Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

The dynamic properties of solutions for a nonlinear shallow water equation

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Abstract

The local well-posedness for the Cauchy problem of a nonlinear shallow water equation is established. The wave-breaking mechanisms, global existence, and infinite propagation speed of solutions to the equation are derived under certain assumptions. In addition, the effects of coefficients λ, β, a, b, and index k in the equation are illustrated.

Keywords: Local well-posedness; Wave-breaking; Global solution; Infinite propagation speed

1 Introduction

We aim to consider the problem

$$\begin{cases} v_t - v_{xxt} + \beta(v_x - v_{xxx}) + \lambda(v - v_{xx}) + (a+b)v^k v_x \\ \quad = bv^{k-1}v_x v_{xx} + av^k v_{xxx}, \\ v(0,x) = v_0(x). \end{cases} \tag{1.1}$$

Here $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, $v(t,x)$ is fluid velocity of water waves, $\lambda \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, k is a positive integer, $\beta(v - v_{xx})$ is the diffusion term, $\lambda(v - v_{xx})$ is the dissipative term, $v_0 \in B^s_{p,r}(\mathbb{R})$ $(s > \max(1 + \frac{1}{p}, \frac{3}{2}))$.

Recently, the Camassa–Holm (CH) equation

$$v_t - v_{xxt} + \beta v_x + 3vv_x = 2v_x v_{xx} + vv_{xxx} \tag{1.2}$$

has attracted much attention. Equation (1.2) admits blow-up phenomena. Replacing v with $v + \beta$ in Eq. (1.2), we obtain

$$v_t - v_{xxt} + \beta(v_x - v_{xxx}) + 3vv_x = 2v_x v_{xx} + vv_{xxx}. \tag{1.3}$$

Taking $k = 1, \lambda = 0, a = 1, b = 2$ in (1.1) gives rise to the Cauchy problem of Eq. (1.3). The solution v to Eq. (1.2) is viewed as a perturbation near β (see [20]). The properties of solutions to the problem with dispersion and dissipative terms are discovered in [15]. Mi et al. [12] investigate the dynamical properties for a generalized CH equation. For a related study of the CH equation and other related partial differential equations, one may refer to references [3, 7, 11, 14, 16].

Taking $k = 1, \lambda = \beta = 0, a = 1, b = 3$ in (1.1) yields the Degasperis–Procesi equation

$$v_t - v_{xxt} + 4vv_x = 3v_x v_{xx} + vv_{xxx}. \tag{1.4}$$

The formation of singularity for solutions to (1.4) is discovered in [17]. Lai and Wu [10] study the local well-posedness for the Cauchy problem of

$$v_t - v_{xxt} + \beta v_x + (a + b)v_x = bv_x v_{xx} + avv_{xxx}, \tag{1.5}$$

where $\beta, a, b \in \mathbb{R}$.

Taking $k = 2, \lambda = \beta = 0, a = 1, b = 3$ in (1.1), we obtain the Novikov equation

$$v_t - v_{xxt} + 4v^2 v_x = 3vv_x v_{xx} + v^2 v_{xxx}. \tag{1.6}$$

Guo [4] studies the persistence properties of solutions to the CH-type equation. Fu and Qu [2] discover blow-up of solutions to Eq. (1.6) in $H^s(\mathbb{R})$ ($s > \frac{5}{2}$). The peakon solutions to the Novikov equation are established in [6].

Himonas and Thompson [8] discover persistence properties for solutions if $\lambda = \beta = 0, a = 1$ in (1.1). The behaviors of solutions [5], global existence of solutions for $a = 1$ [9], and infinite propagation speed of solutions [9, 19] to the problems are investigated. We extend parts of results in [9, 10, 13, 18, 19].

Let $s \in \mathbb{R}, T > 0, p \in [1, \infty]$ and $r \in [1, \infty]$. Thus we set

$$E^s_{p,r}(T) = \begin{cases} C([0, T]; B^s_{p,r}(\mathbb{R})) \cap C^1([0, T]; B^{s-1}_{p,r}(\mathbb{R})), & 1 \le r < \infty, \\ L^\infty([0, T]; B^s_{p,\infty}(\mathbb{R})) \cap \mathrm{Lip}([0, T]; B^{s-1}_{p,\infty}(\mathbb{R})), & r = \infty. \end{cases}$$

Letting $P_1(D) = -\partial_x(1 - \partial_x^2)^{-1}, P_2(D) = (1 - \partial_x^2)^{-1}$, problem (1.1) is turned into

$$\begin{cases} v_t + (av^k + \beta)v_x = P_1(D)[\frac{b}{k+1}v^{k+1} + \frac{3ak-b}{2}v^{k-1}v_x^2] \\ \qquad\qquad + P_2(D)[\frac{(k-1)(ak-b)}{2}v^{k-2}v_x^3 - \lambda v], \\ v(0, x) = v_0(x). \end{cases} \tag{1.7}$$

Now we summarize the main results in this paper.

Theorem 1.1 *Suppose* $1 \le r, p \le \infty$, $v_0 \in B^s_{p,r}(\mathbb{R})$ $(s > \max(1 + \frac{1}{p}, \frac{3}{2}))$. *Then solution* $v \in E^s_{p,r}(T)$ *to problem* (1.1) *is locally well-posed for certain* $T > 0$.

Theorem 1.2 *Suppose* $1 \le r, p \le \infty$, $v_0 \in B^s_{p,r}(\mathbb{R})$ $(s > \max(1 + \frac{1}{p}, \frac{3}{2}))$, $t \in [0, T]$. *Then a solution* v *to problem* (1.1) *blows up in finite time if and only if*

$$\int_0^t \left(1 + \|v_x\|_{L^\infty}\right)^k d\tau = \infty. \tag{1.8}$$

Theorem 1.3 *Suppose $b = a(k+1)$ and $v_0 \in H^s(\mathbb{R})$ ($s > \frac{3}{2}$), $t \in [0,T]$. Then a solution v to problem (1.1) blows up in finite time if and only if*

$$\lim_{t\to T^-} \inf_{x\in\mathbb{R}} v_x(t,x) = -\infty. \tag{1.9}$$

Theorem 1.4 *Suppose $b = a(k+1)$ and $v_0 \in H^s(\mathbb{R})$ ($s \geq 2$) satisfies $\|v_0 - v_{0,xx}\|_{L^2} < \frac{4\lambda}{|a|(k+2)\|v_0\|_{H^1}^{k-1}}$. Then there exists a global solution to problem (1.1) in $H^s(\mathbb{R})$ ($s \geq 2$).*

Theorem 1.5 *Assume $v_0 \in H^s(\mathbb{R})$ ($s \geq 2$), $n_0(x) = v_0 - v_{0,xx} \neq 0$ for all $x \in \mathbb{R}$, $\|n_0\|_{L^2} < (\frac{2^{k+1}\lambda}{|ak-2b|})^{\frac{1}{k}}$ and $b \neq \frac{ak}{2}$. Then a solution v to problem (1.1) is global in $H^s(\mathbb{R})$ ($s \geq 2$).*

Theorem 1.6 *Assume $a > 0$ and let $v_0 \in H^s(\mathbb{R})$ ($s > \frac{5}{2}$) be compactly supported in $[a_0, b_0]$, $t \in [0,T]$. Suppose k is a positive odd number and $b = ak$, or $k = 1, 0 < b < 3a$. Then, the solution $v(t,x)$ to (1.1) satisfies*

$$v(t,x) = \frac{1}{2}L_+(t)e^{-x} \quad \text{for } x \geq p(t,b_0), \qquad v(t,x) = \frac{1}{2}L_-(t)e^{x} \quad \text{for } x \leq p(t,a_0),$$

where $L_+(t)$ and $L_-(t)$ are continuous non-vanishing functions given in (4.1). What is more, $L_+(t) > 0, L_-(t) < 0$ for $t \in [0,T]$. In particular, if $k = 1$, $b = 2a$ or $b = \frac{a}{2}$, then $L_+(t) \leq C_3 e^{(\beta-\lambda)t}$ and $|L_-(t)| \leq C_4 e^{-(\beta+\lambda)t}$.

Remark 1.1 Problem (1.1) is local well-posed in $B^s_{p,r}(\mathbb{R})$ ($s > \max(\frac{3}{2}, 1 + \frac{1}{p})$). $\|v(t)\|_{H^1(\mathbb{R})}$ is bounded if $b = a(k+1)$. Also $\|v(t)\|_{H^2(\mathbb{R})}$ is bounded if $b = \frac{ak}{2}$. Theorem 1.2 improves the result of Theorem 5.1 in [19]. Theorem 1.3 implies that wave-breaking for a solution v occurs if its slope is unbounded. This result improves Theorem 3.1 in [18] and Theorem 5.6 in [19]. From Theorems 1.4, 1.5, and 1.6, we deduce that λ, β, a, b, and k are related to global existence and infinite propagation speed of the solutions. Parts of results in [9, 10, 13, 18, 19] are extended.

2 Proof of Theorem 1.1

We prove Theorem 1.1 in following five steps.

Step 1. Let $v^0 = 0$. Let $(v^i)_{i\in\mathbb{N}} \in C(\mathbb{R}^+; B^\infty_{p,r})$ be smooth and satisfy

$$\begin{cases} (\partial_t + (a(v^i)^k + \beta)\partial_x)v^{i+1} = G, \\ v^{i+1}(0,x) = v_0^{i+1} = S_{i+1}v_0, \end{cases} \tag{2.1}$$

and suppose

$$\begin{aligned} G = {} & P_1(D)\left[\frac{b}{k+1}\left(v^i\right)^{k+1} + \frac{3ak-b}{2}\left(v^i\right)^{k-1}\left(v^i\right)_x^2\right] \\ & + P_2(D)\left[\frac{(k-1)(ak-b)}{2}\left(v^i\right)^{k-2}\left(v^i\right)_x^3 - \lambda v^i\right]. \end{aligned} \tag{2.2}$$

We see $S_{i+1}v_0 \in B^\infty_{p,r}$. Then the solution $v^i \in C(\mathbb{R}^+; B^\infty_{p,r})$ in (2.1) is global for all $i \in \mathbb{N}$ by Lemma 2.5 in [13].

Step 2. It is derived from Lemma 2.4 in [13] that

$$\|v^{i+1}\|_{B^s_{p,r}} \le e^{C_1\int_0^t \|(v^i(\tau))^k\|_{B^s_{p,r}}\,d\tau} \times \left[\|v_0\|_{B^s_{p,r}} + \int_0^t e^{-C_1\int_0^\tau \|(v^i(\xi))^k\|_{B^s_{p,r}}\,d\xi}\|G(\tau,\cdot)\|_{B^s_{p,r}}\,d\tau\right]. \tag{2.3}$$

The notation $a \lesssim b$ means $a \le Cb$ for a certain positive constant C. We acquire the estimates

$$\|G(t,x)\|_{B^s_{p,r}} \lesssim \left(\|v^i\|_{B^s_{p,r}} + 1\right)^k \|v^i\|_{B^s_{p,r}}. \tag{2.4}$$

That is,

$$\|v^{i+1}\|_{B^s_{p,r}} \le C_2 \cdot e^{C_2\int_0^t (\|v^i(\tau)\|_{B^s_{p,r}}+1)^k\,d\tau}\left[\|v_0\|_{B^s_{p,r}} + \int_0^t e^{-C_2\int_0^\tau (\|v^i(\xi)\|_{B^s_{p,r}}+1)^k\,d\xi}\left(\|v^i\|_{B^s_{p,r}}+1\right)^k\|v^i\|_{B^s_{p,r}}\,d\tau\right]. \tag{2.5}$$

One may find certain $T > 0$ which satisfies $2kC_2^{k+1}(1 + \|v_0\|_{B^s_{p,r}})^k T < 1$ and

$$\left(1 + \|v^i(t)\|_{B^s_{p,r}}\right)^k \le \frac{C_2^k(1 + \|v_0\|_{B^s_{p,r}})^k}{1 - 2kC_2^{k+1}(1 + \|v_0\|_{B^s_{p,r}})^k t}. \tag{2.6}$$

Further, we deduce

$$\left(1 + \|v^{i+1}(t)\|_{B^s_{p,r}}\right)^k \le \frac{C_2^k(1 + \|v_0\|_{B^s_{p,r}})^k}{1 - 2kC_2^{k+1}(1 + \|v_0\|_{B^s_{p,r}})^k t},$$

which implies that $(v^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$.

Step 3. Let $m, n \in \mathbb{N}$. From (2.1), we deduce that

$$\begin{aligned}
&\left(\partial_t + \left(a\left(v^{m+n}\right)^k + \beta\right)\partial_x\right)\left(v^{m+n+1} - v^{m+1}\right) \\
&\quad = -a\left(\left(v^{m+n}\right)^k - \left(v^m\right)^k\right)\partial_x v^{m+1} \\
&\qquad + P_1(D)\left[\frac{b}{k+1}\left(\left(v^{m+n}\right)^{k+1} - \left(v^m\right)^{k+1}\right)\right] \\
&\qquad + P_1(D)\left[\frac{3ak-b}{2}\left(\left(v^{m+n}\right)^{k-1}\left(v^{m+n}\right)_x^2 - \left(v^m\right)^{k-1}\left(v^m\right)_x^2\right)\right] \\
&\qquad + P_2(D)\left[\frac{(k-1)(ak-b)}{2}\left(\left(v^{m+n}\right)^{k-2}\left(v^{m+n}\right)_x^3 - \left(v^m\right)^{k-2}\left(v^m\right)_x^3\right)\right] \\
&\qquad + P_2(D)\left[-\lambda\left(v^{m+n} - v^m\right)\right].
\end{aligned} \tag{2.7}$$

Using Lemma 2.4 in [13] yields

$$\|v^{m+n+1} - v^{m+1}\|_{B^{s-1}_{p,r}}$$

$$\leq e^{C\int_0^t \|v^{m+n}\|^k_{B^s_{p,r}}\,d\tau}\biggl[\bigl\|v_0^{m+n+1}-v_0^{m+1}\bigr\|_{B^{s-1}_{p,r}}+C\times\int_0^t e^{-C\int_0^\tau \|v^{m+n}\|^k_{B^s_{p,r}}\,d\xi}$$
$$\times\bigl(\bigl\|v^{m+n}-v^m\bigr\|_{B^{s-1}_{p,r}}\bigl(\bigl\|v^m\bigr\|_{B^s_{p,r}}+\bigl\|v^{m+n}\bigr\|_{B^s_{p,r}}+\bigl\|v^{m+1}\bigr\|_{B^s_{p,r}}+1\bigr)^k\bigr)\,d\tau\biggr]. \tag{2.8}$$

We note that the initial values satisfy

$$v_0^{m+n+1}-v_0^{m+1}=\sum_{q=m+1}^{m+n}\Delta_q v_0.$$

One may find a constant C_{T_1} independent of m to satisfy

$$\bigl\|v^{m+n+1}-v^{m+1}\bigr\|_{L^\infty([0,T];B^{s-1}_{p,r})}\leq C_{T_1}2^{-m}.$$

We obtain the desired results.

Step 4. Following the discussions in Step 4 in Sect. 3.1 in [13], one derives that $v\in E^s_{p,r}(T)$, which is continuous.

Step 5. (Proof of the uniqueness). Suppose $1\leq r,p\leq\infty, s>\max(\frac{3}{2},1+\frac{1}{p})$. Assume v^1 and v^2 satisfy (1.7) with $v_0^1,v_0^2\in B^s_{p,r}$, $v^1,v^2\in L^\infty([0,T];B^s_{p,r})\cap C([0,T];B^{s-1}_{p,r})$. We write $v^{12}=v^1-v^2$. Then

$$v^{12}\in L^\infty\bigl([0,T];B^s_{p,r}\bigr)\cap C\bigl([0,T];B^{s-1}_{p,r}\bigr),$$

which results in

$$\begin{cases}\partial_t v^{12}+(a(v^1)^k+\beta)\partial_x v^{12}=-a((v^1)^k-(v^2)^k)\partial_x v^2+G_1,\\ v^{12}(0,x)=v_0^{12}=v_0^1-v_0^2,\end{cases} \tag{2.9}$$

where

$$\begin{aligned}G_1={}&P_1(D)\biggl[\frac{b}{k+1}\bigl(\bigl(v^1\bigr)^{k+1}-\bigl(v^2\bigr)^{k+1}\bigr)\biggr]\\&+P_1(D)\biggl[\frac{3ak-b}{2}\bigl(\bigl(v^1\bigr)^{k-1}\bigl(v^1\bigr)_x^2-\bigl(v^2\bigr)^{k-1}\bigl(v^2\bigr)_x^2\bigr)\biggr]\\&+P_2(D)\biggl[\frac{(k-1)(ak-b)}{2}\bigl(\bigl(v^1\bigr)^{k-2}\bigl(v^1\bigr)_x^3-\bigl(v^2\bigr)^{k-2}\bigl(v^2\bigr)_x^3\bigr)-\lambda v^{12}\biggr].\end{aligned}$$

Using Lemma 2.4 in [13], we derive the estimates

$$\begin{aligned}&e^{-C\int_0^t \|v^1\|^k_{B^s_{p,r}}\,d\tau}\bigl\|v^{12}\bigr\|_{B^{s-1}_{p,r}}\\&\quad\leq\bigl\|v_0^{12}\bigr\|_{B^{s-1}_{p,r}}\\&\qquad+C\int_0^t e^{-C\int_0^\tau \|v^1\|^k_{B^s_{p,r}}\,d\xi}\bigl\|v^{12}\bigr\|_{B^{s-1}_{p,r}}\bigl(\bigl\|v^1\bigr\|_{B^s_{p,r}}+\bigl\|v^2\bigr\|_{B^s_{p,r}}+1\bigr)^k\,d\tau,\end{aligned}$$

which finishes the proof of the uniqueness.

Remark 2.1 Suppose $b = a(k+1), 1 \le r, p \le \infty$, $v_0 \in B^s_{p,r}(\mathbb{R})$ ($s > \max(1 + \frac{1}{p}, \frac{3}{2})$), $t \in [0, T]$. Then, the solution v to (1.1) satisfies

$$\|v(t)\|_{H^1} \le \|v_0\|_{H^1}.$$

3 Proofs of Theorems 1.2, 1.3, 1.4, and 1.5

3.1 Proof of Theorem 1.2

Taking advantage of the operator Δ_q to (1.7) yields

$$\bigl(\partial_t + \bigl(av^k + \beta\bigr)\partial_x\bigr)\Delta_q v = a\bigl[v^k, \Delta_q\bigr]\partial_x v + \Delta_q G_2(t,x), \tag{3.1}$$

where

$$\begin{aligned} G_2(t,x) = {} & P_1(D)\biggl[\frac{b}{k+1}v^{k+1} + \frac{3ak-b}{2}v^{k-1}v_x^2\biggr] \\ & + P_2(D)\biggl[\frac{(k-1)(ak-b)}{2}v^{k-2}v_x^3 - \lambda v\biggr]. \end{aligned}$$

Applying Lemma 2.3 in [13] gives rise to the estimates

$$\bigl\|a\bigl[v^k, \Delta_q\bigr]\partial_x v\bigr\|_{B^s_{p,r}} \lesssim \|v_x\|^k_{L^\infty}\|v\|_{B^s_{p,r}}$$

and

$$\bigl\|G_2(t,x)\bigr\|_{B^s_{p,r}} \lesssim \bigl(\|v_x\|^k_{L^\infty} + 1\bigr)\|v\|_{B^s_{p,r}}.$$

We derive that

$$\bigl\|v(t)\bigr\|_{B^s_{p,r}} \lesssim \|v_0\|_{B^s_{p,r}} + \int_0^t \bigl(1 + \bigl\|v_x(\tau)\bigr\|_{L^\infty}\bigr)^k \bigl\|v(\tau)\bigr\|_{B^s_{p,r}}\,d\tau.$$

That is,

$$\bigl\|v(t)\bigr\|_{B^s_{p,r}} \lesssim \|v_0\|_{B^s_{p,r}} e^{\int_0^t (1+\|v_x(\tau)\|_{L^\infty})^k\,d\tau}. \tag{3.2}$$

Letting $t \in [0, T^*]$, $T^* < \infty$ and

$$\int_0^t \bigl(1 + \bigl\|v_x(\tau)\bigr\|_{L^\infty}\bigr)^k\,d\tau < \infty, \tag{3.3}$$

we see that $\|v(T^*)\|_{B^s_{p,r}}$ is bounded by using (3.2). It yields a contradiction, ending the proof.

From Remark 2.1, we obtain a blow-up result.

Remark 3.1 If assumption $b = a(k+1)$ is added into Theorem 1.2, then condition in (1.8) is changed into

$$\int_0^t \bigl(1 + \|v_x\|_{L^\infty}\bigr)^2\,d\tau = \infty.$$

3.2 Proof of Theorem 1.3

We only need to prove Theorem 1.3 with $s = 2$ by density argument. Take $b = a(k+1)$. It is deduced from (1.1) that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(v^2 + v_x^2\right)dx + \int_{\mathbb{R}}\lambda\left(v^2 + v_x^2\right)dx = 0, \tag{3.4}$$

which results in

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(v^2 + v_x^2\right)dx \le 0. \tag{3.5}$$

A direct calculation shows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(v_x^2 + v_{xx}^2\right)dx \\
&\quad = a(k+2)\int_{\mathbb{R}} v^k v_x v_{xx}\,dx - \int_{\mathbb{R}}\lambda\left(v_x^2 + v_{xx}^2\right)dx \\
&\qquad - \int_{\mathbb{R}}\left[a(k+1)v^{k-1}v_x v_{xx}^2 + a v^k v_{xxx}v_{xx}\right]dx.
\end{aligned} \tag{3.6}$$

Let $T < \infty$ and $v_x(t,x) \ge -M$ for a certain $M > 0$. We come to the estimate

$$\left\|v(t)\right\|_{H^2} \le \|v_0\|_{H^2} e^{(1+M+\|v_0\|_{H^1})^k t}, \quad \text{for all } t \in [0,T],$$

which yields a contradiction.

3.3 Proof of Theorem 1.4

We take $n = v - v_{xx}$. The first equation in (1.1) is written in the form

$$n_t + \beta n_x + \lambda n + b v^{k-1} v_x n + a v^k n_x = 0. \tag{3.7}$$

We see $b = a(k+1)$ in Theorem 1.4. Multiplying (3.7) by n and applying (3.6) gives rise to

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} n^2\,dx + \lambda\int_{\mathbb{R}} n^2\,dx \lesssim \frac{|a|(k+2)}{4}\|v_0\|_{H^1}^{k-1}\|n\|_{L^2}^3.$$

Taking $\lambda_1 = 2\lambda$ and $M_1 = \frac{|a|(k+2)}{2}\|v_0\|_{H^1}^{k-1}$, we have

$$\frac{d}{dt}\|n\|_{L^2}^2 + \lambda_1\|n\|_{L^2}^2 \le M_1\left(\|n\|_{L^2}^2\right)^{\frac{3}{2}}.$$

It follows that $\|n\|_{L^2} \le e^{-\frac{1}{2}\lambda_1 t}\left(\frac{1}{\|n_0\|_{L^2}} - \frac{M_1}{\lambda_1}\right)^{-1}$ if $\|n_0\|_{L^2} < \frac{\lambda_1}{M_1}$. Then

$$\|v_x\|_{L^\infty} \le \|n\|_{L^2} \le C_2(T).$$

Using Theorem 1.3, we end the proof.

3.4 Proof of Theorem 1.5

We investigate problem

$$\begin{cases} \frac{d}{dt}p(t,x) = av^k(t,p(t,x)) + \beta, \\ p(0,x) = x, \end{cases} \tag{3.8}$$

where $(t,x) \in (0,T) \times \mathbb{R}$.

Lemma 3.1 ([1]) *Let $v \in C([0,T];H^s(\mathbb{R})) \cap C^1([0,T];H^{s-1}(\mathbb{R}))$ $(s \geq 2)$, $(t,x) \in [0,T] \times \mathbb{R}$. It follows that $p \in C^1([0,T] \times \mathbb{R},\mathbb{R})$ to (3.8) is unique and*

$$p_x(t,x) = e^{\int_0^t akv^{k-1}v_x(\tau,p(\tau,x))\,d\tau}. \tag{3.9}$$

Lemma 3.2 *Let $v_0 \in H^s(\mathbb{R})$ $(s \geq 2)$, $(t,x) \in [0,T] \times \mathbb{R}$. Then*

$$n(t,p)(p_x)^{\frac{b}{ak}}(t,x) = n_0 e^{-\lambda t}. \tag{3.10}$$

Moreover, $\|n\|_{L^{\frac{ak}{b}}} = e^{-\lambda t}\|n_0\|_{L^{\frac{ak}{b}}}$. If $b = \frac{ak}{2}$, it holds that

$$\|n\|_{L^2} = e^{-\lambda t}\|n_0\|_{L^2}. \tag{3.11}$$

Proof From (3.10), we acquire that

$$\frac{d}{dt}\left[n(t,p)(p_x)^{\frac{b}{ak}}\right] = -\lambda n(p_x)^{\frac{b}{ak}}. \tag{3.12}$$

That is,

$$n(t,p)(p_x)^{\frac{b}{ak}} = e^{-\lambda t}n_0(x).$$

A direct computation gives rise to

$$\left\|e^{-\lambda t}n_0(x)\right\|_{L^{\frac{ak}{b}}} = \|n\|_{L^{\frac{ak}{b}}}.$$

We note $b = \frac{ak}{2}$. Thus we get (3.11). □

Proof of Theorem 1.5 Multiplying (3.7) by $ne^{2\lambda t}$, we come to

$$\frac{d}{dt}\left(e^{2\lambda t}\int_{\mathbb{R}} n^2\,dx\right) = (ak-2b)e^{2\lambda t}\int_{\mathbb{R}} n^2 v^{k-1}v_x\,dx. \tag{3.13}$$

We derive that

$$\frac{d}{dt}\left(e^{2\lambda t}\int_{\mathbb{R}} n^2\,dx\right) \leq \frac{|ak-2b|}{2^k}e^{-k\lambda t}\left[e^{2\lambda t}\int_{\mathbb{R}} n^2\,dx\right]^{\frac{k+2}{2}}. \tag{3.14}$$

Let $h(t) = e^{2\lambda t}\int_{\mathbb{R}} n^2\,dx$. Bearing in mind that $n_0(x) \not\equiv 0$, $x \in \mathbb{R}$ and (3.10), one deduces that $h(t)$ is positive. Then

$$\frac{d}{dt}\big[h(t)\big]^{-\frac{k}{2}} \geq -\frac{k}{2}\frac{|ak-2b|}{2^k}e^{-k\lambda t}. \tag{3.15}$$

Using the assumption $n_0(x) \not\equiv 0, b \neq \frac{ak}{2}$, $\|n_0\|_{L^2} < (\frac{2^{k+1}\lambda}{|ak-2b|})^{\frac{1}{k}}$, we have $[h(0)]^{-\frac{k}{2}} - \frac{|ak-2b|}{2^{k+1}\lambda} > 0$. We obtain the inequality

$$\left(e^{2\lambda t}\int_{\mathbb{R}} n^2\,dx\right)^{\frac{k}{2}} \leq \left[\|n_0\|_{L^2}^{-k} - \frac{|ak-2b|}{2^{k+1}\lambda}\right]^{-1}.$$

Consequently, we have the estimate

$$\|v_x\|_{L^\infty} \leq \|n\|_{L^2} \leq e^{-\lambda t}\left[\|n_0\|_{L^2}^{-k} - \frac{|ak-2b|}{2^{k+1}\lambda}\right]^{-\frac{1}{k}}.$$

Applying Theorem 1.3, we complete the proof. □

We give a global existence result.

Lemma 3.3 *Let $b = a(k+1)$ or $b = \frac{ak}{2}$, $v_0 \in H^s(\mathbb{R})$ $(s \geq 2)$. Assume $n_0 = v_0 - v_{0,xx}$ does not change sign. It holds that a solution $v(t,x)$ to problem (1.1) exists globally.*

Proof One may assume $n_0(x) > 0$. We use Lemma 3.2 to derive that $n > 0$. Thus

$$v(t,x) = \int_{\mathbb{R}} \frac{1}{2}e^{-|x-\xi|}n(t,\xi)\,d\xi \geq 0.$$

That is,

$$v(t,x) = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}n(t,\xi)\,d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}n(t,\xi)\,d\xi. \tag{3.16}$$

We conclude that

$$v_x(t,x) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}n(t,\xi)\,d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}n(t,\xi)\,d\xi. \tag{3.17}$$

Hence $|v_x| \leq v$.

Applying $b = a(k+1)$ and recalling Remark 2.1, we derive

$$|v_x| \leq |v| \lesssim \big\|v(t)\big\|_{H^1} \lesssim \|v_0\|_{H^1}. \tag{3.18}$$

Taking advantage of $b = \frac{ak}{2}$ and using Lemma 3.2 results in

$$|v_x| \leq |v| \lesssim \|n\|_{L^2} \lesssim \|n_0\|_{L^2}. \tag{3.19}$$

Combining (3.18) or (3.19) with Theorem 1.2, we obtain the desired results. □

4 Proof of Theorem 1.6

Note that $a > 0$. Using $\operatorname{supp} v_0(x) \subset [a_0, b_0]$, we derive that $\operatorname{supp} v_0(x) \subset [p(t,a_0), p(t,b_0)]$. Applying Lemma 3.2 yields that $\operatorname{supp} n(t,x) \subset [p(t,a_0), p(t,b_0)]$, $t \in [0,T]$.

Let

$$L_+(t) = \int_{p(t,a_0)}^{p(t,b_0)} e^{\xi} n(t,\xi)\,d\xi, \qquad L_-(t) = \int_{p(t,a_0)}^{p(t,b_0)} e^{-\xi} n(t,\xi)\,d\xi. \tag{4.1}$$

From (3.16) and (4.1), we have

$$\begin{aligned} v(t,x) &= \frac{1}{2} e^{-x} \left(\int_{-\infty}^{p(t,a_0)} + \int_{p(t,a_0)}^{p(t,b_0)} + \int_{p(t,b_0)}^{x} \right) e^{\xi} n(t,\xi)\,d\xi \\ &\quad + \frac{1}{2} e^{x} \int_{x}^{\infty} e^{-\xi} n(t,\xi)\,d\xi \\ &= \frac{1}{2} e^{-x} L_+(t), \quad x > p(t,b_0). \end{aligned} \tag{4.2}$$

We derive $v = \frac{1}{2} e^{x} L_-(t)$ if $x < p(t,a_0)$. Combining (3.17) with (4.2) gives rise to

$$v = -v_x = v_{xx} = \frac{1}{2} e^{-x} L_+(t), \quad x > p(t,b_0) \tag{4.3}$$

and

$$v = v_x = v_{xx} = \frac{1}{2} e^{x} L_-(t), \quad x < p(t,a_0). \tag{4.4}$$

An application of (4.1) leads to the identity

$$L_+(0) = \int_{a_0}^{b_0} e^{\xi} n_0(\xi)\,d\xi = 0. \tag{4.5}$$

A direct calculation shows

$$\begin{aligned} \frac{d}{dt} L_+(t) &= \int_{-\infty}^{\infty} e^{\xi} n_t(t,\xi)\,d\xi \\ &= -\int_{-\infty}^{\infty} e^{\xi} (\lambda - \beta) n\,d\xi + \int_{-\infty}^{\infty} e^{\xi} \frac{b}{k+1} v^{k+1}\,d\xi \\ &\quad + \frac{3ak - b}{2} \int_{-\infty}^{\infty} e^{\xi} v_x^2 v^{k-1}\,d\xi + \frac{(k-1)(ak-b)}{2} \int_{-\infty}^{\infty} e^{\xi} v_x^3 v^{k-2}\,d\xi. \end{aligned} \tag{4.6}$$

If $b = ak$ and k is a positive odd number, we obtain

$$\frac{d}{dt} L_+(t) + (\lambda - \beta) L_+(t) > 0, \tag{4.7}$$

which is equivalent to the inequality

$$\frac{d[L_+(t) e^{(\lambda-\beta)t}]}{dt} > 0. \tag{4.8}$$

Hence $L_+(t) > 0$, $t \in [0,T)$.

Similarly, we have

$$\frac{d[-L_-(t)e^{(\lambda+\beta)t}]}{dt} > 0. \tag{4.9}$$

Thus, $L_-(t) < 0$, $t \in [0, T)$.

If $k = 1, 0 < b < 3a$, we derive that (4.8) and (4.9) still hold true.

We give the estimates for curve $p(t, b_0)$. Using the assumption $k = 1, b = 2a$ and (3.4) yields

$$\|v\|_{L^\infty} \le \|v\|_{H^1} \le e^{-\lambda t}\|v_0\|_{H^1}. \tag{4.10}$$

Taking $x = b_0$ in (3.8) and integrating (3.8) on $[0, t]$, we come to the estimate

$$\begin{aligned} p(t, b_0) &= b_0 + \int_0^t av(\tau, p)\,d\tau + \beta t \\ &\le \frac{1}{\lambda}C_5 + b_0 + \beta t. \end{aligned} \tag{4.11}$$

We conclude from (4.2) that

$$L_+(t) = 2e^{p(t,b_0)}v\bigl(t, p(t, b_0)\bigr) \le C_3 e^{(\beta-\lambda)t}. \tag{4.12}$$

Similar to the derivation in (4.11), we have

$$\begin{aligned} p(t, a_0) &= a_0 + \int_0^t av(\tau, p)\,d\tau + \beta t \\ &\ge -\frac{1}{\lambda}C_5 + a_0 + \beta t, \end{aligned} \tag{4.13}$$

which, combining with (4.4), implies

$$\bigl|L_-(t)\bigr| \le C_4 e^{-(\beta+\lambda)t}. \tag{4.14}$$

If $k = 1, b = \frac{a}{2}$, it is deduced from (3.11) that $\|v\|_{L^\infty} \le e^{-\lambda t}\|v_0\|_{H^2}$. Similarly, we establish (4.12) and (4.14).

Remark 4.1 If $\operatorname{supp} v_0(x) \subset [a_0, b_0]$ in (1.1), then $n = (1 - \partial_x^2)v(t, x)$ satisfies $\operatorname{supp} n \subset [p(t, a_0), p(t, b_0)]$. Indeed, v does not have compact support. Also $v(t, x)$ is positive if $x \to \infty$ and $v(t, x)$ is negative if $x \to -\infty$.

Acknowledgements

We are grateful to the anonymous referees for a number of valuable comments and suggestions.

Funding

The project is supported by Science Foundation of North University of China (No. 2017030, No. 13011920) and the National Natural Science Foundation of P.R. China (No. 11471263).

Abbreviations

Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
All authors read and approved the final version of the manuscript.

Authors' contributions
All authors contributed equally in this paper. All authors read and approved the final manuscript.

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Received: 3 July 2019 Accepted: 11 October 2019 Published online: 21 October 2019

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