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Ground state sign-changing solutions for a class of double-phase problem in bounded domains

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Abstract
This paper is concerned with the following double-phase problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $N \geq 2$ and $1 < p < q < N$. Assuming that the primitive of $f(x,u)$ is asymptotically q-linear as $|u| \to \infty$ and a weak version of Nehari-type monotonicity condition that the function $u \mapsto \frac{f(x,u)}{|u|^{q-1}}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$ for a.e. $x \in \Omega$, we prove the existence of one ground state sign-changing solution via the constraint variational method and quantitative deformation lemma for the equation. Our results improve and generalize some results obtained by Liu and Dai (J. Differ. Equ. 265(9):4311–4334, 2018).

Keywords: Double-phase problem; Musielak–Orlicz space; Variational method; Ground state sign-changing solutions; Nehari manifold; Perturbation method

1 Introduction and main results

Differential equations and variational problems with double phase operator are a new and interesting topic. It arises from the nonlinear elasticity theory, strongly anisotropic materials, Lavrentiev's phenomenon, and so on (see [2–5]). The study on double-phase problems attracts more and more interest in recent years, and many results have been obtained [1, 6–10]. More precisely, the research is related to the energy functional

$$u \mapsto \int_{\Omega} \left(|\nabla u|^p + a(x)|\nabla u|^q\right) dx, \tag{1}$$

where the integrand switches between two different elliptic behaviors. In [5], energies of the form (1) are used in the context of homogenization and elasticity, and the function a drives the geometry of a composite of two different materials with hardening powers p and q.

In this paper, we are concerned with the existence of sign-changing solutions of the double-phase problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $1 < p < q < N$, and

$$\frac{q}{p} < 1 + \frac{1}{N}, \qquad a : \overline{\Omega} \mapsto [0,+\infty) \quad \text{is Lipschitz continuous,} \tag{2}$$

and $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function satisfying the following assumptions:

(h_1) $f(x,t) = o(|t|^{p-2}t)$ as $t \to 0$ uniformly in $x \in \Omega$;

(h_2) there exist $q < r < p^*$ and some positive constant C such that

$$\big|f(x,t)\big| \leq C\big(1 + |t|^{r-1}\big),$$

where $p^* = \frac{Np}{N-p}$ is the critical exponent.

(h_3) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^q} = +\infty$ uniformly in $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)\,ds$;

(h_4) the function $t \mapsto \frac{f(x,t)}{|t|^{q-1}}$ is nondecreasing on $(-\infty,0) \cup (0,+\infty)$ for a.e. $x \in \Omega$.

The solution of (P) is understand in the weak sense, that is, $u \in W_0^{1,H}(\Omega)$ is a solution of (P) if

$$\begin{aligned} &\int_\Omega \big(|\nabla u|^{p-2}\nabla u \cdot \nabla v + a(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v\big)\,dx \\ &\quad = \int_\Omega f(x,u)v\,dx, \quad \forall v \in W_0^{1,H}(\Omega), \end{aligned}$$

where $W_0^{1,H}(\Omega)$ will be defined in Sect. 2.

Note that energy functional φ associated with (P) is defined by

$$\varphi(u) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\right)dx - \int_\Omega F(x,u)\,dx.$$

It is a well-known consequence of (h_1) and (h_2) that $\varphi \in C^1(W_0^{1,H}(\Omega),\mathbb{R})$ and the critical points of φ are weak solutions of (P). Furthermore, if $u \in W_0^{1,H}(\Omega)$ is a solution of (P) and $u^\pm \neq 0$, then u is a sign-changing solution of (P), where

$$u^+(x) := \max\big\{u(x),0\big\} \quad \text{and} \quad u^-(x) := \min\big\{u(x),0\big\}.$$

To facilitate the narrative, we set

$$\begin{aligned} \mathbb{M}_0 &:= \big\{u \in W_0^{1,H}(\Omega) : u^\pm \neq 0, \langle\varphi'(u),u^+\rangle = \langle\varphi'(u),u^-\rangle = 0\big\}, \\ \mathbb{N}_0 &:= \big\{u \in W_0^{1,H}(\Omega) : u \neq 0, \langle\varphi'(u),u\rangle = 0\big\}, \end{aligned}$$

and put

$$m_0 := \inf_{u\in\mathbb{M}_0} \varphi(u), \qquad n_0 := \inf_{u\in\mathbb{N}_0} \varphi(u).$$

Let us recall some previous results that led us to the present research. The first result is due to Perera and Squassina [6], who considered the following form of (P) with the q-superlinear nonlinearity:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) \\ \quad = \lambda|u|^{p-2}u + |u|^{r-2}u + h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_1}$$

Applying the Morse theory, they proved that (P_1) has a nontrivial solution by assuming that either

(T_1) $\lambda \notin \{\lambda_k\}_{k=1}^{\infty}$; or

(T_2) for some $\delta > 0$, $\frac{|t|^r}{r} + H(x,t) \leq 0$ for a.e. $x \in \Omega$ and $|t| \leq \delta$; or

(T_3) $\frac{|t|^r}{r} + H(x,t) \geq c|t|^s$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$ for some $s \in (p,q)$ and $c > 0$.

Recently, Liu and Dai [1] investigated the sign-changing ground state solution of (P) under (h_1), (h_2), (h_3), and

(h_4)′ the function $t \mapsto \frac{f(x,t)}{|t|^{q-1}}$ is strictly increasing on $(-\infty, 0) \cup (0, +\infty)$.

Additionally, Liu and Dai [9] also obtained the existence of at least three ground state solutions of (P) by using the strong maximum principle for the homogeneous double-phase problem.

It is a well-known consequence of (h_4)′ that there is unique $t_u > 0$ such that $t_u u \in \mathbb{N}_0$ for every $u \in W_0^{1,H}(\Omega) \setminus \{0\}$, which implies that φ has at most one minimizer on $\mathbb{M}_0$. Moreover, (h_4)′ plays a crucial role in [1]. In fact, condition (h_4)′ implies that every minimizer of φ on $\mathbb{M}_0$ is a critical point. However, if $t \mapsto \frac{f(x,t)}{|t|^{q-1}}$ is nonstrictly increasing, then t_u and minimizer of φ on $\mathbb{M}_0$ may not be unique, and their arguments become invalid.

Motivated by the aforementioned works, in the present paper, our goal is to generalize the results mentioned to (P) under a weaker assumption. Precisely, we obtain following results.

Theorem 1.1 *Assume that* (h_1)–(h_4) *hold. Then problem* (P) *has a sign-changing solution* $u_0 \in \mathbb{M}_0$ *such that*

$$\varphi(u_0) = \inf_{u \in \mathbb{M}_0} \varphi(u).$$

Furthermore, suppose that

$$\frac{1}{q} f(x,t)t - F(x,t) > 0, \quad \forall x \in \Omega, t \neq 0, \tag{3}$$

then u_0 *has precisely two nodal domains.*

Theorem 1.2 *Assume that* (h_1)–(h_4) *hold. Then* $m_0 \geq 2n_0$.

The rest of this paper is organized as follows. In Sect. 2, we present some necessary preliminary knowledge on space $W_0^{1,H}(\Omega)$. In Sect. 3, we give some preliminary lemmas needed for the proofs of our main results. We complete the proofs of Theorems 1.1–1.2 in Sect. 4.

2 Preliminaries

To discuss problem (P), we need some facts on the space $W_0^{1,H}(\Omega)$, which is called the Musielak–Orlicz–Sobolev space. For this reason, we recall some properties involving the Musielak–Orlicz spaces, which can be found in [10–14] and references therein.

Denote by $N(\Omega)$ the set of all generalized N-functions. For $1 < p < q$ and $0 \le a(\cdot) \in L^1(\Omega)$, we define

$$H(x,t) = t^p + a(x)t^q, \quad (x,t) \in \Omega \times [0,+\infty).$$

It is clear that $H \in N(\Omega)$ is locally integrable and

$$H(x,2t) \le 2^q H(x,t), \quad (x,t) \in \Omega \times [0,+\infty),$$

which is called condition $(\triangle_2)$.

The Musielak–Orlicz space $L^H(\Omega)$ is defined by

$$L^H(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega H\big(x,|u|\big)\,dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$|u|_H = \inf\left\{ \lambda > 0 : \int_\Omega H\left(x, \left|\frac{u}{\lambda}\right|\right) dx \le 1 \right\}.$$

The Musielak–Orlicz–Sobolev space $W^{1,H}(\Omega)$ is defined by

$$W^{1,H}(\Omega) = \big\{ u \in L^H(\Omega) : |\nabla u| \in L^H(\Omega) \big\}$$

and is equipped with the norm

$$\|u\| = |u|_H + |\nabla u|_H. \tag{4}$$

We denote by $W_0^{1,H}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,H}(\Omega)$. With these norms, the spaces $L^H(\Omega)$, $W_0^{1,H}(\Omega)$ and $W^{1,H}(\Omega)$ are separable reflexive Banach spaces; see [10] for the details.

Proposition 2.1 ([1, Proposition 2.1]) *Set* $\rho_H(u) = \int_\Omega (|u|^p + a(x)|u|^q)\,dx$. *For* $u \in L^H(\Omega)$, *we have*:

- (i) *For* $u \ne 0$, $|u|_H = \lambda \Leftrightarrow \rho_H(\frac{u}{\lambda}) = 1$;
- (ii) $|u|_H < 1 (= 1; > 1) \Leftrightarrow \rho_H(u) < 1 (= 1; > 1)$;
- (iii) *If* $|u|_H \ge 1$, *then* $|u|_H^p \le \rho_H(u) \le |u|_H^q$;
- (iv) *If* $|u|_H \le 1$, *then* $|u|_H^q \le \rho_H(u) \le |u|_H^p$.

Proposition 2.2 ([11, Propositions 2.15 and 2.18])

- (1) *If* $1 \le \vartheta \le p^*$, *then the embedding from* $W_0^{1,H}(\Omega)$ *to* $L^\vartheta(\Omega)$ *is continuous. In particular, if* $\vartheta \in [1, p^*)$, *then the embedding* $W_0^{1,H}(\Omega) \hookrightarrow L^\vartheta(\Omega)$ *is compact.*

(2) *Assume that* (2) *holds. Then the Poincaré's inequality holds, that is, there exists a positive constant* C_0 *such that*

$$|u|_H \leq C_0 |\nabla u|_H, \quad u \in W_0^{1,H}(\Omega).$$

By this lemma there exists $c_\vartheta > 0$ such that

$$|u|_\vartheta \leq c_\vartheta \|u\|, \quad \forall u \in W_0^H(\Omega),$$

where $|u|_s$ denotes the usual norm in $L^\vartheta(\Omega)$ for $1 \leq \vartheta < p^*$. It follows from (2) of Proposition 2.2 that $|\nabla u|_H$ is an equivalent norm in $W_0^{1,H}(\Omega)$. We will use the equivalent norm in the following discussion and write $\|u\| = |\nabla u|_H$ for simplicity.

To discuss problem (P), we need to define a functional in $W_0^{1,H}(\Omega)$:

$$J(u) = \int_\Omega \left(\frac{1}{p} |\nabla u|^p + \frac{a(x)}{q} |\nabla u|^q \right) dx.$$

We know that (see [15, p. 63, example]) $J \in C^1(W_0^{1,H}(\Omega), \mathbb{R})$ and the double-phase operator $-\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)$ is the derivative operator of J in the weak sense. We denote $L = J' : W_0^{1,H}(\Omega) \to (W_0^{1,H}(\Omega))^*$. Then

$$\langle L(u), v \rangle = \int_\Omega \left(|\nabla u|^{p-2} \nabla u \cdot \nabla v + a(x) |\nabla u|^{q-2} \nabla u \cdot \nabla v \right) dx$$

for all $u, v \in W_0^{1,H}(\Omega)$. Here $(W_0^{1,H}(\Omega))^*$ denotes the dual space of $W_0^{1,H}(\Omega)$, and $\langle \cdot, \cdot \rangle$ denotes the pairing between $W_0^{1,H}(\Omega)$ and $(W_0^{1,H}(\Omega))^*$. Then we have the following:

Proposition 2.3 ([1, Proposition 3.1]) *Let* $E = W_0^{1,H}(\Omega)$, *and let* L *be as before. Then*

(1) $L : E \to E^*$ *a continuous, bounded, and strictly monotone operator.*

(2) $L : E \to E^*$ *is a mapping of type* $(S)_+$, *that is, if* $u_n \rightharpoonup u$ *in* E *and* $\limsup_{n\to+\infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0$, *then* $u_n \to u$ *in* E.

(3) $L : E \to E^*$ *is a homeomorphism.*

3 Some preliminary lemmas

In this section, we give some preliminary lemmas crucial for proving our results.

Lemma 3.1 *If assumptions* (h_1)–(h_4) *hold, then*

$$\begin{aligned} \varphi(u) \geq{}& \varphi\left(su^+ + tu^-\right) + \frac{1-s^q}{q} \langle \varphi'(u), u^+ \rangle + \frac{1-t^q}{q} \langle \varphi'(u), u^- \rangle \\ &+ \int_\Omega g(s) \left|\nabla u^+\right|^p dx + \int_\Omega g(t) \left|\nabla u^-\right|^p dx, \\ &\forall u = u^+ + u^- \in E, s, t \geq 0, \end{aligned} \tag{5}$$

where $g(\tau) = \frac{1-\tau^p}{p} - \frac{1-\tau^q}{q}$, $\tau \geq 0$.

Proof By condition (h_4) we have

$$\begin{aligned}
&\frac{1-t^q}{q}\tau f(x,\tau)+F(x,t\tau)-F(x,\tau)\\
&\quad=\int_t^1 f(x,\tau)s^{q-1}\tau\,ds-\int_t^1 f(x,\tau s)\tau\,ds\\
&\quad=\int_t^1\left[\frac{f(x,\tau)}{|\tau|^{q-1}}-\frac{f(x,\tau s)}{|\tau s|^{q-1}}\right]s^{q-1}|\tau|^{q-1}\tau\,ds\\
&\quad\geq 0,\quad t\geq 0,\tau\in\mathbb{R}\setminus\{0\}.
\end{aligned}\tag{6}$$

Clearly, $g(t)\geq g(1)=0$ for any $t\geq 0$. Hence from (6) it follows that

$$\begin{aligned}
&\varphi(u)-\varphi\big(su^++tu^-\big)\\
&\quad=\int_\Omega\left(\frac{1}{p}\big|\nabla u^+\big|^p+\frac{a(x)}{q}\big|\nabla u^+\big|^q\right)dx-\int_\Omega F\big(x,u^+\big)\,dx\\
&\qquad+\int_\Omega\left(\frac{1}{p}\big|\nabla u^-\big|^p+\frac{a(x)}{q}\big|\nabla u^-\big|^q\right)dx-\int_\Omega F\big(x,u^-\big)\,dx\\
&\qquad-\int_\Omega\left(\frac{s^p}{p}\big|\nabla u^+\big|^p+\frac{a(x)s^q}{q}\big|\nabla u^+\big|^q\right)dx+\int_\Omega F\big(x,su^+\big)\,dx\\
&\qquad-\int_\Omega\left(\frac{t^p}{p}\big|\nabla u^-\big|^p+\frac{a(x)t^q}{q}\big|\nabla u^-\big|^q\right)dx+\int_\Omega F\big(x,tu^-\big)\,dx\\
&\qquad-\frac{1-s^q}{q}\big\langle\varphi'(u),u^+\big\rangle-\frac{1-t^q}{q}\big\langle\varphi'(u),u^-\big\rangle\\
&\qquad+\frac{1-s^q}{q}\big\langle\varphi'(u),u^+\big\rangle+\frac{1-t^q}{q}\big\langle\varphi'(u),u^-\big\rangle\\
&\quad=\int_\Omega g(s)\big|\nabla u^+\big|^p\,dx+\int_\Omega g(t)\big|\nabla u^-\big|^p\,dx\\
&\qquad+\frac{1-s^q}{q}\big\langle\varphi'(u),u^+\big\rangle+\frac{1-t^q}{q}\big\langle\varphi'(u),u^-\big\rangle\\
&\qquad+\int_\Omega\left[\frac{1-s^q}{q}f\big(x,u^+\big)u^++F\big(x,su^+\big)-F\big(x,u^+\big)\right]dx\\
&\qquad+\int_\Omega\left[\frac{1-t^q}{q}f\big(x,u^-\big)u^-+F\big(x,tu^-\big)-F\big(x,u^-\big)\right]dx\\
&\quad\geq\frac{1-s^q}{q}\big\langle\varphi'(u),u^+\big\rangle+\frac{1-t^q}{q}\big\langle\varphi'(u),u^-\big\rangle\\
&\qquad+\int_\Omega g(s)\big|\nabla u^+\big|^p\,dx+\int_\Omega g(t)\big|\nabla u^-\big|^p\,dx.
\end{aligned}$$

The proof is completed. □

From Lemma 3.1 we immediately have the following two corollaries.

Corollary 3.2 *Assume that (h_1)–(h_4) hold. If $u=u^++u^-\in\mathbb{M}_0$, then*

$$\varphi(u)=\varphi\big(u^++u^-\big)=\max_{s,t\geq 0}\varphi\big(su^++tu^-\big).$$

Corollary 3.3 *Assume that* (h_1)–(h_4) *hold. If* $u \in \mathbb{N}_0$, *then*

$$\varphi(u) = \max_{t\geq 0} \varphi(tu).$$

Lemma 3.4 *Assume that* (h_1)–(h_3) *and* $(h_4)'$ *hold. If* $u \in E$ *and* $u^{\pm} \neq 0$, *then there exists a unique pair* (s_u, t_u) *of positive numbers such that*

$$s_u u^+ + t_u u^- \in \mathbb{M}_0.$$

Proof For any $u \in E$ with $u^{\pm} \neq 0$, we consider the functions $g(s,t), h(s,t) : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ given by

$$g(s,t) = \langle \varphi'(su^+ + tu^-), su^+\rangle \quad \text{and} \quad h(s,t) = \langle \varphi'(su^+ + tu^-), tu^-\rangle.$$

We directly compute that

$$\begin{aligned} g(s,t) &= \langle \varphi'(su^+ + tu^-), su^+\rangle \\ &= \int_\Omega \left(s^p|\nabla u^+|^p + a(x)s^q|\nabla u^+|^q\right) dx - \int_\Omega f(x, su^+)su^+\, dx, \\ h(s,t) &= \langle \varphi'(su^+ + tu^-), tu^-\rangle \\ &= \int_\Omega \left(t^p|\nabla u^-|^p + a(x)t^q|\nabla u^-|^q\right) dx - \int_\Omega f(x, tu^-)tu^-\, dx. \end{aligned} \tag{7}$$

Using assumptions (h_1) and (h_2), we deduce that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that, for all $(x,t) \in \Omega \times \mathbb{R}$,

$$\begin{aligned} &|f(x,t)| \leq \varepsilon|t|^{p-1} + C_\varepsilon|t|^{r-1}, \\ &|F(x,t)| \leq \varepsilon|t|^p + C_\varepsilon|t|^r, \end{aligned} \tag{8}$$

where $r \in [1, p^*)$ was given in (h_2).

Thus, for $s > 0$ sufficiently small, by (8) and Proposition 2.2(2) we have

$$\begin{aligned} g(s,t) &= \int_\Omega \left(s^p|\nabla u^+|^p + a(x)s^q|\nabla u^+|^q\right) dx - \int_\Omega f(x, su^+)su^+\, dx \\ &\geq s^q \int_\Omega \left(|\nabla u^+|^p + a(x)|\nabla u^+|^q\right) dx \\ &\quad - \int_\Omega \left(\varepsilon s^p|u^+|^p + C_\varepsilon s^r|u^+|^r\right) dx \\ &\geq \begin{cases} s^q\|u^+\|^q - \varepsilon c_p^p s^p\|u^+\|^p - C_\varepsilon c_r^r s^r\|u^+\|^r & \text{if } \|u^+\| < 1, \\ s^q\|u^+\|^p - \varepsilon c_p^p s^p\|u^+\|^p - C_\varepsilon c_r^r s^r\|u^+\|^r & \text{if } \|u^+\| > 1, \end{cases} \end{aligned} \tag{9}$$

and

$$\begin{aligned} h(s,t) &= \int_\Omega \left(t^p|\nabla u^-|^p + a(x)t^q|\nabla u^-|^q\right) dx - \int_\Omega f(x, tu^-)tu^-\, dx \\ &\geq t^q \int_\Omega \left(|\nabla u^-|^p + a(x)|\nabla u^-|^q\right) dx \end{aligned}$$

$$
\begin{aligned}
&- \int_\Omega \left(\varepsilon t^p \left|u^-\right|^p + C_\varepsilon t^r \left|u^-\right|^r\right) dx \\
&\geq \begin{cases} t^q \|u^-\|^q - \varepsilon c_p^p t^p \|u^-\|^p - C_\varepsilon c_r^r t^r \|u^-\|^r & \text{if } \|u^-\| < 1, \\ t^q \|u^-\|^p - \varepsilon c_p^p t^p \|u^-\|^p - C_\varepsilon c_r^r t^r \|u^-\|^r & \text{if } \|u^-\| > 1. \end{cases}
\end{aligned}
\tag{10}
$$

By (9), (10), and the arbitrariness of ε, it is easy to prove that $g(s,s) > 0$ and $h(s,s) > 0$ for $s > 0$ small.

Moreover, using (6), we have

$$
\frac{1}{q}\tau f(x,\tau) - F(x,\tau) \geq 0, \quad \tau \in \mathbb{R} \setminus \{0\}. \tag{11}
$$

Hence by (11) and (h_3) we have that, for $s > 1$,

$$
\begin{aligned}
g(s,t) &= \int_\Omega \left(s^p \left|\nabla u^+\right|^p + a(x) s^q \left|\nabla u^+\right|^q\right) dx - \int_\Omega f\left(x, su^+\right) su^+ \, dx \\
&\leq s^q \int_\Omega \left(\left|\nabla u^+\right|^p + a(x)\left|\nabla u^+\right|^q\right) dx - q \int_\Omega F\left(x, su^+\right) dx \\
&= s^q \int_\Omega \left(\left|\nabla u^+\right|^p + a(x)\left|\nabla u^+\right|^q\right) dx - q \int_\Omega \frac{F(x, su^+)}{|su^+|^q} \left|su^+\right|^q dx \\
&= s^q \left(\int_\Omega \left(\left|\nabla u^+\right|^p + a(x)\left|\nabla u^+\right|^q\right) dx - q \int_{u^+ \neq 0} \frac{F(x, su^+)}{|su^+|^q} \left|u^+\right|^q dx\right)
\end{aligned}
\tag{12}
$$

and, for $t > 1$,

$$
\begin{aligned}
g(s,t) &= \int_\Omega \left(t^p \left|\nabla u^-\right|^p + a(x) t^q \left|\nabla u^-\right|^q\right) dx - \int_\Omega f\left(x, tu^-\right) tu^- \, dx \\
&\leq t^q \int_\Omega \left(\left|\nabla u^-\right|^p + a(x)\left|\nabla u^-\right|^q\right) dx - q \int_\Omega F\left(x, tu^+\right) dx \\
&= t^q \int_\Omega \left(\left|\nabla u^-\right|^p + a(x)\left|\nabla u^-\right|^q\right) dx - q \int_\Omega \frac{F(x, tu^-)}{|tu^-|^q} \left|tu^-\right|^q dx \\
&= t^q \left(\int_\Omega \left(\left|\nabla u^-\right|^p + a(x)\left|\nabla u^-\right|^q\right) dx - q \int_{u^- \neq 0} \frac{F(x, tu^+)}{|tu^-|^q} \left|u^-\right|^q dx\right),
\end{aligned}
\tag{13}
$$

which yields that $g(t,t) < 0$ and $h(t,t) < 0$ for $t > 0$ large. Thus there are $0 < T < R$ such that

$$
g(T,T), h(T,T) > 0 \quad \text{and} \quad g(R,R), h(R,R) < 0. \tag{14}
$$

This fact, combined with (7), implies that

$$
g(T,t) = g(T,T) > 0, \qquad g(R,t) = g(R,R) < 0, \quad t \in [r,R],
$$

and

$$
h(T,t) = h(T,T) > 0, \qquad h(R,t) = h(R,R) < 0, \quad t \in [r,R].
$$

So, by the Miranda theorem in [16] we can find $(s_u, t_u) \in (T,R) \times (T,R)$ such that $g(s_u, t_u) = h(s_u, t_u) = 0$. Therefore $s_u u^+ + t_u u^- \in \mathbb{M}_0$.

Next, we prove the uniqueness. Let (s_i, t_i) be such that $s_i u^+ + t_i u^- \in \mathbb{M}_0$, $i = 1, 2$, that is,

$$g(s_1, t_1) = h(s_1, t_1) = g(s_2, t_2) = h(s_2, t_2) = 0. \tag{15}$$

Then from (5), (7), and (15) it follows that

$$\begin{aligned}
\varphi(s_1u^+ + t_1u^-) \geq{}& \frac{s_1^q - s_2^q}{qs_1^q}\langle\varphi'(s_1u^+ + t_1u^-), s_1u^+\rangle \\
&+ \frac{t_1^q - t_2^q}{qt_1^q}\langle\varphi'(s_1u^+ + t_1u^-), t_1u^-\rangle \\
&+ \varphi(s_2u^+ + t_2u^-) \\
&+ \left(\frac{s_1^p - s_2^p}{p} - \frac{s_1^q - s_2^q}{qs_1^q}s_1^p\right)\int_\Omega |\nabla u^+|^p\,dx \\
&+ \left(\frac{t_1^p - t_2^p}{p} - \frac{t_1^q - t_2^q}{qt_1^q}t_1^p\right)\int_\Omega |\nabla u^-|^p\,dx \\
={}& \varphi(s_2u^+ + t_2u^-) \\
&+ \left(\frac{s_1^p - s_2^p}{p} - \frac{s_1^q - s_2^q}{qs_1^q}s_1^p\right)\int_\Omega |\nabla u^+|^p\,dx \\
&+ \left(\frac{t_1^p - t_2^p}{p} - \frac{t_1^q - t_2^q}{qt_1^q}t_1^p\right)\int_\Omega |\nabla u^-|^p\,dx
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\varphi(s_2u^+ + t_2u^-) \geq{}& \frac{s_2^q - s_1^q}{qs_2^q}\langle\varphi'(s_2u^+ + t_2u^-), s_2u^+\rangle \\
&+ \frac{t_2^q - t_1^q}{qt_2^q}\langle\varphi'(s_2u^+ + t_2u^-), t_2u^-\rangle \\
&+ \varphi(s_1u^+ + t_1u^-) \\
&+ \left(\frac{s_2^p - s_1^p}{p} - \frac{s_2^q - s_1^q}{qs_2^q}s_2^p\right)\int_\Omega |\nabla u^+|^p\,dx \\
&+ \left(\frac{t_2^p - t_1^p}{p} - \frac{t_2^q - t_1^q}{qt_2^q}t_2^p\right)\int_\Omega |\nabla u^-|^p\,dx \\
={}& \varphi(s_1u^+ + t_1u^-) \\
&+ \left(\frac{s_2^p - s_1^p}{p} - \frac{s_2^q - s_1^q}{qs_2^q}s_2^p\right)\int_\Omega |\nabla u^+|^p\,dx \\
&+ \left(\frac{t_2^p - t_1^p}{p} - \frac{t_2^q - t_1^q}{qt_2^q}t_2^p\right)\int_\Omega |\nabla u^-|^p\,dx.
\end{aligned} \tag{17}$$

Both (16) and (17) imply that $s_1 = s_2$ and $t_1 = t_2$, which in turn implies that (s_u, t_u) is the unique pair of positive numbers such that $s_u u^+ + t_u u^- \in \mathbb{M}_0$. We end the proof. □

Furthermore we have the following:

Lemma 3.5 *Assume that* (h_1)–(h_3) *and* $(h_4)'$ *hold. Then*

$$m_0 = \inf_{u\in\mathbb{M}_0} \varphi(u) = \inf_{u\in E, u^\pm\neq 0} \max_{s,t\geq 0} \varphi\big(su^+ + tu^-\big).$$

Proof By Corollary 3.2 we conclude that

$$\begin{aligned}\inf_{u\in E, u^\pm\neq 0} \max_{s,t\geq 0} \varphi\big(su^+ + tu^-\big) &\leq \inf_{u\in\mathbb{M}_0} \max_{s,t\geq 0} \varphi\big(su^+ + tu^-\big)\\ &= \inf_{u\in\mathbb{M}_0} \varphi(u) = m_0.\end{aligned} \tag{18}$$

Moreover, for any $u \in E$ with $u^\pm \neq 0$, from Lemma 3.4 we deduce that

$$\max_{s,t\geq 0} \varphi\big(su^+ + tu^-\big) \geq \varphi\big(s_u u^+ + t_u u^-\big) \geq \inf_{u\in\mathbb{M}_0} \varphi(u) = m_0,$$

which implies

$$\inf_{u\in E, u^\pm\neq 0} \max_{s,t\geq 0} \varphi\big(su^+ + tu^-\big) \geq \inf_{u\in\mathbb{M}_0} \varphi(u) = m_0. \tag{19}$$

Therefore the conclusion directly follows from (18) and (19). □

Lemma 3.6 *Assume that* (h_1)–(h_3) *and* $(h_4)'$ *hold. Then* $m_0 > 0$ *can be achieved.*

Proof Firstly, we will show that $m_0 > 0$. Indeed, for every $u \in \mathbb{M}_0$, we have $u \in \mathbb{N}_0$ and $\langle\varphi'(u), u\rangle = 0$. Then by (h_1)–(h_2) and Propositions 2.1 and 2.2 we get

$$\begin{aligned}&\varepsilon c_p^p\|u\|^p + C_\varepsilon c_r^r\|u\|^r\\ &\quad\geq \varepsilon|u|_p^p + C_\varepsilon|u|_r^r\\ &\quad\geq \int_\Omega f(x,u)u\,dx\\ &\quad= \int_\Omega \big(|\nabla u|^p + a(x)|\nabla u|^q\big)\,dx\\ &\quad\geq \begin{cases}\|u\|^q & \text{if } \|u\| < 1,\\ \|u\|^p & \text{if } \|u\| > 1.\end{cases}\end{aligned}$$

Thus, for any $u \in \mathbb{N}_0$ with $\|u\| < 1$, we have that

$$\frac{1}{2}\|u\|^q \leq C_\varepsilon c_r^r\|u\|^r,$$

which implies that

$$\|u\| \geq \left(\frac{1}{2C_\varepsilon c_r^r}\right)^{\frac{1}{r-q}} =: \alpha_0.$$

Therefore we obtain that $m_0 = \inf_{u\in\mathbb{M}_0} \varphi(u) \geq \alpha_0 > 0$.

It remains to prove that $u_0 \in \mathbb{M}_0$ and $\varphi(u_0) = m_0$. Let $\{u_n\} \subset \mathbb{M}_0$ be a sequence of functions such that $\varphi(u_n) \to m_0$ as $n \to +\infty$. Firstly, we claim that $\{u_n\}$ is bounded. Suppose, by contradiction, that $\|u_n\| \to +\infty$ and let $v_n = \frac{u_n}{\|u_n\|}$. Without loss of generality, we may assume that $v_n \rightharpoonup v$ in E. By the Sobolev embedding theorem we have

$$v_n \to v \quad \text{in } L^{\vartheta}(\Omega), 1 \le \vartheta < p^*, \qquad v_n \to v \quad \text{a.e. on } \Omega.$$

If $v = 0$, then $v_n \to 0$ in $L^{\vartheta}(\Omega)$ for $1 \le \vartheta < p^*$. Fix $R > [q(m_0 + 1)]^{\frac{1}{p}} (> 1)$. By (h_1)–(h_2) there exists $C_1 > 0$ such that

$$F(x,t) \le |t|^p + C_1 |t|^r, \quad x \in \Omega, t \in \mathbb{R}.$$

Then we have that

$$\limsup_{n\to\infty} \int_{\Omega} F(x, Rv_n)\,dx \le R^p \lim_{n\to\infty} \left(|v_n|_p^p + C_1 R^r |v_n|_r^r\right) = 0. \tag{20}$$

Let $t_n = \frac{R}{\|u_n\|}$. Hence by (20) and Corollary 3.3 we get that

$$\begin{aligned}
m_0 + o(1) = \varphi(u_n) \\
&\ge \varphi(t_n u_n) \\
&= \varphi(Rv_n) \\
&= \int_{\Omega} \left(\frac{1}{p} R^p |\nabla v_n|^p + \frac{a(x)}{q} R^q |\nabla v_n|^q\right) dx - \int_{\Omega} F(x, Rv_n)\,dx \\
&\ge \frac{1}{q} R^p - \int_{\Omega} F(x, Rv_n)\,dx \\
&\ge \frac{1}{q} R^p + o(1) \\
&> m_0 + 1 + o(1),
\end{aligned}$$

which yields a contradiction. Thus $v \ne 0$.

For $x \in \{y \in \mathbb{R}^N : v(y) \ne 0\}$, it is clear that $\lim_{n\to+\infty} |u_n(x)| = +\infty$. By hypotheses (h_1) and (h_2) we can find $C_2 \in \mathbb{R}$ such that

$$F(x,t) \ge C_2, \quad (x,t) \in \Omega \times \mathbb{R}. \tag{21}$$

Hence by using (21), (h_3), Proposition 2.1, and Fatou's lemma we have

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \frac{m + o(1)}{\|u_n\|^q} = \lim_{n\to+\infty} \frac{\varphi(u_n)}{\|u_n\|^q} \\
&\le \lim_{n\to+\infty} \left[\frac{1}{p} \frac{\int_{\Omega} (|\nabla u_n|^p + a(x)|\nabla u_n|^q)\,dx}{\|u_n\|^q} - \int_{\Omega} \frac{F(x,u_n)}{\|u_n\|^q}\,dx\right] \\
&\le \frac{1}{p} - \lim_{n\to+\infty} \int_{\Omega} \frac{F(x,u_n)}{\|u_n\|^q}\,dx \\
&= \frac{1}{p} - \lim_{n\to+\infty} \int_{\Omega} \frac{F(x,u_n) - C_2}{\|u_n\|^q}\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{p} - \liminf_{n\to+\infty} \int_\Omega \frac{F(x,u_n) - C_2}{\|u_n\|^q}\,dx \\
&= \frac{1}{p} - \liminf_{n\to+\infty} \int_\Omega \frac{F(x,u_n)}{\|u_n\|^q}\,dx \\
&\le \frac{1}{p} - \int_\Omega \liminf_{n\to+\infty} \frac{F(x,u_n(x))}{|u_n(x)|^q}\big|v_n(x)\big|^q\,dx \\
&= -\infty.
\end{aligned}
$$

This contradiction shows that $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $u_n^\pm \rightharpoonup u_0^\pm$ in E. Then $u_n^\pm \to u_0^\pm$ in $L^\vartheta(\Omega)$ for $\vartheta \in [1,p^*)$ and $u_n \to u_0$ a.e. on Ω.

Our next goal is to prove that $u_0 \in \mathbb{M}_0$ and $\varphi(u_0) = m_0$. Firstly, we claim that $\inf_{u\in\mathbb{N}_0}\varphi(u) > 0$. Indeed, for every $u \in \mathbb{N}_0$, we have $\langle\varphi'(u),u\rangle = 0$. Then by (h_1), (h_2), and Propositions 2.1 and 2.2 we get

$$
\begin{aligned}
&\varepsilon c_p^p\|u\|^p + C_\varepsilon c_r^r\|u\|^r \\
&\quad\ge \varepsilon|u|_p^p + C_\varepsilon|u|_r^r \\
&\quad\ge \int_\Omega f(x,u)u\,dx \\
&\quad= \int_\Omega \bigl(|\nabla u|^p + a(x)|\nabla u|^q\bigr)\,dx \\
&\quad\ge \begin{cases} \|u\|^q & \text{if } \|u\| < 1, \\ \|u\|^p & \text{if } \|u\| > 1. \end{cases}
\end{aligned}
$$

Thus, for any $u \in \mathbb{N}_0$ with $\|u\| < 1$, we have that

$$
\frac{1}{2}\|u\|^q \le C_\varepsilon c_r^r\|u\|^r,
$$

which implies that $\|u\| \ge \alpha_0$. This implies that $\inf_{u\in\mathbb{N}_0}\varphi(u) > 0$. Note that $\{u_n\}_{n\in N} \subset \mathbb{M}_0$. Then it is obvious that $\{u_n^\pm\}_{n\in N} \subset \mathbb{N}_0$, that is,

$$
\int_\Omega \bigl(\big|\nabla u_n^\pm\big|^p + a(x)\big|\nabla u_n^\pm\big|^q\bigr)\,dx = \int_\Omega f\bigl(x,u_n^\pm\bigr)u_n^\pm\,dx \quad \text{and} \quad \big\|u_n^\pm\big\| \ge \alpha_0.
$$

By (h_1) and (h_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
\big|f(x,t)\big| \le \varepsilon|t|^{p-1} + C_\varepsilon|t|^{r-1} \tag{22}
$$

for all $(x,t) \in \Omega \times \mathbb{R}$, where $r \in [1,p^*)$ was given in (h_2). Thus

$$
\begin{aligned}
&\min\bigl\{\alpha_0^p,\alpha_0^q\bigr\} \\
&\quad\le \min\bigl\{\big\|u_n^\pm\big\|^p,\big\|u_n^\pm\big\|^q\bigr\} \\
&\quad\le \int_\Omega \bigl(\big|\nabla u_n^\pm\big|^p + a(x)\big|\nabla u_n^\pm\big|^q\bigr)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega f\bigl(x,u_n^{\pm}\bigr)u_n^{\pm}\,dx \\
&\le \varepsilon \int_\Omega \bigl|u_n^{\pm}\bigr|^p\,dx + C_\varepsilon \int_\Omega \bigl|u_n^{\pm}\bigr|^r\,dx.
\end{aligned} \tag{23}
$$

Because of the boundedness of u_n, there is $C_1 > 0$ such that

$$
\min\bigl\{\alpha_0^p,\alpha_0^q\bigr\} \le \varepsilon C_1 + C_\varepsilon \int_\Omega \bigl|u_n^{\pm}\bigr|^r\,dx.
$$

Choosing $\varepsilon = \frac{\min\{\alpha_0^p,\alpha_0^q\}}{2C_1}$, we get

$$
\int_\Omega \bigl|u_n^{\pm}\bigr|^r\,dx \ge \frac{\min\{\alpha_0^p,\alpha_0^q\}}{2C_\varepsilon}.
$$

By the compactness of the embedding $E \hookrightarrow L^r(\Omega)$ for $p<q<r<p^*$ we get

$$
\int_\Omega \bigl|u_0^{\pm}\bigr|^r\,dx \ge \frac{\min\{\alpha_0^p,\alpha_0^q\}}{2C_\varepsilon},
$$

which yields $u_0^{\pm} \not\equiv 0$. Moreover, note that $u_n^{\pm} \to u_0^{\pm}$ in $L^\vartheta(\Omega)$, $\vartheta \in [1,p^*)$. By conditions (h_1) and (h_2), combined with the Hölder inequality and Lebesgue theorem, we have

$$
\begin{aligned}
&\lim_{n\to+\infty} \int_\Omega f\bigl(x,u_n^{\pm}\bigr)u_n^{\pm}\,dx = \int_\Omega f\bigl(x,u_0^{\pm}\bigr)u_0^{\pm}\,dx, \\
&\lim_{n\to+\infty} \int_\Omega F\bigl(x,u_n^{\pm}\bigr)\,dx = \int_\Omega F\bigl(x,u_0^{\pm}\bigr)\,dx.
\end{aligned} \tag{24}
$$

Hence by the weak lower semicontinuity of the norm we conclude that

$$
\begin{aligned}
\bigl\langle \varphi'(u_0),u_0^{\pm}\bigr\rangle &= \int_\Omega \bigl(\bigl|\nabla u_0^{\pm}\bigr|^p + a(x)\bigl|\nabla u_0^{\pm}\bigr|^q\bigr)\,dx - \int_\Omega f\bigl(x,u_0^{\pm}\bigr)u_0^{\pm}\,dx \\
&\le \liminf_{n\to+\infty} \int_\Omega \bigl(\bigl|\nabla u_n^{\pm}\bigr|^p + a(x)\bigl|\nabla u_n^{\pm}\bigr|^q\bigr)\,dx \\
&\quad - \lim_{n\to+\infty} \int_\Omega f\bigl(x,u_n^{\pm}\bigr)u_n^{\pm}\,dx \\
&= \liminf_{n\to+\infty} \bigl\langle \varphi'(u_n),u_n^{\pm}\bigr\rangle = 0,
\end{aligned} \tag{25}
$$

because $u_n^{\pm} \in \mathbb{N}_0$. Thus by Lemma 3.4 there exist $s_0,t_0>0$ such that $s_0u_0^+ + t_0u_0^- \in \mathbb{M}_0$. Consequently, from (24) and Lemma 3.1 we have

$$
\begin{aligned}
m_0 &= \lim_{n\to+\infty} \biggl[\varphi(u_n) - \frac{1}{q}\bigl\langle \varphi'(u_n),u_n\bigr\rangle\biggr] \\
&= \lim_{n\to+\infty} \int_\Omega \biggl(\frac{1}{p}-\frac{1}{q}\biggr)|\nabla u_n|^p\,dx + \lim_{n\to+\infty} \int_\Omega \biggl[\frac{1}{q}f(x,u_n)u_n - F(x,u_n)\biggr]\,dx \\
&\ge \liminf_{n\to+\infty} \int_\Omega \biggl(\frac{1}{p}-\frac{1}{q}\biggr)|\nabla u_n|^p\,dx + \lim_{n\to+\infty} \int_\Omega \biggl[\frac{1}{q}f(x,u_n)u_n - F(x,u_n)\biggr]\,dx \\
&\ge \int_\Omega \biggl(\frac{1}{p}-\frac{1}{q}\biggr)|\nabla u_0|^p\,dx + \int_\Omega \biggl[\frac{1}{q}f(x,u_0)u_0 - F(x,u_0)\biggr]\,dx
\end{aligned}
$$

$$\begin{aligned}
&= \varphi(u_0) - \frac{1}{q}\langle\varphi'(u_0), u_0\rangle \\
&\geq \varphi\bigl(s_0u_0^+ + t_0u_0^-\bigr) + \frac{1-s_0^q}{q}\langle\varphi'(u_0), u_0^+\rangle + \frac{1-t_0^q}{q}\langle\varphi'(u_0), u_0^-\rangle \\
&\quad - \frac{1}{q}\langle\varphi'(u_0), u_0\rangle \\
&= \varphi\bigl(s_0u_0^+ + t_0u_0^-\bigr) - \frac{s_0^q}{q}\langle\varphi'(u_0), u_0^+\rangle - \frac{t_0^q}{q}\langle\varphi'(u_0), u_0^-\rangle \\
&\geq m_0 - \frac{s_0^q}{q}\langle\varphi'(u_0), u_0^+\rangle - \frac{t_0^q}{q}\langle\varphi'(u_0), u_0^-\rangle.
\end{aligned}$$

This shows that

$$\frac{s_0^q}{q}\langle\varphi'(u_0), u_0^+\rangle + \frac{t_0^q}{q}\langle\varphi'(u_0), u_0^-\rangle \geq 0.$$

From this and from (25) we conclude that

$$\langle\varphi'(u_0), u_0^\pm\rangle = 0 \quad \text{and} \quad \varphi(u_0) = m_0. \qquad \square$$

Similarly to the proof of [1, Theorem 1.4], we can prove the following lemma.

Lemma 3.7 *Assume that* (h_1)*–*(h_3) *and* $(h_4)'$ *hold. If* $u_0 \in \mathbb{M}_0$ *and* $\varphi(u_0) = m_0$*, then* u_0 *is a critical point of* φ*.*

Proof It is clear that $\langle\varphi'(u_0^\pm), u_0^\pm\rangle = 0 = \langle\varphi'(u_0), u_0^\pm\rangle$. It follows from assumption $(h_4)'$ that, for $0 < s \neq 1$ and $0 < t \neq 1$,

$$\begin{aligned}
\varphi\bigl(su_0^+ + tu_0^-\bigr) &= \varphi\bigl(su_0^+\bigr) + \varphi\bigl(tu_0^-\bigr) \\
&< \varphi\bigl(u_0^+\bigr) + \varphi\bigl(u_0^-\bigr) \\
&= \varphi(u_0) = m_0.
\end{aligned} \tag{26}$$

If $\varphi'(u_0) \neq 0$, then there exist $\delta > 0$ and $\nu > 0$ such that

$$\|v - u_0\| \leq 3\delta \quad \Rightarrow \quad \bigl\|\varphi'(v)\bigr\| \geq \nu.$$

Let $D = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g(s,t) = su_0^+ + tu_0^-$. By (26) we have

$$\beta = \max_{(s,t)\in\partial D} \varphi\bigl(g(s,t)\bigr) < m_0. \tag{27}$$

Let $\varepsilon := \min\{\frac{m_0-\beta}{4}, \frac{\lambda\delta}{8}\}$ and $B(u,\delta) := \{v \in E : \|v - u\| \leq \delta\}$. Then [17, Lemma 2.3] yields a deformation η such that

(a) $\eta(1, v) = v$ if $\varphi(v) < m_0 - 2\varepsilon$ or $\varphi(v) > m_0 + 2\varepsilon$,
(b) $\eta(1, \varphi^{m_0+\varepsilon} \cap B(u,\delta)) \subset \varphi^{m_0-\varepsilon}$, and
(c) $\varphi(\eta(1, v)) \leq \varphi(v)$ for all $v \in E$,

where $\varphi^{m_0\pm\varepsilon} := \{v \in E : \varphi(v) \leq m_0 \pm \varepsilon\}$.

It is easy to see that

$$\max_{(s,t)\in D} \varphi\bigl(\eta\bigl(1, g(s,t)\bigr)\bigr) < m_0.$$

Next, we show that $\eta(1, g(D)) \cap \mathbb{M}_0 \neq \emptyset$, contradicting the definition of m_0. Let $h(s,t) = \eta(1, g(s,t))$, $\varphi_0(s,t) = \langle \varphi'(su_0^+)u_0^+, \varphi'(su_0^-)u_0^- \rangle$, and $\varphi_1(s,t) = \langle \frac{1}{s}\varphi'(h^+(s,t)), \frac{1}{t}\varphi'(h^-(s,t))\rangle$. Note that

$$\begin{aligned} &\bigl\langle \varphi'\bigl(tu_0^\pm\bigr), u_0^\pm \bigr\rangle > 0 \quad \text{if } 0 < t < 1, \\ &\bigl\langle \varphi'\bigl(tu_0^\pm\bigr), u_0^\pm \bigr\rangle < 0 \quad \text{if } t > 1. \end{aligned}$$

Hence we have that $\deg(\varphi_0, D, 0) = 1$. On the other hand, using (27) and property (a) of η, we have that $g = h$ on ∂D. Hence $\varphi_1 = \varphi_0$ on ∂D and $\deg(\varphi_1, D, 0) = \deg(\varphi_0, D, 0) = 1$. This show that $\varphi_1(s,t) = 0$ for some $(s,t) \in D$, and so $\eta(1, g(s,t)) = h(s,t) \in \mathbb{M}_0$. Therefore u_0 is a critical point of φ. □

4 Sign-changing solutions

For any $\lambda > 0$, let $f_\lambda(x,t) = f(x,t) + \lambda r|t|^{r-2}t$ and

$$\varphi_\lambda(u) = \varphi(u) - \lambda |u|_r^r, \quad u \in E.$$

Similarly, we define

$$\begin{aligned} \mathbb{M}_\lambda &:= \bigl\{ u \in E : u^\pm \neq 0, \bigl\langle \varphi_\lambda'(u), u^+ \bigr\rangle = \bigl\langle \varphi_\lambda'(u), u^- \bigr\rangle = 0 \bigr\}, \\ \mathbb{N}_\lambda &:= \bigl\{ u \in E : u \neq 0, \bigl\langle \varphi_\lambda'(u), u \bigr\rangle = 0 \bigr\}, \end{aligned}$$

and

$$m_\lambda := \inf_{u\in\mathbb{M}_\lambda} \varphi_\lambda(u), \quad n_\lambda := \inf_{u\in\mathbb{N}_\lambda} \varphi_\lambda(u).$$

Lemma 4.1 *Assume that (h_1)–(h_4) hold. Then there exists a constant $\alpha > 0$, which does not depend on $\lambda \in (0,1]$, such that*

$$\varphi_\lambda(u) \geq \alpha, \quad u \in \mathbb{N}_\lambda, \lambda \in (0,1].$$

Proof For any $\varepsilon > 0$, by (h_1), (h_2), and Propositions 2.1 and 2.2, for any $\lambda \in (0,1]$ and $u \in \mathbb{N}_\lambda$, we have

$$\begin{aligned} &\varepsilon c_p^p \|u\|^p + (C_\varepsilon + 1)c_r^r \|u\|^r \\ &\quad \geq \varepsilon |u|_p^p + (C_\varepsilon + 1)|u|_r^r \\ &\quad \geq \int_\Omega f_\lambda(x,u)u\,dx \\ &\quad = \int_\Omega \bigl(|\nabla u|^p + a(x)|\nabla u|^q\bigr)\,dx \\ &\quad \geq \begin{cases} \|u\|^q & \text{if } \|u\| < 1, \\ \|u\|^p & \text{if } \|u\| > 1. \end{cases} \end{aligned}$$

Thus for any $u \in \mathbb{N}_\lambda$ with $\|u\| < 1$, we have that

$$\frac{1}{2}\|u\|^q \le (C_\varepsilon + 1)c_r^r \|u\|^r,$$

which implies that

$$\|u\| \ge \left(\frac{1}{2(C_\varepsilon + 1)c_r^r}\right)^{\frac{1}{r-q}}.$$

The proof is completed. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Clearly, for every $\lambda > 0$, f_λ satisfies conditions (h_1)–(h_3) and $(h_4)'$, and Lemmas 3.6 and 3.7 imply that there exists $u_\lambda \in \mathbb{M}_\lambda$ such that

$$\varphi_\lambda(u_\lambda) = m_\lambda \quad \text{and} \quad \varphi'_\lambda(u_\lambda) = 0. \tag{28}$$

Furthermore, under assumptions (h_1)–(h_3), we easily obtain that $\mathbb{M}_0 \neq \emptyset$. Let $v_0 \in \mathbb{M}_0$. Then $\varphi(v_0) := \kappa > 0$ and $\langle \varphi'(v_0), v_0^{\pm} \rangle = 0$. Therefore by Lemma 3.4 there exist $s_\lambda > 0$ and $t_\lambda > 0$ such that $s_\lambda v_0^+ + t_\lambda v_0^- \in \mathbb{M}_\lambda$. Then from Corollary 3.2 and Lemma 4.1 we have

$$\begin{aligned} \kappa &= \varphi(v_0) \\ &\ge \varphi\left(s_\lambda v_0^+ + t_\lambda v_0^-\right) \\ &\ge \varphi_\lambda\left(s_\lambda v_0^+ + t_\lambda v_0^-\right) \\ &\ge m_\lambda \ge c_*, \quad \lambda \in (0,1). \end{aligned} \tag{29}$$

Hence, we can choose a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ as $n \to +\infty$ and

$$u_{\lambda_n} \in \mathbb{M}_{\lambda_n}, \qquad \varphi_{\lambda_n}(u_{\lambda_n}) = m_{\lambda_n} \to \overline{m}, \qquad \varphi'_{\lambda_n}(u_{\lambda_n}) = 0. \tag{30}$$

Thus we only need to prove the following claims to complete the proof of Theorem 1.1.

Claim 1 *$\{u_{\lambda_n}\}$ is bounded in E.*

Arguing by contradiction, suppose that $\|u_{\lambda_n}\| \to +\infty$ as $n \to +\infty$. We define the sequence $v_n = \frac{u_{\lambda_n}}{\|u_{\lambda_n}\|}$, $n = 1, 2, \dots$. It is clear that $\{v_n\} \subset E$ and $\|v_n\| = 1$ for any $n \in N$. Therefore, going if necessary to a subsequence, we may assume that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } E, \\ &v_n \to v \quad \text{in } L^{\vartheta}(\Omega), 1 \le \vartheta < p^*, \\ &v_n(x) \to v(x) \quad \text{a.e. on } \Omega. \end{aligned} \tag{31}$$

If $v = 0$, then $v_n \to 0$ in $L^{\vartheta}(\Omega)$ for $1 \le \vartheta < p^*$. Fix $R > [q(m_0 + 1)]^{\frac{1}{p}}$. Using conditions (h_1)–(h_2) and the Lebesgue dominated convergence theorem, we deduce that

$$\limsup_{n\to\infty} \int_\Omega F(x, Rv_n)\,dx \le R^p \lim_{n\to\infty} \left(|v_n|_p^p + C_3 R^r |v_n|_r^r\right) = 0 \tag{32}$$

for some constant $C_3 > 0$.

Let $t_n = \frac{R}{\|u_n\|}$. Then by (32) and Corollary 3.3 we get that

$$\begin{aligned} m_{\lambda_n} &= \varphi_{\lambda_n}(u_{\lambda_n}) \ge \varphi_{\lambda_n}(t_n u_{\lambda_n}) = \varphi_{\lambda_n}(Rv_n) \\ &= \int_\Omega \left(\frac{1}{p}R^p|\nabla v_n|^p + \frac{a(x)}{q}R^q|\nabla v_n|^q\right) dx \\ &\quad - \int_\Omega \left(F(x, Rv_n) + \lambda_n R^r |v_n|^r\right) dx \\ &\ge \frac{1}{q}R^p - \int_\Omega \left(F(x, Rv_n) + \lambda_n R^r |v_n|^r\right) dx \\ &= \frac{1}{q}R^p + o(1) > m_0 + 1 + o(1), \end{aligned}$$

which yields a contradiction. Thus $v \ne 0$.

By (h_3) we get

$$\lim_{k\to+\infty} \frac{F(x, u_{\lambda_n}(x))}{\|u_{\lambda_n}\|^q} = \lim_{k\to+\infty} \frac{F(x, u_{\lambda_n}(x))}{|u_{\lambda_n}(x)|^q}\left|v_n(x)\right|^q = +\infty$$

for all $x \in \Omega_0 := \{x \in \Omega : v(x) \ne 0\}$. Therefore, using (21), (30), and Fatou's lemma, we have

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \frac{\varphi_{\lambda_n}(u_{\lambda_n})}{\|u_{\lambda_n}\|^q} \\ &\le \lim_{n\to\infty}\left[\frac{1}{p}\frac{\int_\Omega (|\nabla u_{\lambda_n}|^p + a(x)|\nabla u_{\lambda_n}|^q)\,dx}{\|u_n\|^q} \right. \\ &\quad \left. - \int_\Omega \frac{F(x, u_{\lambda_n}) + \lambda_n |u_{\lambda_n}|^r}{\|u_{\lambda_n}\|^q}\,dx\right] \\ &\le \lim_{n\to\infty}\left[\frac{1}{p}\frac{\int_\Omega (|\nabla u_{\lambda_n}|^p + a(x)|\nabla u_{\lambda_n}|^q)\,dx}{\|u_{\lambda_n}\|^q} - \int_\Omega \frac{F(x, u_{\lambda_n})}{\|u_{\lambda_n}\|^q}\,dx\right] \\ &\le \frac{1}{p} - \lim_{n\to\infty}\int_\Omega \frac{F(x, u_{\lambda_n})}{\|u_{\lambda_n}\|^q}\,dx \\ &= \frac{1}{p} - \lim_{n\to\infty}\int_\Omega \frac{F(x, u_{\lambda_n}) - C_2}{\|u_{\lambda_n}\|^q}\,dx \\ &\le \frac{1}{p} - \liminf_{n\to\infty}\int_{\Omega_0} \frac{F(x, u_{\lambda_n}) - C_2}{\|u_{\lambda_n}\|^q}\,dx \\ &= \frac{1}{p} - \liminf_{n\to\infty}\int_{\Omega_0} \frac{F(x, u_{\lambda_n}(x))}{|u_{\lambda_n}(x)|^q}\left|v_n(x)\right|^q dx \\ &\to -\infty, \end{aligned} \tag{33}$$

which is contradiction. The proof of Claim 1 is complete. Thus there exist a subsequence of $\{\lambda_n\}$, still denoted by $\{\lambda_n\}$, and $u_0 \in E$ such that

$$u_{\lambda_n} \rightharpoonup u_0 \quad \text{in } E.$$

Claim 2 $\varphi(u_0) = m_0$ *and* $\varphi'(u_0) = 0$.

By the Sobolev embedding theorem, $u_{\lambda_n} \to u_0$ in $L^{\vartheta}(\Omega), 1 \le \vartheta < p^*$, and $u_{\lambda_n}(x) \to u_0(x)$ a.e. on Ω. By (h_2) and the Hölder inequality it is easy to directly compute that

$$\begin{aligned}
&\int_\Omega \big|f(x,u_{\lambda_n})\big||u_n - u_0|\,dx \\
&\quad\le \int_\Omega C\big(1 + |u_{\lambda_n}|^{r-1}\big)|u_n - u_0|\,dx \\
&\quad\le C\int_\Omega |u_n|^{r-1}|u_{\lambda_n} - u_0|\,dx + C\int_\Omega |u_n - u_0|\,dx \\
&\quad\le C\left(\int_\Omega |u_{\lambda_n}|^{(r-1)r'}\,dx\right)^{\frac{1}{r'}}\left(\int_\Omega |u_{\lambda_n} - u_0|^r\,dx\right)^{\frac{1}{r}} \\
&\qquad + C\int_\Omega |u_{\lambda_n} - u_0|\,dx \\
&\quad= C\left(\int_\Omega |u_{\lambda_n}|^r\,dx\right)^{\frac{r-1}{r}}\left(\int_\Omega |u_{\lambda_n} - u_0|^r\,dx\right)^{\frac{1}{r}} + C\int_\Omega |u_{\lambda_n} - u_0|\,dx \\
&\quad= C|u_{\lambda_n}|_r^{r-1}|u_{\lambda_n} - u_0|_r + C|u_{\lambda_n} - u_0|_1 \\
&\quad\to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{34}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Then, using (30), (34), and (h_2), we deduce

$$\begin{aligned}
\big\langle L(u_{\lambda_n}) - L(u_0), u_{\lambda_n} - u_0\big\rangle &= \big\langle \varphi'_{\lambda_n}(u_{\lambda_n}) - \varphi'(u_0), u_{\lambda_n} - u_0\big\rangle \\
&\quad + \int_\Omega \big[f(x,u_{\lambda_n}) + \lambda_n r|u_{\lambda_n}|^{r-2}u_{\lambda_n}\big](u_{\lambda_n} - u_0)\,dx \\
&\quad - \int_\Omega f(x,u_0)(u_{\lambda_n} - u_0)\,dx \\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

Since L is of type $(S)_+$, we see that

$$u_{\lambda_n} \to u_0 \quad \text{in } E, \tag{35}$$

and so $u^{\pm}_{\lambda_n} \to u^{\pm}_0$ in E. Thus from (30) it follows that $\varphi(u_0) = \overline{m}$.

Moreover, by Proposition 2.3, (30), and (35) we get

$$\begin{aligned}
\big\langle \varphi'(u_0), \eta\big\rangle &= \big\langle L(u_0), \eta\big\rangle - \int_\Omega f(x,u_0)\eta\,dx \\
&= \lim_{n\to+\infty}\left(\big\langle L(u_{\lambda_n}), \eta\big\rangle - \int_\Omega \big[f(x,u_{\lambda_n}) + \lambda_n r|u_{\lambda_n}|^{r-2}u_{\lambda_n}\big]\eta\,dx\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to+\infty}\langle \varphi'_{\lambda_n}(u_{\lambda_n}), \eta\rangle \\
&= 0, \quad \eta \in E.
\end{aligned} \tag{36}
$$

This shows that $\varphi'(u_0) = 0$. Again from Lemma 4.1 and (35) we have

$$
\begin{aligned}
&\int_\Omega \left[\frac{1}{q} f(x, u_0^\pm)u_0^\pm - F(x, u_0^\pm)\right] dx + \int_\Omega \left(\frac{1}{p} - \frac{1}{q}\right)|\nabla u_0^\pm|^p\, dx \\
&= \lim_{n\to+\infty}\int_\Omega \left[\frac{1}{q} f(x, u_{\lambda_n}^\pm)u_{\lambda_n}^\pm - F(x, u_{\lambda_n}^\pm)\right] dx \\
&\quad + \lim_{n\to+\infty}\int_\Omega \left(\frac{1}{p} - \frac{1}{q}\right)|\nabla u_{\lambda_n}^\pm|^p\, dx + \lim_{n\to+\infty}\frac{\lambda_n(r-q)}{q}|u_{\lambda_n}^\pm|_r^r \\
&= \lim_{n\to+\infty}\left[\varphi_{\lambda_n}(u_{\lambda_n}^\pm) - \frac{1}{q}\langle \varphi'_{\lambda_n}(u_{\lambda_n}^\pm), u_n^\pm\rangle\right] \\
&= \lim_{n\to+\infty}\varphi_{\lambda_n}(u_{\lambda_n}^\pm) \\
&\geq \alpha > 0.
\end{aligned} \tag{37}
$$

This, together with (6) ($t = 0$), shows that $u_0^\pm \neq 0$. Therefore

$$
\varphi'(u_0) = 0, \quad u_0 \in \mathbb{M}_0, \quad \text{and} \quad \varphi(u_0) = \overline{m} \geq m_0.
$$

Next, we will prove that $\varphi(u_0) = m_0$. Let ε be any positive number. Since $m_0 = \inf_{u\in\mathbb{M}_0}\varphi(u)$, there exists $v_\varepsilon \in \mathbb{M}_0$ such that $\varphi(v_\varepsilon) < m_0 + \varepsilon$. Then (h_3) implies that there exists $M_\varepsilon > 1$ such that, for $s \geq M_\varepsilon$ or $t \geq M_\varepsilon$,

$$
\begin{aligned}
\varphi_{\lambda_n}(sv_\varepsilon^+ + tv_\varepsilon^-) &= \int_\Omega \left(\frac{s^p}{p}|\nabla v_\varepsilon^+|^p + \frac{s^q}{q}|\nabla v_\varepsilon^+|^q\right) dx - \int_\Omega F(x, sv_\varepsilon^+)\, dx \\
&\quad - \lambda_n s^r \int_\Omega |v_\varepsilon^+|^r\, dx \\
&\quad + \int_\Omega \left(\frac{t^p}{p}|\nabla v_\varepsilon^-|^p + \frac{t^q}{q}|\nabla v_\varepsilon^-|^q\right) dx - \int_\Omega F(x, tv_\varepsilon^-)\, dx \\
&\quad - \lambda_n t^r \int_\Omega |v_\varepsilon^-|^r\, dx \\
&\leq \int_\Omega \left(\frac{s^p}{p}|\nabla v_\varepsilon^+|^p + \frac{s^q}{q}|\nabla v_\varepsilon^+|^q\right) dx - \int_\Omega F(x, sv_\varepsilon^+)\, dx \\
&\quad + \int_\Omega \left(\frac{t^p}{p}|\nabla v_\varepsilon^-|^p + \frac{t^q}{q}|\nabla v_\varepsilon^-|^q\right) dx - \int_\Omega F(x, tv_\varepsilon^-)\, dx \\
&< 0.
\end{aligned} \tag{38}
$$

In view of Lemma 3.4, there exists a pair (s_n, t_n) of positive numbers such that $s_n v_\varepsilon^+ + t_n v_\varepsilon^- \in \mathbb{M}_{\lambda_n}$, which, together with (38), implies $0 < s_n, t_n < M_\varepsilon$. Thus from Lemma 3.1 and $\langle \varphi'(v_\varepsilon), v_\varepsilon^\pm\rangle = 0$ we have

$$
\begin{aligned}
m_0 + \varepsilon &> \varphi(v_\varepsilon) = \varphi_{\lambda_n}(v_\varepsilon) + \lambda_n |v_\varepsilon|_r^r \\
&\geq \varphi_{\lambda_n}(s_n v_\varepsilon^+ + t_n v_\varepsilon^-) + \frac{1 - s_n^q}{q}\langle \varphi'_{\lambda_n}(v_\varepsilon), v_\varepsilon^+\rangle + \frac{1 - t_n^q}{q}\langle \varphi'_{\lambda_n}(v_\varepsilon), v_\varepsilon^-\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\int_\Omega g(s_n)\left|\nabla v_\varepsilon^+\right|^p dx+\int_\Omega g(t_n)\left|\nabla v_\varepsilon^-\right|^p dx\\
&\geq m_{\lambda_n}-\frac{1+K_\varepsilon^q}{q}\left|\left\langle\varphi'_{\lambda_n}(v_\varepsilon),v_\varepsilon^+\right\rangle\right|-\frac{1+K_\varepsilon^q}{q}\left|\left\langle\varphi'_{\lambda_n}(v_\varepsilon),v_\varepsilon^-\right\rangle\right|\\
&= m_{\lambda_n}-\frac{(1+K_\varepsilon^q)r\lambda_n}{q}\left|v_\varepsilon^+\right|_r^r-\frac{(1+K_\varepsilon^q)r\lambda_n}{q}\left|v_\varepsilon^-\right|_r^r,
\end{aligned}
$$

which yields

$$
\overline{m}=\lim_{n\to+\infty} m_{\lambda_n}\leq m_0+\varepsilon. \tag{39}
$$

Since $\varepsilon>0$ is arbitrary, we have $\overline{m}\leq m_0$. Thus $\overline{m}=m_0$, that is, $\varphi(u_0)=m_0$.

Now we show that u_0 has exactly two nodal domains. Let $u_0=u_1+u_2+u_3$, where

$$
\begin{aligned}
&u_1\geq 0,\qquad u_2\leq 0,\qquad \Omega_1\cap\Omega_2=\emptyset,\\
&u_1|_{\Omega\setminus(\Omega_1\cup\Omega_2)}=u_2|_{\Omega\setminus(\Omega_1\cup\Omega_2)}=u_3|_{\Omega_1\cup\Omega_2},\\
&\Omega_1:=\left\{x\in\Omega\,|\,u_1(x)>0\right\},\qquad \Omega_2:=\left\{x\in\Omega\,|\,u_1(x)<0\right\},
\end{aligned} \tag{40}
$$

and Ω_i $(i=1,2)$ are connected open subsets of Ω.

Setting $v=u_1+u_2$, we see that $v^+=u_1$ and $v^-=u_2$, that is, $v^\pm\neq 0$. Noting that $\varphi'(u_0)=0$, by a simple computation we have

$$
\left\langle\varphi'(v),v^+\right\rangle=\left\langle\varphi'(v),v^-\right\rangle=0. \tag{41}
$$

By Lemma 3.1 and again by (40) and (41) we conclude that

$$
\begin{aligned}
m_0 &= \varphi(u_0)=\varphi(u_0)-\frac{1}{q}\left\langle\varphi'(u_0),u_0\right\rangle\\
&= \varphi(v)+\varphi(u_3)-\frac{1}{q}\left(\left\langle\varphi'(v),v\right\rangle+\left\langle\varphi'(u_3),u_3\right\rangle\right)\\
&\geq \sup_{s,t\geq 0}\biggl[\varphi\left(sv^++tv^-\right)+\frac{1-s^q}{q}\left\langle\varphi'(v),v^+\right\rangle+\frac{1-t^q}{q}\left\langle\varphi'(v),v^-\right\rangle\\
&\quad+\int_\Omega g(s)\left|\nabla v^+\right|^p dx+\int_\Omega g(t)\left|\nabla v^-\right|^p dx\biggr]+\varphi(u_3)-\frac{1}{q}\left\langle\varphi'(u_3),u_3\right\rangle\\
&\geq \sup_{s,t\geq 0}\varphi\left(sv^++tv^-\right)+\varphi(u_3)-\frac{1}{q}\left\langle\varphi'(u_3),u_3\right\rangle\\
&\geq m_0+\int_\Omega\left(\frac{1}{q}f(x,u_3)u_3-F(x,u_3)\right)dx+\left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega|\nabla u_3|^p dx\\
&\geq m_0+\int_\Omega\left(\frac{1}{q}f(x,u_3)u_3-F(x,u_3)\right)dx,
\end{aligned}
$$

which, together with (3), shows that $u_3=0$. Therefore u_0 has exactly two nodal domains. □

Proof of Theorem 1.2 By Theorem 1.1 there exists $u_0\in\mathbb{M}_0$ such that $\varphi(u_0)=m_0$. Since $u_0^\pm\in\mathbb{N}_0$, we have $m_0=\varphi(u_0)=\varphi(u_0^+)+\varphi(u_0^-)\geq 2n_0$. □

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