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On a weak solution matching up with the double degenerate parabolic equation

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Abstract

The well-posedness of weak solutions to a double degenerate evolutionary $p(x)$-Laplacian equation

$$u_t = \operatorname{div}\left(b(x,t)\left|\nabla A(u)\right|^{p(x)-2}\nabla A(u)\right),$$

is studied. It is assumed that $b(x,t)|_{(x,t)\in\Omega\times[0,T]} > 0$ but $b(x,t)|_{(x,t)\in\partial\Omega\times[0,T]} = 0$, $A'(s) = a(s) \geq 0$, and $A(s)$ is a strictly monotone increasing function with $A(0) = 0$. A weak solution matching up with the double degenerate parabolic equation is introduced. The existence of weak solution is proved by a parabolically regularized method. The stability theorem of weak solutions is established independent of the boundary value condition. In particular, the initial value condition is satisfied in a wider generality.

MSC: 35K55; 35K92; 35K85; 35R35

Keywords: Double degenerate parabolic equation; Well-posedness; Existence; Initial value

1 Introduction

In this paper, the double degenerate evolutionary $p(x)$-Laplacian equation

$$u_t = \operatorname{div}\left(b(x,t)\left|\nabla A(u)\right|^{p(x)-2}\nabla A(u)\right) + f(x,t,u,\nabla u), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

is considered, in which $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p(x) > 1$ is a $C^1(\overline{\Omega})$ function, $b(x,t) \in C^1(\overline{Q_T})$ satisfies

$$b(x,t) > 0, \quad (x,t) \in \Omega \times [0,T]; \qquad b(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{1.2}$$

$A'(s) = a(s) \geq 0$ and $a(s) \in C^1(\mathbb{R})$, $A(0) = 0$. If $A(s) = s$ and $b(x,t) = 1$, equation (1.1) comes from a new interesting family of fluids, the so-called electrorheological fluids (see [1, 2]), and has been widely studied [2–15] in recent decade. If $b(x,t) = 1$, $p(x) = p > 1$ is a constant, equation (1.1) is a generalization of the following polytropic infiltration equation:

$$u_t = \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) + f(x,t,u,\nabla u), \quad (x,t) \in Q_T, \tag{1.3}$$

where $m > 0$; if $p > 1 + \frac{1}{m}$, we have the slow diffusion case, while for $p < 1 + \frac{1}{m}$, it is the fast diffusion case. There are many papers [16–29] that studied various questions about equation (1.3) with the usual initial boundary value conditions

$$u(x,t) = u_0(x), \quad x \in \Omega, \tag{1.4}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T). \tag{1.5}$$

If $f(x,t,u,\nabla u) = \nabla B(u)$ and $u_0(x) \in L^q(\Omega)$ with $q \geq 1$, the initial-boundary value problem of equation (1.3) was considered in [16]. By modifying the usual Morse iteration and imposing some restrictions on $f(x,t,u,\nabla u)$, the local L^∞-estimates were obtained, and $u_t \in L^2(\mathbb{R}^N \times (\tau,T))$ was proved [16]. When $f(x,t,u,\nabla u) = 0$, the Cauchy problem of equation (1.3) with the initial value $u_0(x) \in L^1(\mathbb{R}^N)$ was studied in [18], the existence and uniqueness of weak solutions were proved, and $u_t \in L^1(\mathbb{R}^N \times (\tau,T))$ was shown for any $\tau > 0$. When the initial value $u_0(x)$ is just a measure, the Cauchy problem was considered in [19]. A more general equation was studied in [17] based on an L^1 initial value condition. The large-time behavior of solutions to equation (1.3) had been studied in [21–24, 26], etc. The extinction, positivity, and the blow-up of solutions had been studied in [25, 27], etc. Of course, there are a lot of papers on the other subjects, such as the regularity, Harnack inequality, and the free boundary problem, etc.; for examples, one can refer to [20, 28, 29], etc. Recently, using some techniques of [18], the existence and uniqueness of weak solutions to the following equation:

$$u_t = \operatorname{div}\bigl(a(x)\bigl|\nabla u^m\bigr|^{p-2}\nabla u^m\bigr), \quad (x,t) \in Q_T, \tag{1.6}$$

had been studied in [30–32], where $a(x)$ satisfies

$$a(x) > 0, \quad x \in \Omega; \qquad a(x) = 0, \quad x \in \partial\Omega. \tag{1.7}$$

Equation (1.6) is always degenerate on the boundary. This is the most characteristic feature of equation (1.6) different from equation (1.3). Let us give a further explanation. For two weak solutions $u(x,t), v(x,t)$ of equation (1.6) with the initial value (1.4) but independent of the boundary value condition (1.5), satisfying

$$\nabla u^m \in L^1\bigl(0,T;L^p(\Omega)\bigr), \qquad \nabla v^m \in L^1\bigl(0,T;L^p(\Omega)\bigr),$$

multiplying by $S_n(u^m - v^m)$ on both sides of equation (1.6) and integrating over $Q_t = \Omega \times (0,t)$, from (1.7), one has

$$\begin{aligned}
&\int_0^t S_n\bigl(u^m - v^m\bigr)(u_t - v_t)\,dx\,dt \\
&\quad = -\iint_{Q_t} a(x)\bigl(\bigl|\nabla u^m\bigr|^{p-2}\nabla u^m - \bigl|\nabla v^m\bigr|^{p-2}\nabla v^m\bigr)\bigl(\nabla u^m - \nabla v^m\bigr)S_n'\bigl(u^m - v^m\bigr)\,dx\,dt \\
&\quad \leq 0.
\end{aligned}$$

Here $S_n(s) \in C^1(\mathbb{R})$ is such that $\lim_{n\to\infty} S_n(s) = \operatorname{sgn}(s)$ is the sign function.

Let $n \to \infty$. Then

$$\int_{\Omega} |u(x,t) - v(x,t)\,dx \le \int_{\Omega} \left|u_0(x) - v_0(x)\right| dx.$$

This inequality shows that the stability of weak solutions of equation (1.6) with the initial value (1.4) can be true, the boundary value condition (1.5) is completely redundant. In other words, for the well-posedness problem of equation (1.6), the degeneracy of $a(x)$ on the boundary (1.7) may take the place for the Dirichlet boundary value condition (1.5).

The main aim of this paper is to generalized the above conclusion to the double degenerate evolutionary $p(x)$-Laplacian equation (1.1). For simplicity, we only discuss the problem when $f(x,t,u,\nabla u) \equiv 0$ in equation (1.1). Since we assume that $A(0) = 0$, $A(s)$ is a strictly monotone increasing function, equation (1.6) is the special case of equation (1.1). However, since the diffusion $b(x,t)$ depends the time variable t and the nonlinearity of $A(s)$, equation (1.1) is more general, and there are some essential difficulties that should be overcome.

2 Basic functional space and a new kind of weak solution

We should emphasize again that $f(x,t,u,\nabla u) \equiv 0$ in what follows. Let us first introduce a basic lemma and the definition of weak solutions.

Lemma 2.1

(i) *The spaces* $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$, *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) *Let* $p_1(x)$ *and* $p_2(x)$ *be real functions with* $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} = 1$ *and* $p_1(x) > 1$. *Then, the conjugate space of* $L^{p_1(x)}(\Omega)$ *is* $L^{p_2(x)}(\Omega)$. *And for any* $u \in L^{p_1(x)}(\Omega)$ *and* $v \in L^{p_2(x)}(\Omega)$, *we have*

$$\left|\int_{\Omega} uv\,dx\right| \le 2\|u\|_{L^{p_1(x)}(\Omega)}\|v\|_{L^{p_2(x)}(\Omega)}.$$

(iii)

$$\begin{aligned}
&\textit{If } \|u\|_{L^{p(x)}(\Omega)} = 1, \quad \textit{then } \int_{\Omega} \|u\|^{p(x)}\,dx = 1,\\
&\textit{If } \|u\|_{L^{p(x)}(\Omega)} > 1, \quad \textit{then } \|u\|_{L^{p(x)}(\Omega)}^{p^-} \le \int_{\Omega} |u|^{p(x)}\,dx \le \|u\|_{L^{p(x)}(\Omega)}^{p^+},\\
&\textit{If } \|u\|_{L^{p(x)}(\Omega)} < 1, \quad \textit{then } \|u\|_{L^{p(x)}(\Omega)}^{p^+} \le \int_{\Omega} |u|^{p(x)}\,dx \le \|u\|_{L^{p(x)}(\Omega)}^{p^-}.
\end{aligned}$$

This lemma can be found in [33, 34]. From here on, $p^+ = \max_{x\in\overline{\Omega}} p(x), p^- = \max_{x\in\overline{\Omega}} p(x)$.

Definition 2.2 A function $u(x,t)$ is said to be a weak solution of equation (1.1) with the initial condition (1.5), if

$$u \in L^{\infty}(Q_T), \qquad \frac{\partial}{\partial t}\int_0^u \sqrt{a(s)}\,ds \in L^2(Q_T), \qquad b(x,t)\left|\nabla A(u)\right|^{p(x)} \in L^1(Q_T), \tag{2.1}$$

and for any function $\varphi \in C_0^1(Q_T)$, the following integral equivalence holds:

$$\iint_{Q_T} \left[\frac{\partial u}{\partial t} \varphi(x,t) + b(x,t) \left| \nabla A(u) \right|^{p(x)-2} \nabla A(u) \cdot \nabla \varphi \right] dx\, dt = 0. \tag{2.2}$$

Initial condition (1.5) is satisfied in the sense of

$$\lim_{t \to 0} \int_{\Omega} \left| \int_0^{u(x,t)} \sqrt{a(s)} - \int_0^{u_0(x)} \sqrt{a(s)}\, ds \right| dx = 0. \tag{2.3}$$

In this paper, we first study the existence of the weak solution.

Theorem 2.3 *If $b(x,t)$ satisfies (1.2) and*

$$\left| \frac{\partial b(x,t)}{\partial t} \right| \leq c b(x,t), \tag{2.4}$$

$A(s)$ is a strictly monotone increasing continuous function, $A(0) = 0$, $u_0(x) \geq 0$,

$$u_0 \in L^{\infty}(\Omega), \qquad b(x,0) u_0(x) \in W^{1,p(x)}(\Omega), \tag{2.5}$$

then there is a nonnegative solution of equation (1.1) with the initial value (1.5).

Theorem 2.4 *If $b(x,t)$ satisfies (1.2), $A(s)$ is a strictly monotone increasing function, $A(0) = 0$, and for large enough n,*

$$n^{1-\frac{1}{p^+}} \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} |\nabla b|^{p(x)}\, dx \right)^{\frac{1}{p^+}} \leq c(T), \tag{2.6}$$

$u(x,t)$ and $v(x,t)$ are two weak solutions of equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively, then

$$\int_{\Omega} \left| u(x,t) - v(x,t) \right| dx \leq c \int_{\Omega} \left| u_0(x) - v_0(x) \right| dx, \quad a.e.\ t \in [0,T). \tag{2.7}$$

From here on, ∇b represents the gradient of the spatial variable x, and for any $t \in [0,T)$,

$$\Omega_{\frac{1}{n}t} = \left\{ x \in \partial\Omega : b(x,t) > \frac{1}{n} \right\}.$$

3 The proof of Theorem 2.3

Without loss the generality, we may assume that $A(s)$ is a C^1 function, $A'(s) = a(s) \geq 0$. Consider the parabolically regularized system

$$u_t = \operatorname{div}\left(\left(b(x,t) + \varepsilon \right) \left| \nabla A(u) \right|^{p(x)-2} \nabla A(u) \right), \quad (x,t) \in Q_T, \tag{3.1}$$

$$u(x,0) = u_0(x) + \varepsilon, \quad x \in \Omega, \tag{3.2}$$

$$u(x,t) = \varepsilon, \quad (x,t) \in \partial\Omega \times (0,T). \tag{3.3}$$

Proof of Theorem 2.3 Similar as in [35, 36], by the monotone convergence method, we can prove that the solution $u_\varepsilon \in L^1(0,T:W^{1,p(x)}(\Omega))$ of the initial-boundary value problem (3.1)–(3.3) is such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le c. \tag{3.4}$$

Multiplying (3.1) by $A(u_\varepsilon) - A(\varepsilon)$ and integrating the result over $Q_t = \Omega \times (0,t)$ for any $t \in [0,T)$, as well as denoting

$$\int_0^r A(s)\,ds = \mathbb{A}(r),$$

we get

$$\begin{aligned} &\int_\Omega \mathbb{A}\bigl(u_\varepsilon(x,t)\bigr)\,dx + \iint_{Q_t} \bigl(b(x,t)+\varepsilon\bigr)\bigl|\nabla A(u_\varepsilon)\bigr|^{p(x)}\,dx\,dt \\ &\quad = \int_\Omega \mathbb{A}\bigl(u_0(x)\bigr)\,dx + A(\varepsilon)\int_\Omega \bigl[u(x,t)-u_0(x)\bigr]\,dx \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\iint_{Q_T} b(x,t)\bigl|\nabla A(u_\varepsilon)\bigr|^{p(x)}\,dx\,dt \\ &\quad \le c\iint_{Q_T}\bigl(b(x,t)+\varepsilon\bigr)\bigl|\nabla A(u_\varepsilon)\bigr|^{p(x)}\,dx\,dt \\ &\quad \le c. \end{aligned} \tag{3.6}$$

Multiplying (3.1) by $[A(u_\varepsilon)-A(\varepsilon)]_t$ and integrating the result over $Q_t = \Omega \times (0,t)$,

$$\begin{aligned} &\iint_{Q_t}\bigl(A(u_\varepsilon)\bigr)_t u_{\varepsilon t}\,dx\,dt + \iint_{Q_t}\bigl(b(x,t)+\varepsilon\bigr)\bigl|\nabla A(u_\varepsilon)\bigr|^{p(x)-2}\nabla A(u_\varepsilon)\nabla\bigl(A(u_\varepsilon)\bigr)_t\,dx\,dt \\ &\quad = 0. \end{aligned} \tag{3.7}$$

Since

$$\bigl|\nabla A(u_\varepsilon)\bigr|^{p(x)-2}\nabla\bigl(A(u_\varepsilon)\bigr)_t = \frac{1}{2}\frac{\partial}{\partial t}\int_0^{|\nabla A(u_\varepsilon)|^2} s^{\frac{p(x)-2}{2}}\,ds,$$

and $|\frac{\partial b(x,t)}{\partial t}| \le cb(x,t)$, we obtain

$$\begin{aligned} &\iint_{Q_t}\bigl(b(x,t)+\varepsilon\bigr)\bigl|\nabla A(u_\varepsilon)\bigr|^{p(x)-2}\nabla A(u_\varepsilon)\nabla\bigl(A(u_\varepsilon)\bigr)_t\,dx\,dt \\ &\quad = \frac{1}{2}\iint_{Q_t}\frac{\partial}{\partial t}\biggl[\bigl(b(x,t)+\varepsilon\bigr)\int_0^{|\nabla A(u_\varepsilon)|^2} s^{\frac{p(x)-2}{2}}\,ds\biggr]\,dx\,dt \\ &\qquad - \frac{1}{2}\iint_{Q_t}\int_0^{|\nabla A(u_\varepsilon)|^2} s^{\frac{p(x)-2}{2}}\,ds\frac{\partial b(x,t)}{\partial t}\,dx\,dt \\ &\quad = \frac{1}{2}\int_\Omega \frac{2}{p(x)}\bigl[\bigl(b(x,t)+\varepsilon\bigr)\bigl|\nabla A(u_\varepsilon)\bigr|^{p(x)} - \bigl(b(x,0)+\varepsilon\bigr)\bigl|\nabla A(u_0)\bigr|^{p(x)}\bigr]\,dx \end{aligned}$$

$$
\begin{aligned}
&+c\iint_{Q_t}\frac{b(x,t)}{p(x)}\left|\nabla A(u_\varepsilon)\right|^{p(x)}dx\,dt\\
&\le c.
\end{aligned}
$$

Thus,

$$\iint_{Q_t}\big(A(u_\varepsilon)\big)_t u_{\varepsilon t}\,dx\,dt=\iint_{Q_t}a(u_\varepsilon)|u_{\varepsilon t}|^2\,dx\,dt\le c. \tag{3.8}$$

By (3.6), $u_\varepsilon \rightharpoonup u$ weakly-* in $L^\infty(Q_T)$. For any $\varphi(x,t)\in C_0^1(Q_T)$, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_T}\frac{\partial}{\partial t}\left(\int_0^{u_\varepsilon}\sqrt{a(s)}\,ds-\int_0^{u}\sqrt{a(s)}\,ds\right)\varphi(x,t)\,dx\,dt\\
&\quad=-\lim_{\varepsilon\to 0}\iint_{Q_T}\int_u^{u_\varepsilon}\sqrt{a(s)}\,ds\varphi_t(x,t)\,dx\,dt\\
&\quad=-\lim_{\varepsilon\to 0}\iint_{Q_T}\sqrt{a(\xi)}(u-u_\varepsilon)\varphi_t(x,t)\,dx\,dt\\
&\quad=0,
\end{aligned}
\tag{3.9}
$$

where $\xi\in(u,u_\varepsilon)$ is the mean value. From (3.9) we can extrapolate that

$$\frac{\partial}{\partial t}\int_0^{u_\varepsilon}\sqrt{a(s)}\,ds\rightharpoonup\frac{\partial}{\partial t}\int_0^{u}\sqrt{a(s)}\,ds\quad\text{in } L^2(Q_T). \tag{3.10}$$

Hence, by (3.6), there exists an n-dimensional vector $\overrightarrow{\zeta}=(\zeta_1,\ldots,\zeta_n)$ such that $\overrightarrow{\zeta}=(\zeta_1,\ldots,\zeta_n)$ and

$$|\overrightarrow{\zeta}|\in L^1\big(0,T;L^{\frac{p(x)}{p(x)-1}}(\Omega)\big),$$

such that

$$b(x,t)\left|A(\nabla u_\varepsilon)\right|^{p(x)-2}\nabla u_\varepsilon\rightharpoonup\overrightarrow{\zeta}\quad\text{in } L^1(Q_T).$$

In order to prove that u is a solution of equation (1.1), we notice that for any function $\varphi\in C_0^1(Q_T)$,

$$\iint_{Q_T}\left[u_{\varepsilon t}\varphi+\big(b(x,t)+\varepsilon\big)\left|\nabla A(u_\varepsilon)\right|^{p(x)-2}\nabla A(u_\varepsilon)\cdot\nabla\varphi\right]dx\,dt=0. \tag{3.11}$$

As $\varepsilon\to 0$, since $b(x,t)$ is a $C^1(\overline{Q_T})$ function with $b(x,t)|_{\partial\Omega\times[0,T]}=0$, $b(x,t)>0,(x,t)\in\Omega\times[0,T]$, we get $c>\max_{\operatorname{supp}\varphi}\frac{|\nabla\varphi|}{b(x,t)}>0$ due to $\varphi\in C_0^\infty(Q_T)$, and accordingly,

$$
\begin{aligned}
&\varepsilon\left|\iint_{Q_T}\left|\nabla A(u_\varepsilon)\right|^{p(x)-2}\nabla A(u_\varepsilon)\cdot\nabla\varphi\,dx\,dt\right|\\
&\quad\le\varepsilon\sup_{\operatorname{supp}\varphi}\frac{|\nabla\varphi|}{b(x,t)}\iint_{Q_T}b(x,t)\big(\left|\nabla A(u_\varepsilon)\right|^{p(x)}+c\big)\,dx\,dt\\
&\quad\to 0,
\end{aligned}
$$

as well as

$$\begin{aligned}
&\iint_{Q_T} \vec{\zeta} \cdot \nabla\varphi \, dx\, dt \\
&= \lim_{\varepsilon\to 0} \iint_{Q_T} b(x,t)\big|\nabla A(u_\varepsilon)\big|^{p(x)-2}\nabla A(u_\varepsilon)\cdot\nabla\varphi dx\, dt \\
&= \lim_{\varepsilon\to 0} \iint_{Q_T} \big(b(x,t)+\varepsilon\big)\big|\nabla A(u_\varepsilon)\big|^{p(x)-2}\nabla A(u_\varepsilon)\cdot\nabla\varphi \, dx\, dt \\
&\quad - \lim_{\varepsilon\to 0} \varepsilon \iint_{Q_T} \big|\nabla A(u_\varepsilon)\big|^{p(x)-2}\nabla A(u_\varepsilon)\cdot\nabla\varphi \, dx\, dt \\
&= \lim_{\varepsilon\to 0} \iint_{Q_T} \big(b(x,t)+\varepsilon\big)\big|\nabla A(u_\varepsilon)\big|^{p(x)-2}\nabla A(u_\varepsilon)\cdot\nabla\varphi \, dx\, dt.
\end{aligned}$$

Now, for any function $\varphi \in C_0^1(Q_T)$,

$$\iint_{Q_T} (u\varphi_t + \vec{\zeta}\cdot\nabla\varphi)\, dx\, dt = 0. \tag{3.12}$$

We shall prove that

$$\begin{aligned}
&\iint_{Q_T} b(x,t)\big|\nabla A(u_\varepsilon)\big|^{p(x)-2}\nabla A(u_\varepsilon)\cdot\nabla\varphi \, dx\, dt \\
&\quad = \iint_{Q_T} \overrightarrow{\zeta} \cdot\nabla\varphi \, dx\, dt.
\end{aligned} \tag{3.13}$$

We choose $0 \leq \psi \in C_0^\infty(Q_T)$ and $\psi = 1$ in $\operatorname{supp}\varphi$, and let $v \in L^\infty(Q_T), b(x,t)|\nabla A(v)|^{p(x)} \in L^1(Q_T)$. Then

$$\begin{aligned}
&\iint_{Q_T} \psi\big(b(x,t)+\varepsilon\big)\big(\big|\nabla A(u_\varepsilon)\big|^{p(x)-2}\nabla A(u_\varepsilon) \\
&\quad - \big|\nabla A(v)\big|^{p(x)-2}\nabla A(v)\big)\cdot\big(\nabla A(u_\varepsilon) - \nabla A(v)\big)\, dx\, dt \\
&\quad \geq 0.
\end{aligned} \tag{3.14}$$

Let $\varphi = \psi A(u_\varepsilon)$ in (3.11). Then

$$\begin{aligned}
&\iint_{Q_T} \psi\big(b(x,t)+\varepsilon\big)\big|\nabla A(u_\varepsilon)\big|^{p(x)}\, dx\, dt \\
&\quad = \iint_{Q_T} \psi_t \mathbb{A}(u_\varepsilon)\, dx\, dt \\
&\quad\quad - \iint_{Q_T} \big(b(x,t)+\varepsilon\big)A(u_\varepsilon)\big|\nabla A(u_\varepsilon)\big|^{p(x)-2}\nabla A(u_\varepsilon)\nabla\psi \, dx\, dt.
\end{aligned} \tag{3.15}$$

Accordingly,

$$\begin{aligned}
&\iint_{Q_T} \psi_t \mathbb{A}(u_\varepsilon)\, dx\, dt - \iint_{Q_T} \big(b(x,t)+\varepsilon\big)A(u_\varepsilon)(\big|\nabla A(u_\varepsilon)\big|^{p(x)-2}\nabla A(u_\varepsilon)\cdot\nabla\psi \, dx\, dt \\
&\quad - \iint_{Q_T} \big(b(x,t)+\varepsilon\big)\psi\big|\nabla A(u_\varepsilon)\big|^{p(x)-2}\nabla A(u_\varepsilon)\nabla A(v)\, dx\, dt
\end{aligned}$$

$$
\begin{aligned}
&- \iint_{Q_T} \bigl(b(x,t)+\varepsilon\bigr)\psi \bigl|\nabla A(v)\bigr|^{p(x)-2}\nabla A(v)\cdot \nabla\bigl(A(u_\varepsilon)-A(v)\bigr)\,dx\,dt \\
&\geq 0. \qquad (3.16)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\iint_{Q_T} \psi_t \mathbb{A}(u_\varepsilon)\,dx\,dt - \iint_{Q_T} \bigl(b(x,t)+\varepsilon\bigr)A(u_\varepsilon)(\bigl|\nabla A(u_\varepsilon)\bigr|^{p(x)-2}\nabla A(u_\varepsilon)\cdot\nabla\psi\,dx\,dt \\
&\quad - \iint_{Q_T} \bigl(b(x,t)+\varepsilon\bigr)\psi\bigl|\nabla A(u_\varepsilon)\bigr|^{p(x)-2}\nabla A(u_\varepsilon)\nabla A(v)\,dx\,dt \\
&\quad - \iint_{Q_T} \psi b(x,t)\bigl|\nabla A(v)\bigr|^{p(x)-2}\nabla A(v)\cdot\bigl(\nabla A(u_\varepsilon)-\nabla A(v)\bigr)\,dx\,dt \\
&\quad - \varepsilon\iint_{Q_T} \psi\bigl|\nabla A(v)\bigr|^{p(x)-2}\nabla A(v)\cdot\bigl(\nabla A(u_\varepsilon)-\nabla A(v)\bigr)\,dx\,dt \\
&\geq 0. \qquad (3.17)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\varepsilon\left|\iint_{Q_T} \psi\bigl|\nabla A(v)\bigr|^{p(x)-2}\nabla A(v)\cdot\bigl(\nabla A(u_\varepsilon)-\nabla A(v)\bigr)\,dx\,dt\right| \\
&\leq \varepsilon \sup_{(x,t)\in Q_T}\frac{|\psi|}{b(x,t)}\iint_{Q_T} b(x,t)\bigl|\nabla A(v)\bigr|^{p(x)-1}\bigl|\nabla A(u_\varepsilon)-\nabla A(v)\bigr|\,dx\,dt \\
&\leq \varepsilon \sup_{(x,t)\in Q_T}\frac{|\psi|}{b(x,t)}\biggl(\iint_{Q_T} b(x,t)\bigl|\nabla A(v)\bigr|^{p(x)}\,dx\,dt \\
&\quad + \iint_{Q_T} b(x,t)\bigl|\nabla A(v)\bigr|^{p(x)-1}\bigl|\nabla A(u_\varepsilon)\bigr|\,dx\,dt\biggr) \qquad (3.18)
\end{aligned}
$$

converges to 0 when $\varepsilon\to 0$, we have

$$
\begin{aligned}
&\iint_{Q_T} \psi_t\mathbb{A}(u)\,dx\,dt - \iint_{Q_T} A(u)\overrightarrow{\zeta}\cdot\nabla\psi\,dx\,dt \\
&\quad - \iint_{Q_T} \psi\overrightarrow{\zeta}\cdot\nabla A(v)\,dx\,dt \\
&\quad - \iint_{Q_T} \psi b(x,t)\bigl|\nabla A(v)\bigr|^{p(x)-2}\nabla A(v)\cdot\bigl(\nabla A(u)-\nabla A(v)\bigr)\,dx\,dt \\
&\geq 0.
\end{aligned}
$$

Let $\varphi=\psi A(u)$ in (3.12). We obtain

$$
\begin{aligned}
&\iint_{Q_T} \psi\overrightarrow{\zeta}\cdot\nabla A(u)\,dx\,dt - \iint_{Q_T}\mathbb{A}(u)\psi_t\,dx\,dt + \iint_{Q_T} A(u)\overrightarrow{\zeta}\cdot\nabla\psi\,dx\,dt \\
&= 0.
\end{aligned}
$$

Accordingly,

$$
\iint_{Q_T} \psi\bigl(\overrightarrow{\zeta}-b(x,t)\bigl|\nabla A(v)\bigr|^{p(x)-2}\nabla A(v)\bigr)\cdot\bigl(\nabla A(u)-\nabla A(v)\bigr)\,dx\,dt\geq 0. \qquad (3.19)
$$

Let $A(v) = A(u) - \lambda\varphi, \lambda > 0, \varphi \in C_0^1(Q_T)$, or equivalently, $v = A^{-1}(A(u) - \lambda\varphi)$. Then

$$\iint_{Q_T} \psi\big(\overrightarrow{\zeta} - b(x,t)\big|\nabla\big(A(u) - \lambda\varphi\big)\big|^{p(x)-2}\nabla\big(A(u) - \lambda\varphi\big)\big) \cdot \nabla\varphi \,dx\,dt \geq 0.$$

If $\lambda \to 0$, then

$$\iint_{Q_T} \psi\big(\overrightarrow{\zeta} - b(x,t)\big|\nabla A(u)\big|^{p(x)-2}\nabla A(u)\big) \cdot \nabla\varphi \,dx\,dt \geq 0.$$

Moreover, if $\lambda < 0$, similarly we can get

$$\iint_{Q_T} \psi\big(\overrightarrow{\zeta} - b(x,t)\big|\nabla A(u)\big|^{p(x)-2}\nabla A(u)\big) \cdot \nabla\varphi \,dx\,dt \leq 0.$$

Thus,

$$\iint_{Q_T} \psi\big(\overrightarrow{\zeta} - b(x,t)\big|\nabla A(u)\big|^{p(x)-2}\nabla A(u)\big) \cdot \nabla\varphi \,dx\,dt = 0.$$

Noticing that $\psi = 1$ on $\operatorname{supp}\varphi$, (3.13) holds.

At last, let us prove the initial value condition (1.4) in the sense of (2.3). For any $0 \leq t_1 < t_2 < T$, by (3.8),

$$\begin{aligned}
&\int_\Omega \left| \int_0^{u_\varepsilon(x,t_2)} \sqrt{a(s)} - \int_0^{u_\varepsilon(x,t_1)} \sqrt{a(s)}\,ds \right| dx \\
&\quad\leq (t_2 - t_1) \int_\Omega \left| \int_0^1 \frac{\partial}{\partial s} \int_0^{u_\varepsilon(x,st_2+(1-s)t_1)} \sqrt{a(s)}\,ds \right| dx \\
&\quad\leq (t_2 - t_1) \int_\Omega \int_0^1 \left| \frac{\partial}{\partial s} \int_0^{u_\varepsilon(x,st_2+(1-s)t_1)} \sqrt{a(s)} \right| ds\,dx \\
&\quad\leq (t_2 - t_1) \int_0^T \int_\Omega \left| \frac{\partial}{\partial t} \int_0^{u_\varepsilon(x,s)} \sqrt{a(s)} \right| ds\,dx\,dt \\
&\quad\leq (t_2 - t_1) \left(\int_0^T \int_\Omega \left|\sqrt{a(u_\varepsilon)}|u_{\varepsilon t}\right|^2 dx\,dt \right)^{\frac{1}{2}} \\
&\quad\leq c(t_2 - t_1).
\end{aligned} \tag{3.20}$$

Thus u satisfies equation (1.1) in the sense of Definition 2.2. □

4 Stability theorem

Proof of Theorem 2.4 Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. For any given positive integer n, let $S_n(s)$ be an odd function, and

$$S_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2s^2e^{1-n^2s^2}, & 0 \leq s \leq \frac{1}{n}, \end{cases}$$

$$H_n(s) = \int_0^s S_n(s)\,ds.$$

Clearly,

$$\lim_{n\to 0} S_n(s) = \operatorname{sgn}(s), \quad s \in (-\infty, +\infty).$$

Denote $\Omega_{\lambda t} = \{x \in \Omega : b(x,t) > \lambda\}$ for any $\lambda > 0$, and define

$$\phi_n(x,t) = \begin{cases} 1, & \text{if } x \in \Omega_{\frac{2}{n}t}, \\ n(b(x,t) - \frac{1}{n}), & \text{if } x \in \Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}, \\ 0, & \text{if } x \in \Omega \setminus \Omega_{\frac{1}{n}t}. \end{cases}$$

By a limiting procedure, we can choose $\phi_n S_n(A(u) - A(v))$ as a test function, and get

$$\begin{aligned} &\int_0^t \int_\Omega \phi_n(x,t) S_n\bigl(A(u) - A(v)\bigr) \frac{\partial(u-v)}{\partial t}\, dx\, dt \\ &\quad + \int_0^t \int_\Omega b(x,t)\bigl(\bigl|\nabla A(u)\bigr|^{p(x)-2}\nabla A(u) - \bigl|\nabla A(v)\bigr|^{p(x)-2}\nabla Av\bigr) \\ &\quad \times \nabla\bigl(A(u) - A(v)\bigr) S_n'\bigl(A(u) - A(v)\bigr)\phi_n(x,t)\, dx\, dt \\ &\quad + \int_0^t \int_\Omega b(x,t)\bigl(\bigl|\nabla A(u)\bigr|^{p(x)-2}\nabla A(u) - \bigl|\nabla A(v)\bigr|^{p(x)-2}\nabla A(v)\bigr) \\ &\quad \times S_n\bigl(A(u) - A(v)\bigr)\nabla\phi_n(x,t)\, dx\, dt \\ &= 0. \end{aligned} \tag{4.1}$$

Thus, since $A(r) \geq 0$ is a monotone increasing function,

$$\begin{aligned} &\lim_{n\to\infty} \int_0^t \int_\Omega \phi_n(x,t) S_n\bigl(A(u) - A(v)\bigr) \frac{\partial(u-v)}{\partial t}\, dx\, dt \\ &= \int_0^t \int_\Omega \operatorname{sgn}\bigl(A(u) - A(v)\bigr) \frac{\partial(u-v)}{\partial t}\, dx\, dt \\ &= \int_0^t \int_\Omega \operatorname{sgn}(u-v) \frac{\partial(u-v)}{\partial t}\, dx\, dt \\ &= \int_\Omega \bigl|u(x,t) - v(x,t)\bigr|\, dx - \int_\Omega \bigl|u_0(x) - v_0(x)\bigr|\, dx. \end{aligned} \tag{4.2}$$

Certainly, we have

$$\begin{aligned} &\int_0^t \int_\Omega b(x,t)\bigl(\bigl|\nabla A(u)\bigr|^{p(x)-2}\nabla A(u) - \bigl|\nabla A(v)\bigr|^{p(x)-2}\nabla A(v)\bigr) \\ &\quad \times \nabla\bigl(A(u) - A(v)\bigr) S'_n\bigl(A(u) - A(v)\bigr)\phi_n(x,t)\, dx\, dt \\ &\geq 0. \end{aligned} \tag{4.3}$$

Denote $q(x) = \frac{p(x)}{p(x)-1}$, for any $t \in [0,T)$, $|\nabla\phi_n(x,t)| = \frac{1}{\lambda}\nabla b(x,t)$ when $x \in \Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}$; elsewhere it is identically set to zero. Then we have

$$\biggl|\int_0^t \int_\Omega b(x,t)\bigl(\bigl|\nabla A(u)\bigr|^{p(x)-2}\nabla A(u)$$

$$
\begin{aligned}
&- \left|\nabla A(v)\right|^{p(x)-2}\nabla A(v)\big) \cdot \nabla \phi_n(x,t) S_n\big(A(u)-A(v)\big)\,dx\,dt \Big| \\
&= \left| \int_0^t \int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x,t)\big(\left|\nabla A(u)\right|^{p(x)-2}\nabla A(u) \right. \\
&\quad \left. - \left|\nabla A(v)\right|^{p(x)-2}\nabla A(v)\big) \cdot \nabla \phi_n g_n\big(A(u)-A(v)\big)\,dx\,dt \right| \\
&\leq \int_0^t n \int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}} t} b(x,t)\left|\nabla A(u)\right|^{p(x)-1} + \left|\nabla A(v)\right|^{p(x)-1} \left|\nabla b S_n\big(A(u)-A(v)\big)\right|\,dx \\
&\leq c \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x,t)\left|\nabla A(u)\right|^{p(x)} \right)^{\frac{1}{q^+}} + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x,t)\left|\nabla A(v)\right|^{p(x)} \right)^{\frac{1}{q^+}} \right] dt \\
&\quad \times \int_0^t n \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x,t)\left|\nabla b(x,t)\right|^{p(x)}\,dx \right)^{\frac{1}{p^+}} dt \\
&\leq c \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x,t)\left|\nabla A(u)\right|^{p(x)} \right)^{\frac{1}{q^+}} + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x,t)\left|\nabla A(v)\right|^{p(x)} \right)^{\frac{1}{q^+}} \right] dt \\
&\quad \times \int_0^t n^{1-\frac{1}{p^+}} \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} \left|\nabla b(x,t)\right|^{p(x)}\,dx \right)^{\frac{1}{p^+}} dt \\
&\leq c \int_0^t \left[\left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x,t)\left|\nabla A(u)\right|^{p(x)} \right)^{\frac{1}{q^+}} \right. \\
&\quad \left. + \left(\int_{\Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}} b(x,t)\left|\nabla A(v)\right|^{p(x)} \right)^{\frac{1}{q^+}} \right] dt,
\end{aligned}
\tag{4.4}
$$

which goes to 0 as $n \to 0$.

Now, let $n \to \infty$ in (4.1). Then

$$
\int_\Omega \left|u(x,t)-v(x,t)\right|\,dx \leq \int_\Omega \left|u_0(x)-v_0(x)\right|\,dx, \quad \forall t \in [0,T). \qquad \square
$$

5 Conclusion

The well-posedness of weak solutions to a double degenerate parabolic equation is studied in this paper. Comparing with the related works in this field, the equation considered in this paper is more general and has wider applications. It includes the nonlinear heat conduction equation, the reaction–diffusion equation, the non-Newtonian fluid equation, and the electrorheological fluid equation, etc. Though the method used in this paper seems quite standard, there are still some essential innovations. For example, the initial value condition is satisfied in a special sense and the stability of weak solutions can be proved without any boundary value condition. Certainly, since we assume that $b(x,t)|_{x\in\Omega} > 0$ and $A(s)$ is a strictly monotone increasing function, it excludes the strongly degenerate hyperbolic–parabolic mixed-type equations. It is well-known that for such equations, only under the entropy conditions, the uniqueness of a weak solution can be true; one can refer to the references [37–41] for the details. Thus, if it is only assumed that $a(s) \geq 0$ or $b(x,t)$ is degenerate in the interior of Ω, proving the uniqueness of a weak solution to equation (1.1) is a quite interesting and challenging problem. By the way, since equation (1.1) is isotropic, generalizing the method used in this paper to an anisotropic parabolic equation also seems very interesting. If $A(s) = s$ and $b(x,t) = b(x)$, some progress has been made in [42, 43] in recent years.

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