it follows that we can find $\eta_2 > 0$ such that $F(n, x) \geq \frac{l-\epsilon}{2} x^2 - \eta_2$. Hence, for any $x \in H_1$,

$$\begin{aligned} I_1(x) &= \frac{1}{2} (A_1 x, x) - \sum_{n=1}^N F(n, x(n)) \\ &\leq \frac{1}{2} (\lambda_1 \varepsilon_1^2 \|z_1\|^2 + \lambda_2 \varepsilon_2^2 \|z_2\|^2) - \frac{l-\epsilon}{2} \|x\|^2 + \eta_2 N \\ &= \frac{\lambda_1 - l + \epsilon}{2} \varepsilon_1^2 \|z_1\|^2 + \frac{\lambda_2 - l + \epsilon}{2} \varepsilon_2^2 \|z_2\|^2 + \eta_2 N. \end{aligned}$$

Thus $I_1(x) \rightarrow -\infty$ as $\|x\|_m \rightarrow +\infty$ for $\lambda_i - l + \epsilon < 0$, $i = 1, 2$. $\square$

With the above preparations, we are in a position to state the proof of Theorem 2.1.

Proof of Theorem 2.1 According to (2.16), we have

$$F(n, x) + \frac{m}{2} |x|^2 \leq \frac{1}{2} (m + \lambda_1 - \tau) |x|^2 + \frac{C}{s} |x|^s.$$

For $x \in H$ and $s > 2$ defined in (F_2) , there exists $C_1 > 0$ such that

$$|x|_s := \left(\sum_{n=1}^N |x(n)|^s \right)^{\frac{1}{s}} \leq C_1 \min\{\|x\|, \|x\|_m\}, \quad \forall x \in H. \quad (2.20)$$

Then

$$\begin{aligned} I_1(x) &= \frac{1}{2} \|x\|_m^2 - \sum_{n=1}^N \left[F(n, x(n)) + \frac{m}{2} |x(n)|^2 \right] \\ &\geq \frac{1}{2} \|x\|_m^2 - \frac{m + \lambda_1 - \tau}{2} \|x\|^2 - \frac{C}{s} |x|_s^s \\ &\geq \frac{\tau}{2(m + \lambda_1)} \|x\|_m^2 - \frac{CC_1^s}{s} \|x\|_m^s. \end{aligned}$$

Recall that, for all $x \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}$, $\|x^\pm\| \leq \frac{1}{\sqrt{m+\lambda_1}} \text{dist}_m(x, \mp \Lambda) \leq \frac{1}{\sqrt{m+\lambda_1}} \varepsilon_0$. Then we can find a constant $c_0 > -\infty$ such that $\inf_{x \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} I_1(x) = c_0$. Lemma 2.7 indicates that there exists a constant $R > 2\varepsilon_0$ such that $I_1(x) < c_0 - 1$ holds for all $x \in H_1$ and $\|x\|_m = R$. Last but not least, define a path $h : [0, 1] \rightarrow H_1$ as follows:

$$h(s) = R \frac{\cos(\pi s)z_1 + \sin(\pi s)z_2}{\|\cos(\pi s)z_1 + \sin(\pi s)z_2\|_m}.$$

Obviously, $h(s) \in H_1$ and $\|h(s)\|_m = R$ when $s \in [0, 1]$, then $I_1(h(s)) < c_0 - 1$. Further,

$$h(0) = R \frac{z_1}{\|z_1\|_m} \in D_\varepsilon^+ \setminus D_\varepsilon^-, \quad h(1) = -R \frac{z_1}{\|z_1\|_m} \in D_\varepsilon^- \setminus D_\varepsilon^+,$$

$$\inf_{x \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} I_1(x) = c_0 > c_0 - 1 > \sup_{s \in [0, 1]} I_1(h(s)).$$

Applying Lemma 1.2, $I_1(x)$ possesses a critical point in $H \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$ which corresponds to a sign-changing solution of BVP (1.1) with (1.2). Besides, there also exist a critical point in $D_\varepsilon^- \setminus \overline{D_\varepsilon^+}$ and a critical point in $D_\varepsilon^+ \setminus \overline{D_\varepsilon^-}$ corresponding to a negative solution and a positive solution of BVP (1.1) with (1.2), respectively. This completes the proof of (i).

With the aid of Remark 1.1 and Lemma 2.6, the verification for the case (ii) is similar to the case (i) and we leave it for the reader. $\square$

Here and hereafter in this section, we study positive solutions and negative solutions for BVP (1.1) with (1.2) by means of the mountain pass lemma.

Consider the following functionals:

$$I_1^\pm(x) = \frac{1}{2} \langle x, x \rangle_0 - \sum_{n=1}^N F(n, x^\pm(n)), \quad \forall x \in H.$$

Referring to [22], we have the following.

Lemma 2.8 *Under the conditions of Theorem 2.2, $I_1^\pm$ are continuously differentiable. In addition, the critical points of I_1^+ (I_1^-) are just the positive (negative) solutions of BVP (1.1) with (1.2).*

Therefore, we have reduced the problem of looking for positive (negative) solutions of BVP (1.1) with (1.2) to that of seeking nonzero critical points of the functional I_1^+ (I_1^-) on H .

Lemma 2.9 *If $\liminf_{|x| \rightarrow \infty} \frac{f(n,x)}{x} > \lambda_1$ for all $n \in [1, N]$, then $I_1^\pm$ satisfy the (PS) condition.*

Proof Here we test the case of I_1^+ at length, the other case can be proved analogously and is omitted.

Let $\{x_k\} \subset H$ be a sequence such that $\{I_1^+(x_k)\}$ is bounded and $I_1^{+'}(x_k) \rightarrow 0$ as $k \rightarrow \infty$. For simplicity, denote $(f^+x)(n) = f(n, x^+(n))$ for all $n \in [1, N]$. Then

$$\|x_k^-\|_0^2 \leq \langle x_k, x_k^- \rangle_0 \leq \langle x_k - K_0 f^+ x_k, x_k^- \rangle_0 = \langle I_1^{+'}(x_k), x_k^- \rangle_0 = o(1) \|x_k^-\|_0,$$

which implies $x_k^- \rightarrow 0$ as $k \rightarrow \infty$. Next we prove that $\{x_k^+\}$ is bounded. If it is not, there is a subsequence of $\{x_k\}$ (still denoted by $\{x_k\}$) satisfying $\rho_k = \|x_k^+\|_0 \rightarrow +\infty$ as $k \rightarrow \infty$. Set $y_k = \frac{x_k^+}{\rho_k}$, then $\|y_k\|_0 = 1$. Moreover, the completeness of H means that there is $y \in H$ such that $y_k \rightarrow y$ as $k \rightarrow \infty$. Let $z_1 > 0$ denote the eigenvector corresponding to λ_1 , then

$$\begin{aligned} \lambda_1 \sum_{n=1}^N x_k(n) z_1(n) &= \sum_{n=1}^N \Delta^4 x_k(n-2) z_1(n) \\ &= \sum_{n=0}^{N+1} \Delta^2 x_k(n-1) \Delta^2 z_1(n-1) \\ &= \langle x_k, z_1 \rangle_0 \\ &= \langle K_0 f^+ x_k, z_1 \rangle_0 + \langle I_1^{+'}(x_k), z_1 \rangle_0 \\ &= \sum_{n=1}^N f(n, x_k^+(n)) z_1(n) + \langle I_1^{+'}(x_k), z_1 \rangle_0. \end{aligned}$$

Dividing both sides by ρ_k , we obtain

$$\lambda_1 \sum_{n=1}^N y_k(n) z_1(n) = \sum_{n=1}^N \frac{f(n, x_k^+(n))}{x_k^+(n)} y_k(n) z_1(n) + o(1). \quad (2.21)$$

It is well known that, for any $n \in [1, N]$, either $x_k^+(n) \rightarrow +\infty$ or $\{x_k^+(n)\}$ is bounded.

If $x_k^+(n) \rightarrow +\infty$, by $\liminf_{|x| \rightarrow \infty} \frac{f(n,x)}{x} > \lambda_1$ for all $n \in [1, N]$, we obtain

$$\min_{n \in [1, N]} \liminf_{|x| \rightarrow \infty} \frac{f(n,x)}{x} > \lambda_1. \quad (2.22)$$

If $\{x_k^+(n)\}$ is bounded, then $\frac{f(n, x_k^+(n))}{\rho_k} \rightarrow 0$ and $y(n) = 0$. While the definition of y_k gives $\|y_k\|_0 = 1$, $y \neq 0$. Hence, it is not difficult to find an n such that $x_k^+(n) \rightarrow +\infty$ and $y(n) > 0$. Passing to the limit about k in (2.21) and making use of (2.22) yield

$$\lambda_1 \sum_{n=1}^N y(n) z_1(n) = \lim_{k \rightarrow \infty} \left(\sum_{n=1}^N \frac{f(n, x_k^+(n))}{x_k^+(n)} y_k(n) z_1(n) \right) > \lambda_1 \sum_{n=1}^N y(n) z_1(n),$$

which raises a contradiction. Therefore, I_1^+ satisfies (PS) condition. The proof of Lemma 2.9 is completed. $\square$

With the help of Lemma 2.8 and Lemma 2.9, we can prove Theorem 2.2 via Lemma 1.1.

Proof of Theorem 2.2 According to $\max_{n \in [1, N]} \limsup_{x \rightarrow 0} \frac{f(n, x)}{x} < \lambda_1$, we have constants $\xi_1 > 0$ and $r > 0$ such that

$$F(n, x) \leq \frac{\lambda_1 - \xi_1}{2} |x|^2, \quad n \in [1, N], |x| \leq r.$$

For all $x \in \partial B_r$, we have

$$\begin{aligned} I_1^+(x) &= \frac{1}{2} \langle x, x \rangle_0 - \sum_{n=1}^N F(n, x^+(n)) \\ &\geq \frac{1}{2} \|x\|_0^2 - \frac{\lambda_1 - \xi_1}{2} \|x\|^2 \\ &\geq \frac{1}{2} \|x\|_0^2 - \frac{\lambda_1 - \xi_1}{2\lambda_1} \|x\|_0^2 \\ &= \frac{\xi_1 r^2}{2\lambda_1} \triangleq \rho > 0. \end{aligned}$$

From $\min_{n \in [1, N]} \liminf_{|x| \rightarrow \infty} \frac{f(n, x)}{x} > \lambda_1$, we can choose a constant $\xi_2 > 0$ such that

$$\min_{n \in [1, N]} \liminf_{|x| \rightarrow \infty} \frac{f(n, x)}{x} > \lambda_1 + \xi_2.$$

Further, there exists a constant $C' > 0$ such that

$$F(n, x) \geq \frac{\lambda_1 + \xi_2}{2} |x|^2 - C'$$

is true for all $x \in \mathbf{R}$. Hence, for v large enough

$$\begin{aligned} I_1^+(vz_1) &= \frac{1}{2} \langle vz_1, vz_1 \rangle_0 - \sum_{n=1}^N F(n, vz_1^+(n)) \\ &\leq \frac{v^2}{2} \|z_1\|_0^2 - \frac{\lambda_1 + \xi_2}{2} v^2 \|z_1\|^2 + C'N \\ &= \frac{v^2}{2} \|z_1\|_0^2 - \frac{\lambda_1 + \xi_2}{2\lambda_1} v^2 \|z_1\|_0^2 + C'N \\ &= -\frac{\xi_2 v^2}{2\lambda_1} \|z_1\|_0^2 + C'N < 0. \end{aligned}$$

Based on Lemma 2.9 and Lemma 1.1, $I_1^+(x)$ possesses a critical point $x_0 \in H$ such that $I_1^{+'}(x_0) = 0$ and $I_1^+(x_0) \geq \rho > 0$. Thus

$$\|x_0^-\|_0^2 \leq \langle x_0, x_0^- \rangle_0 \leq \langle x_0 - K_0 f^+ x_0, x_0^- \rangle_0 = \langle I_1^{+'}(x_0), x_0^- \rangle_0 = 0,$$

which implies that $x_0^- = 0$ and $x_0 = x_0^+ \geq 0$. If $x_0 = 0$, then $I_1^+(x_0) = 0$, which is contradiction with $I_1^+(x_0) \geq \rho > 0$, so $x_0 > 0$. In view of Lemma 2.8, $x_0 > 0$ is a positive solution of problem (1.1) with (1.2).

By a little modification of the case of I_1^+ , we can obtain a negative solution for the case of I_1^- . $\square$

3 Multiple solutions for BVP (1.1) with (1.3)

In this section, we discuss the multiple solutions for BVP (1.1) with (1.3).

Let $M_2 = \{x : [-1, N+2] \rightarrow \mathbf{R} \mid \Delta^i x(-1) = \Delta^i x(N-1), i = 0, 1, 2, 3\}$ equipped with the inner product

$$\langle x, y \rangle_m = \sum_{n=1}^N [\Delta^2 x(n-1) \Delta^2 y(n-1) + mx(n)y(n)].$$

Then M_2 is an N -dimensional Hilbert space and the induced norm is

$$\|x\|_m = \left(\sum_{n=1}^N [|\Delta^2 x(n-1)|^2 + m|x(n)|^2] \right)^{\frac{1}{2}}.$$

Consider the functional $I_2 : H \rightarrow \mathbf{R}$

$$I_2(x) = \frac{1}{2} \sum_{n=1}^N |\Delta^2 x(n-1)|^2 - \sum_{n=1}^N F(n, x(n)). \quad (3.1)$$

For any $x = (x(1), x(2), \dots, x(N))^T \in H$, (3.1) can be rewritten as

$$I_2(x) = \frac{1}{2} (A_2 x, x) - \sum_{n=1}^N F(n, x(n)), \quad (3.2)$$

where A_2 is an $N \times N$ matrix defined as

$$A_2 = \begin{pmatrix} 6 & -4 & 1 & \cdots & 0 & 1 & -4 \\ -4 & 6 & -4 & \cdots & 0 & 0 & 1 \\ 1 & -4 & 6 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 6 & -4 & 1 \\ 1 & 0 & 0 & \cdots & -4 & 6 & -4 \\ -4 & 1 & 0 & \cdots & 1 & -4 & 6 \end{pmatrix}_{N \times N}.$$

Direct computation shows that $\omega_k = 16 \sin^4 \frac{k\pi}{N}$, $k = 0, 1, \dots, N-1$ are the eigenvalues of A_2 . Then when $k = 1, 2, \dots, N-1$, we have

$$\omega_k = \omega_{N-k}, \quad 0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_{\lfloor \frac{N}{2} \rfloor}.$$

Our main results of this section are as follows.

Theorem 3.1 *Suppose that:*

- (F₄) $\max_{n \in [1, N]} \limsup_{x \rightarrow 0} \left| \frac{f(n, x)}{x} \right| < \omega_1$;
- (F₅) *when* $x \neq 0$, $x(f(n, x) + mx) > 0$;
- (F₆) *there exist constants* $\sigma > 1$, $C_3, C_4 > 0$ *such that* $|f(n, x)| \leq C_3|x|^\sigma + C_4$;
- (F₇) $\lim_{|x| \rightarrow \infty} \frac{f(n, x)}{x} = +\infty$;
- (F₈) *when* $x \neq 0$, $xf(n, x) - 2F(n, x) > 0$ *and* $xf(n, x) - 2F(n, x) \rightarrow \infty$ *as* $|x| \rightarrow \infty$.

Then we have:

- (i) *if (F₄)–(F₇) hold, then BVP (1.1) with (1.3) has at least three nontrivial solutions: one sign-changing, one positive and one negative;*
- (ii) *the conclusion is still true if we replace (F₇) with (F₈) in the case (i).*

Now we pay our attention to verifying Theorem 3.1 with the aid of Lemma 1.2.

Motivated by Sect. 2, we consider following boundary value problem to obtain Green's function corresponding to BVP (1.1) with (1.3):

$$\begin{cases} \Delta^4 x(n-2) - m\Delta^2 x(n-1) = h(n), & n \in [1, N], \\ \Delta^i x(-1) = \Delta^i x(N-1), & i = 0, 1, 2, 3, \end{cases} \quad (3.3)$$

where $h : [1, N] \rightarrow \mathbb{R}$.

Similar to Lemma 2.1, we have the following.

Lemma 3.1 *Denote*

$$G_3(n, s) = \begin{cases} \frac{p^{s-n+1+N} + p^{n-s+1}}{(1-p^N)(1-p^2)}, & 0 \leq s \leq n \leq N+1, \\ \frac{p^{n-s+1+N} + p^{s-n+1}}{(1-p^N)(1-p^2)}, & 0 \leq n \leq s \leq N+1, \end{cases}$$

and

$$G_4(n, s) = \begin{cases} \frac{(s-1)(N+1-n)}{N}, & 0 \leq s \leq n \leq N+1, \\ \frac{(n-1)(N+1-s)}{N}, & 0 \leq n \leq s \leq N+1, \end{cases}$$

where P *is defined in Lemma 2.1. Then*

$$x(n) = \sum_{s=1}^N G_j(n, s)h(s), \quad n \in [0, N+1], j = 3, 4,$$

is the unique solution of

$$\begin{cases} -\Delta^2 x(n-1) + r_j x(n) = h(n), & n \in [1, N], j = 3, 4, \\ \Delta^i x(-1) = \Delta^i x(N-1), & i = 0, 1, 2, 3, \end{cases}$$

where r_3, r_4 *are solutions of equation* $r^2 - mr = 0$.

Following Lemma 3.1, we have a lemma.

Lemma 3.2 *The linear discrete fourth-order BVP (3.3) admits a unique solution*

$$x(n) = \sum_{s=1}^N \left(\sum_{j=1}^N G_3(n, j) G_4(j, s) h(s) \right) = \sum_{s=1}^N \left(\sum_{j=1}^N G_4(n, j) G_3(j, s) h(s) \right), \quad n \in [1, N]$$

satisfying $x(-1) = x(N-1)$, $x(0) = x(N)$, $x(1) = x(N+1)$ and $x(2) = x(N+2)$.

Remark 3.1 Green's function $G_j(n, s)$ ($j = 3, 4$) defined by Lemma 3.1 satisfies $G_j(n, s) = G_j(s, n) > 0$, $n, s \in [1, N]$. For $x \in H$, define operators K_m^* , f_m^* and S_m^* as follows:

$$(K_m^* x)(n) = \sum_{s=1}^N \sum_{j=1}^N G_4(n, j) G_3(j, s) x(s),$$

$$(f_m^* x)(n) = f(n, x(n)) - m \Delta^2 x(n-1),$$

$$S_m^* = K_m^* f_m^*,$$

where $S_m^* : H \rightarrow H$ is a completely continuous operator. Then $\{x(n)\}_{n=0}^{N+1}$ is a solution of BVP (1.1) with (1.3) if and only if $x = \{x(n)\}_{n=1}^N \in H$ is a fixed point of S_m^* .

With the aim to look for solutions of BVP (1.1) with (1.3) by Lemma 1.2, we need the following lemma and we omit its proof because of its similarity to Lemma 2.3.

Lemma 3.3 *The functional I_2 defined by (3.1) is Fréchet differentiable on H and $I_2'(x) = x - S_m^*(x)$ for all $x \in H$.*

According to Lemma 1.2 and Remark 1.1, it is time for us to formulate a test.

Lemma 3.4 *Let I_2 be defined as (3.1),*

- (i) I_2 satisfies the (PS) condition under (F_7) ;
- (ii) I_2 satisfies the (C) condition under (F_8) .

Proof (i) The proof is similar to that of Lemma 2.5(i) and thus is omitted.

(ii) Let $\{x_k\} \subset H$ be a sequence, then there exists a constant $\tilde{C} > 0$ such that

$$|I_2(x_k)| \leq \frac{\tilde{C}-1}{2} \quad \text{and} \quad (1 + \|x_k\|_m) \|I_2'(x_k)\|_m < 1.$$

It follows from the Cauchy–Schwarz inequality that

$$\langle I_2'(x_k), x_k \rangle_m \leq \|x_k\|_m \|I_2'(x_k)\|_m \leq (1 + \|x_k\|_m) \|I_2'(x_k)\|_m < 1.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^N [x_k f(n, x_k) - 2F(n, x_k)] &= 2I_2(x_k) - \langle I_2'(x_k), x_k \rangle_m \\ &\leq 2|I_2(x_k)| + \|x_k\|_m \|I_2'(x_k)\|_m \leq \tilde{C}. \end{aligned} \quad (3.4)$$

In view of (F_8) , there exists $\delta > 0$ such that

$$xf(n, x) - 2F(n, x) > \tilde{C}, \quad \forall n \in [1, N] \text{ and } |x| > \delta,$$

which contradicts (3.4). Then $\{x_k\}$ is bounded and the proof is finished. $\square$

Lemma 3.5 *Suppose that (F_4) , (F_5) and (F_6) are true, there exists constant $\tilde{\varepsilon}_0 > 0$ such that, for $0 < \varepsilon < \tilde{\varepsilon}_0$,*

- (i) $S_m^*(\partial D_\varepsilon^-) \subset D_\varepsilon^-$. And if $x \in D_\varepsilon^-$ is a nontrivial critical point of I_2 , then x is a negative solution of BVP (1.1) with (1.3);
- (ii) $S_m^*(\partial D_\varepsilon^+) \subset D_\varepsilon^+$. And if $x \in D_\varepsilon^+$ is a nontrivial critical point of I_2 , then x is a positive solution of BVP (1.1) with (1.3).

Proof (i) Since ω_1 and $\omega_{[\frac{N}{2}]}$ are the minimum and maximum eigenvalues of matrix A_2 , respectively. Then

$$\sqrt{\omega_1 + m} \|x\| \leq \|x\|_m \leq \sqrt{\omega_{[\frac{N}{2}]} + m} \|x\|, \quad (3.5)$$

which leads to

$$\begin{aligned} \|x^+\| &= \inf_{\psi \in -\Lambda} \|x - \psi\| \leq \frac{1}{\sqrt{m + \omega_1}} \inf_{\psi \in -\Lambda} \|x - \psi\|_m \\ &= \frac{1}{\sqrt{m + \omega_1}} \text{dist}_m(x, -\Lambda), \quad \forall x \in H. \end{aligned} \quad (3.6)$$

According to (F_4) and (F_6) , we can find a constant $\tilde{\tau} > 0$ such that

$$|f(n, x) + mx| \leq (m + \omega_1 - \tilde{\tau})|x| + C_2|x|^\sigma, \quad \forall (n, x) \in [1, N] \times \mathbf{R}. \quad (3.7)$$

Let $y = S_m^*(x)$ for any $x \in H$. As a matter of fact, $y^+ = y - y^-$ and $y^- \in -\Lambda$ mean $\text{dist}_m(y, -\Lambda) \leq \|y - y^-\|_m = \|y^+\|_m$. Combining (3.5), (3.6) and (3.7), it yields

$$\begin{aligned} &\text{dist}_m(y, -\Lambda) \|y^+\|_m \\ &\leq \langle y^+, y^+ \rangle_m \\ &= \langle S_m^*(x^+), y^+ \rangle_m \\ &= \sum_{n=1}^N [f(n, x^+(n)) + mx^+(n)] y^+(n) \\ &\leq (m + \omega_1 - \tilde{\tau}) \sum_{n=1}^N |x^+(n)| y^+(n) + C_2 \sum_{n=1}^N |x^+(n)|^\sigma y^+(n) \\ &\leq (m + \omega_1 - \tilde{\tau}) \|x^+\| \|y^+\| + C_2 \|x^+\|^\sigma \|y^+\| \\ &\leq \left(\frac{m + \omega_1 - \tilde{\tau}}{m + \omega_1} \text{dist}_m(x, -\Lambda) + \frac{C_2}{\sqrt{(m + \omega_1)^{\sigma+1}}} (\text{dist}_m(x, -\Lambda))^\sigma \right) \|y^+\|_m, \end{aligned}$$

which implies

$$\text{dist}_m(y, -\Lambda) \leq \frac{m + \omega_1 - \tilde{\tau}}{m + \omega_1} \text{dist}_m(x, -\Lambda) + C'_2 (\text{dist}_m(x, -\Lambda))^\sigma,$$

here $C'_2 = \frac{C_2}{\sqrt{(m+\omega_1)^{\sigma+1}}}$. Let $C'_2 (\text{dist}_m(x, -\Lambda))^{\sigma-1} = \frac{\tilde{\tau}}{2(m+\omega_1)}$, then

$$\text{dist}_m(S_m^*(x), -\Lambda) \leq \frac{2(m + \omega_1) - \tilde{\tau}}{2(m + \omega_1)} \text{dist}_m(x, -\Lambda). \quad (3.8)$$

Obviously, $\frac{2(m+\omega_1)-\tilde{\tau}}{2(m+\omega_1)} < 1$ is correct for all $\tilde{\tau} > 0$. Consequently,

$$S_m^*(x) \in D_\varepsilon^-, \quad \forall x \in \partial D_\varepsilon^-.$$

If $x \in D_\varepsilon^-$ is a nontrivial critical point of I_2 , Lemma 3.3 shows $I'_2(x) = x - S_m^*(x) = 0$. Moreover, (3.8) indicates $x \in -\Lambda \setminus \{0\}$. Following (F_5) and Remark 3.1, we see that $x(n) < 0$ in $n \in [1, N]$. Therefore, x is a negative solution of BVP (1.1) with (1.3).

Item (ii) can be discussed similarly, we omit its proof. $\square$

Now we devote our efforts to completing the proof of our main results in this section.

Proof of Theorem 3.1 Since H is a finite-dimensional Hilbert space, there exists constant $\bar{C} > 0$ such that

$$|x|_{\sigma+1} := \left(\sum_{n=1}^N |x(n)|^{\sigma+1} \right)^{\frac{1}{\sigma+1}} \leq \bar{C} \min\{\|x\|, \|x\|_m\}, \quad \forall x \in H. \quad (3.9)$$

According to (3.7), we have

$$F(n, x) + \frac{m}{2}|x|^2 \leq \frac{1}{2}(m + \omega_1 - \tilde{\tau})|x|^2 + \frac{C_2}{\sigma + 1}|x|^{\sigma+1}.$$

Making use of this, together with (3.5) and (3.9), it follows that

$$\begin{aligned} I_2(x) &= \frac{1}{2} \langle x, x \rangle_m - \sum_{n=1}^N \left[F(n, x(n)) + \frac{m}{2} |x(n)|^2 \right] \\ &\geq \frac{1}{2} \|x\|_m^2 - \frac{m + \omega_1 - \tilde{\tau}}{2} \|x\|^2 - \frac{C_2}{\sigma + 1} |x|_{\sigma+1}^{\sigma+1} \\ &\geq \frac{\tilde{\tau}}{2(m + \omega_1)} \|x\|_m^2 - \frac{C_2 \bar{C}^{\sigma+1}}{\sigma + 1} \|x\|_m^{\sigma+1}. \end{aligned}$$

Due to (3.6), $\|x^\pm\| \leq \frac{1}{\sqrt{m+\omega_1}} \text{dist}(x, \mp \Lambda) \leq \frac{1}{\sqrt{m+\omega_1}} \tilde{\varepsilon}_0$ holds for all $x \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}$. Then there exists $c_1 > -\infty$ such that $\inf_{x \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} I_2(x) = c_1$. Let v_1, v_2 be eigenvectors corresponding to eigenvalues ω_1, ω_2 of matrix A_2 . Denote $H_2 = \text{span}\{v_1, v_2\}$, for all $x \in H_2$. Note that $I_2(x) \rightarrow -\infty$ as $\|x\|_m \rightarrow +\infty$. Thus it is easy to look for a constant $\tilde{R} > 2\tilde{\varepsilon}_0$ such that $I_2(x) < c_1 - 1$ and $\|x\|_m = \tilde{R}$. Define a path $g : [0, 1] \rightarrow H_2$ as

$$g(s) = \tilde{R} \frac{\cos(\pi s)v_1 + \sin(\pi s)v_2}{\|\cos(\pi s)v_1 + \sin(\pi s)v_2\|_m}.$$

Direct calculation gives

$$g(0) = \tilde{R} \frac{v_1}{\|v_1\|_m} \in D_\varepsilon^+ \setminus D_\varepsilon^-, \quad g(1) = -\tilde{R} \frac{v_1}{\|v_1\|_m} \in D_\varepsilon^- \setminus D_\varepsilon^+,$$

and

$$\inf_{x \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} I_2(x) > \sup_{s \in [0,1]} I_2(g(s)).$$

Combining Lemma 3.4(i) with Lemma 3.5, it is easy to see that there is a critical point in $H \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$ corresponding to a sign-changing solution of BVP (1.1) with (1.3). In addition, we have a critical point in $D_\varepsilon^+ \setminus \overline{D_\varepsilon^-}$ corresponding to a positive solution and have a critical point in $D_\varepsilon^- \setminus \overline{D_\varepsilon^+}$ corresponding to a negative solution of BVP (1.1) with (1.3). The proof of (i) is finished.

It follows from Lemma 3.4(ii) and Remark 1.1, the proof for (ii) is analogous. Thus, the proof of Theorem 3.1 is complete. $\square$

4 Examples

To demonstrate the applicability of our theoretical results, three examples are provided.

Example 4.1 Let $N = 4$, we consider BVP

$$\begin{cases} \Delta^4 x(n-2) = \frac{-\frac{14}{5}x(n)}{1+x^2(n)} + 3x(n), & n \in [1, 4], \\ x(-1) = x(0) = 0 = x(5) = x(6). \end{cases} \quad (4.1)$$

It is easy to see that $f(n, x) = \frac{-\frac{14}{5}x}{1+x^2} + 3x$ and $\lambda_1 = 0.3944$, $\lambda_2 = 2.6148$. Then

- (1) $\lim_{x \rightarrow 0} \frac{f(n, x)}{x} = \lim_{x \rightarrow 0} \left(\frac{-\frac{14}{5}}{1+x^2} + 3 \right) = 0.2 < \lambda_1$;
- (2) $\lim_{|x| \rightarrow \infty} \frac{f(n, x)}{x} = \lim_{|x| \rightarrow \infty} \left(\frac{-\frac{14}{5}}{1+x^2} + 3 \right) = 3 > \lambda_2$;
- (3) $\lim_{|x| \rightarrow \infty} [xf(n, x) - 2F(n, x)] = \lim_{|x| \rightarrow \infty} \frac{14}{5} \left(\ln(1+x^2) - \frac{x^2}{1+x^2} \right) = +\infty$.

Hence (4.1) meets all conditions in Theorem 2.1. Therefore, Theorem 2.1 guarantees that (4.1) admits at least a positive solution, a negative solution and a sign-changing solution. By direct calculation, we see that $(0, 0, -0.1566, -0.2993, -0.2993, -0.1566, 0, 0)$ is the negative solution and $(0, 0, 0.1566, 0.2993, 0.2993, 0.1566, 0, 0)$ is the positive solution. In addition, $(0, 0, 2.9287, 1.9285, -1.9285, -2.9287, 0, 0)$ and $(0, 0, -2.9287, -1.9285, 1.9285, 2.9287, 0, 0)$ are two sign-changing solutions.

Example 4.2 Consider the following problem:

$$\begin{cases} \Delta^4 x(n-2) = x^3(n) + \frac{1}{5}x(n), & n \in [1, 5], \\ x(-1) = x(0) = 0 = x(6) = x(7). \end{cases} \quad (4.2)$$

In fact, $f(n, x) = x^3 + \frac{1}{5}x$ and $\lambda_1 = 0.2121$. Furthermore,

- (1) $\lim_{|x| \rightarrow \infty} \frac{f(n, x)}{x} = \lim_{|x| \rightarrow \infty} \left(x^2 + \frac{1}{5} \right) > \lambda_1$;
- (2) $\lim_{x \rightarrow 0} \frac{f(n, x)}{x} = \lim_{x \rightarrow 0} \left(x^2 + \frac{1}{5} \right) = 0.2 < \lambda_1$.

Then Theorem 2.2 means that (4.2) has at least a positive solution and a negative solution. Direct calculation shows that $(0, 0, 0.050496, 0.105201, 0.128054, 0.105201, 0.050496, 0, 0)$ is the positive solution and $(0, 0, -0.050496, -0.105201, -0.128054, -0.105201, -0.050496, 0, 0)$ is the negative solution.

Example 4.3 Consider the following problem:

$$\begin{cases} \Delta^4 x(n-2) = \tau |x(n)|^{\tau-2} x(n) - ax(n), & n \in [1, N], \\ \Delta^i x(-1) = \Delta^i x(N-1), & i = 0, 1, 2, 3, \end{cases} \quad (4.3)$$

where $\tau > 2$ and $0 \leq a < \min\{16 \sin^4 \frac{\pi}{N}, m\}$.

Here $f(n, x) = \tau |x|^{\tau-2} x - ax$, $n \in [1, N]$, so $F(n, x) = \int_0^x f(n, s) ds = |x|^\tau - \frac{a}{2} x^2$, and

$$(1) \lim_{x \rightarrow 0} \left| \frac{f(n, x)}{x} \right| = \lim_{x \rightarrow 0} |\tau |x|^{\tau-2} - a| = |a| = a < 16 \sin^4 \frac{\pi}{N} = \omega_1;$$

$$(2) xf(n, x) + mx^2 = \tau |x|^\tau + (m-a)x^2 > 0 \text{ for } x \neq 0;$$

$$(3) \lim_{|x| \rightarrow \infty} \frac{f(n, x)}{x} = \lim_{|x| \rightarrow \infty} (\tau |x|^{\tau-2} - a) = +\infty;$$

$$(4) xf(n, x) - 2F(n, x) = (\tau - 2)|x|^\tau > 0 \text{ for } x \neq 0, \text{ and}$$

$$xf(n, x) - 2F(n, x) = (\tau - 2)|x|^\tau \rightarrow +\infty \text{ as } |x| \rightarrow \infty.$$

Moreover, it is clear that there exist constants $\sigma > 1$, $C_3, C_4 > 0$ such that

$$|f(n, x)| = |\tau |x|^{\tau-2} x - ax| \leq C_3 |x|^\sigma + C_4.$$

Hence, (4.3) satisfies all conditions of Theorem 3.1, which implies that (4.3) has at least three nontrivial solutions: one positive, one negative and one sign-changing.

In particular, choose $m = 2$, $N = 4$, $\tau = 4$ and $a = 1$, (4.3) can be rewritten as

$$\begin{cases} \Delta^4 x(n-2) = 4x^3(n) - x(n), & n \in [1, 4], \\ \Delta^i x(-1) = \Delta^i x(3), & i = 0, 1, 2, 3. \end{cases} \quad (4.4)$$

Obviously, $(0, 0, 0, 0, 0, 0, 0, 0)$ is the trivial solution. The other nontrivial solutions are the positive solution $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, the negative solution $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and two sign-changing solutions $(\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2})$, $(-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2})$.

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